

## Hydrodynamics of active particles in viscosity gradients

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In this work, we analyze the motion of an active particle, modeled as a spherical squirmer, in linearly varying viscosity fields. In general, the presence of a particle will disturb a background viscosity field and the disturbance generated depends on the boundary conditions imposed by the particle on the viscosity field. We find that, irrespective of the details of the disturbance, active squirmer-type particle tend to align down viscosity gradients (negative viscotaxis). However, the rate of rotation and the swimming speed along the gradient do depend on the details of the interaction of the particle and the background viscosity field. In addition, we explore the relative importance on the dynamics of the local viscosity changes on the surface of active particles versus the (nonlocal) changes in the flow field due to spatially varying viscosity (from that of a homogeneous fluid). We show that the relative importance of local versus nonlocal effects depends crucially on the boundary conditions imposed by the particle on the field. This work demonstrates the dangers in neglecting the disturbance of the background viscosity caused by the particle as well as in using the local effects alone to capture the particle motion.

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### I. INTRODUCTION

Active particles are particles or organisms that convert the stored energy or energy from the surroundings into directed motion [1], and a suspension of active particles can be referred to as active matter. We are concerned here with the micron-sized active particles such as biological swimming microorganisms or the synthetic microrobots. Because of the ubiquity of microorganisms in nature, as well as aided by tremendous progress in microfluidic experimental techniques, there has been an enormous amount of research on the motion of active particles in viscous fluids, including developing an understanding of the physics governing microorganism motility [2–10].

Active particles typically reside in the gradients of heat, light, or chemicals and often react to these fields by reorienting and moving along the gradients, a phenomenon known as *taxis*. By exploiting this tendency, one can sort or control the active matter by imposing external gradients. A relatively unexplored form of taxis is viscotaxis—directed motion in viscosity gradients. Several microorganisms move through viscosity gradients and hence exhibit viscotaxis. For instance *Leptospira* and *Spiroplasma* have been observed to display positive viscotaxis, moving up viscosity gradients [11–14], while *Escherichia coli* displays negative viscotaxis [15]. Green microalgae, *Chlamydomonas reinhardtii*, display complex dynamics in the presence of spatial variations in viscosity. If viscosity gradients are weak, they tend to accumulate in the high viscosity regions due to slower speeds, but in the presence of strong viscosity gradients, they reorient toward the low viscosity regions, displaying negative viscotaxis [16,17]. *Helicobacter pylori*, a bacterium commonly found in our guts, also swims through viscosity gradients as it propels by locally lowering

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the viscosity of the surrounding mucus layer [18,19]. Hence, a physical description of the fluid dynamics of viscotaxis allows us to understand the consequences on the motility of microorganisms in nature, but also presents a mechanism by which to control active matter, both natural and synthetic.

Several researchers have analyzed the motion of particles, passive or active, in viscosity gradients. Initial research in this field focused on particles moving through a fluid that is otherwise homogeneous in the absence of the particle, but the presence of the particle generates a disturbance (inhomogeneity) in fluid properties such as viscosity. For instance, a particle hotter than the surrounding fluid disturbs the temperature and thus the viscosity, thereby generating the gradients. Research of this sort started with the calculation of the force and torque acting on a hot passive particle [20]. It was found that a dipolar temperature distribution on the particle, that causes a similar viscosity distribution in the nearby fluid, induces a coupling between the force (respectively, torque) and the angular velocity (respectively, translational velocity). Such coupling is absent for nonskew particles (e.g., sphere, spheroid, disk, etc.) in the homogeneous Newtonian fluids [21]. Later, the motion of an active particle in viscosity gradients caused by variation in the concentration of nearby nutrients was analyzed [22]. This work modeled active particles as a spherical squirmer with fixed power and assumed a weak dependence of viscosity on the nutrient concentration. In the squirmer model, an active particle propels due to prescribed slip on its surface, which is ultimately a manifestation of the coordinated beating of cilia on the surface of a ciliated organism or the chemical reactions occurring on the surface of a phoretic particle. It was found that the speed of a swimmer can either increase or decrease depending on the relative importance of the advective to the diffusive transport rate of the viscosity. Recently, the motion of a spheroidal squirmer in the nutrient induced viscosity gradients was also analyzed [23]. Accounting for a strong coupling between the nutrient concentration and the viscosity, it was found that nutrient advection and viscosity gradients significantly affect the swimming and feeding performance but not the aspect ratio of the particle.

Recent research explores the motion of active particles in preexisting (or background) viscosity gradients. Such analysis is generally carried out by representing active particles as prototypical model swimmers and often by treating the background viscosity field as linear. Initial research in this field modeled active particles as spheres connected by rods, where each sphere locally sees a constant background viscosity, and is acted on by a fixed active (thrust) force, while hydrodynamic interactions are neglected [24]. Using this approach, it was found that nonchiral (linear) active particles generically move up gradients unless they are uniaxial, in which case they do not exhibit viscotaxis by symmetry. Otherwise, nonlinear or chiral swimmers can even move down the gradients. As *Leptospira* and *Spiroplasma* exhibit nonuniaxial shapes while *E. coli* is chiral, this work possibly explains the viscotaxis of these organisms based on their shape and the resulting hydrodynamics. Later, the motion of spherical squirmers in viscosity gradients was analyzed, taking into account locally varying hydrodynamic forces and modifications in the flow due to viscosity differences, and they were found to move down viscosity gradients [25]. A different model microorganism, Taylor's swimming sheet that propels by passing traveling waves along its surface, moving along or against the viscosity gradients was also analyzed [26]. It was found to speed up in the presence of viscosity gradients irrespective of its direction of propulsion. Recently, the axisymmetric motion of a synthetic helical swimmer crossing a sharp as well as diffuse viscosity interface was analyzed [27]. It was found that a head-first (respectively, tail first) swimmer moving up the gradients experiences the speed reduction (respectively, enhancement) but a swimmer moving down the gradient always undergoes the speed reduction irrespective of its orientation.

