Surface tension and instability in the hydrodynamic white hole of a circular hydraulic jump

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We impose a linearized Eulerian perturbation on a steady, shallow, radial outflow of a liquid (water), whose local pressure function includes both the hydrostatic and the Laplace pressure terms. The resulting wave equation bears the form of a hydrodynamic metric. A dispersion relation, extracted from the wave equation, gives an instability due to surface tension and the cylindrical flow symmetry. Using the dispersion relation, we also derive three known relations that scale the radius of the circular hydraulic jump in the outflow. The first two relations are scaled by viscosity and gravity, with a capillarity-dependent crossover to the third relation, which is scaled by viscosity and surface tension. The perturbation as a high-frequency traveling wave, propagating radially inward against the bulk outflow, is blocked just outside the circular hydraulic jump. The amplitude of the wave also diverges here because of a singularity. The blocking is associated with surface tension, which renders the circular hydraulic jump a hydrodynamic white hole.

I. INTRODUCTION

The hydraulic jump is an abrupt discontinuity in the free-surface height of a flowing liquid, with the postjump height being greater than the prejump height [1]. The circular hydraulic jump is formed by the outward radial flow of a liquid on a horizontal plane, after the flow originates at the point of impingement of a vertically downward liquid jet [2]. The flow in this case is commonly viewed as shallow and axially symmetric. The formation of a hydraulic jump depends on two defining attributes of a liquid, namely, viscosity and surface tension. Various theories have addressed the question of how the circular hydraulic jump forms because of viscosity [2–7] and surface tension [8–10], with experimental evidence in favor of both means [2,8,11–19].

Apart from conventional fluid dynamics, the hydraulic jump is viewed with keen interest from the perspective of the fluid analog of gravity, specifically as a hydrodynamic white hole [6,20–25]. In this respect, the circular hydraulic jump is like a standing event horizon where the velocity of the radial flow equals the local speed of either surface gravity waves [6] or capillary-gravity waves [21]. The hydrodynamic event horizon segregates the supercritical and the subcritical regions of the flow, where the critical condition refers to the matching of the speed of the bulk flow with the speed of the waves. The horizon (which spatially coincides with the circular jump) thus outlines a barrier against the upstream transmission of information, i.e., becomes a hydrodynamic white hole. As the equatorial flow proceeds outwards from its point of origin, its radial velocity is greater than

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the speed of gravity waves, but viscosity and the radial geometry decelerate the flow downstream. When the critical condition is met, the jump and the horizon occur simultaneously [6]. The horizon of a white hole in the circular hydraulic jump has been analyzed in theoretical studies [6,21,25] and demonstrated experimentally [23]. However, the horizon of the white hole by itself is inadequate to explain why a jump should coincide with it [25]. The jump is brought about by energy dissipation (at the discontinuity), which is a general principle, established by Lord Rayleigh [26] (also see [1]). In shallow laboratory flows of normal liquids, mainly viscosity causes the jump at positions of the centimetre order [3,4,27]. For jump positions of smaller length scales, surface tension is the main cause [8], as is known for submicron molten metal droplets [28] and superfluids [15]. In the present work, we revisit the circular hydraulic jump and the associated hydrodynamic horizon from the viewpoint of surface tension.

The outflow that we study here pertains to the standard type-I hydraulic jump, which is physically characterized by a negligible flow height at the outer boundary (the perimeter of the horizontal flow base), where the liquid falls freely off the edge of the base plane [4,12]. In Sec. II we set down the height-averaged equations of such a flow, taken to be shallow. Its local pressure function accounts for the effect of both hydrostatics and surface tension. An Eulerian perturbation of the steady flow establishes the wave equation of a fluid metric in Sec. III. We derive a dispersion relation from the wave equation in Sec. IV. The dispersion relation brings forth two salient results. The first is an instability in the flow because of surface tension and the cylindrical symmetry. Secondly, the dispersion relation unifies three known scaling formulas of the radius of the circular hydraulic jump in a single theoretical framework. The first two formulas depend on viscosity and gravity [4,9] and the third depends on viscosity and surface tension [18]. Viscosity and gravity determine the jump scaling when the wavelength of the perturbation far exceeds the capillary length, whereas surface tension becomes a dominant effect at the cost of gravity when the capillary length is much greater than the wavelength of the perturbation. The crossover from the one regime to the other occurs when the capillary length and the wavelength are evenly matched.

