PHYSICAL REVIEW FLUIDS **6**, 084610 (2021)

Conformal invariance of the 1-point statistics of the zero-isolines of 2*d* scalar fields in inverse turbulent cascades

M. Wacławczyk [*]
Institute of Geophysics, Faculty of Physics, University of Warsaw, Pasteura 5, 02-093 Warsaw, Poland

V. N. Grebenev [†]
Federal Research Center for Information and Computational Technologies, Russian Academy of Sciences, Lavrentjev avenue 6, 630090 Novosibirsk, Russia

M. Oberlack [‡]
Department of Mechanical Engineering, Technische Universität Darmstadt and Otto-Bernd-Strasse 2, Darmstadt 64287, Germany

(Received 27 October 2020; accepted 26 July 2021; published 26 August 2021)

This study concerns conformal invariance of certain statistics in 2*d* turbulence. Namely, there exists numerical evidence by Bernard *et al.* [Nature Phys. **2**, 124 (2006)], that the zero-vorticity isolines $\boldsymbol{x}(l, t)$ for the 2*d* Euler equation with an external force and a uniform friction belong to the class of conformally invariant random curves. Based on this evidence, the CG invariance was formally proven by Grebenev *et al.* [J. Phys. A: Math. Theor. **50**, 435502 (2017)] by a Lie group analysis for the 1-point probability density function (PDF) governed by the inviscid Lundgren-Monin-Novikov (LMN) equations for 2*d* vorticity fields under the zero external force field. In this work we consider the first equation from the LMN chain for 2*d* scalar fields under Gaussian white-in-time forcing and large-scale friction. With this, the flow can be kept in a statistically steady state and the analysis is performed for the stationary LMN. Specifically, for the inviscid case we prove the CG invariance of the 1-point statistics of the zero-isolines $\boldsymbol{x}(l)$ of a scalar field, i.e., the CG invariance of the probability $f_1(\boldsymbol{x}(l), \phi)d\phi$ that a random curve $\boldsymbol{x}(l)$ passes through the point $\boldsymbol{x}$ with $\phi = 0$ for $l = l_1$. We show an example, where the proposed transformations represent a change between PDF's describing homogeneous and nonhomogeneous fields. Possible implications of this result are discussed.

DOI: 10.1103/PhysRevFluids.6.084610

I. INTRODUCTION

Invariance under scaling transformations is a remarkable feature of the Navier-Stokes system in 2*d*. It is closely related to the concept of self-similarity which implies the investigated fields show the same statistical properties at different scales. After a scale transformation, a scalar field $\Phi(\boldsymbol{x})$ becomes $\lambda^s\Phi(\lambda\boldsymbol{x})$, where λ is a scaling factor and s is a coefficient which in the first place is arbitrary in the Navier-Stokes system if viscosity is neglected.

The presence of the inverse cascade in 2*d* turbulence allows to obtain a stationary state where energy, injected at the forcing scale is transported towards larger scales where it is extracted by

*marta.waclawczyk@igf.fuw.edu.pl
†Corresponding author: vngrebenev@gmail.com
‡oberlack@fdy.tu-darmstadt.de

the large-scale friction. In this region of the spectral space, Kolmogorov's scaling laws are satisfied exactly. However, in many physical systems the existing scale invariance may be extended to the conformal invariance [1], a scaling which depends on position $\boldsymbol{x} \to \boldsymbol{\lambda}(\boldsymbol{x})$. In the context of $2d$ turbulence, such possibilty was first suggested by Polyakov [2]. The first example is the CG invariance of the statistics of zero-vorticity isolines, based on the $2d$ Euler equation with an external Gaussian white-in-time forcing on small scales and a uniform friction in a flow domain, see Refs. [3] and [4]. Therein the equations were solved on a torus with periodic boundary conditions and the system was kept in a statistically stationary state. It was shown numerically that zero-vorticity isolines belong to the class of conformally invariant curves that can be conformally mapped into a one-dimensional Brownian walk called Schramm-Löwner evolution (SLE_κ) curves [5], with diffusion coefficient $\kappa = 6$, where κ classifies the conformal invariant random curves. It is interesting that SLE_6 first appeared in the classical model of critical percolation and corresponds to a self-avoiding random walk [6]. The second example refers to statistical properties of turbulent inverse cascades in a class of models describing a scalar field transported by a two-dimensional flow; see Refs. [7,8]. These works include several cases of physical models, such as surface quasigeostrophic (SQG) turbulence which describes a rotating stably stratified fluid with a uniform potential vorticity, the asymptotic case of the Hasegawa-Mima equation for drift waves in magnetized plasma [9], the Charney and Oboukhov equation for waves in rotating fluids [10]. It was shown that the zero-isolines of the scalar field are statistically equivalent to conformal invariant curves within the resolution of numerics. In particular, the zero-temperature isolines in SQG model belong to the same universality class SLE_κ with $\kappa = 4$ at large scales (in the inverse cascade).

The exact mathematical result, as to whether the conformal invariance exists in the statistics of two-dimensional turbulence, was first obtained in Ref. [11]. Therein we performed a Lie group analysis of the first equation from the infinite Lundgren-Monin-Novikov (LMN) hierarchy for the probability density functions (PDFs) of vorticity, with the assumptions of zero viscosity and forcing. We proved that the CG is generally broken for the first LMN equation or the $f_1(\boldsymbol{x}, \omega, t)$—equation, apart from points $\boldsymbol{x} \in D \subset \mathbb{R}^2$ on the associated characteristic with zero-vorticity or in general on the level set $\{\boldsymbol{x} \in \mathbb{R}^2 : \omega = \omega(\boldsymbol{x}) = 0\}$. Notice that the characteristics of the LMN chain exhibit a direct analogy to the Lagrangian description of turbulence [12] and the characteristic equations describe the dynamics of the statistics of a class of fluid particles moving in a conditional velocity field. The Lagrangian point of view was further addressed in Ref. [13], where the CG invariance both for the Lagrangian path and the 1-point PDF of vorticity, i.e., $f_1(\boldsymbol{x}, \omega, t)$ taken along the zero-vorticity characteristics was established. The CG invariance of the normalization and reduction properties, the separation and coincidence properties of the PDFs were also proven. The above-mentioned findings were expanded in Ref. [14] to a broader class of hydrodynamic models generalized to large-scale friction for scalar fields. We proved that the zero-scalar characteristics of the equations are conformally invariant in the presence of large-scale friction, while viscosity, in general, breaks CG.

To link the results obtained in Refs. [11,13,14] with the results of Refs. [3,7], we presently consider the first equation from the LMN chain for $2d$ scalar fields ϕ under Gaussian white-in-time forcing and large-scale friction. With this, the system is kept in a statistically steady state which leads to the stationary LMN equations. The invariance of such stationary setting with nonzero forcing is considered for the first time in this work. Moreover, while our previous contributions were focused on the Lie group analysis and infinitesimal forms of transformations, in the present paper we consider their corresponding global forms. We are interested in the CG invariance of the 1-point statistics of the zero-isolines $\boldsymbol{x}(l)$ of scalar fields or the probability measure $f_1(\boldsymbol{x}, \phi)d\phi$ calculated on $\boldsymbol{x}(l)$. To show the invariance we will use the group of transformations G which was derived in Ref. [11] for the vorticity field and was generalized for scalar fields in Ref. [14]. The group G acts conformally with respect to the spatial variable $\boldsymbol{x}$ and transforms invariantly only "a fragment" of the first LMN equation, i.e., the $f_1(\boldsymbol{x}, \omega, t)|_{\omega=0}$ equation.

We show that with the use of conformal invariance some statistics of anisotropic field can be determined based on solution of the LMN for the isotropic case. This is another new contribution of the present paper. As our considerations are restricted to the first LMN equation, the 2-point PDF which enters this equation, should be treated as an a priori determined probability density.

In the following Sec. II equations governing the $2d$ turbulent flow are introduced. Next, in Sec. III the classical scaling invariance is discussed. Its extension towards conformal invariance is presented in Sec. IV. In Sec. IV A we consider the first equation of the inviscid LMN chain for scalar fields under a white-in-time forcing term with the Gaussian statistics to link it to the inverse cascade template [8]. We demonstrate that the group G transforms this equation invariantly, too. Then we prove the invariance of the probability measure under the conformal transformations of D. In section IV B, we show that, as in the case of the characteristic lines (see Ref. [14]), the CG invariance of the 1-point statistics of the zero-isolines of scalar fields is broken if the viscous term is included into the equation. An example of transformation is introduced in Sec. V. A discussion and summary of results are given in Sec. VI.

II. GOVERNING EQUATIONS

We consider a class of hydrodynamic models in $2d$ for a scalar variable $\Phi(\boldsymbol{x}, t)$, governed by the following equation

$$\frac{\partial \Phi}{\partial t} + \boldsymbol{u} \cdot \nabla \Phi = \nu \nabla^2 \Phi - \alpha \Phi + L(\boldsymbol{x}, t), \tag{1}$$

where ν denotes molecular viscosity or diffusivity, α is the Ekman friction coefficient with the dimension $1/\tau$ where τ is a timescale and $L(\boldsymbol{x}, t)$ is a white-in-time random Gaussian forcing term with nonzero second-order cumulants

$$\langle L(\boldsymbol{x}, t) L(\boldsymbol{x}', t')\rangle = 2Q(\boldsymbol{x}, \boldsymbol{x}')\delta(t - t'), \tag{2}$$

where δ is the Dirac δ function. The term $-\alpha\Phi$ in Eq. (1) represents the frictional damping, responsible for the removal of energy at large scales. For instance, relevant to geophysical applications is the rotating flow subject to the Ekman friction. For a single scalar field, the form of the friction is the same as given in Eq. (1); see Ref. [15].