A viscosity gradient can arise due to the spatial variation of a scalar like the temperature, salt or nutrient concentration in a fluid that is ultimately coupled to viscosity. However, the introduction of a particle will generally disturb a preexisting viscosity field to satisfy boundary conditions for the scalar field on the surface of the particle (because the properties of the particle are different from those of the fluid). For instance, even at steady-state a particle in an otherwise linear temperature field will disturb the background field unless the thermal conductivity of the particle and the fluid are identical. The disturbance generated by the particle tends to vanish far from the particle, however,

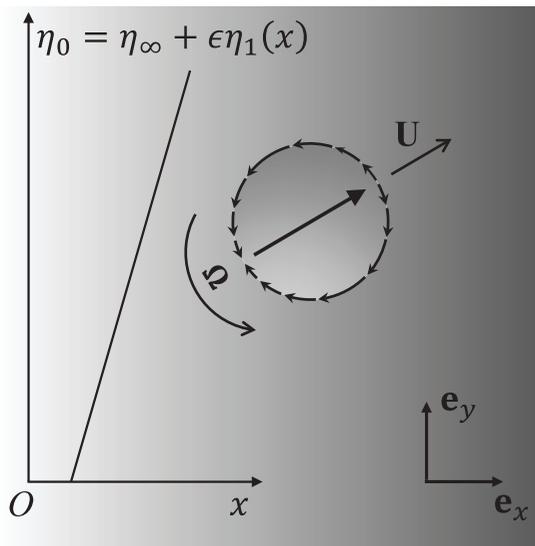


FIG. 1. A schematic showing the motion of an active particle in linear viscosity fields and the associated coordinate system. The radius of the particle is  $a$ . It translates and rotates with the velocities  $\mathbf{U}$  and  $\boldsymbol{\Omega}$ , respectively. The ambient viscosity  $\eta_0$  increases linearly with  $x$ .

near the particle, it is the same order of magnitude as that of the preexisting (background) viscosity field. Despite this, most previous work on active particles in viscosity gradients have neglected the disturbance caused by the particle to the ambient viscosity field [16,24,25,27]. In this work we aim to address this gap and investigate the effect of the disturbance of the viscosity induced by the presence of the particle and the consequence on viscotaxis. Furthermore, several previous studies used only the local effects of changes in viscosity not the (nonlocal) modification of the flow (from that in a homogeneous fluid), to study the particle motion in viscosity gradients [24,27]. These nonlocal effects may be as important as the complementary local effects. In fact, the nonlocal effects have been observed to be more important than the local effects for particles (passive and active) in non-Newtonian fluids [28–31]. In light of this, we perform a quantitative comparison between local and nonlocal effects for the motion of an active particle in viscosity gradients.

We organize the paper as follows. We provide the mathematical formulation associated with the motion of active particles in viscosity gradients in Sec. II, and describe the solution methodology in Sec. III. We then discuss the influence of disturbance viscosity in Sec. IV, and distinguish between the local and nonlocal effects, and quantify these effects in Sec. V. We finally provide few concluding remarks in Sec. VI.

## II. SWIMMING IN VISCOSITY GRADIENTS

We consider the motion of an active particle in an otherwise quiescent Newtonian fluid (see Fig. 1 for a schematic). In the absence of the particle, the fluid viscosity,  $\eta_0(\mathbf{x})$ , is nonuniform due to corresponding spatial variations in the temperature, salt, or nutrient concentration. The particle disturbs the ambient (or background) viscosity field as its properties (like thermal conductivity) are usually different from those of fluid. We denote this disturbance viscosity by  $\eta'(\mathbf{x})$  and hence the viscosity in the presence of particle is  $\eta(\mathbf{x}) = \eta_0(\mathbf{x}) + \eta'(\mathbf{x})$ . We are interested in leading-order effects and so simply prescribe a linear variation of the ambient viscosity field,  $\eta_0 = \eta_\infty + \eta_\infty \frac{x}{L}$ , where the variation itself occurs on the macroscopic length scale  $L$ . Usually, this length scale is much larger than the particle size  $a$ , i.e.,  $\epsilon = \frac{a}{L} \ll 1$ . This means that in the vicinity of the particle,

the ambient viscosity varies weakly,  $\eta_0 = \eta_\infty + \frac{\eta_\infty a}{L} \frac{x}{a} = \eta_\infty + \epsilon \eta_\infty x/a = \eta_\infty + \epsilon \eta_1$ , where  $\eta_1 = \eta_\infty x/a$ . Noting the absolute value of the viscosity does not affect the dynamics of an active particle with a prescribed gait [5], we choose an arbitrary point in the fluid as the origin in the laboratory frame of reference, and we denote the position vector with respect to the origin and the unit vectors in the Cartesian coordinate system, respectively, by  $\mathbf{x} = (x, y, z)$ ,  $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ . In this frame,  $\mathbf{x} = \mathbf{x}_c$  denotes the center of the particle. Also,  $\eta_\infty$  is some reference viscosity while  $\epsilon$  characterizes the deviation from this reference viscosity.

Neglecting fluid inertia, the flow is governed by the incompressible Stokes equations

$$\nabla \cdot \boldsymbol{\sigma} = \mathbf{0}, \quad (1)$$

$$\nabla \cdot \mathbf{u} = 0, \quad (2)$$

where  $\mathbf{u}$  is the velocity field and  $\boldsymbol{\sigma}$  the stress tensor defined as

$$\boldsymbol{\sigma} = -p\mathbf{I} + \eta_\infty \dot{\boldsymbol{\gamma}} + \boldsymbol{\tau}_{NN}, \quad (3)$$

where  $\dot{\boldsymbol{\gamma}} = \nabla \mathbf{u} + (\nabla \mathbf{u})^T$  and we have defined

$$\boldsymbol{\tau}_{NN} = (\eta(\mathbf{x}) - \eta_\infty) \dot{\boldsymbol{\gamma}}, \quad (4)$$

which captures the difference in stress due to changes in viscosity from the reference value  $\eta_\infty$ .

The transport of the scalar (e.g., temperature, salt, or nutrient concentration) that determines the viscosity is governed by an advection-diffusion equation. For weak variations of the scalar, the change in the viscosity is linearly proportional to the change in the scalar which means that the viscosity transport is also governed by a similar advection-diffusion equation. The relative order of magnitude of the advective to the diffusive transport of viscosity is characterized by the Péclet number  $Pe$  which can be written as  $Pe = RePr$ . For swimming in water, the Reynolds number  $Re \sim 10^{-5}$ – $0.1$  [5], the Prandtl number  $Pr \sim 7$ – $10^3$  [32–34] and hence  $Pe \sim 10^{-4}$ – $10^2$ . Here, the low and relatively high ( $\sim 0.1$ ) values of  $Re$  occur, respectively, for the micron sized organism like *E. coli* and larger organism like *Paramecium* while the low and high values of  $Pr$ , respectively, are due to the high and low diffusive scalar like the temperature and the salt or nutrient concentration. The Péclet number is small for the motion of the small to moderate sized organisms like *E. coli* or human spermatozoon in water with temperature or salt variations. In this case, the viscosity transport is simply governed by a Laplace equation,

$$\nabla^2 \eta = 0. \quad (5)$$

Here, the Reynolds number  $Re$  is the ratio of the inertial to the viscous forces while the Prandtl number  $Pr$  is the ratio of the diffusivities of the momentum to the viscosity transport.

