

Representing rectangular jet dynamics through azimuthal Fourier modes

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Rectangular propulsion nozzles offer thrust-vectoring and air-frame-integration advantages over their more commonly studied circular counterparts. However, they display many distinguishing features which violate assumptions, such as azimuthal homogeneity, typically used in prediction tools for circular jets. In the present paper, we examine the utility of an azimuthal Fourier decomposition for rectangular Mach 1.3 jets of aspect ratio (AR) 1, 4, and 8 using large-eddy simulations, with a circular jet of the same equivalent diameter for reference. The simulations manifest key features of rectangular jets, including higher spreading rates and shorter potential cores with increasing AR, axis switching (AR=4), and azimuthal variation in peak acoustic intensity (AR=8). We show that, after projection on a cylindrical frame, a sine-cosine ansatz for the azimuthal Fourier series affords a more convenient representation of nonaxisymmetric flow features than the commonly used complex exponential ansatz. Fluctuation magnitudes of the higher azimuthal modes show rapid reduction in amplitude, similar to those observed in circular jets, especially if an acoustic fluctuation field based on momentum potential theory is chosen instead of pressure fluctuations. The leading modes differ, however, from those of a circular jet in two important aspects, namely, the mechanisms represented by the sine and cosine coefficients of the first azimuthal mode and the rate of streamwise decay of all modes with increasing AR. These differences are traced to the near- and far-field rectangular jet asymmetry by examining azimuthal inhomogeneity, the implications of which are assessed with a generalized expression for acoustic intensity based on energies of leading modes. The significant simplicity of circular plumes is recovered as a special case of the analysis. Invocation of the twofold mirror symmetry property of rectangular jets eases the prediction procedure so that only two extra terms, representing mechanisms unique to rectangular jets, specifically preferential flapping in the minor axis direction and coupling of axisymmetric and second azimuthal modes, are sufficient to recover the advantages of azimuthal decomposition.

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I. INTRODUCTION

Increasing engine power requirements for commercial and military aircraft have continued to raise concern about acoustic radiation from jets. This has motivated new research on noise

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production and propagation mechanisms in both perfectly and imperfectly expanded conditions. Several review articles provide overview snapshots of the state-of-the-art research [1–5]. Major recent advances have leveraged the better understanding to derive low-rank models of turbulent fluctuations [6–9] and relatively simple tools for noise predictions [2,10–13]. These efforts are interrelated since the growth and decay of large scale turbulent structures, which are the dominant dynamic feature of jet turbulence, have also been linked to the peak turbulent mixing noise emitted in the downstream shallow angle direction [1].

Most studies in this vein have addressed jets issuing from circular nozzles, which display inherent symmetries that can be exploited for model simplification. A cylindrical coordinate system is a natural choice for these jets and it simplifies the application of the azimuthal periodicity property of the statistics through an azimuthal Fourier series decomposition, offering several advantages. For example, Michalke and Fuchs [14] showed that the fluctuating pressure field of a turbulent circular jet could be represented to reasonable accuracy with only the first three azimuthal modes, since they contain most of the fluctuation energy, allowing for significant data reduction by a truncation of the Fourier series beyond the leading terms. Moreover, Schmidt *et al.* [7] used spectral proper orthogonal decomposition (SPOD) to show that the axisymmetric and first azimuthal mode exhibit pronounced low-rank behavior for a range of acoustically significant frequencies. In addition to these advantages, an azimuthal homogeneity arising out of the polar symmetry of circular jets further simplifies their modeling [6,15].

Besides data driven methods like Fourier analysis or proper orthogonal decomposition (POD), the linear Navier-Stokes operator can also be used to model the dominant dynamic features of the jet [16]. Gudmundsson and Colonius [8] used the linear framework of the parabolized stability equations (PSE) to model coherent fluctuations in the jet and demonstrated close agreement with the near-field low frequency turbulent statistics for the first three azimuthal modes. The assumption of mean flow homogeneity in the azimuthal direction, used in several earlier linear analyses, can also be relaxed as shown by Lajús *et al.* [17] who used Floquet theory to model the axisymmetric instability waves of corrugated jets. While such studies lay the groundwork for a linear analysis of the azimuthal modes of more complex rectangular jet flow fields, the turbulence statistics of the azimuthal Fourier modes in rectangular jets have not yet been analyzed.

In addition to modeling turbulent fluctuations in the jet, the azimuthal Fourier basis is also particularly suited to the problem of jet aeroacoustics because higher azimuthal modes have lower acoustic efficiency [18], and thus have a progressively smaller impact on the peak acoustic radiation in the downstream shallow angle direction. This is also corroborated by experimental results, where the peak jet noise field is found to be predominantly axisymmetric. Cavalieri *et al.* [19] further showed that higher azimuthal Fourier modes ($m = 1$ and 2) of circular jets contribute progressively less with increasing Mach number, and have a relatively minor influence in transonic and supersonic jets.

These favorable properties of azimuthal Fourier modes, coupled with an improved understanding of jet turbulence, have led to the development of a variety of models for jet acoustic radiation. These can be broadly categorized into dynamic and kinematic models. The former reduce the far-field acoustic radiation based solely on the dominant dynamical features of the jet. For instance, Sinha *et al.* [20] were able to capture the two leading azimuthal modes of the far-field acoustics of a circular supersonic jet with appreciable accuracy using both a PSE framework established by Gudmundsson and Colonius [8], as well as the leading SPOD modes of the jet. Kinematic models, on the other hand, employ carefully chosen statistics within the core of the jet, with some empiricism, to model the acoustic source field and have received significant attention in recent years [2,10,13,21]. Some of these kinematic models have shown that the statistics of the fully turbulent flow can yield erroneous far-field acoustics [22]. Given the acoustic significance of the axisymmetric azimuthal mode, better results are obtained when quantities such as the amplitude envelope [11], two point coherence length [22], and cross spectral density are evaluated for the axisymmetric azimuthal mode alone instead of the full turbulent data. Thus, azimuthal mode decomposition of fluctuation data is a key component of most acoustic models for circular jets.