Lastly, in Sec. V, on applying the WKB approximation, we design the perturbation to be a traveling wave of high frequency. By this we show that the hydrodynamic event horizon, where the hydraulic jump is also located, blocks a wave that travels from the subcritical flow region towards the circular jump, against the bulk outflow. The amplitude of this traveling wave diverges as the jump radius is approached, thereby demonstrating that the jump is like a white hole to any approaching wave from the subcritical region. To support our theory, we provide photographic evidence from an experiment by Kate *et al.* [16].

II. HEIGHT-AVERAGED FLOW EQUATIONS

Cylindrical coordinates, (r, ϕ, z), are suitable for a shallow radial flow of liquid on a plane [1]. The flow, being axially symmetric, is independent of the azimuthal coordinate, ϕ. Being also shallow, the flow allows a vertical averaging of the flow variables [4,5] through the height of the flow, under the boundary conditions that velocities vanish at $z = 0$ (the no-slip condition), and vertical gradients of velocities vanish at the free surface of the flow (the no-stress condition) [4,5,7,27]. The boundary conditions are applied under the operative assumption that the vertical component of the velocity is small compared to its radial component, and the vertical variation of the radial velocity (through the shallow layer of fluid) is much greater than its radial variation [4]. Quantities with the z coordinate are averaged thus through the flow height, with the double z derivative approximated as $\partial^2/\partial z^2 \equiv -1/h^2$ [4], in which h is the free-surface height of the flow.

In terms of h and the vertically averaged radial velocity, v, the time-dependent continuity equation of the shallow-water circular flow is [6,29]

$$\frac{\partial h}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(rvh) = 0. \tag{1}$$

Likewise, the Navier-Stokes equation of the time-dependent radial component of the flow is [6,29]

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} + \frac{1}{\rho}\frac{\partial P}{\partial r} = -\frac{\nu v}{h^2}, \tag{2}$$

whose right hand side has been approximated by $\nu\nabla^2 v \simeq -\nu v/h^2$ for a thin layer of the flow [4]. Here ν is the kinematic viscosity. The solutions of Eqs. (1) and (2), $h(r,t)$ and $v(r,t)$, can be known upon prescribing a function for the pressure, P, in Eq. (2). Since we are concerned with the effect of surface tension, in addition to the usual hydrostatic pressure, we account for the surface pressure, as given by Laplace's formula [1,8,10]. Together, these two effects give

$$P = h\rho g - \frac{\sigma}{r}\frac{\partial}{\partial r}\left[\frac{r}{\sqrt{1+(\partial h/\partial r)^2}}\frac{\partial h}{\partial r}\right], \tag{3}$$

with the first term on the right hand side being the hydrostatic pressure, containing the liquid density, ρ, and the acceleration due to gravity, g. The second term on the right hand side of Eq. (3) is the contribution that surface tension, σ, makes to the pressure. With P set down in terms of h and r, the coupled system of Eqs. (1) and (2) is now closed.

In the steady state of the shallow radial flow, whereby $\partial/\partial t \equiv 0$, the solutions of Eqs. (1) and (2) are $h_0(r)$ and $v_0(r)$, with the subscript "0" standing for steady values. The integral of Eq. (1) in the steady limit gives $rv_0h_0 = Q/2\pi$, in which Q is the steady volumetric flow rate. In the absence of surface tension ($\sigma = 0$), the behavior of the steady solutions and the conditions to form the circular hydraulic jump are known [4,6]. The hydraulic jump is a discontinuity in the solutions of $h_0(r)$ and $v_0(r)$. From the perspective of fluid analogs of gravity, an event horizon exists at the radius of the hydraulic jump, $r = r_J$, where $v_0^2(r_J) = gh_0(r_J)$ [6]. Hereafter, we proceed to examine how the fluid event horizon and the hydraulic jump are affected by surface tension.

III. PERTURBATION AND A HYDRODYNAMIC METRIC

We define a variable, $f = rvh$, in which $v \equiv v(r,t)$ and $h \equiv h(r,t)$ for the vertically averaged radial flow [6,29]. Under steady conditions, $f = f_0 = Q/2\pi$, as Eq. (1) shows. Next, we apply a time-dependent radial perturbation on the flow that Eqs. (1) and (2) describe. The perturbations in v and h are set, respectively, as $v(r,t) = v_0(r) + v'(r,t)$ and $h(r,t) = h_0(r) + h'(r,t)$. With f' denoting a perturbation in f, we linearize according to $f = f_0 + f'$ and get

$$f' = r(v_0h' + h_0v'). \tag{4}$$

Under this Eulerian perturbation scheme, the fluctuation about the steady continuity condition is obtained from Eq. (1) as [6]

$$\frac{\partial h'}{\partial t} = -\frac{1}{r}\frac{\partial f'}{\partial r}, \tag{5}$$

which relates h' to f'. A similar relation, connecting v' and f', is then derived by using Eq. (5) in Eq. (4). This is [6]

$$\frac{\partial v'}{\partial t} = \frac{v_0}{f_0}\left(\frac{\partial f'}{\partial t} + v_0\frac{\partial f'}{\partial r}\right). \tag{6}$$