Far away from the particle, the flow decays to zero while the viscosity approaches the ambient viscosity

$$\mathbf{u} \rightarrow \mathbf{0}, \quad \eta \rightarrow \eta_0(\mathbf{x}) \quad \text{as} \quad r = |\mathbf{r}| \rightarrow \infty, \quad (6)$$

where  $\mathbf{r} = \mathbf{x} - \mathbf{x}_c$ . On the particle, the fluid velocity is same as the particle surface velocity as a consequence of the kinematic and no-slip boundary conditions. The particle itself is propelling with a translational velocity  $\mathbf{U}$  and an angular velocity  $\boldsymbol{\Omega}$  due to the activity that is embedded in the slip velocity on the particle surface  $\mathbf{u}^s$ . Thus, the fluid velocity on the particle is

$$\mathbf{u} = \mathbf{U} + \boldsymbol{\Omega} \times \mathbf{r} + \mathbf{u}^s \quad \text{on} \quad S_p, \quad (7)$$

where  $S_p$  denotes the particle surface. We neglect the particle inertia and assume the particle is neutrally buoyant. This implies the hydrodynamic force and torque acting on the particle are zero:

$$\mathbf{F} = \int_{S_p} \mathbf{n} \cdot \boldsymbol{\sigma} dS = \mathbf{0}, \quad (8)$$

$$\mathbf{L} = \int_{S_p} \mathbf{r} \times (\mathbf{n} \cdot \boldsymbol{\sigma}) dS = \mathbf{0}, \quad (9)$$

where  $\mathbf{n}$  is an unit normal to the surface that points into the fluid.

We model the active particle as a spherical squirmer of radius  $a$  that has only the tangential squirming modes [35–37]. This model is a good representation of the ciliated organisms like *Paramecium* or *Opalina* which generate slip through a synchronous motion of a large number of cilia on their surface. In this model, the slip velocity  $\mathbf{u}^s$  is given by

$$\mathbf{u}^s = - \sum_{n=1}^{\infty} B_n W_n(\mathbf{p} \cdot \mathbf{e}_r) \mathbf{p} \cdot (\mathbf{I} - \mathbf{e}_r \mathbf{e}_r), \quad W_n(x) = \frac{2}{n(n+1)} P_n'(x). \quad (10)$$

Here  $\mathbf{p}$  is the orientation of the particle,  $\mathbf{e}_r = \mathbf{r}/r$ ,  $P_n$  is the Legendre polynomial of degree  $n$ , and  $B_n$  are the squirming modes. In homogeneous Newtonian fluids, only the  $B_1$  mode contributes to the swimming speed while the  $B_2$  mode leads to the slowest decaying flow field and hence it determines the far-field representation of the swimmer. The ratio  $\alpha = B_2/B_1$  can be used to distinguish the three types of swimmers. So-called pusher swimmers, like *E. coli*, push the fluid along their axis while drawing fluid from their sides. Pullers, like *Chlamydomonas*, do the opposite, in that they pull the fluid along their axis while ejecting fluid from their sides. Neutral swimmers like *Volvox carteri* exhibit the flow signature similar to that of a potential dipole. Puller, pusher and neutral swimmers are characterized by  $\alpha > 0$ ,  $< 0$ , and  $= 0$ , respectively.

### III. SOLUTION METHOD

We apply the reciprocal theorem to the unknown flow of the present problem and the auxiliary flow due to a rigid body motion of a spherical particle in a homogeneous fluid of viscosity  $\eta_\infty$ . After making use of the boundary conditions on the particle surface in both problems, we find the swimming velocity of the active particle to be

$$\mathbf{U} = \hat{\mathbf{R}}_{\mathbf{F}\mathbf{U}}^{-1} \cdot \left[ \int_{S_p} \mathbf{u}^s \cdot (\mathbf{n} \cdot \hat{\boldsymbol{\tau}}_{\mathbf{U}}) dS + \mathbf{F}_{NN}^l + \int (\nabla \cdot \boldsymbol{\tau}_{NN}) \cdot \hat{\mathbf{G}}_{\mathbf{U}} dV \right]. \quad (11)$$

Here  $\mathbf{U} = [\mathbf{U} \ \boldsymbol{\Omega}]^T$  is a six-dimensional vector containing the particle's translational and angular velocity while  $\mathbf{F}_{NN}^l = [\mathbf{F}_{NN}^l \ \mathbf{L}_{NN}^l]^T$  is also a six-dimensional vector consisting of the force and torque due to the stress  $\boldsymbol{\tau}_{NN}$ :

$$\mathbf{F}_{NN}^l = \int_{S_p} \mathbf{n} \cdot \boldsymbol{\tau}_{NN} dS, \quad (12)$$

$$\mathbf{L}_{NN}^l = \int_{S_p} \mathbf{r} \times (\mathbf{n} \cdot \boldsymbol{\tau}_{NN}) dS. \quad (13)$$

We note that the variables with caret are associated with the auxiliary flow problem. Specifically,  $\hat{\mathbf{G}}_{\mathbf{U}} \cdot \hat{\mathbf{U}}$ ,  $\hat{\boldsymbol{\tau}}_{\mathbf{U}} \cdot \hat{\mathbf{U}}$ , respectively, are the flow field, and the stress tensor due to the rigid body motion of a sphere in a homogeneous fluid. The sphere experiences the hydrodynamic force or torque,  $\hat{\mathbf{F}} = -\hat{\mathbf{R}}_{\mathbf{F}\mathbf{U}} \cdot \hat{\mathbf{U}}$ . A detailed derivation may be found elsewhere [38]. Equation (11) explicitly divides the dynamics of an active particle in fluid with spatially varying viscosity into three contributions: the first term in the brackets on the right-hand side (when multiplied with the mobility) gives the translational and rotational velocity of the active particle in a homogeneous fluid with constant

viscosity; the second term represents the effect on the dynamics due to changes in the force and torque from that in a homogeneous fluid; the final term accounts for changes in the dynamics due to changes in the flow field that arise as a result of differences in the stress tensor from that of a homogeneous fluid with constant viscosity.