As in Ref. [8], we assume that the components of the velocity field $\boldsymbol{u} = (u, v)$ read

$$u(\boldsymbol{x}, t) = \beta \int d\boldsymbol{x}' \Phi(\boldsymbol{x}', t) \frac{(y - y')}{|\boldsymbol{x} - \boldsymbol{x}'|^m}, \tag{3}$$

$$v(\boldsymbol{x}, t) = -\beta \int d\boldsymbol{x}' \Phi(\boldsymbol{x}', t) \frac{(x - x')}{|\boldsymbol{x} - \boldsymbol{x}'|^m}, \tag{4}$$

where β is a specific model constant, which can be set to 1 by rescaling of the system and $m > 1$ is an integer. Different values of m correspond to different physical models, that is, for $m = 2$, the scalar Φ is a vorticity in $2d$ turbulence, for $m = 3$, Eqs. (1), (3), and (4) describe the surface quasi-geostrophic model, with the variable Φ being a temperature, finally $m = 6$ is the asymptotic limit of an equation which describes large-scale flows of a rotating shallow fluid flow [10] or a certain regime of plasma flows [9].

In the present work we will focus on the statistical approach and instead of the instantaneous scalar $\Phi(\boldsymbol{x}, t)$ we will rather consider its probability density function. The following notation will be used: the components of the position vector $\boldsymbol{x}$ are $\boldsymbol{x} = (x, y)$. The sample space variable of a scalar field at the point $\boldsymbol{x}$ is denoted by ϕ. If 2-point statistics are considered, the second point will be denoted as $\boldsymbol{x}' = (x', y')$ and the corresponding sample space variable of the scalar will be denoted by ϕ'. The transport equation for the 1-point pdf $f_1(\boldsymbol{x}, \phi, t)$ was derived in Ref. [16], see also Ref. [17], and can be written as

$$\frac{\partial f_1(\boldsymbol{x}, \phi, t)}{\partial t} + \nabla_{\boldsymbol{x}} \cdot [\langle \mathbf{u}(\boldsymbol{x}) | \boldsymbol{x}, \phi\rangle f_1(\boldsymbol{x}, \phi, t)] = \mathcal{F}, \tag{5}$$

where $\langle \mathbf{u}(\boldsymbol{x})|\boldsymbol{x},\phi\rangle$ is the conditional velocity field with components [17]

$$\langle u(\boldsymbol{x},t)|\boldsymbol{x},\phi,t\rangle = -\int d\mathbf{x}' d\phi' \phi' \frac{y-y'}{|\boldsymbol{x}-\boldsymbol{x}'|^m} f_2(\phi,\boldsymbol{x},\phi',\boldsymbol{x}',t) f_1^{-1}(\boldsymbol{x},\phi,t), \tag{6}$$

$$\langle v(\boldsymbol{x},t)|\boldsymbol{x},\phi,t\rangle = \int d\mathbf{x}' d\phi' \phi' \frac{x-x'}{|\boldsymbol{x}-\boldsymbol{x}'|^m} f_2(\phi,\boldsymbol{x},\phi',\boldsymbol{x}',t) f_1^{-1}(\boldsymbol{x},\phi,t). \tag{7}$$

The random forcing, large-scale friction, and viscous transport are encoded in the term $\mathcal{F}$ in Eq. (5)

$$\mathcal{F} = \alpha \frac{\partial}{\partial\phi}(\phi f_1) - \nu \frac{\partial}{\partial\phi}\left(\int d\phi' \int dx' \delta(\boldsymbol{x}-\boldsymbol{x}')\phi' \Delta^2_{\boldsymbol{x}'} f_2\right) + Q(\boldsymbol{x},\boldsymbol{x})\frac{\partial^2 f_1}{\partial\phi^2}, \tag{8}$$

where $Q(\boldsymbol{x},\boldsymbol{x})$ is the amplitude of the forcing which was calculated from the 2-point correlation function $Q(\boldsymbol{x},\boldsymbol{x}')$, see Eq. (2), by setting $\boldsymbol{x}'=\boldsymbol{x}$. In the case of isotropic forcing, Q is a function of the relative distance between the points $Q(|\boldsymbol{x}-\boldsymbol{x}'|)$, hence, after setting $\boldsymbol{x}'=\boldsymbol{x}$ we obtain $Q(0)$.

Equation (5) is coupled with the normalisation formulas, the separation and coincidence equations [17]:

$$\int d\phi f_1 = 1, \quad \int d\phi' f_2(\boldsymbol{x},\phi,\boldsymbol{x}',\phi',t) = f_1(\phi,\boldsymbol{x},t), \tag{9}$$

$$\lim_{|\boldsymbol{x}-\boldsymbol{x}'|\to\infty} f_2(\boldsymbol{x},\phi,\boldsymbol{x}',\phi',t) = f_1(\boldsymbol{x},\phi,t) f_1(\boldsymbol{x}',\phi',t), \tag{10}$$

$$\lim_{|\boldsymbol{x}-\boldsymbol{x}'|\to 0} f_2 = \delta(\phi-\phi')f_1. \tag{11}$$

Even in the absence of viscosity, that is, setting $\nu=0$ in Eq. (8) the system can reach a statistically stationary state, in which the statistics are independent of time. In such a case, the energy injected by the forcing at scale l_f is transported in the inverse cascade towards large scales $L \gg l_f$ and extracted due to the nonzero friction α [18].

III. CLASSICAL SCALING INVARIANCE

It is known that the transport Eq. (1) is invariant under a set of symmetry transformations, that is, such transformations of dependent and independent variables which do not change the functional form of the equation. The full set of symmetries of $2d$ Navier-Stokes equations was derived in Ref. [19]. We will not consider this full set in detail, but we will rather focus on the scaling symmetry. Its possible extension towards conformal transformation under certain conditions will be discussed in the following section.

At the beginning let us consider the following set of new variables, denoted by the symbol *

$$\boldsymbol{x}^* = \lambda \boldsymbol{x}, \quad t^* = \zeta t, \quad \Phi^* = \lambda^s \Phi, \quad \boldsymbol{u}^* = \lambda^{s+3-m}\boldsymbol{u}, \tag{12}$$

where $\lambda,\ \zeta \in \mathbb{R}_+$, and $s\in\mathbb{R}$ are arbitrary constants and the form of the transformed velocity $\boldsymbol{u}^*$ is determined by Eqs. (3) and (4). We can also include transformations of friction and forcing into the analysis, by treating α and L as free parameters of the equation. In such a case, we deal with the equivalence transformations [20] of the equation in a given class, which is a change of variables that maps the equation into another equation in the same class, rather than the symmetry transformations. Let us set

$$\alpha^* = \zeta^{-1}\alpha, \quad L^* = \lambda^{2s+2-m}L. \tag{13}$$

We will next write Eq. (1) in the new variables, first for the zero-viscosity case $\nu=0$:

$$\frac{\partial\Phi^*}{\partial t^*} + \boldsymbol{u}^*\cdot\nabla^*\Phi^* = -\alpha^*\Phi^* + L^*(\boldsymbol{x}^*,t^*). \tag{14}$$

Substituting Eqs. (12) and (13) into Eq. (14) we obtain

$$\frac{\lambda^s}{\zeta}\frac{\partial \Phi}{\partial t} + \lambda^{2s+2-m}\boldsymbol{u}\cdot\nabla\,\Phi = -\frac{\lambda^s}{\zeta}\alpha\Phi + \lambda^{2s+2-m}L(\boldsymbol{x},t). \tag{15}$$

This equation will reduce to Eq. (1), written in the original "old" variables $\boldsymbol{x}$, t, $\boldsymbol{u}$, ϕ, α, L provided that $\zeta = \lambda^{-s-2+m}$. It is to note that the coefficient s remains arbitrary. With this the scaling group Eq. (12) is formally specified and constitutes a two-parameter group.

However, if now viscosity is taken into account, then we have

$$\lambda^{2s+2-m}\left(\frac{\partial \Phi}{\partial t} + \boldsymbol{u}\cdot\nabla\,\Phi\right) = \lambda^{s-2}\nu\nabla^2\Phi + \lambda^{2s+2-m}[-\alpha\Phi + L(\boldsymbol{x},t)], \tag{16}$$

and invariance is obtained upon setting $s-2 = 2s+2-m$, hence, $s = m-4$, which results in

$$\lambda^{m-6}\frac{\partial \Phi}{\partial t} + \lambda^{m-6}\boldsymbol{u}\cdot\nabla\,\Phi = \lambda^{m-6}\nu\Delta^*\Phi^* - \lambda^{m-6}\alpha\Phi + \lambda^{m-6}L(\boldsymbol{x},t). \tag{17}$$

After dividing both sides by λ^{m-6} this equation will reduce to Eq. (1) written in the "old" variables. As it is seen, the presence of viscosity restricts the value of the coefficient s and formally Eq. (17) only admits a one-parameter group.