Now we perturb v and h in Eq. (2) to a linear order about the steady state. Then taking the time derivative of the resulting linearized equation, and applying both Eqs. (5) and (6) in it, we finally get a wave equation

$$\frac{\partial}{\partial t}\left(v_0\frac{\partial f'}{\partial t}\right) + \frac{\partial}{\partial t}\left(v_0^2\frac{\partial f'}{\partial r}\right) + \frac{\partial}{\partial r}\left(v_0^2\frac{\partial f'}{\partial t}\right) + \frac{\partial}{\partial r}\left[v_0\left(v_0^2 - gh_0 + \frac{gh_0l^2\Gamma}{r^2}\right)\frac{\partial f'}{\partial r}\right]$$

$$= -\frac{\nu v_0}{h_0^2}\left(\frac{\partial f'}{\partial t} + 3v_0\frac{\partial f'}{\partial r}\right) - v_0gh_0rl^2\frac{\partial}{\partial r}\left[\frac{1}{r\{1+(dh_0/dr)^2\}^{3/2}}\frac{\partial^3 f'}{\partial r^3} - \frac{\Gamma}{r^2}\frac{\partial^2 f'}{\partial r^2}\right], \tag{7}$$

in which l is the capillary length, $l = \sqrt{\sigma/(\rho g)}$ [1], and

$$\Gamma = \frac{1}{[1 + (dh_0/dr)^2]^{3/2}}\left[1 + \frac{3 dh_0/dr}{1 + (dh_0/dr)^2}\frac{d^2 h_0}{dr^2}\right]. \tag{8}$$

The capillary length, l, introduced in Eq. (7), captures the effect of surface tension. The importance of surface tension can be gauged by comparing the capillary length with the wavelength of the perturbation. We show this in Sec. IV.

If the terms depending on viscosity and surface tension in Eq. (7) are to vanish ($\nu = 0$ and $\sigma = 0$), then the resulting wave equation in f' is rendered compactly as $\partial_\alpha(\mathsf{f}^{\alpha\beta}\partial_\beta f') = 0$, in which the Greek indices run from 0 to 1, with 0 standing for t and 1 standing for r. From the terms on the left hand side of Eq. (7), we can then read the symmetric matrix,

$$\mathsf{f}^{\alpha\beta} = v_0\begin{bmatrix} 1 & v_0 \\ v_0 & v_0^2 - gh_0 \end{bmatrix}. \tag{9}$$

The basic principle of establishing a hydrodynamic metric and an analog horizon rests on an equivalence between Eq. (9) and the d'Alembertian for a scalar field in curved geometry (see [6] and relevant references therein for details). The d'Alembertian has the form [24]

$$\triangle \psi \equiv \frac{1}{\sqrt{-\mathsf{g}}}\partial_\alpha(\sqrt{-\mathsf{g}}\,\mathsf{g}^{\alpha\beta}\partial_\beta\psi). \tag{10}$$

Under the identification that $\mathsf{f}^{\alpha\beta} = \sqrt{-\mathsf{g}}\,\mathsf{g}^{\alpha\beta}$, and therefore $\mathsf{g} = \det(\mathsf{f}^{\alpha\beta})$, one can prove the existence of a white hole horizon for surface waves when $v_0^2 = gh_0$ [6,20]. This produces a fluid analog of a general relativistic problem [20].

By disregarding both viscosity and surface tension in a normal liquid, the symmetry of the metric implied by Eq. (9) may be preserved, but the absence of viscosity (and surface tension) will also prevent the formation of a physical inner solution within the jump radius [4]. This difficulty is overcome by accounting for viscosity in the radial outflow [4], even though it has to invalidate the hydrodynamic metric. The precise condition of the analog horizon is thus lost. Nevertheless, the basic properties of surface waves are not affected overmuch [20], and the most important feature to emerge from the analogy of a white hole horizon remains qualitatively unchanged, namely, that a disturbance propagating upstream from the subcritical flow region (where $v_0^2 < gh_0$) cannot penetrate through the horizon into the supercritical region of the flow (where $v_0^2 > gh_0$), both in the presence of viscosity [6] and surface tension [21].