While much of the literature cited above has focused on circular jets, interest in rectangular jets has grown in recent years. Among their advantages are higher rates of entrainment [23,24], thrust vectoring potential, and easier airframe integration. The near-field dynamics of rectangular jets, however, are more complex than those of circular jets, and can include axis switching [25,26], preferential flapping in the minor axis direction [27,28], warping of azimuthal vortex rings [29,30], and different forms of streamwise vorticity generation [26]. These features can manifest their signature in the far-field acoustics as well. In addition to the variables that affect the dynamics and acoustics of circular jets such as operating conditions, rectangular jets are characterized by their aspect ratio (AR) [31] and upstream duct geometry [26].

Bridges [31] performed a detailed comparison of the acoustics of circular and rectangular jets. Key findings were that the far-field acoustics of low AR rectangular jets were essentially axisymmetric and closely matched those of a similarly sized circular jet. However, at high AR such as $AR = 8$, several distinct features emerged relative to circular jets. While qualitative features such as superdirectivity were retained, the minor axis direction exhibited 2-dB excess far-field acoustic radiation compared to the major axis direction, marking a loss in the acoustic axisymmetry.

The rectangular jet aeroacoustic field is also affected by imperfect expansion in a manner distinct from circular jets. For example, Veltin and McLaughlin [32] found the minor axis plane of an $AR = 1.75$ jet to be 3 dB louder than the major axis plane in the sideline direction due to the shock component of the noise. This disparity, however, did not significantly affect the peak noise in the downstream shallow angle direction which remained axisymmetric for this AR. Heeb *et al.* [33] also measured noise in the downstream shallow angle direction to be axisymmetric for an $AR=2$ shock-containing jet. Thus, although there are a variety of flow field differences, the peak noise from low AR jets in the downstream shallow angle direction is similar to that of circular jets for a range of operating conditions [34].

Despite qualitative similarities such as superdirectivity and acoustic axisymmetry (for low AR), many of the sophisticated analyses tools developed for the circular jet acoustics have not been extended to rectangular jets. One of the difficulties encountered is that, unlike the azimuthal Fourier modes for circular jets, there is no obvious set of spatial functions in which to expand the instantaneous statistics of rectangular jets. Kinzie and McLaughlin [35] address the related problem for elliptical jets by considering Mathieu functions, which have an elliptical variation in the azimuthal direction, in line with the shape of the nozzle exit. The sharp corners of rectangular jets pose an even greater difficulty in the isolation of such functions. Additionally, the form of the plume can change quite drastically downstream of the nozzle exit due to axis switching, rendering spatial functions derived for a specific nozzle exit shape less useful in these regions. Even if such functions are found, they may not possess the many simplifying characteristics of Fourier modes, such as mutual orthogonality and decreasing acoustic efficiency.

Techniques to deploy an azimuthal Fourier decomposition for rectangular jets have the potential to vastly simplify their analyses and facilitate adaptation of key modeling procedures from those evolved for circular jets. The utility of any azimuthal mode decomposition for rectangular jets can be broken down into a test for two features. First, a rapid convergence of the instantaneous statistics of the jet in the azimuthal Fourier space is desired, since this fosters a simplified error bound truncation of the Fourier series beyond the dominant modes. Unlike for circular jets, this property is not guaranteed for rectangular jets since a projection on a cylindrical coordinate system, necessary for the azimuthal Fourier decomposition, may not be well suited due to a lack of polar symmetry. This is especially true for the region near the nozzle exit where such jets are predominantly rectangular in cross section; sharp corners immediately downstream of the nozzle exit may require infinitely many Fourier modes for a complete representation. Nonetheless, instabilities arising at the corners of rectangular jets in the near nozzle region smooth out mean flow gradients [36], which may help ameliorate this difficulty. Recently, Chakrabarti *et al.* [37] demonstrated promising preliminary results in reconstructing the instantaneous statistics of an $AR = 4$ rectangular jet using the leading azimuthal modes.

The second, related, desired feature is an ability to reconstruct the overall sound pressure levels (OASPLs) using the leading Fourier modes. In circular jets, azimuthal homogeneity of the statistics enables a reconstruction of the root mean square (RMS) of fluctuations as the sum of squares of the RMS of individual modes; this greatly reduces the complexity of modeling [15,19]. The lack of azimuthal statistical homogeneity in rectangular jets introduces additional complication, since it implies that constructive interference between all pairs of azimuthal modes must be accounted for while reconstructing the RMS. However, some simplification is afforded by leveraging mirror symmetry of rectangular jets along the minor and major axis planes. Verifying the applicability of such simplifying techniques for rectangular jets is thus nontrivial.

