Letter

Symmetry analysis of the turbulent dissipation rate

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A core attribute of any turbulent flow is the rate at which it dissipates energy ε. In his classic study from 1935, Taylor invoked rotational symmetry to transform the original cumbersome expression for ε into a remarkably simple formula but for which it would be practically impossible to compute ε in most experiments. Taylor's analysis, though ingenious, leaves it unclear if the formula truly conforms with rotational symmetry. We use the rigorous approach of Lie groups and show that Taylor's formula indeed holds for rotational symmetry. Further, we find that the formula is surprisingly robust—it holds, as is, for a distinctly different symmetry: reflectional symmetry. Additionally, we highlight that the widely used tests for identifying flow symmetries can yield misleading results. With rigor, precision, and clarity, the machinery of Lie groups delineates the underlying symmetries that dictate turbulent flows.

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I. INTRODUCTION

The governing equations of turbulent flows, the Navier-Stokes equations, are a set of coupled, nonlinear, partial differential equations. As a result, mathematical elucidation of turbulent flows remains a famously long-standing unsolved problem in classical physics. This challenge is also reflected in the fact that proving the existence and uniqueness of solutions of the Navier-Stokes equations is one of the (as yet unsolved) Clay millennium problems [1]. At the same time, most flows around us, in nature and in technology, are turbulent flows. Applications abound.

Amid these exacting circumstances, progress in turbulence research has been driven in no small measure by semiempirical theories, phenomenological theories, and idealized theories [2,3]. Of crucial import in such theories is a cardinal attribute of turbulent flows: the turbulent dissipation rate (meaning, the mean dissipation rate of the turbulent kinetic energy per unit mass) $\langle\varepsilon\rangle$, where $\langle\,\rangle$ denotes an ensemble average. For example, $\langle\varepsilon\rangle$ underpins the celebrated scaling laws of Kolmogorov, such as the two-thirds law [4] and the four-fifths law [5] (one of the very few exact relations in turbulence), as well as modern research on intermittency and anomalous scaling [6,7].

Starting from the incompressible Navier-Stokes equations, we get

$$\varepsilon = \nu(\nabla\boldsymbol{u} : \nabla\boldsymbol{u} + \nabla\boldsymbol{u} : \nabla\boldsymbol{u}^{\mathrm{T}}), \tag{1}$$

where ν is the kinematic viscosity and $\boldsymbol{u}$ is the velocity field. Taking an ensemble average and casting the tensor components in index notation (with the Einstein summation convention for repeated

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indices) yields

$$\langle\varepsilon\rangle = \nu\left(\left\langle\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}\right\rangle + \left\langle\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}\right\rangle\right). \tag{2}$$

It is clear that 12 gradient moments are needed in order to determine $\langle\varepsilon\rangle$. This is a formidable task, particularly in experiments. Typically, to determine the value of $\langle\varepsilon\rangle$ at any spatial point, all three velocity components must be measured at different locations (spaced in the three coordinate axis directions) in close vicinity of that point. These would yield the velocity gradients, from which the gradient moments must be computed and averaged over a long time (for a statistically stationary flow).

In 1935, Taylor [8] showed that under certain conditions (which we discuss shortly) the cumbersome Eq. (2) can be expressed as

$$\langle\varepsilon\rangle = 15\nu\langle(\partial u_1/\partial x_1)^2\rangle. \tag{3}$$

The extraordinary simplification is writ large. One needs to measure only one derivative of one velocity component. The practically infeasible task of measuring $\langle\varepsilon\rangle$ now becomes a relatively straightforward one. In other words, Taylor's formula makes it possible to put to the empirical test the predictions from the different theories of turbulent flows. In this Letter, we train our focus on the key ideas that underly this remarkable formula.

II. TAYLOR'S FORMULA: THE 1935 DERIVATION

At the heart of Taylor's analysis resides symmetry. Focusing on incompressible flows, he introduced a crucial symmetry: The statistical properties are invariant to arbitrary rotations of the reference frame. The introduction of this rotational symmetry marked the beginning of studies on "isotropic turbulence" [2,9], the bedrock upon which the modern edifice of turbulence research is built.