Separating Eq. (11) for the translational velocity and the angular velocity we have

$$\mathbf{U} = \hat{\mathbf{R}}_{FU}^{-1} \cdot \left[ \int_{S_p} \mathbf{u}^s \cdot (\mathbf{n} \cdot \hat{\mathbf{T}}_U) dS + \int_{S_p} \mathbf{n} \cdot \boldsymbol{\tau}_{NN} dS + \int (\nabla \cdot \boldsymbol{\tau}_{NN}) \cdot \hat{\mathbf{G}}_U dV \right], \quad (14)$$

$$\boldsymbol{\Omega} = \hat{\mathbf{R}}_{L\Omega}^{-1} \cdot \left[ \int_{S_p} \mathbf{u}^s \cdot (\mathbf{n} \cdot \hat{\mathbf{T}}_\Omega) dS + \int_{S_p} \mathbf{r} \times (\mathbf{n} \cdot \boldsymbol{\tau}_{NN}) dS + \int (\nabla \cdot \boldsymbol{\tau}_{NN}) \cdot \hat{\mathbf{G}}_\Omega dV \right]. \quad (15)$$

Here,  $\hat{\mathbf{G}}_U \cdot \hat{\mathbf{U}}$ ,  $\hat{\mathbf{T}}_U \cdot \hat{\mathbf{U}}$  (respectively,  $\hat{\mathbf{G}}_\Omega \cdot \hat{\boldsymbol{\Omega}}$ ,  $\hat{\mathbf{T}}_\Omega \cdot \hat{\boldsymbol{\Omega}}$ ) are the flow field and the stress tensor due to the translation (respectively, rotation) of a sphere in a homogeneous fluid with velocity  $\hat{\mathbf{U}}$  (respectively,  $\hat{\boldsymbol{\Omega}}$ ). The translating particle experiences the hydrodynamic force  $\hat{\mathbf{F}} = -\hat{\mathbf{R}}_{FU} \cdot \hat{\mathbf{U}}$ , while the rotating particle experiences the hydrodynamic torque  $\hat{\mathbf{L}} = -\hat{\mathbf{R}}_{L\Omega} \cdot \hat{\boldsymbol{\Omega}}$ . For a sphere,  $\hat{\mathbf{R}}_{FU} = 6\pi\eta_\infty a \mathbf{I}$  and  $\hat{\mathbf{R}}_{L\Omega} = 8\pi\eta_\infty a^3 \mathbf{I}$ .

We perform a regular perturbation in  $\epsilon$  and expand any variable  $f(\epsilon)$  as  $f = f_0 + \epsilon f_1 + \epsilon^2 f_2 + \dots$ . As  $\eta(\mathbf{x}) - \eta_\infty = \epsilon \eta_1 + \eta' \sim O(\epsilon)$ ,  $\boldsymbol{\tau}_{NN} = (\eta(\mathbf{x}) - \eta_\infty) \dot{\boldsymbol{\gamma}} \sim O(\epsilon)$ , at leading order, we have a swimmer moving through a homogeneous Newtonian fluid of viscosity  $\eta_\infty$ . The velocity of such swimmer is well known:

$$\mathbf{U}_0 = \frac{1}{6\pi\eta_\infty a} \int_{S_p} \mathbf{u}^s \cdot (\mathbf{n} \cdot \hat{\mathbf{T}}_U) dS = \frac{2}{3} B_1 \mathbf{p} \equiv \mathbf{U}_N, \quad (16)$$

$$\boldsymbol{\Omega}_0 = \frac{1}{8\pi\eta_\infty a^3} \int_{S_p} \mathbf{u}^s \cdot (\mathbf{n} \cdot \hat{\mathbf{T}}_\Omega) dS = \mathbf{0}. \quad (17)$$

At first order, we have

$$\mathbf{U}_1 = \frac{1}{6\pi\eta_\infty a} \int_{S_p} \mathbf{n} \cdot \boldsymbol{\tau}_{NN,1} dS + \frac{1}{6\pi\eta_\infty a} \int (\nabla \cdot \boldsymbol{\tau}_{NN,1}) \cdot \hat{\mathbf{G}}_U dV, \quad (18)$$

$$\boldsymbol{\Omega}_1 = \frac{1}{8\pi\eta_\infty a^3} \int_{S_p} \mathbf{r} \times (\mathbf{n} \cdot \boldsymbol{\tau}_{NN,1}) dS + \frac{1}{8\pi\eta_\infty a^3} \int (\nabla \cdot \boldsymbol{\tau}_{NN,1}) \cdot \hat{\mathbf{G}}_\Omega dV. \quad (19)$$

Here  $\epsilon \boldsymbol{\tau}_{NN,1} = (\eta(\mathbf{x}) - \eta_\infty) \dot{\boldsymbol{\gamma}}_0 = (\epsilon \eta_1 + \eta') \dot{\boldsymbol{\gamma}}_0$ , where  $\dot{\boldsymbol{\gamma}}_0 = \nabla \mathbf{u}_0 + (\nabla \mathbf{u}_0)^T$  is the rate of strain tensor associated with the leading-order flow  $\mathbf{u}_0$ . We see that the variation of viscosity from the constant reference viscosity  $\eta_\infty$  changes the traction locally on the surface of the active particle but also modifies the flow from its value in the homogeneous fluid  $\mathbf{u}_0$ . Both these changes alter the swimming velocity from that in the homogeneous fluid  $\mathbf{U}_0$ ,  $\boldsymbol{\Omega}_0$  represented by an extra velocity  $\mathbf{U}_1$ ,  $\boldsymbol{\Omega}_1$  [see Eqs. (18) and (19)]. The effect of locally modifying the viscosity from that of a constant viscosity  $\eta_\infty$  to  $\epsilon \eta_1 + \eta'$  alone (without the effect of the associated change in the flow from  $\mathbf{u}_0$ ) is referred to as the local effect in line with the literature on the motion of particles in shear-thinning fluids and is given by the surface integral terms in Eqs. (18) and (19). The complementary effect due to modifying the flow from the homogeneous fluid value  $\mathbf{u}_0$  (without the associated change in the viscosity on the surface of the particle) is referred to as the nonlocal effect and this is given by the volume integral terms in Eqs. (18) and (19).

