Letter

Nonlinear shallow water dynamics with odd viscosity

Gustavo M. Monteiro[1] and Sriram Ganeshan[1,2]
[1]*Department of Physics, City College, City University of New York, New York, New York 10031, USA*
[2]*CUNY Graduate Center, New York, New York 10031, USA*

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In this Letter, we derive the Korteweg–de Vries (KdV) equation corresponding to the surface dynamics of a shallow depth (h) two-dimensional fluid with odd viscosity (ν_o) subject to gravity (g) in the long-wavelength weakly nonlinear limit. In the long-wavelength limit, the odd viscosity term plays the role of surface tension albeit with opposite signs for the right and left movers. We show that there exist two regimes with a sharp transition point within the applicability of the KdV dynamics, which we refer to as weak ($|\nu_o| < \sqrt{gh^3}/6$) and strong ($|\nu_o| > \sqrt{gh^3}/6$) parity-breaking regimes. While the "weak" parity-breaking regime results in minor qualitative differences in the soliton amplitude and velocity between the right and left movers, the "strong" parity-breaking regime on the contrary results in solitons of depression (negative amplitude) in one of the chiral sectors.

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Introduction. In conventional fluids, viscosity is often associated with dissipation. However, there exist viscosity coefficients that perform no work, i.e., the internal forces are transverse to the fluid motion, as shown in Fig. 1. Because of that, these nondissipative viscosity coefficients cannot be invariant under parity symmetry, only showing up in chiral fluids [1]. Parity-breaking phenomena in two-dimensional fluids such as odd viscosity effects have been at the center of investigation in diverse platforms. Examples of quantum systems where odd viscous effects are important include electron fluids in mesoscopic systems [2–4], quantum Hall fluids [5–26], and chiral superfluids and superconductors [27].

In classical fluids, odd viscosity shows up in polyatomic gases [28–31], chiral active matter [32–34], vortex dynamics in two dimensions (2D) [35–38], and chiral active fluids [32–34]. For incompressible flows, it has been shown by one of the authors that odd viscosity effects are absent when the fluid is spread on the entire plane or confined in rigid domains with no-slip boundary conditions [39]. In other words, the velocity profile is independent of the odd viscosity. Nevertheless, the signature of this parity-breaking coefficient is present in surface waves and in the interface between two fluids governed by kinematic and no-stress boundary conditions, which explicitly depends on the odd viscosity. The dynamical surface problem in the presence of odd viscosity results in an oscillating boundary layer where the vorticity is confined within some thickness of $\delta \propto \sqrt{\nu_e}$ [40,41] (where ν_e is the kinematic shear viscosity) for the dissipative case and $\delta \propto c_s^{-1}$ (where c_s is the sound velocity) for the nondissipative compressible case [42]. In the limit of a very thin boundary layer, that is, $\nu_e \to 0$ or $c_s \to \infty$, both the fluid pressure at the edge and the surface vorticity diverge as $1/\sqrt{\nu_e}$ or c_s, but the quantity $\tilde{p} = p - \nu_o \rho\, \omega$ remains finite. We refer to $\tilde{p}$ as modified pressure, where ν_o is the odd viscosity and the variables p, ω, and ρ are the fluid pressure, vorticity, and constant background density, respectively. This cancellation of divergences allows us to write the dynamical surface problem with odd viscosity as an effective irrotational system where all the effects of odd viscosity and the boundary layer can be absorbed into a modified pressure term at the edge. In short, for an irrotational flow, that is, $\boldsymbol{v} = \boldsymbol{\nabla}\theta$, the effective boundary dynamics can be

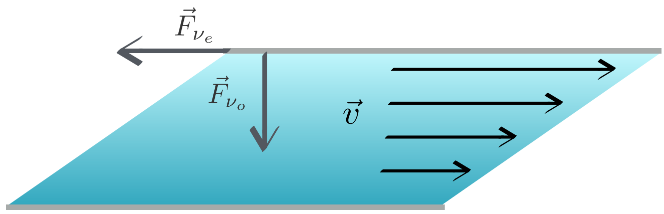

FIG. 1. A shearing flow $\vec{v}$ is subjected to a dissipative force opposite to the direction of the fluid motion due to shear viscosity whereas the odd viscosity ν_o generates an in-plane force normal to the fluid motion. The normal force to the motion is nondissipative.