The present paper thus examines ways of using azimuthal decomposition approaches to facilitate the description of rectangular jet evolution for a range of aspect ratios. Specifically, we examine the statistics of the leading azimuthal modes of rectangular jets and assess their features relative to those of a circular jet. To that end, large-eddy simulations (LESs) are performed for rectangular jets of AR 1, 4, and 8 along with a circular jet of equivalent diameter. The details of the numerical algorithm and flow conditions used are described in Sec. II. Key differences between the rectangular and circular jets, including critical mean flow parameters and the far-field acoustics, are highlighted in Sec. III. Although the pressure field is typically employed for such procedures, we rely primarily on an acoustic variable derived from a physics-based fluid thermodynamic (FT) component (vortical, acoustic, and entropic) decomposition of the turbulent fluctuation field [38]. The FT decomposition provides greater insights into the acoustic dynamics and improves the convergence of modal decomposition techniques [39]; a summary of this decomposition and dominant features of the individual FT components of the rectangular jets are presented in Sec. IV. The effectiveness of azimuthal decomposition is investigated by mapping fluctuation fields onto a cylindrical mesh. The consequences of the departure from axisymmetry are then evaluated in Sec. V by examining the dynamics of higher azimuthal Fourier modes of the acoustic fluctuations. The implications of the azimuthal inhomogeneity of rectangular jets as well as some specific dynamic features associated with their near-field asymmetry are discussed in Sec. VI. Using these considerations, a reduced order model for rectangular jets is developed in Sec. VII using only three leading azimuthal modes. Finally, a summary of the paper is presented in Sec. VIII.

II. METHODOLOGY

The nondimensionalized compressible Navier-Stokes equations cast in the strong conservation form are solved in a curvilinear coordinate (ξ, η, ζ) frame with J being the Jacobian of the curvilinear transformation given by $(J = \frac{\partial(\xi, \eta, \zeta, \tau)}{\partial(x, y, z, t)})$ [40]:

$$\frac{\partial}{\partial \tau} \left(\frac{\vec{q}}{J} \right) + \frac{\partial \hat{F}}{\partial \xi} + \frac{\partial \hat{G}}{\partial \eta} + \frac{\partial \hat{H}}{\partial \zeta} = \frac{1}{\text{Re}} \left[\frac{\partial \hat{F}_v}{\partial \xi} + \frac{\partial \hat{G}_v}{\partial \eta} + \frac{\partial \hat{H}_v}{\partial \zeta} \right] \quad (1)$$

where \vec{q} refers to a vector composed of the conserved variables. Thus, $\vec{q} = [\rho, \rho u, \rho v, \rho w, \rho E]^T$. The vectors \hat{F} , \hat{G} , and \hat{H} correspond to the inviscid fluxes and the vectors \hat{F}_v , \hat{G}_v , and \hat{H}_v constitute the viscous fluxes. For example,

$$F = \begin{bmatrix} \rho \hat{U} \\ \rho u \hat{U} + \xi_x p \\ \rho v \hat{U} + \xi_y p \\ \rho w \hat{U} + \xi_z p \\ (\rho E + p) \hat{U} \end{bmatrix} \quad \text{and} \quad F_v = \begin{bmatrix} 0 \\ \xi_{x_i} \sigma_{i1} \\ \xi_{x_i} \sigma_{i1} \\ \xi_{x_i} \sigma_{i1} \\ \xi_{x_i} (u_j \sigma_{ij} - \Theta_i) \end{bmatrix}. \quad (2)$$

The contravariant velocity component is given by $\hat{U} = \xi_x u + \xi_y v + \xi_z w$ and the specific energy density is

$$E = \frac{T}{\gamma(\gamma - 1)M_j^2} + \frac{1}{2}(u^2 + v^2 + w^2). \quad (3)$$

The deviatoric stress (σ_{ij}) and heat flux vector (Θ_i) are given by

$$\sigma_{ij} = \mu \left(\frac{\partial \xi_k}{\partial x_j} \frac{\partial u_i}{\partial \xi_k} + \frac{\partial \xi_k}{\partial x_i} \frac{\partial u_j}{\partial \xi_k} - \frac{2}{3} \frac{\partial \xi_l}{\partial x_k} \frac{\partial u_k}{\partial \xi_l} \delta_{ij} \right), \quad (4)$$

$$\Theta_i = -\frac{1}{(\gamma - 1)M_j^2} \left(\frac{\mu}{\text{Pr}} \right) \frac{\partial \xi_j}{\partial x_i} \frac{\partial T}{\partial \xi_j}. \quad (5)$$

Stokes's hypothesis is assumed for the bulk viscosity coefficient, i.e., $\lambda = -2/3\mu$. Further details of the derivations may be found in Vinokur [41]. Flow velocities and density are normalized by the jet exit velocity (u_j) and density (ρ_j), respectively, while the pressure is nondimensionalized by $\rho_j U_j^2$; the perfect gas relationship then reads ($p = \rho T / \gamma M_j^2$), where M_j refers to the jet exit Mach number. A constant Prandtl number of 0.72 is assumed along with a constant ratio for the specific heats ($\gamma = 1.4$). Sutherland's law is used to model the temperature dependence of viscosity.

To ensure consistency with the literature for circular jets, and an even basis of comparison, the length scale is chosen as the diameter of a circular nozzle with an equal exit area as the rectangular nozzles being studied (equivalent diameter D_e). The nondimensional length of the major axis (l) as a function of the AR is given by

$$\frac{l}{D_e} = \sqrt{\frac{\pi(\text{AR})}{4}}. \quad (6)$$

For reference, Fig. 1(a) shows the nozzle exit cross sections of the jets being studied. In order to preserve a constant exit area, the minor axis length of the nozzle progressively decreases with increasing AR, the impact of which is assessed with the results.

The flow parameters are chosen to be representative of conditions examined in the literature. For concreteness, perfectly expanded conditions are chosen at a design Mach number of 1.3 and a Reynolds number 10^6 , patterned after extensive experimental [42–44] and numerical studies [45–48]. Furthermore, the goal of evaluating distinctions between the different ARs and the circular nozzles necessitates similar nozzle exit conditions for all cases, since these can affect the downstream growth of the plume [49–51]. For this reason, a uniform laminar nozzle exit velocity profile is chosen for all cases, similar to those of Gaitonde [45] and Gaitonde and Samimy [46]. This approach has previously provided good agreement with the circular jet data of Samimy *et al.* [42]. Additionally, lower inflow turbulence levels enhance phenomena specific to rectangular jets, such as axis switching [23,49], and thus provide a suitable testbed for the azimuthal Fourier analysis. The far-field and downstream boundaries are at $10D_e$ and $40D_e$ from the nozzle exit, respectively. The mesh is gradually stretched out starting at $5D_e$ from the individual shear layers in the cross stream direction and $35D_e$ from the nozzle exit in the streamwise direction to provide a suitable sponge condition. A method of characteristics boundary condition is used on the outflow boundary [52].