This scaling symmetry is also "transferred" to equations describing the statistics of Eq. (1) and the PDF Eq. (5); see also Ref. [21]. In this case we will consider the following transformations of the independent variables:

$$\boldsymbol{x}^* = \lambda\boldsymbol{x}, \quad \boldsymbol{x}'^* = \lambda\boldsymbol{x}', \quad t^* = \lambda^2 t, \quad \phi^* = \lambda^{m-4}\phi, \quad \phi'^* = \lambda^{m-4}\phi', \tag{18}$$

whereas the PDF functions will be transformed as

$$f_1^*(\boldsymbol{x}^*,\phi^*,t^*) = \frac{1}{\lambda^{m-4}}f_1(\boldsymbol{x},\phi,t), \quad f_2^*(\boldsymbol{x}^*,\phi^*,\boldsymbol{x}'^*,\phi'^*,t^*) = \frac{1}{\lambda^{2m-8}}f_2(\boldsymbol{x},\phi,\boldsymbol{x}',\phi',t). \tag{19}$$

The above form assures the invariance of the probability measures

$$f_1^* d\phi^* = \frac{1}{\lambda^{m-4}}\lambda^{m-4}f_1 d\phi = f_1 d\phi, \quad f_2^* d\phi^* d\phi'^* = f_2 d\phi d\phi'.$$

Moreover, the transport Eq. (5) is invariant under the change of variables Eqs. (18) and (19), provided that

$$\alpha^* = \lambda^{-2}\alpha, \quad Q^* = \lambda^{2m-10}Q. \tag{20}$$

After introducing Eqs. (18)–(20) into Eq. (5) written for the new variables, we find each term in this equation is multiplied by λ^{2-m}. Hence, after dividing both sides by this factor the original equation written in the "old" variables is obtained.

This scaling invariance has some important implications. For example, once the solution of Eq. (5) in the original variables is known, the statistics of the rescaled field and $\partial f_1^*/\partial t^*$ are also determined, as

$$\frac{\partial f_1^*}{\partial t^*} = \lambda^{2-m}\frac{\partial f_1}{\partial t}.$$

The transformed PDF f_1^* describes 1-point statistics of the transformed (rescaled) field. However, the structure of the field remains the same after the transformation. In particular, if the scalar field is isotropic, the transformed field is also structured as such. We will further argue that this is not necessarily the case of conformal transformations, where, under certain conditions, solution of Eq. (5) for a homogeneous field could determine solution for an inhomogeneous one, only after proper rescaling of the variables. That is to say, without the need of solving the equation for the transformed field.

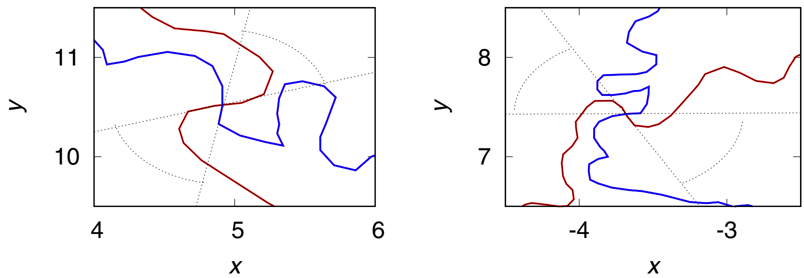

FIG. 1. Left plot: Two intersecting curves. Right plot: Conformal transformation of the curves.

IV. CONFORMAL INVARIANCE OF THE 1-POINT STATISTICS OF THE ZERO-ISOLINES

The conformal invariance is often understood as a generalisation of the scaling invariance, such that the coefficients in Eqs. (18) are not constants any more, but become functions of space. In particular,

$$\boldsymbol{x}^* = \boldsymbol{\lambda}(\boldsymbol{x}), \tag{21}$$

where $\boldsymbol{\lambda}(\boldsymbol{x})$ is a vector with components $\boldsymbol{\lambda} = (\mathcal{X}, \mathcal{Y})$, such that the following Cauchy-Riemann conditions are satisfied:

$$\frac{\partial \mathcal{X}}{\partial x} = \frac{\partial \mathcal{Y}}{\partial y}, \quad \frac{\partial \mathcal{X}}{\partial y} = -\frac{\partial \mathcal{Y}}{\partial x}. \tag{22}$$

Equations (22) assure that the transformation is angle-preserving. Note that with $\partial \mathcal{X}/\partial x = \text{const}$ and $\partial \mathcal{X}/\partial y = 0$, Eq. (21) reduces to the classical scaling. The case $\partial \mathcal{X}/\partial y = \text{const}$ corresponds to scaling and rotation transformation. Finally, the general case of nonconstant gradients $\partial \mathcal{X}/\partial x$ and/or $\partial \mathcal{X}/\partial y$ allow to rescale the space by a factor that depends on the position. A nontrivial example of a conformal transformation reads

$$x^* = \mathcal{X}(x, y, a) = \frac{x + a(x^2 + y^2)}{1 + 2ax + a^2(x^2 + y^2)}, \tag{23}$$

$$y^* = \mathcal{Y}(x, y, a) = \frac{y}{1 + 2ax + a^2(x^2 + y^2)}, \tag{24}$$

where a is an arbitrary constant. Figure 1 presents two intersecting curves and their conformal transformation. Note that the angle between the curves at their intersection point remains unchanged after the transformation.

In this work we will be particularly interested in the statistics of zero-lines $\boldsymbol{x}(l)$ of the scalar fields Φ and invariance of their probability measure $f_1(\boldsymbol{x}, \phi)d\phi|_{\phi=0}$ for $\boldsymbol{x} \in \boldsymbol{x}(l)$, under the transformations of the curve $\boldsymbol{x}(l)$, as there is numerical evidence which indicate that such lines, which are boundaries of large clusters, are conformally invariant [3,7]. We call the measure conformally invariant if it is invariant with respect to a conformal transformation $F : D \mapsto D^*$, that is, $\mu_D(\boldsymbol{x}) = \mu_{D^*}(\boldsymbol{\lambda}(\boldsymbol{x}))$, see, e.g., Ref. [4].

We consider the group of transformations G which was derived by the Lie group analysis in Ref. [11] for the vorticity field and was generalized for scalar fields in Ref. [14]. This analysis was performed for the so-called infinitesimal forms of transformations. In this work we will only consider the transformations in their equivalent global forms [14], i.e.,

$$x^* = \mathcal{X}(x, y, a), \quad y^* = \mathcal{Y}(x, y, a), \tag{25}$$

$$x'^* = \mathcal{X}(x, y, a) + \gamma^{1/3}(\boldsymbol{x})\left[(x' - x)\frac{\partial \mathcal{X}}{\partial x} + (y' - y)\frac{\partial \mathcal{X}}{\partial y}\right], \tag{26}$$

$$y'^* = \mathcal{Y}(x, y, a) + \gamma^{1/3}(\boldsymbol{x})\left[(x' - x)\frac{\partial \mathcal{Y}}{\partial x} + (y' - y)\frac{\partial \mathcal{Y}}{\partial y}\right], \tag{27}$$

$$\phi^* = \gamma^{-1}(\boldsymbol{x})\phi, \tag{28}$$

$$\phi'^* = \gamma^{-m/6}(\boldsymbol{x})\phi', \tag{29}$$

$$f_1^* = \gamma(\boldsymbol{x})f_1, \tag{30}$$

$$f_2^* = \gamma^{(6+m)/6}(\boldsymbol{x})f_2. \tag{31}$$

Here,

$$\gamma(\boldsymbol{x}) = \left[\left(\frac{\partial \mathcal{X}}{\partial x}\right)^2 + \left(\frac{\partial \mathcal{X}}{\partial y}\right)^2\right]^{-1} = \left[\left(\frac{\partial \mathcal{Y}}{\partial x}\right)^2 + \left(\frac{\partial \mathcal{Y}}{\partial y}\right)^2\right]^{-1}, \tag{32}$$

where $\mathcal{X}(x, y, a)$ and $\mathcal{Y}(x, y, a)$ are conjugate harmonic functions [11] which satisfy the Cauchy-Riemann conditions Eq. (22). It is often more convenient to consider the transformations in the complex plane, where $F(z, a) = \mathcal{X}(x, y, a) + i\mathcal{Y}(x, y, a)$ is a conformal mapping of the variable $z = x + iy$. G acts as the conformal group on a domain $D \subset \mathbb{R}^2$. The transformation of $Q^*(\boldsymbol{x}^*, \boldsymbol{x}^*)$ is to be determined later.

It will be shown in the following that after introducing Eqs. (25)–(31) into the first LMN Eq. (5) written in the new variables

$$\frac{\partial f_1^*(\boldsymbol{x}^*, \phi^*, t^*)}{\partial t^*} + \boldsymbol{\nabla}^*_{\boldsymbol{x}^*} \cdot [\langle \mathbf{u}^*(\boldsymbol{x}^*)|\boldsymbol{x}^*, \phi^*\rangle f_1^*(\boldsymbol{x}^*, \phi^*, t^*)] = \mathcal{F}^*(\boldsymbol{x}^*, \phi^*, t^*), \tag{33}$$

we will obtain

$$\frac{\partial f_1(\boldsymbol{x}, \phi, t)}{\partial t} + \boldsymbol{\nabla}_{\boldsymbol{x}} \cdot [\langle \mathbf{u}(\boldsymbol{x})|\boldsymbol{x}, \phi\rangle f_1(\boldsymbol{x}, \phi, t)] = \mathcal{F}(\boldsymbol{x}, \phi, t) + \mathcal{G}(\boldsymbol{x}, \phi, t), \tag{34}$$

where $\mathcal{G}$ is a function (to be determined later), such that

$$\mathcal{G}(\boldsymbol{x}, 0, t) = 0. \tag{35}$$

Equality (35) implies that after setting $\phi = 0$ in Eq. (34), that is, if the evolution equation for $f_1^*(0, \boldsymbol{x}^*, t^*)$ is to be considered, one obtains the invariance, as in such a case from Eq. (34), the first LMN Eq. (5) for $\phi = 0$ is recovered

$$\frac{\partial f_1(\boldsymbol{x}, \phi, t)\big|_{\phi=0}}{\partial t} + \boldsymbol{\nabla}_{\boldsymbol{x}} \cdot [\langle \mathbf{u}(\boldsymbol{x})|\boldsymbol{x}, \phi\rangle f_1(\boldsymbol{x}, \phi, t)]\big|_{\phi=0} = \mathcal{F}(\boldsymbol{x}, 0, t). \tag{36}$$

This means exactly that once $\partial f_1(0, \boldsymbol{x}, t)/\partial t$ is known, the solution for $\partial f_1^*(0, \boldsymbol{x}^*, t^*)/\partial t^*$ is also determined, although it is not necessarily so for arbitrary $\phi \neq 0$.