Next, we outline how Taylor derived Eq. (3). For later reference, we recast the derivation in the notation of linear transformations and group theory. To account for rotational symmetry, he performed three successive finite rotations, $\varphi = \{\pi/2, \pi/2, \pi/4\}$, about different rotation axes, yielding the following transformations of the velocity gradients,

$$\begin{aligned}
\nabla'\boldsymbol{u}' &= \mathrm{R}^{\mathrm{T}}\left(\frac{\pi}{2}, \mathbf{e}_3\right)\cdot\nabla\boldsymbol{u}\cdot\mathrm{R}\left(\frac{\pi}{2}, \mathbf{e}_3\right),\\
\nabla'\boldsymbol{u}' &= \mathrm{R}^{\mathrm{T}}\left(\frac{\pi}{2}, \mathbf{e}_1\right)\cdot\nabla\boldsymbol{u}\cdot\mathrm{R}\left(\frac{\pi}{2}, \mathbf{e}_1\right),\\
\nabla'\boldsymbol{u}' &= \mathrm{R}^{\mathrm{T}}\left(\frac{\pi}{4}, \mathbf{e}_3\right)\cdot\nabla\boldsymbol{u}\cdot\mathrm{R}\left(\frac{\pi}{4}, \mathbf{e}_3\right),
\end{aligned} \tag{4}$$

where $\mathbf{e}_i$ is the unit vector of the coordinate axis i, R is the rotation matrix (where its arguments specify the rotation angle and the rotation axis), and the primed variables indicate transformed variables. Using the above transformations, he then proceeded to explicitly compute and individually enforce the finite rotation invariance condition, namely

$$\left\langle\left(\frac{\partial u'_j}{\partial x'_i}\right)^2\right\rangle = \left\langle\left(\frac{\partial u_j}{\partial x_i}\right)^2\right\rangle, \quad i, j = 1, 2, 3, \tag{5}$$

with no summation over indices, leading to a set of 45 linearly independent equations to be solved.

Taylor then introduced another symmetry by assuming that the turbulence is statistically homogeneous, i.e., invariant to spatial translation. Starting with the incompressibility constraint,

$$\nabla\cdot\boldsymbol{u} = 0, \tag{6}$$

homogeneity yields the condition

$$\langle \nabla \boldsymbol{u} : \nabla \boldsymbol{u}^{\mathrm{T}} \rangle = 0. \tag{7}$$

Using Eq. (7) with Eq. (2) reduces the number of derivative moments in Eq. (2) from 12 to nine,

$$\langle \varepsilon \rangle = \nu \left\langle \left(\frac{\partial u_i}{\partial x_j} \right)^2 \right\rangle \equiv \nu \langle \omega^2 \rangle, \tag{8}$$

where ω is the magnitude of the vorticity. (In Taylor's derivation, considerations of homogeneity appear in a different form [10].)

By solving the two constraints Eqs. (6) and (7) together with the nonlinear system Eq. (5), after some lengthy calculations Taylor obtained the following conditions for statistical isotropy:

$$\begin{aligned} \left\langle \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i} \right\rangle &= -\frac{1}{2} \left\langle \left(\frac{\partial u_1}{\partial x_1} \right)^2 \right\rangle, \quad i \neq j = 1, 2, 3, \\ \left\langle \left(\frac{\partial u_i}{\partial x_j} \right)^2 \right\rangle &= 2 \left\langle \left(\frac{\partial u_1}{\partial x_1} \right)^2 \right\rangle, \quad i \neq j = 1, 2, 3, \end{aligned} \tag{9}$$

with no summation over indices.

Finally, using Eq. (9) with Eq. (8), the problem of determining the turbulent dissipation rate for homogeneous isotropic turbulence reduces to measuring a single derivative moment and we arrive at the Taylor formula, Eq. (3):

$$\langle \varepsilon \rangle = 15 \nu \left\langle \left(\frac{\partial u_1}{\partial x_1} \right)^2 \right\rangle.$$

III. REVISITING TAYLOR'S DERIVATION

Taylor, with his signature wizardry, transformed the intractable Eq. (2) into the elegant Eq. (3). This ingenious derivation, however, leaves some essential aspects unclear, which still need to be clarified.

Specifically, Eq. (3) is predicated on two equations: Eqs. (8) and (9). Equation (8) is a clear consequence of homogeneity and incompressibility. Equation (9), on the other hand, needs careful evaluation. Taylor derived Eq. (9) by adroitly invoking a specific choice of rotation angles and axes, $\{\mathrm{R}(\pi/2, \mathbf{e}_3) \rightarrow \mathrm{R}(\pi/2, \mathbf{e}_1) \rightarrow \mathrm{R}(\pi/4, \mathbf{e}_3)\}$, in no particular order. It is, however, not evident whether Eq. (9) remains valid for any set of rotation angles and axes. Neither is it clear how many such rotations are needed to exhaustively obtain all the relations between the derivative moments that conform to rotational symmetry, meaning invariance to arbitrary rotations of the reference frame. To wit, a general approach is necessary to derive Eq. (9) for rotational symmetry. To that task we now turn our attention.