expressed as a Laplace equation for the velocity potential θ in the bulk and Bernoulli's equation at the boundary with the modified pressure [40]. The variational principle for this boundary dynamics was later formulated in terms of a geometric action which resulted in the odd-viscosity-induced effective pressure at the boundary [41]. In the limit of infinitely deep fluid the weakly nonlinear dynamics within a small angle approximation was shown to be governed by the novel *chiral-Burgers* equation [40].

In this Letter, we study the shallow depth limit of the weakly nonlinear surface dynamics with odd viscosity and gravitational force (confining potential) (see the schematic in Fig. 2). We assume that the boundary layer is the shortest length scale and the effective dynamics is irrotational, using the hydrodynamic equations from Ref. [41] as the starting point. We show that, for later times and long wavelengths, the weakly nonlinear dynamics is given by the integrable Kortweg–de Vries (KdV) equation with the kinematic odd viscosity ν_o entering the coefficient of the dispersive term,

$$\pm\eta_t + \sqrt{gh}\,\eta_x + \frac{3}{2}\sqrt{\frac{g}{h}}\eta\,\eta_x + \sqrt{gh^5}\left(\frac{1}{6} \pm \frac{\nu_o}{\sqrt{gh^3}}\right)\eta_{xxx} = 0. \tag{1}$$

Here, $\eta(x,t)$ is the boundary shape profile, h is the average depth of the fluid, g is the acceleration of gravity, and the subscripts t and x refer to partial derivatives with respect to such variables. The positive sign corresponds to right-movers whereas the negative sign refers to left-movers. The manifestation of odd viscosity in the above equation is similar to that of the surface tension although with different signs for the left and right movers.

The odd viscosity entering the KdV equation has major consequences due to its parity-breaking effects. We show that there exist two regimes with a sharp transition point within the applicability of the KdV dynamics, which we refer to as weak ($|\nu_o| < \sqrt{gh^3}/6$) and strong ($|\nu_o| > \sqrt{gh^3}/6$) parity-breaking regimes. In the weak parity-breaking regime, the left- and right-moving solitons slightly

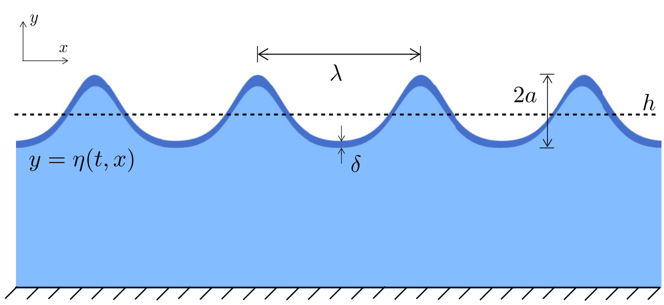

FIG. 2. Schematics of the shallow fluid dynamics with free surface. Here, a denotes the amplitude of the edge profile and δ is the boundary layer thickness. The vertical direction is exaggerated to highlight these features.

differ in amplitude and speed. In the strong parity-breaking regime, one of the sectors becomes solitonic waves of depression. At the critical value ($|\nu_o| = \sqrt{gh^3}/6$), one of the sectors becomes unstable and higher-order derivative terms become important. The parity-breaking KdV dynamics discussed here is in stark contrast to the parity-preserving case of shallow water KdV dynamics without odd viscosity.