We will also show an example of the original and the transformed PDF's f_1 and f_2, related by Eqs. (30) and (31) for $\phi = 0$, which satisfy the normalization, separation, and coincidence conditions Eqs. (9)–(11).

To prove the invariance of Eq. (5) for $\phi = 0$ we will first determine how different terms will be transformed after introducing Eqs. (25)–(31). With Eqs. (25)–(27) we have

$$|\boldsymbol{x}^* - \boldsymbol{x}'^*| = \gamma^{-1/6}(\boldsymbol{x})|\boldsymbol{x} - \boldsymbol{x}'|. \tag{37}$$

To transform the integrals Eqs. (6) and (7) we first note that

$$-\frac{y^* - y'^*}{|\boldsymbol{x}^* - \boldsymbol{x}'^*|^m} = \gamma^{(m+2)/6}\left[\frac{\partial \mathcal{X}}{\partial x}\left(-\frac{y - y'}{|\boldsymbol{x} - \boldsymbol{x}'|^m}\right) + \frac{\partial \mathcal{X}}{\partial y}\left(\frac{x - x'}{|\boldsymbol{x} - \boldsymbol{x}'|^m}\right)\right], \tag{38}$$

$$\frac{x^* - x'^*}{|\boldsymbol{x}^* - \boldsymbol{x}'^*|^m} = \gamma^{(m+2)/6}\left[-\frac{\partial \mathcal{X}}{\partial y}\left(-\frac{y-y'}{|\boldsymbol{x}-\boldsymbol{x}'|^m}\right) + \frac{\partial \mathcal{X}}{\partial x}\left(\frac{x-x'}{|\boldsymbol{x}-\boldsymbol{x}'|^m}\right)\right], \tag{39}$$

where we also used the Cauchy-Riemann conditions Eq. (22). After calculating the determinant of the Jacobi matrix the following transformed increment from the integrals Eqs. (6) and (7) is derived (see Appendix A):

$$dx'^* dy'^* d\phi'^* = dx' dy' d\phi' [\gamma(x,y)]^{-(m+2)/6}.$$

Substituting also the transformed PDFs Eqs. (30) and (31) into Eqs. (6) and (7) written for the new variables we obtain the transformed conditional velocity components as functions of the original variables $(\boldsymbol{x}, \phi, t)$, i.e.,

$$\langle u^*(\boldsymbol{x}^*, t^*)|\boldsymbol{x}^*, \phi^*, t^*\rangle f_1^* = \gamma(\boldsymbol{x})\left[\frac{\partial \mathcal{X}}{\partial x}\langle u(\boldsymbol{x}, t)|\boldsymbol{x}, \phi, t\rangle + \frac{\partial \mathcal{X}}{\partial y}\langle v(\boldsymbol{x}, t)|\boldsymbol{x}, \phi, t\rangle\right] f_1, \tag{40}$$

$$\langle v^*(\boldsymbol{x}^*, t^*)|\boldsymbol{x}^*, \phi^*, t^*\rangle f_1^* = \gamma(\boldsymbol{x})\left[-\frac{\partial \mathcal{X}}{\partial y}\langle u(\boldsymbol{x}, t)|\boldsymbol{x}, \phi, t\rangle + \frac{\partial \mathcal{X}}{\partial x}\langle v(\boldsymbol{x}, t)|\boldsymbol{x}, \phi, t\rangle\right] f_1. \tag{41}$$

Finally, the transformed divergence $\nabla^*_{\boldsymbol{x}^*}$ applied to an arbitrary vector function $\boldsymbol{\mathcal{H}}(\boldsymbol{x}, \phi, t)$ reads

$$\nabla^*_{\boldsymbol{x}^*} \cdot \boldsymbol{\mathcal{H}} = \gamma(\boldsymbol{x})\begin{bmatrix} \frac{\partial \mathcal{X}}{\partial x}\frac{\partial}{\partial x} + \frac{\partial \mathcal{X}}{\partial y}\frac{\partial}{\partial y} + \phi\left(\frac{\partial \mathcal{X}}{\partial x}\frac{\partial \ln\gamma}{\partial x} + \frac{\partial \mathcal{X}}{\partial y}\frac{\partial \ln\gamma}{\partial y}\right)\frac{\partial}{\partial \phi} \\ \frac{\partial \mathcal{X}}{\partial x}\frac{\partial}{\partial y} - \frac{\partial \mathcal{X}}{\partial y}\frac{\partial}{\partial x} + \phi\left(\frac{\partial \mathcal{X}}{\partial x}\frac{\partial \ln\gamma}{\partial y} - \frac{\partial \mathcal{X}}{\partial y}\frac{\partial \ln\gamma}{\partial x}\right)\frac{\partial}{\partial \phi}\end{bmatrix} \cdot \boldsymbol{\mathcal{H}}(\boldsymbol{x}, \phi, t). \tag{42}$$

We will first assume that $\mathcal{F} = 0$ in Eq. (5). The case $\mathcal{F} \neq 0$ will be considered in detail in the following subsections. Calculation of the divergence $\nabla^*_{\boldsymbol{x}^*} \cdot (\langle \boldsymbol{u}^*|\phi^*, \boldsymbol{x}^*, t^*\rangle f_1^*)$ is tedious, although straightforward, and is presented in Appendix B. We only note that we make use of the fact that $\mathcal{X}$ and $\mathcal{Y}$ are harmonic functions, what follows from the Cauchy-Riemann conditions Eq. (22):

$$\frac{\partial^2 \mathcal{X}}{\partial x^2} + \frac{\partial^2 \mathcal{X}}{\partial y^2} = 0, \quad \frac{\partial^2 \mathcal{Y}}{\partial x^2} + \frac{\partial^2 \mathcal{Y}}{\partial y^2} = 0.$$

The final result reads

$$\nabla^*_{\boldsymbol{x}^*} \cdot [\langle \boldsymbol{u}^*(\boldsymbol{x}^*, t^*)|\boldsymbol{x}^*, \phi^*, t^*\rangle f_1^*] = \gamma(\boldsymbol{x})\nabla_{\boldsymbol{x}} \cdot [\langle \boldsymbol{u}(\boldsymbol{x}, t)|\boldsymbol{x}, \phi, t\rangle f_1] + \mathcal{G}(\boldsymbol{x}, \phi, t), \tag{43}$$

where

$$\mathcal{G}(\boldsymbol{x}, \phi, t) = \frac{\phi}{\gamma(\boldsymbol{x})}\left[\frac{\partial \gamma}{\partial x}\frac{\partial \langle u(\boldsymbol{x}, t)|\boldsymbol{x}, \phi, t\rangle f_1}{\partial \phi} + \frac{\partial \gamma}{\partial y}\frac{\partial \langle v(\boldsymbol{x}, t)|\boldsymbol{x}, \phi, t\rangle f_1}{\partial \phi}\right]. \tag{44}$$

Introducing Eqs. (43) and (44) into the LMN equation written in the new variables, see Eq. (33), we find, for $\phi = 0$, that $\mathcal{G} = 0$ follows and the LHS transforms as LHS $\to$ LHS* such that LHS$^* = \gamma(\boldsymbol{x})$LHS, hence we have

$$\gamma(\boldsymbol{x})\frac{\partial f_1(\boldsymbol{x}, \phi, t)\big|_{\phi=0}}{\partial t} + \gamma(\boldsymbol{x})\nabla_{\boldsymbol{x}} \cdot [\langle \mathbf{u}(\boldsymbol{x})|\boldsymbol{x}, \phi\rangle f_1(\boldsymbol{x}, \phi, t)]\big|_{\phi=0} = 0, \tag{45}$$

which reduces to Eq. (36) for zero-friction, zero forcing, and zero-viscosity.