IV. DISPERSION, INSTABILITY, AND JUMP SCALING

Looking closely at Eq. (7), we discern in it the form of the wave equation. Consequently, we can extract a dispersion relation from it. An approximation makes the dispersion relation stand out clearly. We know that the steady free-surface height of the flow varies slowly with r for the greater part of the flow. Therefore, we approximate $dh_0/dr \simeq 0$, which gives $\Gamma \simeq 1$ and simplifies Eq. (7) to

$$\frac{\partial^2 f'}{\partial t^2} + 2\frac{\partial}{\partial r}\left(v_0\frac{\partial f'}{\partial t}\right) + \frac{1}{v_0}\frac{\partial}{\partial r}\left[v_0(v_0^2 - gh_0)\frac{\partial f'}{\partial r}\right]$$
$$= -\frac{\nu}{h_0^2}\left(\frac{\partial f'}{\partial t} + 3v_0\frac{\partial f'}{\partial r}\right) - \frac{\sigma h_0 r}{\rho}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial^3 f'}{\partial r^3} - \frac{1}{r^2}\frac{\partial^2 f'}{\partial r^2} + \frac{1}{r^3}\frac{\partial f'}{\partial r}\right). \tag{11}$$

With respect to the steady background flow, Eq. (11) becomes

$$\frac{\partial^2 f'}{\partial t^2} = gh_0\frac{\partial^2 f'}{\partial r^2} - \frac{\nu}{h_0^2}\frac{\partial f'}{\partial t} - \frac{\sigma h_0}{\rho}\left(\frac{\partial^4 f'}{\partial r^4} - \frac{2}{r}\frac{\partial^3 f'}{\partial r^3} + \frac{3}{r^2}\frac{\partial^2 f'}{\partial r^2} - \frac{3}{r^3}\frac{\partial f'}{\partial r}\right), \tag{12}$$

which is the wave equation (for gravity waves) when $\nu = 0$ and $\sigma = 0$. The solution, $f'(r,t) \sim \exp[i(kr - \omega t)]$, applied to Eq. (12), gives a quadratic equation,

$$(\omega - kv_{\mathrm{B}})^2 = \left(gh_0 - \frac{3\sigma h_0}{\rho r^2}\right)k^2 + \frac{\sigma h_0 k^4}{\rho} - \frac{i\nu}{h_0^2}(\omega - kv_{\mathrm{B}}) + \frac{i\sigma h_0}{\rho}\left(\frac{2k^3}{r} - \frac{3k}{r^3}\right), \qquad (13)$$

in which v_{B} stands for the bulk motion of the liquid. The outcome due to viscosity is well known and has been reported in detail previously [6]. Hence we set $\nu = 0$ in our present study and devote our attention fully to the effect of σ in the wave equation. This gives us

$$\omega = kv_{\mathrm{B}} + \sqrt{gh_0}\left[1 - \frac{3l^2}{r^2} + l^2 k^2 + \frac{il^2}{r^2}\left(2kr - \frac{3}{kr}\right)\right]^{1/2}k. \qquad (14)$$

In the reference frame of $v_{\mathrm{B}} = 0$, we can view Eq. (14) in the form $\omega = \sqrt{gh_0}(A + iB)^{1/2}k = \sqrt{gh_0}(X + iY)k$, in which A, B, X, and Y are all real quantities. These are related among themselves by $X = \pm[(\sqrt{A^2 + B^2} + A)/2]^{1/2}$ and $Y = \pm[(\sqrt{A^2 + B^2} - A)/2]^{1/2}$.

In Eq. (14), the terms with l^2/r^2 have arisen because of the cylindrical symmetry of the shallow flow. For water, $l = 0.27$ cm and $r \sim 10$ cm, which makes $l^2/r^2 \ll 1$. Since B contains only a single term with l^2/r^2 and A is at least $O(1)$, as we see in Eq. (14), it is clear that $B \ll A$. Hence, by a binomial expansion, we can approximate $Y \simeq \pm B/(2\sqrt{A})$. Since both signs are admissible, with $A \sim 1$ and $B \sim l^2/r^2$, we realize that the amplitude of f' can grow as

$$|f'(r,t)| \sim \exp\left(\frac{l^2 k \sqrt{gh_0}t}{r^2}\right), \qquad (15)$$

on a time scale of $r^2/(l^2 k \sqrt{gh_0})$. Since $r^2 \gg l^2$, this is a long time scale for the growth of an instability in the flow. Apropos of this, surface ripples [11], capillary-gravity waves [13], and instability due to surface tension [7,13] are known for normal liquids. Similar features have also been observed in superfluids [15], which we stress here because we have set $\nu = 0$.