Before commencing with the analysis, we note that elucidating the underpinnings of Eq. (9) has broader implications besides deriving Taylor's formula. Indeed, Eq. (9)—Taylor's conditions of statistical isotropy— is considered a standard test of rotational symmetry for an incompressible, homogeneous flow [2,11]. In particular, it has been used to analyze empirical data so as to assess rotational symmetry in several turbulent flows [e.g., grid-generated flow, isotropic-periodic-box flow, wake flow, jet flow, wall-bounded flows, and atmospheric flows (see Sec. 23.2 in Ref. [2] and Refs. [12,13])] and, recently, even transitional pipe flow [14]. Often, only a subset of Eq. (9) is tested, the most commonly tested conditions being

$$K_1 \equiv \frac{2\left\langle \left(\frac{\partial u_1}{\partial x_1} \right)^2 \right\rangle}{\left\langle \left(\frac{\partial u_2}{\partial x_1} \right)^2 \right\rangle} = 1 \quad \text{and} \quad K_2 \equiv \frac{2\left\langle \left(\frac{\partial u_1}{\partial x_1} \right)^2 \right\rangle}{\left\langle \left(\frac{\partial u_3}{\partial x_1} \right)^2 \right\rangle} = 1. \tag{10}$$

IV. TURBULENCE POSSESSING ROTATIONAL SYMMETRY

A turbulent flow will be said to have rotational symmetry if its statistical properties are invariant to proper rotations [i.e., the special orthogonal group or SO(3) in $\mathbb{R}^3$] about an arbitrary rotation axis with an arbitrary rotation angle. An element of the SO(3) subgroup is parametrized by three parameters, namely, the rotation angle and two components of the rotation axis vector. In the theory of Lie groups, these parameters are called group parameters. A central theme in the theory of Lie groups is that a finite transformation can always be generated through a succession of infinitesimal transformations about the group identity. On that basis Lie made an important observation in 1873 [15]:

Theorem 1. If a function satisfies an infinitesimal invariance condition it must satisfy the finite invariance condition as well.

To see this, consider the effect of an infinitesimal transformation $\mathrm{F}[s, \widetilde{\mathbf{x}}]$ on some function or functional $\Psi[\mathbf{x}]$. A function $\Psi[\mathbf{x}]$ is said to be invariant under the Lie group $\mathrm{T}^s : \{x^j = \mathrm{F}^j[s, \widetilde{\mathbf{x}}], j = 1, \ldots, n\}$ if and only if

$$\Psi[\mathbf{x}] = \Psi[\mathrm{F}[s, \widetilde{\mathbf{x}}]] = \Psi[\widetilde{\mathbf{x}}], \tag{11}$$

where, for invariance, the group parameter s must vanish for the function to read the same in the new coordinates $\widetilde{\mathbf{x}}$. This finite invariance condition generally depends on the group parameter s nonlinearly and is difficult to deal with for most practical purposes.

Taylor used the above finite invariance condition [see Eq. (5)] in his formulation where he performed three separate rotations given by Eq. (4) to deal with the nonlinear dependence on the group parameter φ. Instead, if we now make the definition

$$\xi^j[\mathbf{x}] = \left[\frac{\partial \mathrm{F}^j}{\partial s}\right]_{s=0}, \quad j = 1, 2, \ldots, n, \tag{12}$$

then one can write down the Lie series representation of the function Ψ as

$$\Psi[\widetilde{\mathbf{x}}] = \Psi[\mathbf{x}] + s(\chi\Psi[\mathbf{x}]) + \frac{s^2}{2!}(\chi(\chi\Psi[\mathbf{x}])) + \frac{s^3}{3!}(\chi(\chi(\chi\Psi[\mathbf{x}]))) + \cdots \tag{13}$$

with the group operator $\chi := \xi^j[\mathbf{x}]\partial/\partial x^j$ and the Lie derivative of Ψ is $\chi\Psi$. Lie's great advance was to replace the nonlinear invariance condition [Eq. (11)] with the linear infinitesimal invariance condition:

$$\chi\Psi[\mathbf{x}] = 0. \tag{14}$$

The importance of Eq. (14) is that if a function satisfies the infinitesimal invariance condition, then it must also satisfy the finite invariance condition [15].

We are now ready to begin our analysis. First, we introduce the infinitesimal rotation formula,

$$\mathrm{R}(\varphi, \boldsymbol{n}) = \mathbb{1} + \varphi \mathrm{Q}(\boldsymbol{n}), \quad \mathrm{Q} = \begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix}, \tag{15}$$

where φ is the infinitesimal rotation angle, $\boldsymbol{n} = (n_1, n_2, n_3)^{\mathrm{T}}$ is the axis of rotation, and $\mathrm{Q} \equiv \mathrm{Q}(\mathbf{n})$ is the generator of infinitesimal rotations. Our goal is to derive the necessary and sufficient conditions for the statistical property $\langle \nabla \boldsymbol{u} \otimes \nabla \boldsymbol{u} \rangle$ to be simultaneously independent of φ and $\mathbf{n}$. To see why we study $\langle \nabla \boldsymbol{u} \otimes \nabla \boldsymbol{u} \rangle$, note that Taylor's derivation focuses on the symmetry conditions for all the combinations of the nine components of the velocity gradient tensor $\nabla \boldsymbol{u}$ taken two at a time. This is encapsulated in his consideration of the finite rotation invariance condition [Eq. (5)] that includes all such combinations of quadratic terms. Using tensor notation, since all such combinations are contained in the quantity $\langle \nabla \boldsymbol{u} \otimes \nabla \boldsymbol{u} \rangle$, we only need to study the symmetry properties of this quantity.