The Roe scheme is used for the inviscid fluxes [53] with a third-order reconstruction using the monotonic upstream-centered scheme for conservation laws (MUSCL) approach [54]. Viscous terms are discretized with a second-order central differencing technique. Following Pulliam and Chaussee [55], the temporal integration is performed using an approximately factored second-order Beam Warming scheme with two subiterations. The rectangular jets are simulated using stretched Cartesian meshes, shown for example for the $\text{AR} = 4$ jet in Figs. 1(b)–1(d). The circular reference jet mesh has been presented in the literature [46]. The streamwise ($\Delta x = 0.023D_e$ at the nozzle exit) and cross-streamwise mesh spacing ($\Delta y = 0.005D_e$ along the lipline) are chosen based on numerous prior studies using the same solution procedure at similar Mach (1.3) and Reynolds (10^6) numbers. The meshes employed for the rectangular jets comprise 15×10^6 points while that for

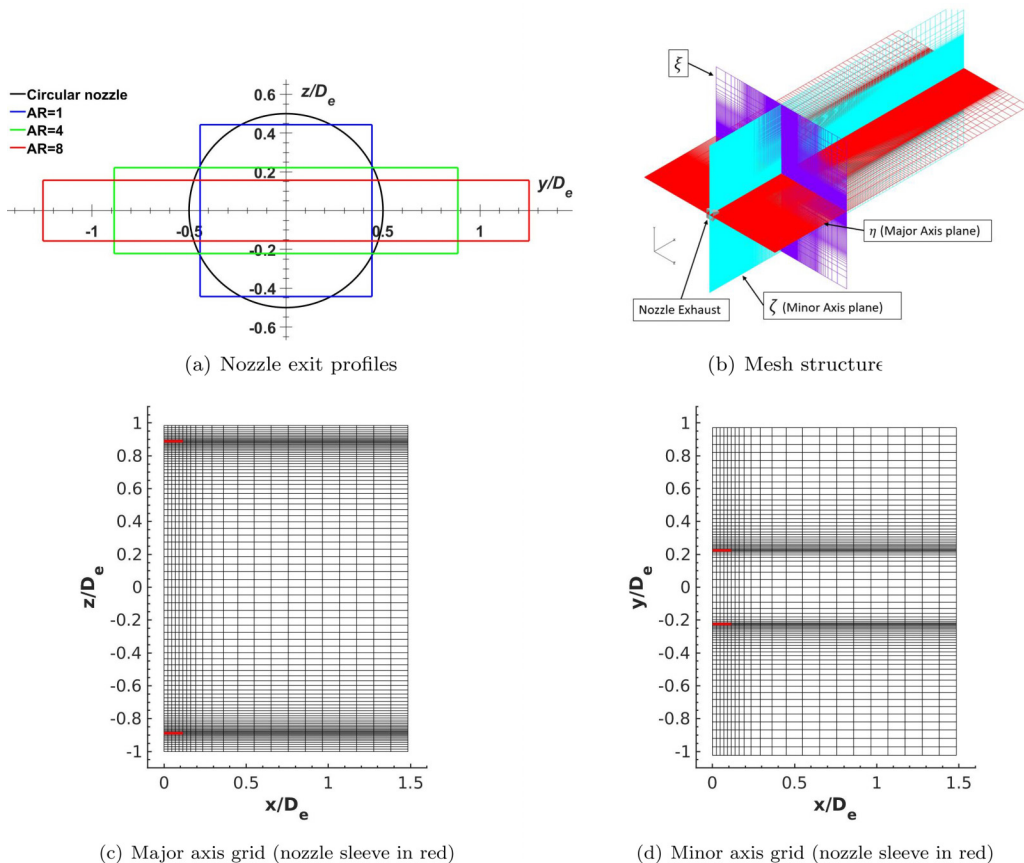


FIG. 1. Characteristics of inflow and spatial discretization.

the circular jet uses 18×10^6 . Note that the internal structure of the nozzle, such as considered in Chakrabarti *et al.* [56], is not simulated since differences between plume development cannot then be ascribed solely to the nozzle exit profiles of interest. Prior mesh-converged results using similar grid sizes have successfully reproduced experimental measurements of mean centerline velocity fields, near-acoustic field, turbulent kinetic energy, and shallow-angle and far-field noise for both circular [45,48] and rectangular jets [56].

III. OVERALL FLOW FIELD FEATURES

Figure 2 shows instantaneous snapshots of each simulated flow field using isosurfaces of the Q criterion ($Q = 1$) colored by the streamwise velocity component. The wide spectrum of scales resolved by the LES can be clearly discerned. The figures also show increased jet spreading rate with AR, similar to the observations of Grinstein [23] and Gutmark and Grinstein [24], who linked this behavior to higher entrainment than in circular jets; this also results in large scale structures spreading further from the axis in higher AR jets.