Going back to Eq. (14) in the reference frame of $v_{\mathrm{B}} = 0$, and neglecting l^2/r^2 in it, we get a comoving dispersion relation,

$$\omega \simeq \sqrt{gh_0}(1 + l^2 k^2)^{1/2}k. \qquad (16)$$

The foregoing equation is actually the long-wavelength limiting case of the dispersion relation for capillary-gravity waves, $\omega^2 = [gk + (\sigma/\rho)k^3]\tanh(kh_0)$, for $kh_0 \ll 1$ [which approximates to $\tanh(kh_0) \simeq kh_0$] [1]. In the limit of $k \ll h_0^{-1}$, the wavelength $\lambda \gg h_0$, which is the case of long wavelengths in shallow-water flows. Thus this condition is implicit in Eq. (16) and all the equations that lead to it, starting with Eqs. (1) and (2).

From Eq. (16) we derive some familiar scaling relations for the radius of the circular hydraulic jump. First, for $kl \ll 1$, i.e., $\lambda \gg l$, Eq. (16) gives the phase velocity of gravity waves, $v_{\mathrm{p}} = \omega/k \simeq \sqrt{gh_0}$. Now, viscosity affects the bulk motion, which is seen by comparing the first term on the left hand side of Eq. (2) with the viscosity term on the right hand side. The time scale on which viscosity decelerates the flow is $t_{\mathrm{visc}} \sim h_0^2/\nu$ [6]. The deceleration of an advanced layer of the flow by viscosity can be known upstream if a traveling wave carries the information against the flow. But this information travels at the speed of surface gravity waves, $\sqrt{gh_0}$, something that is known from Eq. (16) when $kl \ll 1$. Therefore, the supercritical region, where $v_0 > \sqrt{gh_0}$, remains uninformed about the viscous deceleration downstream [6]. The flow thus proceeds radially outwards without hindrance in the supercritical region till v_0 becomes comparable with $\sqrt{gh_0}$, and only then does the information about an obstacle ahead catch up with the fluid. Defining a dynamic time scale $t_{\mathrm{dyn}} \sim r/v_0$ (the time scale of the bulk motion) and setting $t_{\mathrm{visc}} \simeq t_{\mathrm{dyn}}$, with the additional requirements, $v_0 \simeq v_{\mathrm{p}} \simeq \sqrt{gh_0}$ and $rv_0 h_0 = Q/2\pi$, deliver a scale of the jump radius as

$$r_{\mathrm{J}} \sim Q^{5/8}\nu^{-3/8}g^{-1/8}, \qquad (17)$$

due originally to Bohr *et al.* [4], with some later refinements [17]. To derive the scaling relation in Eq. (17), Bohr *et al.* [4] matched steady inner and outer solutions through a standing shock at the hydraulic jump. On the other hand, the crux of our chain of reasoning to arrive at Eq. (17) is that the hydraulic jump forms when the two time scales, $t_{\rm visc}$ and $t_{\rm dyn}$, match each other closely and when the Froude number $\mathcal{F} = v_0/\sqrt{gh_0} \simeq 1$. In matching the time scales, $t_{\rm visc}$ brings in viscosity as a physical means for the jump to form, for which the unity of $\mathcal{F}$ (implying only the horizon) is not enough [25]. Under the combined effect of all these conditions, a layer of fluid arriving late is halted by an obstacle formed by a layer of fluid ahead, slowed abruptly by viscosity. But the fluid cannot accumulate indefinitely, and also the continuity of the fluid flow is to be maintained. The fluid layer arriving late, therefore, slides over the slowly flowing layer ahead, causing a sudden increase in the flow height—a hydraulic jump [6].

Once a scaling relation is known for $r_{\rm J}$, as in Eq. (17), the height of the jump is then scaled as $h_{\rm J} \sim Q^{1/4}\nu^{1/4}g^{-1/4}$. In the postjump region of the flow, this height remains nearly unchanged, which allows the postjump flow height, H, to be set as $H \sim h_{\rm J}$. Now, Eq. (17) scales $r_{\rm J}$ in terms of the free parameters of the flow, Q, ν, and g, but, in terms of H, Eq. (17) can be recast into a scaling relation,

$$r_{\rm J} \sim Q^{3/4}\nu^{-1/4}g^{-1/4}H^{-1/2}, \tag{18}$$

due to Rojas *et al.* [9]. As opposed to Eq. (17), the jump radius in Eq. (18) is not scaled by the free parameters of the flow, but by the depth of the liquid downstream of the jump.