Now, from the tensor transformation rule

$$\nabla' \boldsymbol{u}' = \mathrm{R}^{\mathrm{T}}(\varphi, \boldsymbol{n}) \cdot \nabla \boldsymbol{u} \cdot \mathrm{R}(\varphi, \boldsymbol{n}), \tag{16}$$

we can express the transformation $\nabla \boldsymbol{u} \otimes \nabla \boldsymbol{u} \rightarrow \nabla \boldsymbol{u}' \otimes \nabla \boldsymbol{u}'$ in a compact form as

$$\nabla' \boldsymbol{u}' \otimes \nabla' \boldsymbol{u}' = \nabla \boldsymbol{u} \otimes \nabla \boldsymbol{u} + \varphi[\nabla \boldsymbol{u} \otimes \nabla \boldsymbol{u}, \mathrm{Q}(\boldsymbol{n}) \oplus \mathrm{Q}(\boldsymbol{n})], \tag{17}$$

where $[a, b] := ab - ba$ is the commutator and $\oplus$ is the tensor sum with the property $a \oplus b := a \otimes \mathbb{1}_b + \mathbb{1}_a \otimes b$. Also note the linear dependence on the infinitesimal angle of rotation φ. And, though not immediately apparent, there is linear dependence on axis n_1, n_2, and n_3 as well. Next, taking the ensemble average of Eq. (17), we arrive at

$$\langle \nabla' \boldsymbol{u}' \otimes \nabla' \boldsymbol{u}' \rangle = \langle \nabla \boldsymbol{u} \otimes \nabla \boldsymbol{u} \rangle + \varphi \langle [\nabla \boldsymbol{u} \otimes \nabla \boldsymbol{u}, \mathrm{Q}(\mathbf{n}) \oplus \mathrm{Q}(\mathbf{n})] \rangle, \tag{18}$$

which is the first-order Lie series representation of $\langle \nabla \boldsymbol{u} \otimes \nabla \boldsymbol{u} \rangle$; see Eq. (13). Consequently, the invariance condition $\langle \nabla' \boldsymbol{u}' \otimes \nabla' \boldsymbol{u}' \rangle = \langle \nabla \boldsymbol{u} \otimes \nabla \boldsymbol{u} \rangle$ is satisfied if and only if all terms depending on (φ, n_1, n_2, n_3) jointly vanish. We can then state the infinitesimal condition of rotational symmetry of $\langle \nabla \boldsymbol{u} \otimes \nabla \boldsymbol{u} \rangle$ as

$$\langle [\nabla \boldsymbol{u} \otimes \nabla \boldsymbol{u}, \mathrm{Q}(\mathbf{n}) \oplus \mathrm{Q}(\mathbf{n})] \rangle = \mathbf{0}. \tag{19}$$

Note that invariance to φ is now guaranteed.

For $\dim(\mathrm{Q}) = m = 3$, Eq. (19) is a system of $m^2 \times m^2$ equations that are linear in n_1, n_2, and n_3. However, the number of independent equations, excluding the m^2 diagonal ones, is $m^2(m^2 - 1)/2$. Therefore, there are $m^2 + m^2(m^2 - 1)/2 = m^2(m^2 + 1)/2$ linearly independent equations. In $\mathbb{R}^3$, there is a total of 45 independent equations. However, each of these equations is linear in n_1, n_2, and n_3. In order to guarantee independence to the rotation axis, each coefficient of n_1, n_2, and n_3 must be identically zero. As an example, consider the (1,2)-matrix entry of Eq. (19):

$$\begin{aligned} &-\left\langle \frac{\partial u_1}{\partial x_1}\frac{\partial u_1}{\partial x_3} \right\rangle n_1 + \left[\left\langle \frac{\partial u_1}{\partial x_2}\frac{\partial u_1}{\partial x_3} \right\rangle + \left\langle \frac{\partial u_1}{\partial x_2}\frac{\partial u_3}{\partial x_1} \right\rangle + \left\langle \frac{\partial u_1}{\partial x_1}\frac{\partial u_3}{\partial x_2} \right\rangle \right] n_2 \\ &+ \left[\left\langle \left(\frac{\partial u_1}{\partial x_1}\right)^2 \right\rangle - \left\langle \left(\frac{\partial u_1}{\partial x_2}\right)^2 \right\rangle - \left\langle \frac{\partial u_1}{\partial x_2}\frac{\partial u_2}{\partial x_1} \right\rangle - \left\langle \frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_2} \right\rangle \right] n_3 = 0. \end{aligned} \tag{20}$$