A. CG invariance for nonzero friction and forcing

We now consider the stationary form of Eq. (5) with nonzero friction and forcing terms. We will set, for the time being, $\nu = 0$. Then, the transformed form of this equation reads

$$\nabla^*_{\boldsymbol{x}^*} \cdot [\langle \mathbf{u}^*(\boldsymbol{x}^*)|\boldsymbol{x}^*, \phi^*\rangle f_1^*(\boldsymbol{x}^*, \phi^*, t^*)] = \alpha\frac{\partial}{\partial \phi^*}(\phi^* f_1^*) + Q^*(\boldsymbol{x}^*, \boldsymbol{x}^*)\frac{\partial^2 f_1^*}{\partial \phi^{*2}}. \tag{46}$$

For the transformations described by Eqs. (28) and (30), we have

$$\frac{\partial}{\partial \phi^*} = \gamma(\boldsymbol{x})\frac{\partial}{\partial \phi}, \qquad \frac{\partial^2}{\partial \phi^{*2}} = \gamma^2(\boldsymbol{x})\frac{\partial^2}{\partial \phi^2}$$

and $\phi^* f_1^* = \phi f_1$, hence

$$\mathcal{F}^* = \gamma(\boldsymbol{x})\alpha\frac{\partial}{\partial \phi}(\phi f_1) + Q^*(\boldsymbol{x}^*, \boldsymbol{x}^*)\gamma^3(\boldsymbol{x})\frac{\partial^2 f_1}{\partial \phi^2}. \tag{47}$$

The amplitude $Q(\boldsymbol{x}, \boldsymbol{x})$ is transformed under the actions of G. In such a case, we deal with the equivalence transformations [20]. We recall that after the transformation the LHS of the LMN equation scales as $\gamma(\boldsymbol{x})$LHS; see Eq. (45). The same will be true for the RHS provided that

$$Q^*(\boldsymbol{x}^*, \boldsymbol{x}^*) = \gamma^{-2}(\boldsymbol{x})Q(\boldsymbol{x}, \boldsymbol{x}). \tag{48}$$

Since $\gamma^{-2}(\boldsymbol{x})$ is a positive number, then Q^* looks like an amplitude of the transformed forcing term. Therefore the forcing term L scales as $L^* = \gamma^{-1}(\boldsymbol{x})L$.

With this the transformed LMN Eq. (46), calculated at $\phi^* = 0$, in the original variables reads

$$\gamma(\boldsymbol{x})\nabla_{\boldsymbol{x}} \cdot [\langle \mathbf{u}(\boldsymbol{x})|\boldsymbol{x}, \phi\rangle f_1(\boldsymbol{x}, \phi, t)]\big|_{\phi=0} = \gamma(\boldsymbol{x})\alpha\frac{\partial}{\partial \phi}(\phi f_1)\Big|_{\phi=0} + \gamma(\boldsymbol{x})Q\frac{\partial^2 f_1}{\partial \phi^2}\Big|_{\phi=0}. \tag{49}$$

Hence, after dividing both sides by $\gamma(\boldsymbol{x}) \neq 0$, the original Eq. (5) for $\nu = 0$, evaluated at $\phi = 0$ is obtained.

In particular, the stationary form of Eq. (5) is invariantly transformed and holds at $\boldsymbol{x}^* = \boldsymbol{x}^*(l_1^*)$ after the transformation, where $\boldsymbol{x}^*(l^*)$ is the transformed zero-isoline $\boldsymbol{x}(l)$. The invariance of the probability measure $f_1 d\phi$, i.e., that the zero-isoline $\boldsymbol{x}(l)$ passes through the point $\boldsymbol{x}$, for $l = l_1$ follows from Eqs. (28) and (30):

$$f_1^*(\boldsymbol{x}^*(l_1^*), \phi^*)d\phi^*\Big|_{\phi^*=0} = f_1[(\boldsymbol{x}(l_1), \phi)]d\phi\Big|_{\phi=0}, \tag{50}$$

which present the value at $\phi = 0$. Hence, we proved the CG invariance of the stationary 1-point statistics of the zero-isoline $\boldsymbol{x}(l)$ of a scalar field in an inviscid flow.

B. CG invariance is broken for the viscous flow

In this section, we consider Eq. (5) in the presence of a nonzero viscosity. The following calculations are analogous to those presented in Ref. [14] for the characteristic equations. Considering a nonzero viscosity $\mathcal{F}$ is extended to Eq. (8). The second *RHS* term in Eq. (8) contributes to the direct cascade and removes energy at small scales due to viscosity. This term can also be reformulated as follows:

$$\mathcal{M} = \nu\frac{\partial}{\partial \phi}\left(\int d\phi' \int d\boldsymbol{x}'\delta(\boldsymbol{x} - \boldsymbol{x}')\phi'\Delta_{\boldsymbol{x}'} f_2\right) = \nu \lim_{|\boldsymbol{x}'-\boldsymbol{x}| \to 0}\frac{\partial}{\partial \phi}\int d\phi'\phi'\Delta_{\boldsymbol{x}'} f_2.$$

To determine $\mathcal{M}^*$, we first note that from Eq. (29) the infinitesimal $d\phi'^*$ is transformed according to

$$d\phi'^* = \gamma^{-m/6}(\boldsymbol{x})d\phi' \tag{51}$$

and the transformed module $|\boldsymbol{x}^* - \boldsymbol{x}'^*|$ is rescaled by the factor $\gamma^{-1/6}(\boldsymbol{x})$ as given in Eq. (37). Hence, if $|\boldsymbol{x}^* - \boldsymbol{x}'^*| \to 0$, then we get that $|\boldsymbol{x} - \boldsymbol{x}'| \to 0$. Further, using the definition Eqs. (26) and (27), we find

$$\nabla^*_{\boldsymbol{x}'^*} = \gamma^{2/3}(\boldsymbol{x})\begin{bmatrix} \frac{\partial \mathcal{X}}{\partial x}\frac{\partial}{\partial x} + \frac{\partial \mathcal{X}}{\partial y}\frac{\partial}{\partial y} \\ \frac{\partial \mathcal{X}}{\partial x}\frac{\partial}{\partial y} - \frac{\partial \mathcal{X}}{\partial y}\frac{\partial}{\partial x} \end{bmatrix}, \tag{52}$$

hence the Laplacian operator $\Delta^*_{x'^*}$ reads

$$\Delta^*_{x'^*} = \gamma^{4/3}(\boldsymbol{x})\left[\left(\frac{\partial \mathcal{X}}{\partial x}\right)^2 + \left(\frac{\partial \mathcal{X}}{\partial y}\right)^2\right]\Delta_{x'} = \gamma^{1/3}(\boldsymbol{x})\Delta_{x'}, \tag{53}$$

where we used Eq. (32). Substituting all the above transformed terms into formula for $\mathcal{M}^*$ and using Eqs. (28), (29), and (31), we obtain

$$\mathcal{M}^* = \nu \lim_{|\boldsymbol{x}'^*-\boldsymbol{x}^*|\to 0} \frac{\partial}{\partial \phi^*}\int d\phi'^*\phi'^*\Delta^*_{x'^*} f_2^* = \nu\gamma^{(14-m)/6} \lim_{|\boldsymbol{x}'-\boldsymbol{x}|\to 0} \frac{\partial}{\partial \phi}\int d\phi'\phi'\Delta_{x'} f_2. \tag{54}$$

Apparently, Eq. (54) or $\mathcal{M}^*$ scales differently than the LHS of Eq. (33) with $\mathcal{M}^* = \gamma^{(14-m)/6}(\boldsymbol{x})\mathcal{M}$ and, hence, viscosity is symmetry breaking with respect to the CG, except $m = 8$. The same result was obtained in Ref. [14] in the case of the characteristic lines. We suspect that this could be the reason why the CG was not observed at small scale turbulence in the previous studies [3,4,7,8], as it seems that $m = 8$ does not correspond to any known physical model. This situation is somehow analogous to that presented in Sec. III, for the classical scaling, where the presence of nonzero viscosity determined the coefficient s in Eqs. (12). Apparently, in case of CG viscous term also restricts the power coefficient to the only possible value with $m = 8$.

V. EXAMPLE OF TRANSFORMATION OF PDF's

In this section we will consider an example of PDF's transformed according to Eqs. (30) and (31). This example describes transformation of a PDF of a homogeneous field to the PDF of a nonhomogeneous field, both calculated at $\phi = 0$. The aim of this example is to demonstrate how the proposed transformations could potentially be used for predictions of turbulence statistics in $2d$.

We will consider a possibly simple, analytical form of the PDF to perform further calculations. A good candidate for the 2-point PDF is the bivariate Gaussian function, which was also considered in Ref. [22] as a simple model of $2d$ turbulence, which correctly reproduces some gross statistical features of the field, although it fails to model the energy transfer across scales. Analysis of real, numerical or experimental data of $2d$ turbulence is left for further work. The 2-point bivariate Gaussian PDF reads

$$f_2(\boldsymbol{x},\phi,\boldsymbol{x}',\phi') = \frac{1}{2\pi\sigma(\boldsymbol{x})\sigma(\boldsymbol{x}')\sqrt{1-\rho^2}}\exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{\phi^2}{\sigma^2(\boldsymbol{x})} - 2\rho\frac{\phi\phi'}{\sigma(\boldsymbol{x})\sigma(\boldsymbol{x}')} + \frac{\phi'^2}{\sigma^2(\boldsymbol{x}')}\right)\right], \tag{55}$$

where $\rho(\boldsymbol{x},\boldsymbol{x}')$ is a correlation function, which equals 1 if fluctuations at the two points are perfectly correlated and 0 if they are statistically independent. Here, $\sigma(\boldsymbol{x})$ stands for the standard deviation of the fluctuations of ϕ at point $\boldsymbol{x}$. The PDF which enters the LMN equation calculated at $\phi = 0$ reads

$$f_2(\boldsymbol{x},0,\boldsymbol{x}',\phi') = \frac{1}{2\pi\sigma(\boldsymbol{x})\sigma(\boldsymbol{x}')\sqrt{1-\rho^2}}\exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{\phi'^2}{\sigma^2(\boldsymbol{x}')}\right)\right]. \tag{56}$$

We assume the following, simplified form of the correlation function:

$$\rho^2(\boldsymbol{x},\boldsymbol{x}') = \begin{cases} 1 - \frac{|\boldsymbol{x}-\boldsymbol{x}'|^n}{\sigma^2(\boldsymbol{x})\sigma^2(\boldsymbol{x}')} & \text{if } \frac{|\boldsymbol{x}-\boldsymbol{x}'|^n}{\sigma^2(\boldsymbol{x})\sigma^2(\boldsymbol{x}')} < 1, \\ 0 & \text{otherwise,} \end{cases}$$

where the power n is so far unspecified.