Incompressible fluids with odd viscosity. The hydrodynamic equations for incompressible fluids with odd viscosity consist of the Newton's second law, together with the incompressibility condition, that is,

$$\partial_t v_i + v_j \partial_j v_i = \frac{1}{\rho} \partial_j T_{ji} - \partial_i (gy), \tag{2}$$

$$\partial_i v_i = 0. \tag{3}$$

Here, v_i are the components of the flow velocity, ρ is the constant and uniform fluid density, and the summation over repeated indices is assumed ($i, j = 1, 2$). The term gy is the external gravitational potential and, to have a closed system of equations, one needs to define a constitutive relation expressing the stress tensor T_{ij} in terms of the velocity of the fluid. For a fluid with only odd viscosity, we have

$$T_{ij} = -p\delta_{ij} + \nu_o \rho \, (\partial_i^* v_j + \partial_i v_j^*). \tag{4}$$

The first term of the stress tensor (4) is the standard isotropic pressure term. The second term in Eq. (4) is the odd viscosity term. The coefficient ν_o is known as kinematic odd viscosity (or Hall viscosity). In writing this term we introduced the notation $a_i^* \equiv \epsilon_{ij} a_j$ so that the "starred" vector $\mathbf{a}^*$ is just a vector $\mathbf{a}$ rotated by 90° clockwise.

Under the incompressibility condition (3), we have $\Delta v_i^* = \partial_i \omega$ and using this identity Eq. (2) becomes

$$\partial_t v_i + v_j \partial_j v_i = -\partial_i \left(\frac{\tilde{p}}{\rho} + gy \right). \tag{5}$$

This differs from the ordinary Euler equation in the definition of the modified pressure [1,39]

$$\tilde{p} := p - \nu_o \rho \, \omega, \tag{6}$$

where $\omega = \partial_i v_i^*$ is the fluid vorticity. From Eqs. (3) and (5), one can see that the fluid energy density ($\frac{1}{2} v_i^2$) is conserved for any real value of ν_o.

Since the fluid density is constant, there is no equation of state and the pressure is completely determined by the flow. The curl of the Euler equation is the equation for the flow vorticity, which does not depend on the pressure. This vorticity equation together with the incompressibility condition (3) completely determines the fluid flow, up to boundary conditions. Therefore, the presence of odd viscosity will only change the velocity flow if the boundary conditions depend on ν_o, otherwise, it only modifies how the pressure depends on velocity flow, which is not an easily accessible quantity in experiments [39].

Irrotational limit of the free surface dynamics and boundary layer approximation. Bulk equations of motion (2) must be supplemented with boundary conditions. The fluid free surface is a dynamical interface $y = \eta(t, x)$ between two fluids where we impose one kinematic and two dynamical boundary conditions. The kinematic boundary condition states that the velocity of the fluid normal to the boundary is equal to the rate of change of the boundary shape. The pair of dynamical boundary conditions imposes that there are no normal and tangent forces acting on an element of the fluid surface. Hence, they can be rewritten as

$$\partial_t \eta = (v_y - v_x \partial_x \eta)|_{y=\eta(t,x)}, \tag{7}$$

$$n_i T_{ij}|_{y=\eta(t,x)} = 0, \tag{8}$$

where n_i are the components of the normal vector to the surface $y = \eta(t, x)$ and the stress tensor is given by Eq. (4).

The presence of two dynamical boundary conditions (DBCs) along with incompressibility requires a singular boundary layer where the vorticity is confined. The role of this singular boundary layer is to ensure that there are no tangent forces on this interface. It has been shown that regardless of the boundary layer mechanism (dissipative or compressible), the normal component of the DBC is universal and geometric in nature [40–42]. Assuming that the boundary layer is stable and is confined to short length scales, the free-surface problem can be written as an effective description of the fluid with the effects of boundary layer encoded in the odd viscosity modified pressure term,

$$\tilde{p}|_{y=\eta(t,x)} = \frac{2\nu_o\rho}{\sqrt{1+(\partial_x\eta)^2}}\partial_x v_n. \tag{9}$$

Here, v_n is the velocity component which is normal to the boundary, taken at $y = \eta(t, x)$.