In this case, the three invariance constraints are

$$\left\langle \frac{\partial u_1}{\partial x_1}\frac{\partial u_1}{\partial x_3} \right\rangle = 0. \tag{21a}$$

$$\left\langle \frac{\partial u_1}{\partial x_2}\frac{\partial u_1}{\partial x_3} \right\rangle + \left\langle \frac{\partial u_1}{\partial x_2}\frac{\partial u_3}{\partial x_1} \right\rangle + \left\langle \frac{\partial u_1}{\partial x_1}\frac{\partial u_3}{\partial x_2} \right\rangle = 0, \tag{21b}$$

$$\left\langle \left(\frac{\partial u_1}{\partial x_1}\right)^2 \right\rangle - \left\langle \left(\frac{\partial u_1}{\partial x_2}\right)^2 \right\rangle - \left\langle \frac{\partial u_1}{\partial x_2}\frac{\partial u_2}{\partial x_1} \right\rangle - \left\langle \frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_2} \right\rangle = 0, \tag{21c}$$

In group theory such equations are called determining equations.

The whole system Eq. (19) is thus an easily solvable linear system of equations. They can be efficiently solved using a numerical solver. Using MATHEMATICA, we obtain

$$\left\langle \left(\frac{\partial u_1}{\partial x_1}\right)^2 \right\rangle = \left\langle \left(\frac{\partial u_2}{\partial x_2}\right)^2 \right\rangle = \left\langle \left(\frac{\partial u_3}{\partial x_3}\right)^2 \right\rangle, \tag{22a}$$

$$\left\langle \frac{\partial u_1}{\partial x_2}\frac{\partial u_2}{\partial x_1} \right\rangle = \left\langle \frac{\partial u_1}{\partial x_3}\frac{\partial u_3}{\partial x_1} \right\rangle = \left\langle \frac{\partial u_2}{\partial x_3}\frac{\partial u_3}{\partial x_2} \right\rangle, \tag{22b}$$

$$\left\langle \frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_2} \right\rangle = \left\langle \frac{\partial u_1}{\partial x_1}\frac{\partial u_3}{\partial x_3} \right\rangle = \left\langle \frac{\partial u_2}{\partial x_2}\frac{\partial u_3}{\partial x_3} \right\rangle, \tag{22c}$$

$$\left\langle\left(\frac{\partial u_1}{\partial x_2}\right)^2\right\rangle=\left\langle\left(\frac{\partial u_1}{\partial x_3}\right)^2\right\rangle=\left\langle\left(\frac{\partial u_2}{\partial x_1}\right)^2\right\rangle=\left\langle\left(\frac{\partial u_3}{\partial x_1}\right)^2\right\rangle=\left\langle\left(\frac{\partial u_2}{\partial x_3}\right)^2\right\rangle=\left\langle\left(\frac{\partial u_3}{\partial x_2}\right)^2\right\rangle$$
$$=\left\langle\left(\frac{\partial u_1}{\partial x_1}\right)^2\right\rangle-\left\langle\frac{\partial u_1}{\partial x_2}\frac{\partial u_2}{\partial x_1}\right\rangle-\left\langle\frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_2}\right\rangle, \tag{22d}$$

with all other derivative moments being identically zero. These conditions hold for an incompressible turbulent flow with rotational symmetry. If we now invoke homogeneity [Eq. (7)], we arrive at Taylor's conditions of statistical isotropy [Eq. (9)].

V. TURBULENCE POSSESSING REFLECTIONAL SYMMETRY

Seeking a different perspective on Eq. (9), we now analyze a case not considered in Taylor's analysis: turbulence possessing reflectional symmetry, meaning $\langle\nabla\boldsymbol{u}\otimes\nabla\boldsymbol{u}\rangle$ remains invariant to reflections about any arbitrary plane through the origin.

In analyzing reflectional symmetry, we first note that the idea of infinitesimal transformations, in principle, does not apply to reflections, or, more generally, to improper rotations, $\mathrm{SO}^-(3)$. This is because an improper rotation cannot be obtained from a succession of infinitesimal transformations, i.e., one cannot obtain a determinant -1 transformation via a succession of infinitesimal rotations. Thus, we turn to finite transformations.