The Gaussian 1-point PDF f_1 is obtained from Eq. (55) after integrating over the sample space ϕ',

$$f_1 = \int f_2 d\phi' = \frac{1}{\sqrt{2\pi}\sigma(\boldsymbol{x})}\exp\left(-\frac{\phi^2}{\sigma^2(\boldsymbol{x})}\right). \tag{57}$$

As the standard deviation $\sigma(\boldsymbol{x})$ is not constant, Eq. (55) is a 2-point PDF of a nonhomogeneous field.

Let us now consider the following, homogeneous PDF f_2^*, in the transformed variables which depends on the distance $|\boldsymbol{x}^* - \boldsymbol{x}'^*|$ but not $\boldsymbol{x}^*$ and $\boldsymbol{x}'^*$ separately,

$$f_2^*(\boldsymbol{x}^*, \phi^*, \boldsymbol{x}'^*, \phi'^*) = \frac{1}{2\pi\sqrt{1-\rho^{*2}}} \exp\left[-\frac{1}{2(1-\rho^{*2})}(\phi^{*2} - 2\rho^*\phi^*\phi'^* + \phi'^{*2})\right], \tag{58}$$

where $\sigma = 1$ and

$$\rho^{*2}(\boldsymbol{x}^*, \boldsymbol{x}'^*) = \begin{cases} 1 - |\boldsymbol{x}^* - \boldsymbol{x}'^*|^n & \text{if} \quad |\boldsymbol{x}^* - \boldsymbol{x}'^*|^n < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Analogous to Eq. (56), at $\phi^* = 0$ we have

$$f_2^*(\boldsymbol{x}^*, 0, \boldsymbol{x}'^*, \phi'^*) = \frac{1}{2\pi\sqrt{1-\rho^{*2}}} \exp\left[-\frac{\phi'^{*2}}{2(1-\rho^{*2})}\right], \tag{59}$$

and, after integrating over ϕ'^* we obtain the 1-point PDF:

$$f_1^* = \int f_2^* d\phi'^* = \frac{1}{\sqrt{2\pi}} \exp(-\phi^{*2}). \tag{60}$$

For such a homogeneous form of 2-point PDF, the second, integral term in Eq. (5) equals zero, hence, the condition $\mathcal{F} = 0$ defines the stationary situation.

We note that both, f_2 and f_2^*, as classical bivariate Gaussian functions, satisfy properties of 2-point PDF's Eqs. (9) and (11). They are normalized, moreover at $|\boldsymbol{x} - \boldsymbol{x}'| \to 0$, $f_2 = f_1\delta(\phi - \phi')$ and at $|\boldsymbol{x} - \boldsymbol{x}'| \to \infty$, $f_2 = f_1(\boldsymbol{x}, \phi) f_1(\boldsymbol{x}', \phi')$.

We now introduce transformation of variables, as defined by Eqs. (25)–(30) and assume that

$$\sigma(\boldsymbol{x}) = \gamma(\boldsymbol{x})$$

is described by Eq. (32). Hence, from Eq. (28) $\phi^* = \phi/\gamma(\boldsymbol{x})$. Substituting to Eq. (60) and comparing with Eq. (57) we obtain

$$f_1^* = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\phi^2}{\gamma^2(\boldsymbol{x})}\right) = \gamma(\boldsymbol{x}) f_1, \tag{61}$$

which is exactly the transformation Eq. (30). We also have, from Eq. (29), that $\phi'^* = \phi'/\gamma(\boldsymbol{x})^{m/6}$. Moreover, the global form of the transformation of $|\boldsymbol{x}^* - \boldsymbol{x}'^*|$, cf. Eq. (37), reads $|\boldsymbol{x} - \boldsymbol{x}'| = |\boldsymbol{x}^* - \boldsymbol{x}'^*|/\gamma(\boldsymbol{x})^{1/6}$. With this, the 2-point correlation function ρ^* transforms as follows:

$$\rho^{*2} = \begin{cases} 1 - |\boldsymbol{x} - \boldsymbol{x}'|^n/\gamma^{n/6}(\boldsymbol{x}) & \text{if} \quad |\boldsymbol{x} - \boldsymbol{x}'|^n/\gamma^{n/6}(\boldsymbol{x}) < 1, \\ 0 & \text{otherwise.} \end{cases}$$

After introducing this into Eq. (59) we obtain

$$f_2^*|_{\phi^*=0} = \frac{\gamma^{n/12}(\boldsymbol{x})}{2\pi|\boldsymbol{x} - \boldsymbol{x}'|^{n/2}} \exp\left[-\frac{\gamma(\boldsymbol{x})^{n/6}}{2|\boldsymbol{x} - \boldsymbol{x}'|^n} \frac{\phi'^2}{\gamma^{m/3}(\boldsymbol{x})}\right]. \tag{62}$$

To obtain the proper scaling of the 2-point PDF, as defined in Eq. (31), we require the power coefficient $n/12$ to be equal to $(m+6)/6$, hence $n = 2m + 12$. For such a case we obtain

$$f_2^*|_{\phi^*=0} = \frac{\gamma^{(m+6)/6}(\boldsymbol{x})}{2\pi\gamma(\boldsymbol{x}')\gamma(\boldsymbol{x})} \underbrace{\frac{\gamma(\boldsymbol{x})\gamma(\boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|^{n/2}}}_{1/\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2} \underbrace{\frac{\gamma^2(\boldsymbol{x})\gamma^2(\boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|^n}}_{1/(1-\rho^2)} \frac{\phi'^2}{\gamma^2(\boldsymbol{x}')}\right],$$

where above we grouped some terms, divided and multiplied by $\gamma(\boldsymbol{x})$ and $\gamma(\boldsymbol{x}')$, where necessary. The last step is to note that, according to the definition of ρ^2 above, $1-\rho^2 = |\boldsymbol{x}-\boldsymbol{x}'|^n/\gamma^2(\boldsymbol{x})\gamma^2(\boldsymbol{x}')$, hence, Eq. (62) can be rewritten as

$$f_2^*\big|_{\phi^*=0} = \frac{\gamma^{(m+6)/6}(\boldsymbol{x})}{2\pi\gamma(\boldsymbol{x})\gamma(\boldsymbol{x}')\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)}\frac{\phi'^2}{\gamma^2(\boldsymbol{x}')}\right] = \gamma^{(m+6)/6}(\boldsymbol{x}) f_2\big|_{\phi=0}.$$

With this it follows that the transformation of f_2, as given by Eq. (31), is realizable, provided that the 2-point correlation has a specific form. We consider only the first LMN equation and f_2 can be understood as an externally specified function. For example, for vorticity field $m = 2$, we have $n = 2m + 12 = 16$. For such large power coefficients ρ is nearly constant for small $|\boldsymbol{x}-\boldsymbol{x}'|$ and next decreases fast to 0. It is somewhat akin to the correlation function of a flow composed by statistically independent point vortices [23] provided that, in the numerical approximation, the vorticity of a single vortex is constant within a small region $|\boldsymbol{x}-\boldsymbol{x}'| < \epsilon$. In such a case we have $\rho = 1$ for $|\boldsymbol{x}-\boldsymbol{x}'| < \epsilon$ and $\rho = 0$ otherwise. We considered here a particular, analytical example of 2-point PDF in the form of bivariate Gaussian function. In real $2d$ turbulence f_2 has a different form and the corresponding 2-point PDF can also vary. Finally, we believe the above considerations provide only one of infinity many possibilities and they could be extended to a more general form of ρ, possibly in some approximate sense.

The conclusion that follows is that, for the given example, the first inviscid LMN Eq. (5) in $*$ variables, calculated for $\phi^* = 0$ transforms into inviscid LMN equation for a nonhomogeneous field, calculated at $\phi = 0$.

However, no conclusions can be drawn for $f_1(\phi, \boldsymbol{x}, t)$ if $\phi \neq 0$, as CG invariance is broken for such a case.

VI. CONCLUSIONS AND OUTLOOK

A question posed in Ref. [4] was whether the conformal invariance of statistics of the zero-scalar isolines could be explained by symmetry analysis of the underlying equations. If this is the case, it is worthwhile to consider the Lundgren-Monin-Novikov equations for the PDF of the scalar. We considered the first equation from the infinite LMN hierarchy for a model with an arbitrary coefficient m. The value $m = 2$ refers to vorticity field in $2d$, $m = 3$ describes the SQG model, and $m = 6$ corresponds to the large-scale flows in a rotating shallow fluid.

We investigated how this equation is transformed under the action of G derived in Refs. [11,14]. We then established that the CG invariance can be retained in the presence of large-scale friction and forcing under the restriction $\phi = 0$; however, it is broken if the viscous term is included into the equation.