The basic premise of Eqs. (17) and (18), both free of the surface tension, σ, is that $kl \ll 1$ in Eq. (16). In the opposite limit of $kl \gg 1$, Eq. (16) approximates to $\omega \simeq \sqrt{(\sigma h_0)/\rho}\,k^2$. Forcing the condition, $k \sim h_0^{-1}$, one gets the phase velocity, $v_{\rm p} = \omega/k \simeq \sqrt{\sigma/(\rho h_0)}$. Thereafter, using the condition $v_0 \simeq v_{\rm p} \simeq \sqrt{\sigma/(\rho h_0)}$ and following the same line of reasoning that led to Eq. (17), one arrives at a scaling relation,

$$r_{\rm J} \sim Q^{3/4}\rho^{1/4}\nu^{-1/4}\sigma^{-1/4}, \tag{19}$$

due to Bhagat *et al.* [18]. The noteworthy aspect of Eq. (19) is that it is free of gravity, g, but depends on the surface tension, σ. The scaling formula of Eq. (19) can also be derived by forcing the condition, $H \simeq l$, in Eq. (18) [18].

Looking at Eq. (16), we realize that Eqs. (17) and (18), both dependent on gravity but free of surface tension, are valid in the limit of $kl \ll 1$ (or $\lambda \gg l$). In contrast, Eq. (19), free of gravity but dependent on surface tension, is valid in the opposite limit of $kl \gg 1$ (or $\lambda \ll l$). The crossover from the former regime to the latter happens when $\lambda \sim l$. The different scaling relations here show that a shallow flow can generally accommodate the effects of both gravity and capillarity [19].

Our derivation of the scaling formulas in Eqs. (17), (18), and (19) is based on the matching of viscous and dynamical time scales when the flow becomes critical. An alternative approach to the same end is through a first-order dynamical system in the steady flow. A relevant critical point of this dynamical system is a spiral [2,4,7]. However, a physical flow must be single valued and cannot have a spiral profile. Therefore, close to the spiral critical point, an inner solution is matched to an outer solution through a jump discontinuity [4,7]. The inner and outer solutions are uniquely characterized by the inner and outer boundary conditions, respectively [7]. Moreover, the matching of these solutions at the jump implies that its location is also determined by the boundary conditions. This is seen in both radial flows [7] and channel flows [27].

V. WAVE BLOCKING AT THE ANALOG HORIZON

We subject the liquid outflow to a high-frequency traveling wave under the WKB approximation [6]. When both $\nu = 0$ and $\sigma = 0$, the traveling wave does not destabilize the flow [6]. However, just outside the event horizon, viscosity causes a large divergence in the amplitude of the wave that propagates against the outward bulk flow [6]. Since the destabilizing effect of viscosity is known already [6], we ignore the viscosity-dependent terms (i.e., set $\nu = 0$) in Eq. (11). Thereafter, what

remains of Eq. (11) is subjected to a solution of the form $f'(r, t) = p(r) \exp(-i\omega t)$, which leads to

$$\left(v_0^2 - gh_0\right)\frac{d^2p}{dr^2} + \left[\frac{1}{v_0}\frac{d}{dr}\left(v_0^3 - v_0 gh_0\right) - 2i\omega v_0\right]\frac{dp}{dr} - \left(\omega^2 + 2i\omega\frac{dv_0}{dr}\right)p$$
$$= -\frac{\sigma h_0}{\rho}\left(\frac{d^4p}{dr^4} - \frac{2}{r}\frac{d^3p}{dr^3} + \frac{3}{r^2}\frac{d^2p}{dr^2} - \frac{3}{r^3}\frac{dp}{dr}\right). \tag{20}$$

For the spatial part of the solution, we prescribe $p(r) = e^s$, with $s \equiv s(r)$ given by a converging power series [6],

$$s(r) = \sum_{n=-1}^{\infty} \frac{\tilde{k}_n(r)}{\omega^n}. \tag{21}$$

The convergence is ensured if the frequency, ω, is high, so that any term in the power series of Eq. (21) becomes much smaller than its preceding term, i.e., $\omega^{-(n+1)}\tilde{k}_{n+1} \ll \omega^{-n}\tilde{k}_n$. This condition is physically satisfied if the wavelength is smaller than a characteristic length scale of the flow, which, in this instance, is the jump radius itself. Hence, under the WKB approximation, only the first two terms are significant, with the former contributing to the phase of the travelling perturbation and the latter to its amplitude [6].