A. Reflectional symmetry from rotational symmetry

Given an infinitesimal group representation, by exponentiation we obtain the corresponding finite group representation. For example, the finite rotation group representation of Eq. (15) is

$$\mathrm{R}(\varphi,\mathbf{n})=\mathbb{1}+(\sin\varphi)\mathrm{Q}+(1-\cos\varphi)\mathrm{Q}^2, \tag{23}$$

with the property that $\mathrm{R}(\varphi,\mathbf{n})\cdot\mathbf{n}\equiv\mathbf{n}$, showing that a proper rotation leaves the rotation axis $\mathbf{n}$ invariant. [Note that the matrix Q has the properties: $\mathrm{Q}_{ij}:=-\varepsilon_{ijk}n^k$ and $(\mathrm{Q}^2)_{ij}=\mathrm{Q}_{i\ell}\mathrm{Q}_{\ell j}=n_in_j-\delta_{ij}$.] For $\varphi=\pi$, we have

$$\mathrm{R}(\pi,\mathbf{n})=\mathbb{1}+2\mathrm{Q}^2\equiv-\mathbb{1}+2\mathbf{n}\otimes\mathbf{n}, \tag{24}$$

which is a projective plane in SO(3). Similarly, any improper rotation is given by

$$\widetilde{\mathrm{R}}(\varphi,\mathbf{n})=-\mathbb{1}+(\sin\varphi)\mathrm{Q}-(1+\cos\varphi)\mathrm{Q}^2, \tag{25}$$

with the property that $\widetilde{\mathrm{R}}(\varphi,\mathbf{n})\cdot\mathbf{n}\equiv-\mathbf{n}$, showing that an improper rotation inverts the axis of rotation. For $\varphi=0$, we have

$$\widetilde{\mathrm{R}}(0,\mathbf{n})=\mathbb{1}-2\mathbf{n}\otimes\mathbf{n}\quad\text{(reflection in plane }\perp\text{ to }\mathbf{n}). \tag{26}$$

From Eqs. (24) and (26) it is clear that the set of all reflections about a plane perpendicular to the rotation axis $\mathbf{n}$ and passing through the origin is related to proper rotations via

$$\widetilde{\mathrm{R}}(0,\mathbf{n})=-\mathrm{R}(\pi,\mathbf{n}). \tag{27}$$

Equation (27) has an important consequence for reflectional and rotational symmetry of even-rank tensors. Consider a generic even-rank tensor $S^{(2d)}$ of rank $2d$. Invoking Eq. (27), we can express the action of the reflection matrix $\widetilde{\mathrm{R}}(0,\mathbf{n})$ on this object as

$$\begin{aligned}S'^{(2d)}_{i_1\cdots i_{2d}}&=\widetilde{\mathrm{R}}_{i_1j_1}(0,\mathbf{n})\cdots\widetilde{\mathrm{R}}_{i_{2d}j_{2d}}(0,\mathbf{n})S^{(2d)}_{j_1\cdots j_{2d}}\\&=(-1)^{2d}\mathrm{R}_{i_1j_1}(\pi,\mathbf{n})\cdots\mathrm{R}_{i_{2d}j_{2d}}(\pi,\mathbf{n})S^{(2d)}_{j_1\cdots j_{2d}}\\&=\mathrm{R}_{i_1j_1}(\pi,\mathbf{n})\cdots\mathrm{R}_{i_{2d}j_{2d}}(\pi,\mathbf{n})S^{(2d)}_{j_1\cdots j_{2d}}.\end{aligned} \tag{28}$$

Clearly, an even-rank tensor transforms identically for all proper rotations $\mathrm{R}(\pi, \mathbf{n})$ with rotation center at the origin and all reflections $\widetilde{\mathrm{R}}(0, \mathbf{n})$ about planes perpendicular to the rotation axis $\mathbf{n}$ and passing through the origin. This implies that the tensor $S^{(2d)}$ has the same finite invariance condition under the subset of proper rotations $\mathrm{R}(\pi, \mathbf{n})$ and under the subset of improper rotations $\widetilde{\mathrm{R}}(0, \mathbf{n})$.

The above result bears directly on the conditions of reflectional symmetry for $\langle \nabla \mathbf{u} \otimes \nabla \mathbf{u} \rangle$, which we note is a rank-4 tensor. For this tensor, the analysis of the previous section shows that the conditions given by Eq. (22) ensure rotational symmetry with arbitrary rotation angle and axis. Considering the rotation angle π and axis $\mathbf{n}$, it follows that if the velocity derivative moments obey Eq. (22), $\langle \nabla \mathbf{u} \otimes \nabla \mathbf{u} \rangle$ manifests rotational symmetry under the transformation $\mathrm{R}(\pi, \mathbf{n})$. In turn, by virtue of Eq. (28), this symmetry maps onto reflectional symmetry under the transformation $\widetilde{\mathrm{R}}(0, \mathbf{n})$. We therefore conclude that the rotational symmetry conditions, Eq. (22), are identical to the reflectional symmetry conditions for velocity derivative moments. And, if we now invoke homogeneity [Eq. (7)], we again arrive at Taylor's conditions of statistical isotropy [Eq. (9)].