If the properties of the particle are same as those of the fluid, the particle does not produce any disturbance ( $\eta' = 0$ ) and it propels with the first order velocities [25]

$$\mathbf{U}_1 = -\frac{aB_2}{5} (\mathbf{I} - 3\mathbf{p}\mathbf{p}) \cdot \nabla \left( \frac{\eta_1}{\eta_\infty} \right), \quad (20)$$

$$\boldsymbol{\Omega}_1 = -\frac{1}{2} \mathbf{U}_N \times \nabla \left( \frac{\eta_1}{\eta_\infty} \right). \quad (21)$$

The effect of the ambient viscosity is to turn the particle to align it against the viscosity gradient (negative viscotaxis). In this steady-state orientation, relative to their speed in the homogeneous fluid, the pushers speed up, the pullers slow down and the neutral swimmers do not experience any speed change. We next demonstrate the significance of the disturbance viscosity.

#### IV. EFFECT OF THE DISTURBANCE VISCOSITY

Recall the viscosity field satisfies the Laplace equation and  $\eta = \eta_0 + \eta'$ . As the ambient viscosity  $\eta_0$  is linear in position, the disturbance viscosity  $\eta'$  must also satisfy a Laplace equation

$$\nabla^2 \eta' = 0, \quad (22)$$

and also decays to zero far from the particle

$$\eta' = 0 \quad \text{as} \quad r \rightarrow \infty. \quad (23)$$

The general solution of Eq. (22), satisfying Eq. (23), is given by

$$\eta' = \sum_{k=0}^{\infty} \sum_{m=0}^k r^{-k-1} (A_{k,m} \cos m\theta + B_{k,m} \sin m\theta) P_k^m(\cos \phi), \quad (24)$$

where  $A_{k,m}$ ,  $B_{k,m}$  are the constant coefficients,  $\phi$ ,  $\theta$  are the polar and azimuthal angles in the spherical coordinate system located at the center of the particle while  $P_k^m$  is the associated Legendre polynomial of degree  $k$  and order  $m$ . Here, the terms proportional to  $1/r$ ,  $1/r^2$ , respectively, are the source and source-dipole, essentially the Green's functions of the Laplace equation and its derivative. The exact structure of the disturbance viscosity and its influence on the swimming velocities depend on the boundary condition on the particle surface. This surface can be impermeable (respectively, insulating) to the nutrient or the salt concentration (respectively, to the temperature) that is responsible for the viscosity variations. In this case, a no-flux condition for the viscosity holds on the particle surface,  $\mathbf{n} \cdot \nabla \eta = 0$  at  $r = a$ . Alternatively, the particle surface can be at a constant temperature which translates to a constant viscosity condition on the particle surface,  $\eta = \eta_p = \text{const.}$  at  $r = a$ . We hereby analyze these two cases separately.

##### A. No-flux

The no-flux condition can be simplified to

$$\left. \frac{\partial \eta'}{\partial r} \right|_{r=a} = -\frac{\epsilon \eta_{\infty}}{a} \sin \phi \cos \theta. \quad (25)$$

The disturbance viscosity that satisfies this constraint is given by

$$\eta' = \frac{\epsilon a^2 \eta_{\infty}}{2r^3} (x - x_c) = \frac{\epsilon a^3}{2r^3} \eta_1 (x - x_c). \quad (26)$$

The no-flux condition prevents the occurrence of the source term and the solution is a source-dipole. This disturbance viscosity has the same angular dependence as that of the ambient viscosity, however, unlike the ambient viscosity, the disturbance viscosity decays with position (see Fig. 2). This leads to a swimming velocity due to the disturbance viscosity of a similar form as that due to the ambient viscosity but with a reduced magnitude. The change in the velocity of the particle due to the disturbance viscosity alone (denoted by a prime) is given by

$$\mathbf{U}'_1 = -\frac{aB_2}{60} (\mathbf{I} - 3\mathbf{pp}) \cdot \nabla \left( \frac{\eta_1}{\eta_{\infty}} \right), \quad (27)$$

$$\boldsymbol{\Omega}'_1 = -\frac{1}{8} \mathbf{U}_N \times \nabla \left( \frac{\eta_1}{\eta_{\infty}} \right). \quad (28)$$

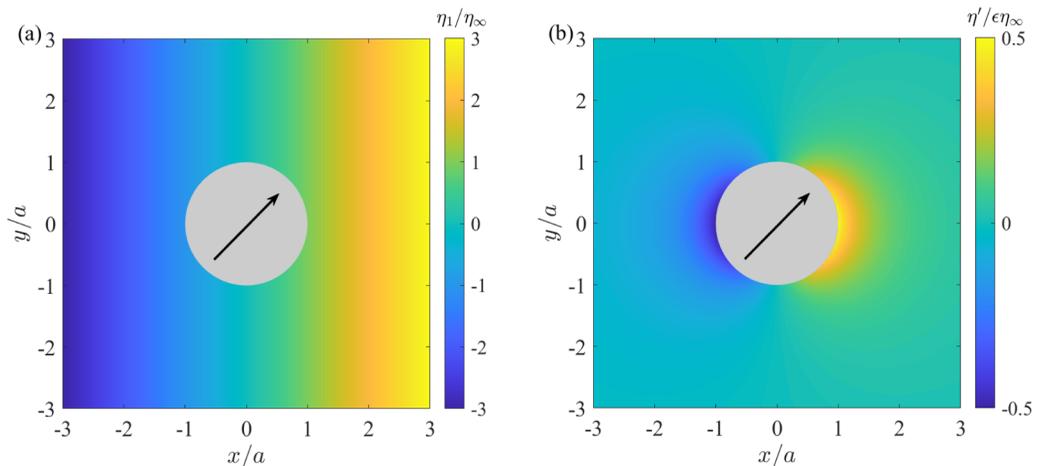


FIG. 2. (a) The ambient viscosity relative to the reference viscosity and (b) the disturbance viscosity due to the no-flux condition at the instant when the particle's center is at the origin in the laboratory frame, i.e.,  $\mathbf{x}_c = \mathbf{0}$ . These viscosity distributions do not depend on the orientation of the particle.