This effective free-surface dynamics, that is, Eqs. (2), (3), and (9), can be expressed in the form of an action principle, as shown in Ref. [41]. For irrotational flows, i.e., $\boldsymbol{v} = \boldsymbol{\nabla}\theta$, this action simplifies and can be thought of as an odd viscosity extension of the Luke's variational principle [43]. Following Ref. [41], the hydrodynamic action becomes

$$\begin{aligned} S = &-\iint dt\,dx \int_{-h}^{\eta(t,x)} dy\left[\theta_t + \frac{1}{2}\left(\theta_x^2+\theta_y^2\right)\right] \\ &-\iint dt\,dx\left[\frac{1}{2}g\eta^2 - \nu_o\,\eta_t\tan^{-1}(\eta_x)\right], \end{aligned} \tag{10}$$

where $\eta(t, x)$ is the top surface shape function. Here and in the following, the subscripts t, x, and y refer to partial derivatives with respect to such variables. The domain of the fluid is bounded by a finite depth at the bottom. The bulk and boundary hydrodynamic equations are the equations of motion for the action (10), with respect to the variables θ and η (for more details, check Ref. [41]). The bulk equation for the irrotational system is simply the Laplace equation for the potential $\Delta\theta = 0$ defined in the domain $-h < y < \eta(t, x)$. The boundary conditions at the top and bottom of the fluid domain can be written as

$$\theta_y = 0, \quad y = -h, \tag{11}$$

$$\eta_t + \eta_x\theta_x = \theta_y, \quad y = \eta(t,x), \tag{12}$$

$$\theta_t + \frac{\theta_x^2+\theta_y^2}{2} + g\eta = \frac{2\nu_o}{\sqrt{1+\eta_x^2}}\,\partial_x\left[\frac{\eta_t}{\sqrt{1+\eta_x^2}}\right], \quad y = \eta(t,x). \tag{13}$$

Here, Eq. (12) is the kinematic boundary condition and Eq. (13) is the odd viscosity modified dynamic boundary condition on the top surface, whereas Eq. (11) simply states that $v_n = 0$ at the bottom surface. Note that the boundary condition at the bottom surface is a slip boundary condition.

Linear waves in the long-wavelength limit. Before we dive into the nonlinear dynamics of the shallow fluid regime, let us focus on the linearized free-surface problem with finite depth. In this limit, we drop all the quadratic terms in Eqs. (12) and (13) and evaluate the derivatives of θ at $y = 0$. For the monochromatic surface profile $\eta(x, t) = a\cos(kx - \Omega t)$ as an input, we find the velocity potential to be

$$\theta(x,y,t) = \frac{a\,\Omega}{k\sinh(kh)}\cosh[k(y+h)]\sin(kx-\Omega t),$$

with the surface dispersion relation determined by Eq. (13),

$$\Omega = \tanh kh\left[-\nu_o k^2 \pm \sqrt{\nu_o^2k^4 + gk\coth(kh)}\right]. \tag{14}$$

In the absence of gravity ($g = 0$) and in the deep ocean limit ($h \to \infty$), we have that $\tanh kh \approx k/|k|$ and we recover the known odd-viscosity-dominated dispersion $\Omega = \{0, -2\nu_o k|k|\}$. The weakly nonlinear dynamics for this system was discussed in Ref. [40].

Shallow waves, on the other hand, arise when the fluid depth h is much smaller than the characteristic wavelengths of the system. In other words, they are characterized by $kh \ll 1$. In this approximation, the leading terms in the dispersion (14) are given by

$$\Omega \approx \sqrt{\frac{g}{h}}\left[\pm kh - \left(\frac{\nu_o}{\sqrt{gh^3}} \pm \frac{1}{6}\right)(kh)^3\right]. \tag{15}$$

The first term is the usual shallow water gravity wave dispersion, whereas the second one is the odd viscosity contribution to the long-wavelength dispersion relation. *Prima facie* it seems that the odd viscosity is qualitatively similar to the surface tension effect for the shallow water surface dispersion. However, the physical manifestation of odd viscosity is completely different, since the coefficient of the cubic term can develop a relative sign change between the left mover and right mover for $|\nu_o| > \frac{1}{6}\sqrt{gh^3}$, what indicates a strong parity-breaking phenomena.