With surface tension included, the highest derivative in Eq. (20) is of the quartic order. In applying the WKB approximation, we, therefore, adopt an iterative approach. First, we set $\sigma = 0$, and write all $\tilde{k}_n$ in $s(r)$ as k_n, with the latter implying the solution series of $s(r)$ without surface tension. Then, accounting for the first two terms in the series of $s(r)$, along with the approximation that $\omega k_{-1} \gg k_0$, we gather all the coefficients of ω^2 (the highest order of ω) to arrive at [6]

$$k_{-1} = i \int \frac{1}{v_0 \mp \sqrt{gh_0}}\, dr. \tag{22}$$

Solving likewise for the coefficients of ω gives us

$$k_0 = -\frac{1}{2}\ln(v_0\sqrt{gh_0}) + C, \tag{23}$$

in which C is an integration constant. The convergence of $s(r) \simeq \omega k_{-1} + k_0$ can be verified self-consistently from Eqs. (22) and (23) by showing that $\omega k_{-1} \gg k_0$ [6].

Now we take up Eq. (20) with $\sigma \neq 0$, and in it we apply $s(r)$ as given in Eq. (21). The highest order of ω in terms that are explicitly without σ is ω^2, and the highest order of ω in terms that explicitly have σ is ω^4. Gathering the former from the left hand side and the latter from the right hand side gives

$$\left(v_0^2 - gh_0\right)\left(\frac{d\tilde{k}_{-1}}{dr}\right)^2 - 2iv_0\frac{d\tilde{k}_{-1}}{dr} - 1 \simeq -\frac{\sigma h_0}{\rho}\left(\frac{dk_{-1}}{dr}\right)^4\omega^2. \tag{24}$$

In our iterative approach we have approximated $\tilde{k}_{-1} \simeq k_{-1}$ on the right hand side of Eq. (24), where σ is explicitly present. This approximation is valid for small values of σ, whereby the capillary length, l, will be much smaller than the wavelength of the traveling perturbation in the shallow-water flow. Solving the quadratic form of $d\tilde{k}_{-1}/dr$ in Eq. (24), we get

$$\tilde{k}_{-1} \simeq k_{-1} \pm i \int \frac{\omega^2 l^2 \sqrt{gh_0}}{2(v_0 \mp \sqrt{gh_0})^4}\, dr. \tag{25}$$

The second term on the right hand side of Eq. (25) adds a surface-tension-dependent correction to what we already know from Eq. (22). This correction is of the order of ω^2, and appears to be dominant over k_{-1}. This, however, is not really the case. Noting that the wavelength $\lambda(r) = 2\pi(v_0 \mp \sqrt{gh_0})/\omega$, we immediately see that the correction term in Eq. (25) is subdominant to k_{-1}, when $l \ll \lambda$. This validates our iterative method self-consistently.

After ω^2, the next order is of ω in all the terms that are explicitly free of σ, while terms that explicitly bear σ come with ω^3 as the next higher order, following ω^4. Terms with ω on the left hand side and ω^3 on the right hand side lead to

$$
2\left[\left(v_0^2 - gh_0\right)\frac{d\tilde{k}_{-1}}{dr} - iv_0\right]\frac{d\tilde{k}_0}{dr} + \frac{1}{v_0}\frac{d}{dr}\left[v_0\left(v_0^2 - gh_0\right)\frac{d\tilde{k}_{-1}}{dr}\right] - 2i\frac{dv_0}{dr}
$$
$$
\simeq -\frac{\sigma\omega^2 h_0}{\rho}\left[4\left(\frac{dk_{-1}}{dr}\right)^3\frac{dk_0}{dr} + 6\left(\frac{dk_{-1}}{dr}\right)^2\frac{d^2k_{-1}}{dr^2} - \frac{2}{r}\left(\frac{dk_{-1}}{dr}\right)^3\right] \simeq \frac{2\sigma\omega^2 h_0}{\rho r}\left(\frac{dk_{-1}}{dr}\right)^3, \quad (26)
$$

in which, on the right hand side, we ultimately retain only the term that is most significant. Adopting the same line of reasoning, as has been done following Eq. (24), gives

$$
\tilde{k}_0 \simeq k_0 \mp \int \frac{\omega^2 l^2}{gh_0 r(\mathcal{F} \mp 1)^3}\,dr. \tag{27}
$$

The second term on the right hand side of Eq. (27) adds a correction to k_0, as given in Eq. (23). We stress once again that $\omega^2 l^2$ renders the correction term subdominant to k_0.