Combining the contribution of the ambient and the disturbance viscosities, the change in the swimming velocity of a particle with no-flux condition is

$$\mathbf{U}_1 = -\frac{13}{60}aB_2(\mathbf{I} - 3\mathbf{p}\mathbf{p}) \cdot \nabla \left( \frac{\eta_1}{\eta_\infty} \right), \quad (29)$$

$$\mathbf{\Omega}_1 = -\frac{5}{8}\mathbf{U}_N \times \nabla \left( \frac{\eta_1}{\eta_\infty} \right). \quad (30)$$

We see that the disturbance viscosity does not change the physics predicted by considering only the ambient viscosity, it merely increases the rate at which the particle rotates to align opposite to the viscosity gradients and it also increases the speed change experienced by the pushers and pullers in the steady-state orientation (pushers swim faster and pullers swim slower while neutral swimmers propel at the same speed compared to that in the homogeneous fluid).

### B. Constant viscosity

The constant viscosity condition simplifies to

$$\eta'|_{r=a} = \eta_c - \epsilon\eta_\infty \sin\phi \cos\theta, \quad (31)$$

where  $\eta_c = \eta_p - \eta_\infty - \epsilon\eta_\infty x_c/a$ . This constraint has two parts, the first part is constant,  $\eta_c$ , while the second varies spatially over the particle surface. The constant is enforced through a source term while the spatial variation is satisfied through a source-dipole term. Putting these two together, the disturbance viscosity is

$$\eta' = \frac{\eta_c a}{r} - \frac{\epsilon a^2 \eta_\infty}{r^3}(x - x_c) = \frac{\eta_c a}{r} - \frac{\epsilon a^3}{r^3}\eta_1(x - x_c). \quad (32)$$

We see that the source dipole here is a factor ( $= -2$ ) times the source dipole due to the no-flux condition in Eq. (26), hence the swimming velocity due to the source dipole here should be the same factor times the swimming velocity due to the no-flux condition. The source term is spherically symmetric [see Fig. 3(a)], so it has no preferred direction. Hence the source term will induce only a translational velocity along  $\mathbf{p}$  and no angular velocity. Overall, the change in the velocity of the

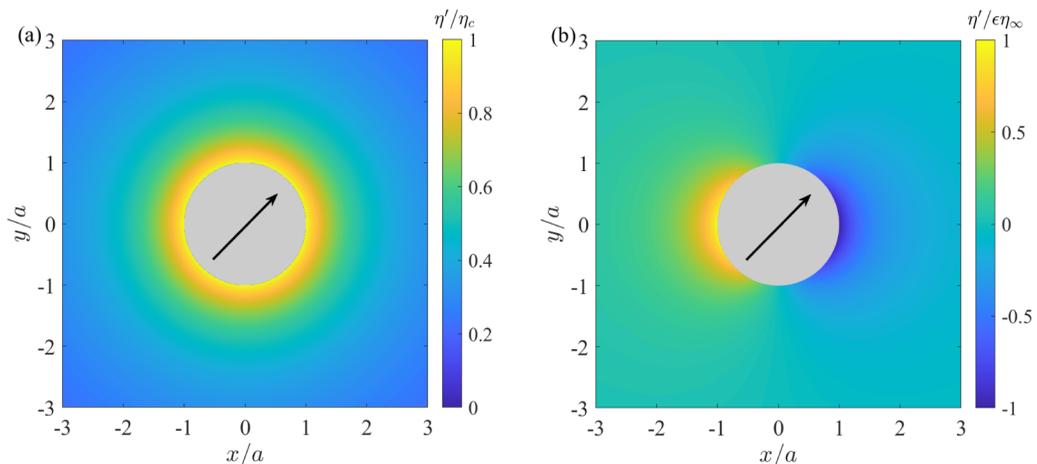


FIG. 3. (a) The source and (b) the source-dipole part of the disturbance viscosity due to a constant viscosity condition at the instant when the particle center is at the origin in the laboratory frame, i.e.,  $\mathbf{x}_c = \mathbf{0}$ . These viscosity distributions do not depend on the orientation of the particle.

particle due to the disturbance viscosity alone (with the constant viscosity boundary condition) is given by

$$\mathbf{U}'_1 = \frac{1}{12} \frac{\eta_c}{\epsilon \eta_\infty} \mathbf{U}_N + \frac{aB_2}{30} (\mathbf{I} - 3\mathbf{pp}) \cdot \nabla \left( \frac{\eta_1}{\eta_\infty} \right), \quad (33)$$

$$\boldsymbol{\Omega}'_1 = \frac{1}{4} \mathbf{U}_N \times \nabla \left( \frac{\eta_1}{\eta_\infty} \right). \quad (34)$$

Again, summing up the contributions of the ambient and the disturbance viscosities, the change in the velocity of a particle with the constant viscosity boundary condition is

$$\mathbf{U}_1 = \frac{1}{12} \frac{\eta_c}{\epsilon \eta_\infty} \mathbf{U}_N - \frac{aB_2}{6} (\mathbf{I} - 3\mathbf{pp}) \cdot \nabla \left( \frac{\eta_1}{\eta_\infty} \right), \quad (35)$$

$$\boldsymbol{\Omega}_1 = -\frac{1}{4} \mathbf{U}_N \times \nabla \left( \frac{\eta_1}{\eta_\infty} \right). \quad (36)$$

The disturbance viscosity does not change the reorientation process predicted by considering only the ambient viscosity, it only decreases the rate at which the particle turns to align itself against the viscosity gradients. The disturbance viscosity, however, modifies the speed changes experienced by the swimmer in the steady-state orientation in comparison to those predicted by considering the ambient viscosity alone. A neutral swimmer propels faster than that in the homogeneous fluid if it is cooler than the ambient fluid ( $\eta_c > 0$ ). A pusher or puller swims faster or slower than that in the homogeneous fluid depending on the magnitude of  $\alpha = B_2/B_1$  and how hot or cold the particle is relative to the ambient fluid.

## V. LOCAL VERSUS NONLOCAL EFFECTS

Instead of analyzing local and nonlocal effects simultaneously, as we did in the previous section, in this section we look at each term separately and then quantify their relative importance. We do so because (as we show below) local effects are typically easy to predict, but alone may or may not qualitatively capture the dynamics.

TABLE I. The first-order local, nonlocal, and total velocities of the swimmer. Here,  $\boldsymbol{\delta} = \mathbf{I} - 3\mathbf{pp}$ .