The shallow wave condition naturally introduces a power expansion in kh. Formally, it is convenient to define an expansion parameter $\varepsilon \ll 1$, such that $kh = \sqrt{\varepsilon}\,\bar{k}$ and $\bar{k}$ is a dimensionless wave number. Since an expansion in powers of kh can be translated into a derivative expansion for the fluid dynamics, this rescaling is equivalent to the redefinition $x = \frac{h}{\sqrt{\varepsilon}}X$, where X is the dimensionless horizontal coordinate. In the same way, we can define $y = hY$, with Y being the dimensionless vertical coordinate. In this counting scheme, we have that $\partial_x \sim O(\varepsilon^{1/2})$, whereas $\partial_y \sim O(1)$.

The wave dynamics dictated by Eq. (15) evolves according to two distinct timescales. The linear term in the dispersion relation scales with $\sqrt{\varepsilon}$ and governs the splitting of an initial disturbance into right-moving and left-moving wave packets. The first timescale, which we denote by T, is the characteristic time in which left and right movers are so far apart, we can study them separately. In other words, for sufficiently later times, the only role of T is to account for the boost of the center of mass and it only appears in the combination $X - T$, for right movers, or $X + T$, for left movers. On the other hand, the cubic term scales as $\varepsilon^{3/2}$ and gives rise to a dispersive group velocity of the boosted wave packet. This effect becomes relevant at much later times, in comparison to T, and introduces a second timescale, which governs the time evolution of the boosted wave packet and which we denote by τ. This means that both variables θ and η evolve according to this double timescale, such that, the time derivative becomes

$$\partial_t = \sqrt{\frac{g\varepsilon}{h}}\partial_T + \sqrt{\frac{g\varepsilon^3}{h}}\partial_\tau.$$

In the following, we derive the full nonlinear shallow water dynamics with odd viscosity and discuss how the parity-breaking effects manifest in the nonlinear dynamics. In particular we show that $\nu_o = \frac{1}{6}\sqrt{gh^3}$ manifests as a critical point that separates two qualitatively different regimes of nonlinear dynamics.

Nonlinear shallow depth waves. The Korteweg–de Vries (KdV) equation arises in the study of shallow water waves of long wavelengths and small amplitudes. In the following analysis, we show how the KdV equation corresponding to the shallow depth limit is modified by the presence of the odd viscosity term. The counting scheme for the KdV equation is chosen such that the small amplitude regime corresponds to $\eta \sim O(\varepsilon)$ [44], that is, η is of the same order as ∂_x^2. Thus, we can rescale the boundary shape in terms of h as $\eta = \varepsilon h \mathfrak{y}$.

The KdV regime happens for sufficiently later times, so that right-moving and left-moving solutions are independent and well separated. Here, we restrict ourselves to only right-moving propagation, since the analysis for the left movers follows similarly. Hence, let us assume θ and η of the form

$$\theta(t,x,y) = \sqrt{\varepsilon h^3 g}\,\vartheta(\tau,\sigma,Y;\varepsilon), \tag{16}$$

$$\eta(t,x) = \varepsilon h\,\mathfrak{y}(\tau,\sigma;\varepsilon), \tag{17}$$

with $\sigma = X - T$. Under these conditions, the bulk equation of motion and the boundary condition at the flat bottom become

$$\varepsilon\,\vartheta_{\sigma\sigma} + \vartheta_{YY} = 0, \quad -1 < Y < \varepsilon\mathfrak{y}, \tag{18}$$

$$\vartheta_Y = 0, \quad Y = -1. \tag{19}$$

Let us denote $\vartheta(\tau,\sigma,-1;\varepsilon)$ by $\phi(\tau,\sigma;\varepsilon)$. This way, the solution of Eq. (18) with the condition (19) can be written as

$$\vartheta(\tau,\sigma,Y;\varepsilon) = \sum_{n=0}^{\infty} \frac{(-\varepsilon)^n(1+Y)^{2n}}{(2n)!}\partial_\sigma^{2n}\phi. \tag{20}$$