The mean flow field facilitates a more systematic assessment of the spatial development with implications to the unsteady dynamics. For instance, in circular jets, the potential core length is correlated with the length scale across which coherent structures grow and decay prior to turbulent breakdown and is significant for jet aeroacoustic modeling [11]. Figure 3 shows contours of mean streamwise velocity at several axial stations for each jet. The square jet ($AR = 1$) cross section

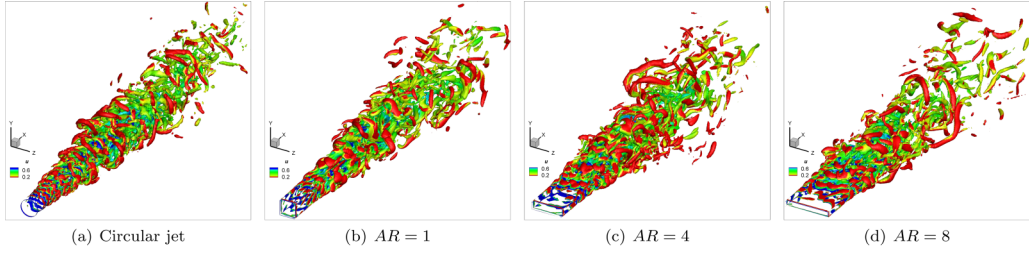


FIG. 2. Isosurfaces of the Q criterion ($Q = 1$) colored by streamwise velocity.

undergoes a 45° rotation downstream of the nozzle exit, so that the diagonals are reoriented along the initially flat sides of the jet; this is most clearly evident at $x = 5D_e$. Similar phenomena observed in triangular jets have been associated with corners in nozzle shapes [24]. For square jets, the effect is more pronounced for lower inflow turbulence levels [57]. By $x = 10D_e$, however, the square jet ($AR = 1$) displays an axisymmetric cross section and is not very different from the circular jet at the same axial station. The jets from the higher AR nozzles, however, develop differently, since they combine the effects of the corners in the nozzle geometry with the influence of unequal characteristic lengths as in elliptic jets [29]. Crucially, these jets do not evolve to an axisymmetric form. Axis switching is evident in the $AR = 4$ and 8 jets, resulting in a predominantly elliptical mean flow cross section that is wider along the minor axis plane by $x/D_e = 10$ and 12, respectively. Axis switching of high AR rectangular and elliptic jets is also more pronounced for low nozzle exit turbulence levels as noted in the previous section.

The collapse of the potential core is examined using the mean streamwise velocity. Values along the jet centerline are shown in Fig. 4(a), along with the experimental data of Samimy *et al.* [42] for the circular jet; the undulations in mean streamwise velocity near the nozzle exit are caused by

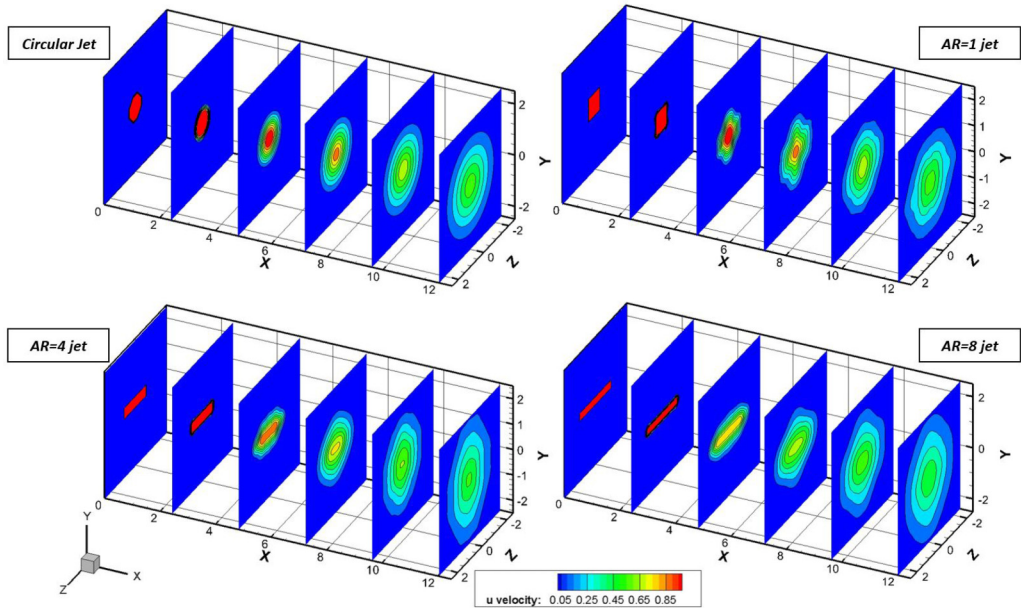


FIG. 3. Contours of mean streamwise velocity at several axial stations showing the spatial development of the jets.