Velocity	Boundary condition	Local	Nonlocal	Total
$\boldsymbol{\Omega}_1$	No-flux	$-\frac{3}{4}\mathbf{U}_N \times \nabla\left(\frac{\eta_1}{\eta_\infty}\right)$	$\frac{1}{8}\mathbf{U}_N \times \nabla\left(\frac{\eta_1}{\eta_\infty}\right)$	$-\frac{5}{8}\mathbf{U}_N \times \nabla\left(\frac{\eta_1}{\eta_\infty}\right)$
	Constant viscosity	$\mathbf{0}$	$-\frac{1}{4}\mathbf{U}_N \times \nabla\left(\frac{\eta_1}{\eta_\infty}\right)$	$-\frac{1}{4}\mathbf{U}_N \times \nabla\left(\frac{\eta_1}{\eta_\infty}\right)$
$\mathbf{U}_1$	No-flux	$-\frac{1}{15}aB_2\boldsymbol{\delta} \cdot \nabla\left(\frac{\eta_1}{\eta_\infty}\right)$	$-\frac{3}{20}aB_2\boldsymbol{\delta} \cdot \nabla\left(\frac{\eta_1}{\eta_\infty}\right)$	$-\frac{13}{60}aB_2\boldsymbol{\delta} \cdot \nabla\left(\frac{\eta_1}{\eta_\infty}\right)$
	Constant viscosity	$\mathbf{0}$	$\frac{\eta_c}{12\epsilon\eta_\infty}\mathbf{U}_N$	$\frac{\eta_c}{12\epsilon\eta_\infty}\mathbf{U}_N$
			$-\frac{1}{6}aB_2\boldsymbol{\delta} \cdot \nabla\left(\frac{\eta_1}{\eta_\infty}\right)$	$-\frac{1}{6}aB_2\boldsymbol{\delta} \cdot \nabla\left(\frac{\eta_1}{\eta_\infty}\right)$

Analyzing the results in Table I, we see that the local, nonlocal and total effects depend only on the first two squirming modes, which means that the higher ( $n > 2$ ) modes do not contribute. Also, we find by comparing the local effects with the total response that the results are qualitatively similar for a particle imposing the no-flux condition to the viscosity field. However, for an active particle with a constant-viscosity boundary condition it is the nonlocal effects that are dominant. In fact, the local effects for such particle are identically zero. This is because the viscosity field  $\epsilon\eta_1 + \eta'$  near such particle is a constant ( $=\eta_c$ ) and this reduces the local effects to the force or torque acting on the particle in homogeneous fluid which is identically zero. In the sections below we unpack and parse these results to develop a more mechanistic understanding of the results in this table.

### A. Reorientation

For simplicity, we consider a particle with only  $B_1$  squirming mode and oriented orthogonal to the viscosity gradients along the  $\mathbf{e}_y$  direction.

#### 1. Local effects

The torque due to local changes in the viscosity is given by

$$\mathbf{L}_{NN}^l = \frac{2B_1}{a} \int_{S_p} [\eta(\mathbf{x}) - \eta_\infty] \mathbf{r} \times \mathbf{p} dS. \quad (37)$$

From this we see that the viscosity at the surface of the particle simply acts as a weight on the moment arm, and that an asymmetry in the viscosity at the surface will lead to a reorientation of the particle toward the lower viscosity. The viscosity at the surface has a contribution from both the ambient and the disturbance  $\eta(\mathbf{x}) - \eta_\infty = \epsilon\eta_1 + \eta'$ . We know from Eq. (32) that on the surface of the particle  $\eta' = -\epsilon\eta_1 + \text{const}$ . to satisfy a constant viscosity boundary condition, and it is clear from Eq. (37) that a constant viscosity on the surface of the particle clearly produces no net torque. In contrast, Eq. (26) shows that to satisfy the no-flux boundary condition,  $\eta' = \frac{1}{2}\epsilon\eta_1 + \text{const}$ . meaning the asymmetry of the ambient viscosity at the surface of the particle, and thus the torque, is enhanced by a 3/2 factor [see Fig. 2(b)]. An application of the divergence theorem to Eq. (37) allows one to simply resolve the torque as

$$\mathbf{L}_{NN}^l = -\frac{9}{2}\mathbf{U}_N \times \int_{V_p} \epsilon \nabla \eta_1 dV = -6\pi a^3 \mathbf{U}_N \times \epsilon \nabla \eta_1, \quad (38)$$

and multiplication by the mobility leads to the result in Table I.

## 2. Nonlocal effects

Recall the nonlocal effect is given by  $\int (\nabla \cdot \boldsymbol{\tau}_{NN,1}) \cdot \hat{\mathbf{G}}_{\Omega} dV$ . We use the expression for  $\hat{\mathbf{G}}_{\Omega}$  to simplify this integral to

$$\int \left( \frac{a^3}{r^3} \right) \mathbf{r} \times \nabla \cdot [(\epsilon\eta_1 + \eta')\dot{\boldsymbol{\gamma}}_0] dV. \quad (39)$$

As the strain-rate tensor due to  $B_1$  mode is divergence-free, the nonlocal effect finally simplifies to

$$\int \left( \frac{a^3}{r^3} \right) \mathbf{r} \times \nabla(\epsilon\eta_1 + \eta') \cdot \dot{\boldsymbol{\gamma}}_0 dV. \quad (40)$$

We examine the nonlocal effect by analyzing the kernel of the integral that represents the nonlocal effects,  $\nabla(\epsilon\eta_1 + \eta') \cdot \dot{\boldsymbol{\gamma}}_0$ . The contribution of ambient viscosity to this kernel that is not along the radial direction is

$$\frac{\epsilon\eta_{\infty}}{a} \frac{2a^3}{r^5} B_1(x - x_c) \mathbf{e}_y + \frac{\epsilon\eta_{\infty}}{a} \frac{2a^3}{r^5} B_1(y - y_c) \mathbf{e}_x. \quad (41)$$

Half of this kernel (the first term) has left-right mirror symmetry while the other half has front-back mirror symmetry. Each half of this kernel induces an angular velocity of equal magnitude but opposite direction making the net nonlocal effect due to the ambient viscosity to be zero. The source part of the disturbance viscosity as well as its gradient do not have any directional preference or any asymmetries. Hence, the nonlocal effect due to the source is zero. The source dipole part of the disturbance viscosity is proportional to  $x$  and it decays with the position. So, the gradient of the source dipole has two parts. One is along the gradient of the ambient viscosity and hence, like the ambient viscosity, this part induces zero angular velocity. The other part is in the radially inward direction due to the decay of source-dipole with position and also it is proportional to  $x$ . The latter characteristic introduces a left-right mirror symmetry to the kernel that consequently induces a nonzero angular velocity. As the disturbance viscosity always has a source-dipole component, its nonlocal effect is nonzero regardless of the boundary condition on the particle surface. See the fourth column in Table I.