Plugging Eq. (20) into Eqs. (12) and (13) and neglecting terms of $O(\varepsilon^2)$ or higher, we obtain

$$\phi_\sigma = \mathfrak{y} + \varepsilon\left(\phi_\tau + \tfrac{1}{2}\phi_\sigma^2 + \tfrac{1}{2}\phi_{\sigma\sigma\sigma} + 2\bar{\nu}_o\mathfrak{y}_{\sigma\sigma}\right), \tag{21}$$

$$\mathfrak{y}_\sigma - \phi_{\sigma\sigma} = \varepsilon\left[\tfrac{1}{2}\mathfrak{y}_{\sigma\sigma\sigma} - \mathfrak{y}_\tau + \tfrac{2}{3}\phi_{\sigma\sigma\sigma\sigma} + \partial_\sigma(\phi_\sigma\mathfrak{y})\right]. \tag{22}$$

Here, we denoted $\bar{\nu}_o = \nu_o/\sqrt{gh^3}$. Equation (21) allow us to perturbatively express ϕ_σ in terms of $\mathfrak{y}$. Substituting this expression into Eq. (22), the leading order equation for the right-moving surface wave in the boosted reference frame becomes

$$\mathfrak{y}_\tau + \tfrac{3}{2}\mathfrak{y}\,\mathfrak{y}_\sigma + \left(\tfrac{1}{6} + \bar{\nu}_o\right)\mathfrak{y}_{\sigma\sigma\sigma} = 0, \tag{23}$$

which is simply the well-known KdV equation. In terms of the dimensionful variables t, x, and $\eta(t,x)$, it becomes Eq. (1).

As previously mentioned, we could repeat the same analysis for the left-moving solitons. For $\xi = X + T$, we obtain

$$\mathfrak{y}_\tau - \tfrac{3}{2}\mathfrak{y}\,\mathfrak{y}_\xi - \left(\tfrac{1}{6} - \bar{\nu}_o\right)\mathfrak{y}_{\xi\xi\xi} = 0. \tag{24}$$

Under reflection about the y axis (parity operation), $\mathfrak{y} \to \mathfrak{y}$, $\xi \to -\sigma$, and Eq. (24) becomes

$$\mathfrak{y}_\tau + \tfrac{3}{2}\mathfrak{y}\,\mathfrak{y}_\sigma + \left(\tfrac{1}{6} - \bar{\nu}_o\right)\mathfrak{y}_{\sigma\sigma\sigma} = 0. \tag{25}$$

The odd viscosity term breaks parity symmetry of the problem [45], since the left-moving soliton under reflection about the y axis does not behave as the right-moving soliton. The odd viscosity term entering the KdV equation is similar to the presence of surface tension [46]. Within this analogy of odd viscosity as the surface tension, the left mover and right mover will have opposite signs of surface tension due to the parity-breaking effects of odd viscosity. In other words, odd viscosity in the KdV regime acts as a chirality-dependent surface tension term.

Soliton solution. In the following we analyze the role of odd viscosity in the single soliton solution of Eqs. (23) and (24). Although multisoliton solutions also show the same qualitative behavior, they will not be discussed in this Letter. The single soliton solution corresponding to the left and right movers can be written as

$$\mathfrak{y}(\tau,x_\pm) = 8\bar{k}^2\left(\tfrac{1}{6} \mp \bar{\nu}_o\right)\text{sech}^2\left[\bar{k}x_\pm \pm 4\bar{k}^3\tau\left(\tfrac{1}{6} \mp \bar{\nu}_o\right)\right], \tag{26}$$

where $\bar{k}$ is the dimensionless wave number and we denoted $x_+ = \xi$ and $x_- = \sigma$ in order to shorten the notation. Moreover, $\tau = 0$ was chosen such that the soliton center of mass is at $x_\pm = 0$. Note that