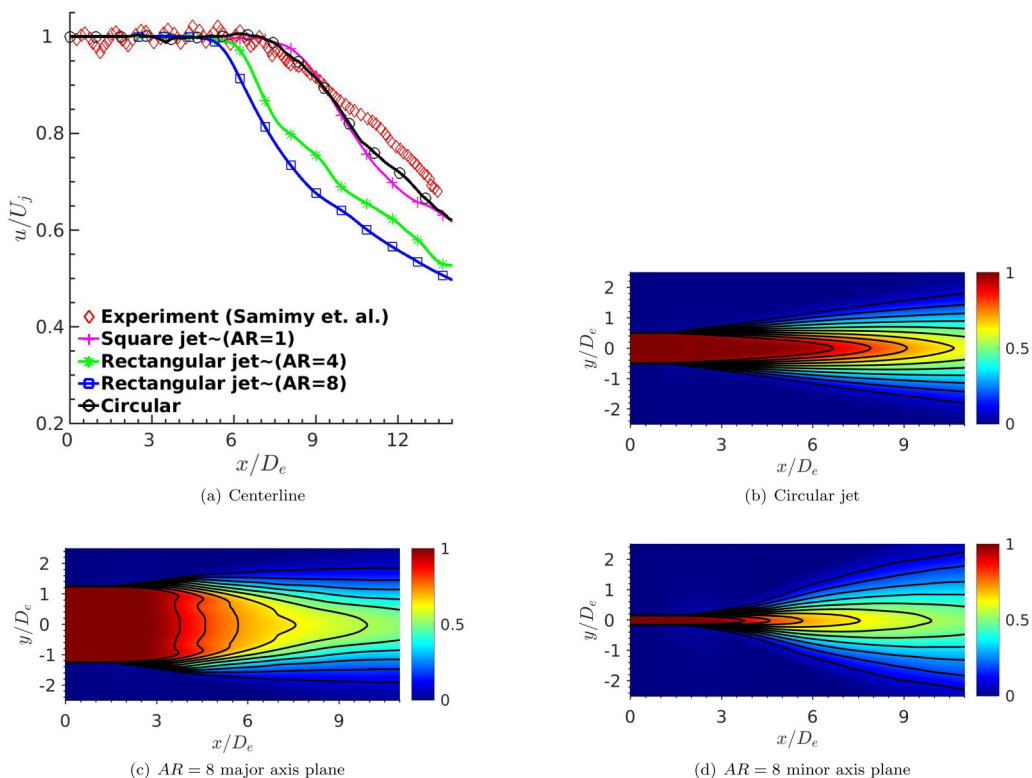


FIG. 4. Effect of the AR on the length of the potential core shown using streamwise mean velocity.

a weak shock train in the core of the jet and result from a slight mismatch between the reservoir and ambient conditions [43]. Similar to previously reported computational studies where the inflow conditions are not known [46,47,58], the experimental core length is employed as a reference to highlight the decay rate downstream of the potential core collapse. The potential core length of the rectangular jets is smaller than that of the circular jet, and decreases with AR. This is most evident for the AR = 8 jet the potential core length of which (based on 90% of the jet exit velocity) is about $3D_e$ shorter than the circular jet. The reason is evident from the cross-sectional plane data of Fig. 4: increasing AR while maintaining the nozzle exit area results in a shortening of the minor axis length [Fig. 1(a)] and thus the distance between the shear layers on the longer nozzle sides. These growing shear layers converge at relatively shorter streamwise distances from the nozzle exit. This effect was investigated by Chakrabarti *et al.* [56] in the context of an AR = 2 rectangular jet and by Chakrabarti *et al.* [37] for an AR = 4 jet. These results are consistent with those of Krothapalli *et al.* [59], who modeled the decay of the mean centerline streamwise velocity of a low speed, high AR jet, with power laws for three distinct regions. For such jets, shear layers on the minor axis plane converge to the centerline within a shorter streamwise distance compared to those on the major axis plane, resulting in a streamwise velocity decay that follows the same power law as a planar jet. The axisymmetric power law for streamwise velocity decay was recovered further downstream. While the present, much higher velocity, rectangular jets do not exhibit these distinct decay regions, the convergence effect of the shear layers on the minor axis plane towards the centerline is similar and manifests as a decrease in the potential core length.

For subsequent reference, the OASPLs are examined to delineate the effect of the AR on acoustic intensity and directivity. The Ffowcs-Williams and Hawkins (FWH) analogy [60] is used in the frequency domain as in Mendez *et al.* [61]. Data over 100 characteristic times (defined based on

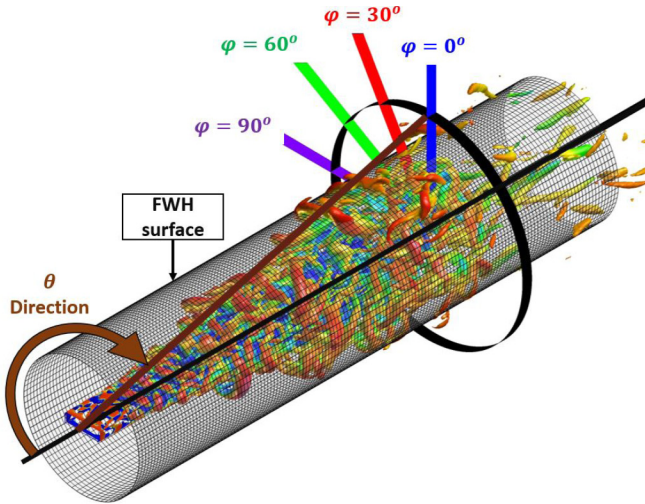
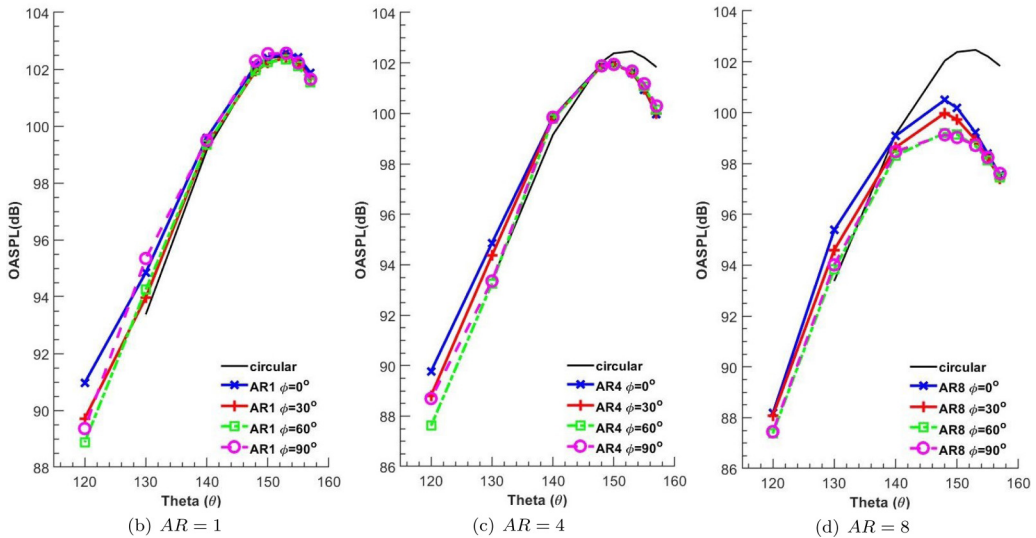