We presented an example of 1- and 2-point PDFs which transform according to Eqs. (25)–(31). It represents a transition from PDF's describing a homogeneous into PDF's describing a nonhomogeneous field, calculated at $\phi^* = 0$ and $\phi = 0$, respectively. The 2-point PDF's can be treated as an external function entering the first LMN Eq. (5). Its form was specified *a priori* as bivariate Gaussian, with prescribed 2-point correlation function. It follows from the proposed transformations, that in the considered case, once solution for the homogeneous field is calculated, its nonhomogeneous counterpart is also known at $\phi = 0$. As it follows, the statistic which remains invariant after the CG transformation is the 1-point probability measure of ϕ taking the value $\phi = 0$, or the probability that the zero-scalar isoline $\boldsymbol{x}(l)$ passes through the selected point $\boldsymbol{x}$.

In future, it remains to show the CG invariance of the n-point statistics of the zero-vorticity isolines $\boldsymbol{x}(l)$ by considering the n-point PDF $f_n(\boldsymbol{x}(l_1), \phi = 0, \ldots, \boldsymbol{x}(l_n), \phi_n = 0)$ of the vorticity field along a zero-vorticity random curve $\boldsymbol{x}(l)$.

ACKNOWLEDGMENTS

M.O. was partially supported by the German Research Foundation (DFG) under Grant No. OB 96/48-1. The authors acknowledge Z. Wacławczyk for drawing the teaser figure.

APPENDIX A: JACOBI MATRIX

To transform the increment $d\mathbf{x}'^{*}d\phi'^{*}$ in the integrals Eqs. (6) and (7), according to relations Eqs. (26), (27), and (29), we calculate the determinant of the Jacobi matrix,

$$detJ = \gamma^{2/3}\begin{vmatrix} \frac{\partial\mathcal{X}}{\partial x} & \frac{\partial\mathcal{X}}{\partial y} & 0 \\ \frac{\partial\mathcal{Y}}{\partial x} & \frac{\partial\mathcal{Y}}{\partial y} & 0 \\ 0 & 0 & \gamma^{-m/6} \end{vmatrix} = \gamma^{2/3-m/6}\left[\left(\frac{\partial\mathcal{X}}{\partial x}\right)^2+\left(\frac{\partial\mathcal{Y}}{\partial y}\right)^2\right] = \gamma^{-1/3-m/6}, \tag{A1}$$

where we also made use of the Cauchy-Riemann conditions, Eq. (22) and the definition of $\gamma(\boldsymbol{x})$ from Eq. (32).

APPENDIX B: DIVERGENCE

To calculate the divergence of the conditional velocity $\nabla^{*}_{\boldsymbol{x}^*}\cdot[\langle \boldsymbol{u}^*(\boldsymbol{x}^*,t^*)|\boldsymbol{x}^*,\phi^*,t^*\rangle f_1^*]$ introducing the form of transformed variables Eqs. (25)–(31) we apply the operator Eq. (42) to Eqs. (40) and (41). Using the product rule we obtain

$$\begin{aligned}
&\nabla^{*}_{\boldsymbol{x}^*}\cdot[\langle \boldsymbol{u}^*|\boldsymbol{x}^*,\phi^*,t^*\rangle f_1^*]\\
&= \gamma\left(\frac{\partial\mathcal{X}}{\partial x}\right)^2[\langle u|\boldsymbol{x},\phi,t\rangle f_1]\frac{\partial\gamma}{\partial x} + \gamma^2\frac{\partial\mathcal{X}}{\partial x}[\langle u|\boldsymbol{x},\phi,t\rangle f_1]\frac{\partial^2\mathcal{X}}{\partial x^2} + \gamma^2\left(\frac{\partial\mathcal{X}}{\partial x}\right)^2\frac{\partial[\langle u|\boldsymbol{x},\phi,t\rangle f_1]}{\partial x}\\
&\quad + \cancel{\gamma\left(\frac{\partial\mathcal{X}}{\partial x}\right)\left(\frac{\partial\mathcal{X}}{\partial y}\right)[\langle v|\boldsymbol{x},\phi,t\rangle f_1]\frac{\partial\gamma}{\partial x}} + \gamma^2\frac{\partial\mathcal{X}}{\partial x}[\langle v|\boldsymbol{x},\phi,t\rangle f_1]\frac{\partial^2\mathcal{X}}{\partial x\partial y}\\
&\quad + \cancel{\gamma^2\left(\frac{\partial\mathcal{X}}{\partial x}\right)\left(\frac{\partial\mathcal{X}}{\partial y}\right)\frac{\partial[\langle v|\boldsymbol{x},\phi,t\rangle f_1]}{\partial x}} + \cancel{\gamma\left(\frac{\partial\mathcal{X}}{\partial x}\right)\left(\frac{\partial\mathcal{X}}{\partial y}\right)[\langle u|\boldsymbol{x},\phi,t\rangle f_1]\frac{\partial\gamma}{\partial y}}\\
&\quad + \gamma^2\frac{\partial\mathcal{X}}{\partial y}[\langle u|\boldsymbol{x},\phi,t\rangle f_1]\frac{\partial^2\mathcal{X}}{\partial x\partial y} + \cancel{\gamma^2\left(\frac{\partial\mathcal{X}}{\partial x}\right)\left(\frac{\partial\mathcal{X}}{\partial y}\right)\frac{\partial[\langle u|\boldsymbol{x},\phi,t\rangle f_1]}{\partial y}}\\
&\quad + \left(\frac{\partial\mathcal{X}}{\partial y}\right)^2[\langle v|\boldsymbol{x},\phi,t\rangle f_1]\frac{\partial\gamma}{\partial y} + \gamma^2\frac{\partial\mathcal{X}}{\partial y}[\langle v|\boldsymbol{x},\phi,t\rangle f_1]\frac{\partial^2\mathcal{X}}{\partial y^2} + \gamma^2\left(\frac{\partial\mathcal{X}}{\partial y}\right)^2\frac{\partial[\langle v|\boldsymbol{x},\phi,t\rangle f_1]}{\partial x}\\
&\quad - \cancel{\gamma\left(\frac{\partial\mathcal{X}}{\partial x}\right)\left(\frac{\partial\mathcal{X}}{\partial y}\right)[\langle u|\boldsymbol{x},\phi,t\rangle f_1]\frac{\partial\gamma}{\partial y}} - \gamma^2\frac{\partial\mathcal{X}}{\partial x}[\langle u|\boldsymbol{x},\phi,t\rangle f_1]\frac{\partial^2\mathcal{X}}{\partial y^2}\\
&\quad - \cancel{\gamma^2\left(\frac{\partial\mathcal{X}}{\partial x}\right)\left(\frac{\partial\mathcal{X}}{\partial y}\right)\frac{\partial[\langle u|\boldsymbol{x},\phi,t\rangle f_1]}{\partial y}} + \gamma\left(\frac{\partial\mathcal{X}}{\partial x}\right)^2[\langle v|\boldsymbol{x},\phi,t\rangle f_1]\frac{\partial\gamma}{\partial y}\\
&\quad + \gamma^2\frac{\partial\mathcal{X}}{\partial x}[\langle v|\boldsymbol{x},\phi,t\rangle f_1]\frac{\partial^2\mathcal{X}}{\partial x\partial y} + \gamma^2\left(\frac{\partial\mathcal{X}}{\partial x}\right)^2\frac{\partial[\langle v|\boldsymbol{x},\phi,t\rangle f_1]}{\partial y}\\
&\quad + \gamma\left(\frac{\partial\mathcal{X}}{\partial y}\right)^2[\langle u|\boldsymbol{x},\phi,t\rangle f_1]\frac{\partial\gamma}{\partial x} + \gamma^2\frac{\partial\mathcal{X}}{\partial y}[\langle u|\boldsymbol{x},\phi,t\rangle f_1]\frac{\partial^2\mathcal{X}}{\partial x\partial y} + \gamma^2\left(\frac{\partial\mathcal{X}}{\partial y}\right)^2\frac{\partial[\langle u|\boldsymbol{x},\phi,t\rangle f_1]}{\partial x}\\
&\quad - \cancel{\gamma\left(\frac{\partial\mathcal{X}}{\partial x}\right)\left(\frac{\partial\mathcal{X}}{\partial y}\right)[\langle v|\boldsymbol{x},\phi,t\rangle f_1]\frac{\partial\gamma}{\partial x}} - \gamma^2\frac{\partial\mathcal{X}}{\partial y}[\langle v|\boldsymbol{x},\phi,t\rangle f_1]\frac{\partial^2\mathcal{X}}{\partial x^2}
\end{aligned}$$