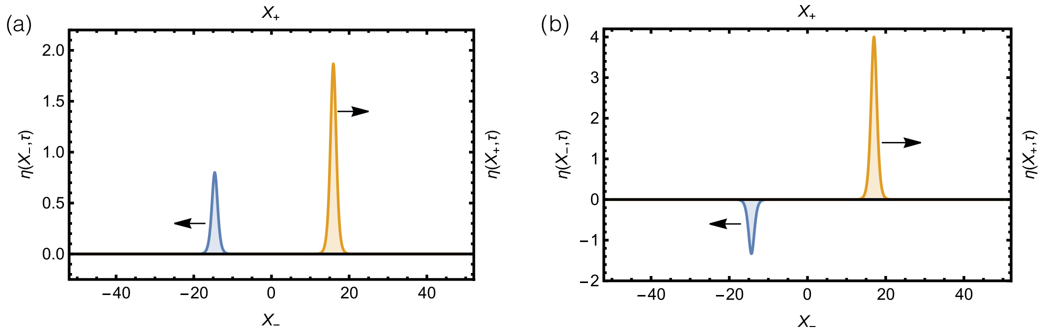

FIG. 3. (a) Left- (blue) and right- (yellow) moving soliton in the weak parity-breaking regime with parameters $|\bar{\nu}_o| = \frac{1}{15}$, $\bar{k} = 1$, and $\tau = 1$. (b) Left- (blue) and right- (yellow) moving soliton in the strong parity-breaking regime with parameters $|\bar{\nu}_o| = \frac{1}{3}$, $\bar{k} = 1$, and $\tau = 1$.

$(\frac{1}{6} \mp \bar{\nu}_o)$ enters both the amplitude and the wave speed. Therefore, the odd viscosity modification to the KdV soliton dynamics can be separated into three regimes depending on the value of $(\frac{1}{6} \mp \bar{\nu}_o)$.

"*Weak*" *parity-breaking regime* ($|\bar{\nu}_o| < \frac{1}{6}$): In this case, $(\frac{1}{6} \mp \bar{\nu}_o) > 0$, with left- and right-moving solitons, only differ in the magnitude of the amplitude and velocity as shown in Fig. 3. We refer to this as the *weak parity-breaking regime*.

"*Strong*" *parity-breaking regime* ($|\bar{\nu}_o| > \frac{1}{6}$): In this case, $(\frac{1}{6} \mp \bar{\nu}_o)$ have opposite signs. We call this the *strong parity-breaking regime*, because the difference between the left- and right-moving solitons is more striking, that is, one sector has positive amplitude, whereas the other corresponds to solitonic waves of depression or depletion as shown in Fig. 3.

"*Critical*" *dynamics* ($|\bar{\nu}_o| = \frac{1}{6}$): At these critical points, the dispersive term in one of the sectors vanishes and we end up with the inviscid Burger's equation for such a sector. In fact, it is known that solutions of the inviscid Burger's equation are subjected to a blow-up time, in which the spatial derivative of η becomes infinite and higher-order derivative terms become important. For this particular case, the scaling presented here breaks down for one of the sectors and terms with $\eta_{\sigma\sigma\sigma\sigma\sigma}$ or $\eta_{\xi\xi\xi\xi\xi}$ become necessary to avoid the gradient catastrophe [44].

Discussion and outlook. In this Letter, we derived the parity-broken generalization of the Korteweg–de Vries equation for a shallow depth fluid with odd viscosity and subjected to gravity in the long-wavelength weakly nonlinear limit. The presence of odd viscosity manifests weak and strong parity-breaking regimes in the two chiral sectors of the KdV dynamics. The odd viscosity term plays the role of surface tension albeit with opposite signs for the right and left movers. For a fluid with surface tension ($\mathcal{T}$) and no odd viscosity, the coefficients $+\frac{\nu_o}{\sqrt{gh^3}}$ for right movers and $-\frac{\nu_o}{\sqrt{gh^3}}$ for left movers in Eq. (1) are replaced by the same surface tension term $-\frac{\mathcal{T}}{2\rho g h^2}$ [46]. In future work, we aim to specialize this result to chiral active fluids, where odd viscous effects have been observed in free-surface dynamics [34]. In order to make contact with experiments, we will numerically study the Cauchy initial value problem of an initial perturbation that evolves into left- and right-moving solitons and quantify conditions under which weak and strong parity-breaking KdV dynamics emerges.

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