 (a) Schematic of azimuthal (ϕ) and polar (θ) angles along with the FWH surface


FIG. 5. Variation of far-field OASPL with polar angle for jets of various AR.

the jet exit velocity and nozzle equivalent diameter) were interpolated on a cylindrical surface three equivalent diameters away from the axis of each jet. The FWH surface extends to the downstream end of the domain, where eddies are dissipated out using sponge zones, precluding the need for end caps discussed by Mendez *et al.* [61]. The far-field acoustic spectra (sound pressure levels) are obtained at several geometric points around the jet and the OASPL is then obtained by integrating across the frequency band of interest ($St_{D_e} \in [0.1, 1]$). A validation exercise for this procedure may be found in Chakrabarti *et al.* [37].

The azimuthal and polar angles with respect to the nozzle exit are shown schematically in Fig. 5(a). Figures 5(b)–5(d) display the variation of the OASPL with polar angle (θ) at various azimuthal angles (ϕ) for each jet simulated. The square jet [Fig. 5(b)] is acoustically similar to the circular jet, as evidenced by its axisymmetric noise intensity and close correspondence with the peak

radiation intensity and direction of the latter. With increasing AR, however, two key trends become apparent. First, the axisymmetry of the far-field acoustic radiation is distorted; for the AR = 8 jet, a 2-dB variation in noise intensity is observed around the azimuth [Fig. 5(d)]. The minor axis plane of this jet is louder than the major axis plane, matching the observations of several previous studies on rectangular jets [31,62]. Second, with increasing AR, the peak acoustic radiation decreases slightly and its direction moves progressively to lower polar angles (towards the sideline direction), similar to the results of Bridges [31].

IV. FLUCTUATION FIELDS EXAMINED

The most common variable used for azimuthal mode decomposition of circular jets is typically the pressure fluctuation, p' [7,11,14,20,63]. In regions far from the turbulent fluctuations, the relationship of p' to the acoustics is direct. In the near-acoustic field, however, p' is characterized by both acoustic and hydrodynamic content [64,65]. A direct observable consequence is the scale disparity between the pressure fluctuations in the near versus far fields of jets [10]. This property of p' can affect conclusions on the convergence of azimuthal modes, and motivates approaches to isolate an acoustic component for analysis. p' splittings have been derived through several data driven techniques [65–67]. An alternative definition of an acoustic field, that avoids user-defined criteria, has recently been obtained through the physics-based decomposition provided by momentum potential theory [38]. The approach, summarized below, splits the turbulent field into FT components, i.e., hydrodynamic, acoustic, and thermal fluctuations. Application to LES of circular jet flows [48] reveals several attractive properties. The acoustic component displays a jittering, spatiotemporally modulated wave-packet form with radial decay rate, far-field power spectral density, and phase speeds that are consistent with those expected from the literature. More pertinent to this paper, the acoustic component also yields better convergence of parameters for kinematic far-field noise models than p' [39]. In order to extract the FT components, a Helmholtz decomposition of the “momentum-density” field ($\rho\mathbf{u}$) is used to define (vector) acoustic (irrotational and isentropic fluctuations, $-\nabla\psi'_A$), thermal (irrotational and isobaric fluctuations, $-\nabla\psi'_T$), and vortical/hydrodynamic solenoidal fluctuation (\mathbf{B}') components. The vector fields satisfy the following governing equations [38]:

$$\nabla^2\psi'_A = -\frac{1}{c^2}\frac{\partial p'}{\partial t}, \quad (7)$$

$$\nabla^2\psi' = -\frac{\partial \rho'}{\partial t}, \quad (8)$$

$$\mathbf{B}' = \rho\mathbf{u} - \overline{\rho\mathbf{u}} - \nabla\psi', \quad (9)$$

$$\psi'_T = \psi' - \psi'_A. \quad (10)$$

The spatiotemporally resolved fluctuation fields from the LES are used as the source fields of the Poisson equations (7) and (8) to obtain the acoustic potential (ψ'_A) as well as the total irrotational potential (ψ'). The latter, along with the LES data for the momentum-density field ($\rho\mathbf{u}$), is then used to evaluate the hydrodynamic fluctuations per Eq. (9). Finally, the thermal fluctuation potential (ψ'_T) is obtained using Eq. (10). Algorithmic details and additional properties of the decomposition may be found in [48,68].

Figure 6 shows instantaneous isosurfaces of the the acoustic, hydrodynamic, and pressure fluctuations for each case. Here and below, the streamwise components of acoustic and hydrodynamic fluctuations ($\frac{\partial\psi'_A}{\partial x}$ and B'_x) are plotted; these are the largest components, and facilitate better contrast because of their positive and negative values. For simplicity, $\frac{\partial\psi'_A}{\partial x}$ will be referred to as the acoustic fluctuations (or component). The magnitudes of the isolevels are chosen to elicit the main components of the jet; these confirm that even in the rectangular cross-section cases, the hydrodynamic fluctuation magnitude is much larger than the acoustic component. The circular jet results