## B. Speed changes in the steady-state orientation

The steady-state orientation is opposite to the ambient viscosity gradients. For the ambient viscosity,  $\eta_0 = \eta_{\infty} + \epsilon\eta_{\infty}x/a$ , this orientation is along  $-\mathbf{e}_x$ . In this configuration, the flow and the viscosity fields are axisymmetric, the axis of symmetry is along the swimmer's orientation. To understand the speed changes in the steady-state orientation, we consider a swimmer with first two squirming modes as the swimming velocity in viscosity gradients anyways depends on only these two squirming modes.

### 1. Local effects

The force due to local variation in the viscosity is given by

$$\mathbf{F}_{NN}^l = \int_{S_p} \mathbf{n} \cdot \boldsymbol{\tau}_{NN} dS = \int_{S_p} (\epsilon\eta_1 + \eta') \mathbf{n} \cdot \dot{\boldsymbol{\gamma}}_0 dS. \quad (42)$$

For a particle with constant viscosity condition, the viscosity  $\epsilon\eta_1 + \eta'$  is constant ( $=\eta_c$ ) near the particle and this reduces the local force to the force due to viscous traction in homogeneous Newtonian fluid which is identically zero. In homogeneous fluids, the force due to pressure and viscous traction are both zero to enforce the force-free condition on the particle. However, near a particle with a no-flux boundary condition, the viscosity  $\epsilon\eta_1 + \eta'$  is  $\frac{3}{2}\epsilon\eta_1 + \text{const}$ . Here, the ambient viscosity  $\eta_1$  is front-back mirror symmetric as  $\eta_1 \propto x$ . The flow and hence the shear-rate  $\dot{\boldsymbol{\gamma}}_{0,rx}$  due to  $B_1$  mode in the homogeneous fluid are front-back symmetric. This symmetry together with the

mirror-symmetry in the viscosity induce a mirror-symmetry in the traction  $\mathbf{n} \cdot \boldsymbol{\tau}_{NN,1}$ , which means that for every traction in front of the particle, there is an equal in magnitude but oppositely directed traction on the back of the particle making the net local force to vanish. In contrast, the flow and the shear-rate  $\dot{\gamma}_{0,rx}$  due to  $B_2$  mode in the homogeneous fluid are front-back mirror symmetric. This mirror-symmetry along with a similar symmetry in the viscosity makes the traction front-back symmetric, meaning that for every traction in front of the particle, there is an equal traction behind the particle resulting in a finite local force. Hence, while the local force acting on a particle with constant viscosity condition is zero, this force acting on a particle with no-flux condition depends only on  $B_2$  mode. Again, a multiplication of this force with mobility gives the local velocity reported in the third column of Table I.

## 2. Nonlocal effects

We examine the nonlocal effects by analyzing the kernel of the integral that represents the nonlocal effects,  $\nabla \cdot \boldsymbol{\tau}_{NN,1} \cdot \hat{\mathbf{G}}_U$  or  $\nabla \cdot \boldsymbol{\tau}_{NN,1}$ . Through a linear decomposition of the first order problem, we can show that the term  $\nabla \cdot \boldsymbol{\tau}_{NN,1}$  modifies the flow from that in the homogeneous fluid and therefore, is responsible for the nonlocal effect [31]. If the kernel and hence, the modified flow are front-back mirror symmetric, then the nonlocal effect being the integral of the kernel vanishes. As usual, the flow due to the  $B_1$  mode is front-back symmetric in the homogeneous fluid, while the flow due to the  $B_2$  mode is mirror-symmetric. Meanwhile, the particle imposes a source-dipole disturbance on the ambient viscosity. Inspecting the kernel,  $\nabla \cdot \boldsymbol{\tau}_{NN,1}$ , we see that the symmetry of the flow coupled with the symmetry of the disturbance viscosity leads to an integral that has mirror symmetry and thus zero for the  $B_1$  mode, while nonzero for the  $B_2$  mode thereby generating a finite nonlocal effect. In contrast, if the particle also imposes source term disturbance (with spherical symmetry) on the viscosity, then the contribution is opposite (only due to the  $B_1$  mode). As a particle generates a source-dipole irrespective of the boundary condition, the nonlocal effect depends on the  $B_2$  mode regardless of the boundary condition on the particle. However, since a particle with a constant viscosity condition additionally generates the source, the nonlocal effect on such particle also depends on the  $B_1$  mode. See the fourth column in Table I.

## VI. CONCLUSIONS

Spatial variations of viscosity in fluids may arise due to changes in temperature, salt or nutrient concentration. Insertion of a particle into the fluid tends to disturb the background viscosity field due to boundary conditions imposed by the particle on the viscosity field. In general we find that active squirmer-type particles reorient down the viscosity gradient (negative viscotaxis) and they propel in the steady-state orientation with a speed different from that in the homogeneous fluid. The specific boundary conditions on the viscosity field at the surface of the particle do not qualitatively affect the reorientation process, only changing the rate at which the steady-state orientation is achieved. On the contrary, the speed changes experienced by the swimmer while propelling along the steady-state orientation are significantly dependent on the boundary conditions on the particle. For active particles that impose a no-flux condition, pushers speed up, the pullers slow down and the neutral swimmers do not change their speed. In contrast, the speed changes experienced by a particle that imposes a constant-viscosity boundary condition are more complex. Even the neutral swimmer speeds up if it is cold relative to the ambient fluid while the speed changes of a pusher or a puller depend on how strong the pusher or puller is (the ratio of the squirming modes,  $B_2/B_1$ ) and how hot or cold the particle is relative to the ambient fluid. This suggests the additional degrees of freedom, the particle temperature and the squirming modes, that one can explore in controlling the active matter.

We also quantified the relative importance of the local and nonlocal effects, where the local effects are a consequence of locally modifying the viscosity but not the flow while the nonlocal effects are due to changing the flow from that in the homogeneous fluid without altering the

viscosity. We found that the local effects are sufficient to determine the motion of a particle imposing a no-flux boundary condition but it is the nonlocal effects that dominate the (change in) dynamics of a particle imposing a constant-viscosity boundary condition. Hence, one should exercise caution when using a simple local effects based theory to study the motion of active particles in viscosity gradients.

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