$$
\begin{aligned}
&\cancel{-\gamma^2\left(\frac{\partial\mathcal{X}}{\partial x}\right)\left(\frac{\partial\mathcal{X}}{\partial y}\right)\frac{\partial[\langle v|\boldsymbol{x},\phi,t\rangle f_1]}{\partial x}} + \gamma\phi\frac{\partial\mathcal{X}}{\partial x}\left(\frac{\partial\mathcal{X}}{\partial x}\frac{\partial\ln\gamma}{\partial x} + \cancel{\frac{\partial\mathcal{X}}{\partial y}\frac{\partial\ln\gamma}{\partial y}}\right)\frac{\partial[\langle u|\boldsymbol{x},\phi,t\rangle f_1]}{\partial\phi}\\
&+\gamma\phi\frac{\partial\mathcal{X}}{\partial y}\left(\cancel{\frac{\partial\mathcal{X}}{\partial x}\frac{\partial\ln\gamma}{\partial x}} + \frac{\partial\mathcal{X}}{\partial y}\frac{\partial\ln\gamma}{\partial y}\right)\frac{\partial[\langle v|\boldsymbol{x},\phi,t\rangle f_1]}{\partial\phi}\\
&-\gamma\phi\frac{\partial\mathcal{X}}{\partial y}\left(\cancel{\frac{\partial\mathcal{X}}{\partial x}\frac{\partial\ln\gamma}{\partial y}} - \frac{\partial\mathcal{X}}{\partial y}\frac{\partial\ln\gamma}{\partial x}\right)\frac{\partial[\langle u|\boldsymbol{x},\phi,t\rangle f_1]}{\partial\phi}\\
&+\gamma\phi\frac{\partial\mathcal{X}}{\partial x}\left(\frac{\partial\mathcal{X}}{\partial x}\frac{\partial\ln\gamma}{\partial y} - \cancel{\frac{\partial\mathcal{X}}{\partial y}\frac{\partial\ln\gamma}{\partial x}}\right)\frac{\partial[\langle v|\boldsymbol{x},\phi,t\rangle f_1]}{\partial\phi},
\end{aligned}
\tag{B1}
$$

where the terms which cancel are struck through. Using the definition of $\gamma(\boldsymbol{x})$, cf. Eq. (32), we can further rewrite the first terms from lines 2, 3, 4, and 5 as

$$
[\langle u|\boldsymbol{x},\phi,t\rangle f_1]\frac{\partial\gamma}{\partial x} + [\langle v|\boldsymbol{x},\phi,t\rangle f_1]\frac{\partial\gamma}{\partial y}.
$$

We next calculate derivatives of γ, using the definition Eq. (32). With this the above formula reads

$$
-2\gamma^2[\langle u|\boldsymbol{x},\phi,t\rangle f_1]\left(\frac{\partial\mathcal{X}}{\partial x}\frac{\partial^2\mathcal{X}}{\partial x^2} + \frac{\partial\mathcal{X}}{\partial y}\frac{\partial^2\mathcal{X}}{\partial x\partial y}\right) - 2\gamma^2[\langle v|\boldsymbol{x},\phi,t\rangle f_1]\left(\frac{\partial\mathcal{X}}{\partial x}\frac{\partial^2\mathcal{X}}{\partial x\partial y} + \frac{\partial\mathcal{X}}{\partial y}\frac{\partial^2\mathcal{X}}{\partial y^2}\right).
$$

Mixed derivatives will now cancel with the second terms in lines 3, 4, 7, and 8 from Eq. (B1). Terms with $\partial^2\mathcal{X}/\partial x^2$ and $\partial^2\mathcal{X}/\partial y^2$ will be subtracted from second terms in lines 2 and 5 in Eq. (B1). In this way all the first and second terms in lines 2–9 reduce to

$$
-\gamma^2\frac{\partial\mathcal{X}}{\partial x}[\langle u|\boldsymbol{x},\phi,t\rangle f_1]\left(\frac{\partial^2\mathcal{X}}{\partial x^2} + \frac{\partial^2\mathcal{X}}{\partial y^2}\right) - \gamma^2\frac{\partial\mathcal{X}}{\partial y}[\langle v|\boldsymbol{x},\phi,t\rangle f_1]\left(\frac{\partial^2\mathcal{X}}{\partial x^2} + \frac{\partial^2\mathcal{X}}{\partial y^2}\right) = 0,
$$

which equals zero due to the fact that $\mathcal{X}$ is a harmonic function. We are hence left with the third terms in lines 2, 5, 7, and 8, which can be added and rewritten as

$$
\gamma(\boldsymbol{x})\frac{\partial[\langle u|\boldsymbol{x},\phi,t\rangle f_1]}{\partial x} + \gamma(\boldsymbol{x})\frac{\partial[\langle v|\boldsymbol{x},\phi,t\rangle f_1]}{\partial x},
\tag{B2}
$$

where we again made use of Eq. (32).

Finally, the remaining terms in lines 10–13 can be combined to give

$$
\phi\frac{\partial\ln\gamma}{\partial x}\frac{\partial[\langle u|\boldsymbol{x},\phi,t\rangle f_1]}{\partial\phi} + \phi\frac{\partial\ln\gamma}{\partial y}\frac{\partial[\langle v|\boldsymbol{x},\phi,t\rangle f_1]}{\partial\phi}.
$$

With this, the transformed divergence of the conditional velocity reads

$$
\begin{aligned}
\boldsymbol{\nabla}^*_{\boldsymbol{x}^*}\cdot[\langle\boldsymbol{u}^*|\boldsymbol{x}^*,\phi^*,t^*\rangle f_1^*] &= \gamma(\boldsymbol{x})\frac{\partial[\langle u|\boldsymbol{x},\phi,t\rangle f_1]}{\partial x} + \gamma(\boldsymbol{x})\frac{\partial[\langle v|\boldsymbol{x},\phi,t\rangle f_1]}{\partial x}\\
&\quad + \frac{\phi}{\gamma(\boldsymbol{x})}\left\{\frac{\partial\gamma}{\partial x}\frac{\partial[\langle u|\boldsymbol{x},\phi,t\rangle f_1]}{\partial\phi} + \frac{\partial\gamma}{\partial y}\frac{\partial[\langle v|\boldsymbol{x},\phi,t\rangle f_1]}{\partial\phi}\right\}.
\end{aligned}
\tag{B3}
$$

[1] Y. Nakayama, Scale invariance vs conformal invariance, Phys. Rep. **569**, 1 (2015).

[2] A. M. Polyakov, The theory of turbulence in two dimensions, Nucl. Phys. B **396**, 367 (1993).

[3] D. Bernard, G. Boffetta, A. Celani, and G. Falkovich, Conformal invariance in two-dimensional turbulence, Nat. Phys. **2**, 124 (2006).

[4] G. Falkovich, Conformal invariance in hydrodynamic turbulence, Russ. Math. Surv. **62**, 497 (2007).

[5] O. Schramm, Scaling limits of loop-erased random walks and uniform spanning trees, Israel J. Math. **118**, 221 (2000).

[6] K. Wilson, Renormalization group and critical phenomenon, Phys. Rev. B **4**, 3184 (1971).
[7] D. Bernard, G. Boffetta, A. Celani, and G. Falkovich, Inverse Turbulent Cascades and Conformally Invariant Curves, Phys. Rev. Lett. **98**, 024501 (2007).
[8] G. Falkovich and S. Musacchio, Conformal invariance in inverse turbulent cascades, arXiv:1012.3868.
[9] A. Hasegawa and K. Mima, Pseudo-three-dimensional turbulence in magnetized nonuniform plasma, Phys. Fluids **21**, 87 (1978).
[10] W. Horton and A. Hasegawa, Quasi–two–dimensional dynamics of plasmas and fluids, Chaos **4**, 227 (1994).
[11] V. N. Grebenev, M. Wacławczyk and M. Oberlack, Conformal invariance of the Lungren-Monin-Novikov equations for vorticity fields in 2D turbulence, J. Phys. A: Math. Theor. **50**, 435502 (2017)
[12] S. Thalabard and J. Bec, Turbulence of generalized flows in two dimensions, J. Fluid Mech. **883**, A49 (2020).
[13] V. N. Grebenev, M. Wacławczyk, and M. Oberlack, Conformal invariance of the zero-vorticity Lagrangian path in 2D turbulence, J. Phys. A: Math. Theor. **52**, 335501 (2019).
[14] M. Wacławczyk, V. N. Grebenev, and M. Oberlack, Conformal invariance of characteristic lines in a class of hydrodynamic models, Symmetry **12**, 1482 (2020).
[15] G. Boffetta, A. Celani, S. Musacchio, and M. Vergassola, Intermittency in two-dimensional Ekman-Navier-Stokes turbulence, Phys. Rev. E **66**, 026304 (2002).
[16] E. A. Novikov, Kinetic equations for a vortex field, Dokl. Akad. Nauk SSSR **177**, 299 (1967).
[17] R. Friedrich, A. Daitche, O. Kamps, J. Lülff, M. Michel Voßkuhle and M. Wilczek, The Lundgren-Monin-Novikov hierarchy: Kinetic equations for turbulence, C. R. Phys. **13**, 929 (2012).
[18] G. Boffetta, and R. E. Ecke, Two-dimensional turbulence, Annu. Rev. Fluid Mech. **44**, 427 (2012).
[19] V. K. Andreev, O. V. Kaptsov, V. V. Pukhnachev and A. A. Rodionov, *Application of the Group Theoretical Methods in Hydrodynamics* (Kluwer Acdemic Publishers, Dordrech, Boston, London, 1998).
[20] L. V. Ovsyannikov, *Group Analysis of Differential Equations* (Nauka, Moscow, 1978).
[21] M. Wacławczyk, N. Staffolani, M. Oberlack, A. Rosteck, M. Wilczek, R. Friedrich, Statistical symmetries of the Lundgren-Monin-Novikov hierarchy, Phys. Rev. E **90**, 013022 (2014)
[22] R. Friedrich, M. Voßkuhle, O. Kamps and M. Wilczek, Two-point vorticity statistics in the inverse cascade of two-dimensional turbulence, Phys. Fluids **24**, 125101 (2012);
[23] D. Boyer, Critical behavior of vorticity in two-dimensional turbulence, Phys. Rev. E **60**, 6769 (1999)