In the traveling perturbation, which can now be written as $f'(r,t) \simeq \exp(\omega\tilde{k}_{-1} + \tilde{k}_0 - i\omega t)$, we see that $\tilde{k}_{-1}$ contributes to the phase and $\tilde{k}_0$ contributes to the amplitude. Since we are concerned with the stability of the traveling wave, we extract its amplitude, which, expressed in full, is

$$
|f'(r,t)| \sim (v_0\sqrt{gh_0})^{-1/2}\exp\left[\mp\int \frac{\omega^2 l^2}{gh_0 r(\mathcal{F} \mp 1)^3}\,dr\right]. \tag{28}
$$

The upper sign in Eq. (28) pertains to a wave that propagates upstream against the outward radial bulk flow of liquid. We look at this case closely. In the subcritical region of the flow, where $\mathcal{F} < 1$, the integrand in Eq. (28) is negative. As the wave approaches the singularity, which is owed entirely to gravity and where $\mathcal{F} = 1$, the integrand diverges. With the negative sign outside the integral, the overall outcome is $|f'(r,t)| \longrightarrow \infty$; i.e., the wave suffers an instability. The very opposite of all this occurs just inside the singularity. Here, with $\mathcal{F} > 1$, the integral acquires a negative sign overall, which results in $|f'(r,t)| \longrightarrow 0$. This discontinuity in the inward propagation of the wave is forced by the term with surface tension in Eq. (28). Since this happens at the analog event horizon, we can say that the horizon acts like an impenetrable barrier (a fluid analog of a white hole) against incoming waves from the subcritical region.

The foregoing theoretical claim receives support from an experiment carried out by Kate *et al.* [16]. The experiment was on the interaction of two adjacent hydraulic jumps formed by normally impinging water jets, of which one was static and the other was mobile [16].[1] The photograph in Fig. 1, taken by Kate, Das, and Chakraborty [16], shows clearly that, when one water jet is moved close to the other one, the water trapped along the stagnation line between the circular hydraulic jumps created by the two jets is raised to a greater height than the rest of the flow. A steady archlike upwash fountain thus comes to stand by itself (like a standing wall of water). The jump formed by the moving water jet is like a subcritical disturbance propagating upstream towards the jump formed due to the static jet. This disturbance is blocked by the static jump, which is an unyielding fluid white hole. Consequently, as the disturbance approaches the static circular jump, the free-surface height of the water increases dramatically due to the accumulation of water. This agrees with what we have concluded from Eq. (28), namely, the blocking of a wave approaching the singularity from the subcritical region. Furthermore, our conjecture is that the steady upwash fountain between the

---

[1]This experiment [16] supported a theory of the formation of circular hydraulic jumps due to viscosity [6]. Since surface tension has as much of a role to play as viscosity to form circular hydraulic jumps, we refer to the same experiment in support of our present study. In a qualitative sense, both viscosity and surface tension are responsible for blocking waves against the bulk flow at the event horizon.
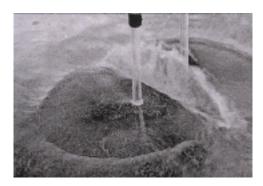
FIG. 1. Oblique view of a steady arch-shaped upwash fountain between two circular hydraulic jumps. The close approach of one jump to the other is like a collision of two white holes. The photograph is by the courtesy of Kate, Das, and Chakraborty [16].

two contiguous hydraulic jumps could be the fluid analog of the compression and bulging of the spacetime geometry between two colliding general relativistic white holes. As a caveat we point out that, in the experiment of Kate *et al.* [16], the disturbance propagating upstream is not axisymmetric about the static jump, and so the standing wall of water between the two jumps is not axisymmetric either.

The derivation of the time-averaged energy flux of the perturbation in a two-dimensional radial flow has been established in a previous study [6]. By the same method, energy fluctuations of the first-order disappear upon time averaging, but second-order terms survive to contribute to the time-averaged energy flux, F. With this contribution, we can show that $F \sim \langle |f'(r, t)|^2 \rangle$. Since $|f'(r, t)|$ diverges just outside the analog event horizon for a wave propagating against the radial outflow, F will also exhibit a similar divergence about the same spatial location [6].

VI. CONCLUDING REMARKS

This theoretical study on the effect of surface tension in type-I hydraulic jumps has revealed two types of instabilities. One, as in Eq. (15), results from surface tension and the cylindrical geometry of the shallow flow. The other, as in Eq. (28), is the combined outcome of gravity and surface tension. Gravity waves define the location of the singularity in Eq. (28), but the divergence just outside the horizon singularity occurs because of surface tension. Surface tension is known to cause other instabilities as well. For instance, the breaking of the axial symmetry of the steady circular hydraulic jump is an instability for which surface tension is responsible [7,14].

The scaling formula proposed by Bhagat *et al.* [18], as in Eq. (19), has been the subject of close scrutiny because of its exclusion of gravity [10,19]. However, such scaling has been argued to be valid for developing hydraulic jumps in the capillary regime [19]. In any case, it is known that surface tension is much more significant than gravity for circular jumps of small radii [8]. Femtocups created through gravity-free hydraulic jumps of molten metals are a case in point [28].

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