B. Reflectional symmetry

In the analysis above, we obtained reflectional symmetry conditions for $\langle \nabla \boldsymbol{u} \otimes \nabla \boldsymbol{u} \rangle$ by exploiting the link between reflections and proper rotations. This process entailed that the flow possesses rotational symmetry. However, the same conditions can be reached by a direct, brute-force computation of the reflectional symmetry conditions without requiring rotational symmetry, as we outline next.

For a plane of reflection P normal to some arbitrary unit vector $\mathbf{n} = (n_1, n_2, n_3)^{\mathrm{T}}$, we can write

$$\mathrm{P} = \left\{ \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \middle| n_1 x_1 + n_2 x_2 + n_3 x_3 = 0 \right\}. \tag{29}$$

The dimension of P in $\mathbb{R}^3$ is 2. This means that two parameters are needed to completely parametrize a plane of reflection in three dimensions. Using the fact that $\mathbf{n}$ is a unit normal, we then rewrite it as

$$\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \\ \sqrt{1 - n_1^2 - n_2^2} \end{pmatrix}. \tag{30}$$

This guarantees that the reflectional invariance condition consists of a set of linearly independent equations. The reflection matrix corresponding to reflections in a plane P is given by

$$\widetilde{\mathrm{R}}(0, \mathbf{n}) = \mathbb{1} - 2\mathbf{n} \otimes \mathbf{n} := \mathbb{1} - \mathrm{N}^{(2)}, \quad \mathrm{N}^{(2)} = \mathbf{n} \otimes \mathbf{n}. \tag{31}$$

Then, from the tensor transformation rule,

$$\nabla' \boldsymbol{u}' = \widetilde{\mathrm{R}}^{\mathrm{T}}(0, \mathbf{n}) \cdot \nabla \boldsymbol{u} \cdot \widetilde{\mathrm{R}}(0, \mathbf{n}), \tag{32}$$

the transformation $\nabla \boldsymbol{u} \otimes \nabla \boldsymbol{u} \to \nabla' \boldsymbol{u}' \otimes \nabla' \boldsymbol{u}'$, after some simplifications, yields

$$\begin{aligned} \langle \nabla' \boldsymbol{u}' \otimes \nabla' \boldsymbol{u}' \rangle = \langle \nabla \boldsymbol{u} \otimes \nabla \boldsymbol{u} \rangle + (4 \langle [[\nabla \boldsymbol{u}, \mathrm{N}^{(2)}], \mathrm{N}^{(2)}] \otimes [[\nabla \boldsymbol{u}, \mathrm{N}^{(2)}], \mathrm{N}^{(2)}] \rangle \\ + 4 \langle [\nabla \boldsymbol{u}, \mathrm{N}^{(2)}] \otimes [\nabla \boldsymbol{u}, \mathrm{N}^{(2)}] \rangle - 2 \langle [[\nabla \boldsymbol{u} \otimes \nabla \boldsymbol{u}, \mathrm{N}^{(2)} \oplus \mathrm{N}^{(2)}], \mathrm{N}^{(2)} \oplus \mathrm{N}^{(2)}] \rangle). \end{aligned} \tag{33}$$

Finally, the invariance condition $\langle \nabla' \boldsymbol{u}' \otimes \nabla' \boldsymbol{u}' \rangle = \langle \nabla \boldsymbol{u} \otimes \nabla \boldsymbol{u} \rangle$ is satisfied if and only if all terms depending on n_1 and n_2 jointly vanish, which implies

$$\begin{aligned} 2 \langle [[\nabla \boldsymbol{u}, \mathrm{N}^{(2)}], \mathrm{N}^{(2)}] \otimes [[\nabla \boldsymbol{u}, \mathrm{N}^{(2)}], \mathrm{N}^{(2)}] \rangle + 2 \langle [\nabla \boldsymbol{u}, \mathrm{N}^{(2)}] \otimes [\nabla \boldsymbol{u}, \mathrm{N}^{(2)}] \rangle \\ - \langle [[\nabla \boldsymbol{u} \otimes \nabla \boldsymbol{u}, \mathrm{N}^{(2)} \oplus \mathrm{N}^{(2)}], \mathrm{N}^{(2)} \oplus \mathrm{N}^{(2)}] \rangle = 0. \end{aligned} \tag{34}$$

Solving this nonlinear system, which corresponds to the finite condition of reflectional symmetry of $\langle \nabla \boldsymbol{u} \otimes \nabla \boldsymbol{u} \rangle$, we find the same equations as for the rotational symmetry conditions [Eq. (22)].

VI. CONCLUDING REMARKS

In summary, we used the machinery of Lie groups to furnish a rigorous derivation of Taylor's formula for turbulence possessing rotational symmetry. Unlike Taylor's derivation [8], our analysis clearly establishes that the formula is not tied to invariance under a few specific choices of rotation angles and axes but in fact holds for invariance under any arbitrary rotation angles and axes. Additionally, we investigated the case of turbulence possessing reflectional symmetry. Our analysis showed that the necessary and sufficient conditions for $\langle \nabla \boldsymbol{u} \otimes \nabla \boldsymbol{u} \rangle$ to possess reflectional symmetry are identical to those for rotational symmetry. This result has two important implications. First, that Taylor's formula holds for rotational symmetry as well as for reflectional symmetry. Second, that Taylor's conditions of statistical isotropy [Eq. (9)], which are considered a standard test of rotational symmetry for an incompressible, homogeneous flow (cf. Sec. III), hold equally well for reflectional symmetry, and therefore cannot be used to reliably identify either symmetry.

Last, a note on isotropic turbulence and symmetries: In Taylor's formulation [8] and several subsequent studies (e.g., Refs. [16,17]), isotropy implies rotational symmetry. In other studies (e.g., Refs. [2,3,9,11]), isotropy is taken to imply rotational symmetry as well as reflectional symmetry. The different symmetry requirements can entail disparate consequences. By way of illustration, for a flow with reflectional symmetry, the helicity must vanish [18], which implies that flows with helicity—whose examples include not just rotating flows but even grid turbulence [18] (which is typically considered a canonical example of isotropic turbulence)—can be isotropic if only rotational symmetry is required but not if reflectional symmetry is also required. By contrast, our analysis shows that for an incompressible, homogeneous flow, the same Taylor's formula holds under either definition of isotropic turbulence.

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[1] C. L. Fefferman, Existence and smoothness of the Navier-Stokes equation, Clay Millennium Problems (2000), https://www.claymath.org/sites/default/files/navierstokes.pdf.

[2] A. S. Monin and A. M. Yaglom, *Statistical Fluid Mechanics: Mechanics of Turbulence*, Vol. 2 (MIT Press, Cambridge, MA, 1975).

[3] P. A. Davidson, *Turbulence: An Introduction for Scientists and Engineers* (Oxford University Press, Oxford, U.K., 2004).

[4] A. N. Kolmogorov, The local structure of turbulence in incompressible viscous fluid for very large Reynolds numbers, Dokl. Akad. Nauk SSSR **30**, 301 (1941) [reprinted in English in Proc. R. Soc. A **434**, 9 (1991)].

[5] A. N. Kolmogorov, Dissipation of energy in locally isotropic turbulence, Dokl. Akad. Nauk SSSR **32**, 16 (1941) [reprinted in English in Proc. R. Soc. A **434**, 15 (1991)].

[6] U. Frisch, *Turbulence: The Legacy of A. N. Kolmogorov* (Cambridge University Press, Cambridge, U.K., 1995).

[7] G. Falkovich and K. R. Sreenivasan, Lessons from hydrodynamic turbulence, Phys. Today **59**(4), 43 (2006).

[8] G. I. Taylor, Statistical theory of turbulence, Proc. R. Soc. London, Ser. A **151**, 421 (1935).

[9] G. K. Batchelor, *The Theory of Homogeneous Turbulence* (Cambridge University Press, Cambridge, U.K., 1953).

[10] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevFluids.6.L082602 for a discussion on the homogeneity condition in Taylor's analysis.

[11] J. O. Hinze, *Turbulence* (McGraw-Hill, New York, 1987).

[12] L. W. B. Browne, R. A. Antonia, and D. A. Shah, Turbulent energy dissipation in a wake, J. Fluid Mech. **179**, 307 (1987).
[13] S. Almalkie and S. M. de Bruyn Kops, Energy dissipation rate surrogates in incompressible Navier-Stokes turbulence, J. Fluid Mech. **697**, 204 (2012).
[14] R. T. Cerbus, C.-c. Liu, G. Gioia, and P. Chakraborty, Small-scale universality in the spectral structure of transitional pipe flows, Sci. Adv. **6**, eaaw6256 (2020).
[15] B. J. Cantwell, *Introduction to Symmetry Analysis* (Cambridge University Press, Cambridge, U.K., 2002).
[16] M. Lesieur, *Turbulence in Fluids: Stochastic and Numerical Modelling* (Springer, Netherlands, 1987).
[17] W. D. McComb, *Homogeneous, Isotropic Turbulence: Phenomenology, Renormalization and Statistical Closures*, The International Series of Monographs on Physics No. 162 (Oxford University Press, Oxford, U.K., 2014).
[18] H. K. Moffatt and A. Tsinober, Helicity in laminar and turbulent flow, Annu. Rev. Fluid Mech. **24**, 281 (1992).