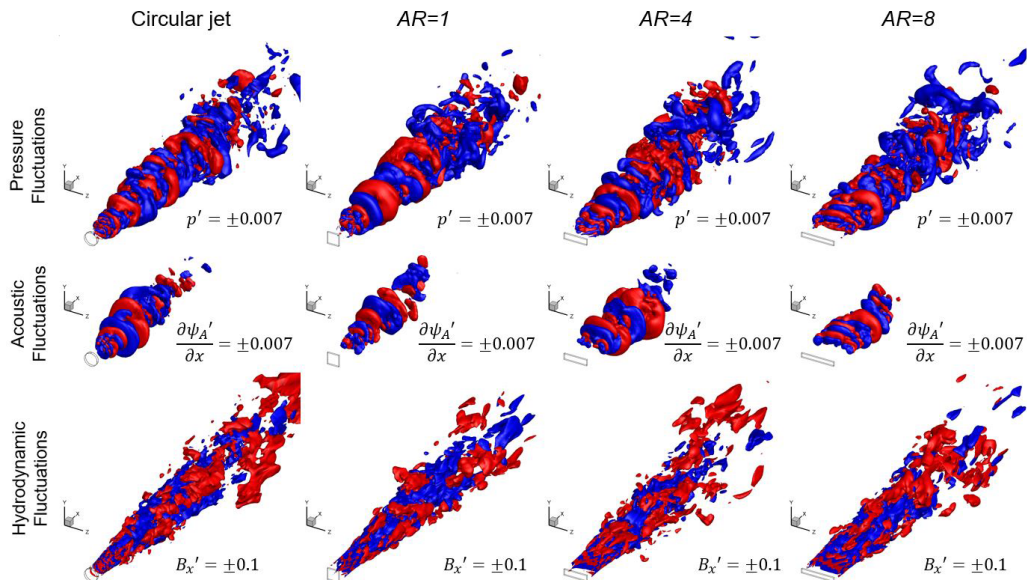


FIG. 6. Isosurfaces of three-dimensional snapshots of the pressure fluctuations (first row) from the LES compared with those of the acoustic (second row) and hydrodynamic fluctuations (third row) extracted from the simulated flow fields.

are similar to those described in the literature [39,48]. The rectangular jets show corresponding features: hydrodynamic fluctuations are dominated by axially extended structures, reflective of a broad spatial wave-number range and chaotic underlying jet turbulence, while acoustic fluctuations exhibit a more organized wave-packet structure with greater coherence and a compact radial support. Indeed, the acoustic fluctuations for all cases resemble a spatially modulated wave packet with an envelope marking the growth, saturation, and decay of the waves downstream of the nozzle exit. The differences in spreading characteristics with change in AR are captured primarily by hydrodynamic fluctuations.

The pressure field shows elements of both hydrodynamic and acoustic components. The initial region has the nature of a spatially growing wave packet, similar to the acoustic fluctuating field. However, further downstream, the disorganized structure more closely resembles the hydrodynamic component. A key distinction between the acoustic fluctuations ($\frac{\partial \psi_A'}{\partial x}$) and p' is thus that the former diminish rapidly beyond the potential core, whereas the latter continue to grow. This has significant implications in statistical convergence of the azimuthal decomposition properties of interest, as discussed further below. In the results below, $\frac{\partial \psi_A'}{\partial x}$ will be used most frequently, though select results with p' will also be put forth for reference purposes.

V. AZIMUTHAL MODES IN RECTANGULAR JETS

The advantage of rapid convergence of fluctuation statistics for circular jets, which allows modeling with only a few modes, motivates an examination of these statistics for near-field rectangular jet behavior.

A. Decomposition ansatz

The Fourier decomposition is introduced in the azimuthal direction (ϕ) in a different manner than for circular jets. Specifically, rather than the more commonly used complex exponential form

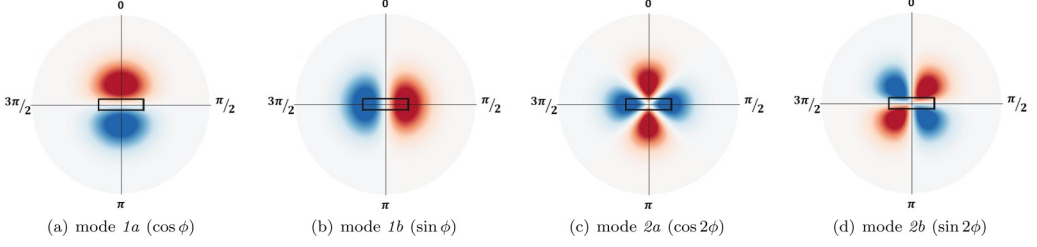


FIG. 7. Polar plots depicting spatial distributions of the two leading azimuthally varying Fourier modes, with nozzle outline for orientation.

[14,18,19],

$$q(x, r, \phi, t) = \sum_{m=-\infty}^{\infty} \hat{q}_m(x, r, t) e^{im\phi}, \quad (11)$$

the following closely related ansatz provides a more convenient representation and interpretation of nonaxisymmetric mechanisms:

$$q(x, r, \phi, t) = q_a^0(x, r, t) + \sum_{m=1}^{\infty} q_a^m(x, r, t) \cos(m\phi) + q_b^m(x, r, t) \sin(m\phi) \quad (12)$$

where $q_a^m(x, r, t)$ and $q_b^m(x, r, t)$ refer to the two-dimensional (2D) spatiotemporal cosine and sine Fourier coefficients, respectively; m refers to the azimuthal mode number; and

$$\begin{aligned} q_a^0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} q(x, r, \phi, t) d\phi, & q_a^m &= \frac{1}{\pi} \int_{-\pi}^{\pi} q(x, r, \phi, t) \cos(m\phi) d\phi, \\ q_b^m &= \frac{1}{\pi} \int_{-\pi}^{\pi} q(x, r, \phi, t) \sin(m\phi) d\phi. \end{aligned} \quad (13)$$

The relationship between the coefficients of Eqs. (11) and (12) is

$$\hat{q}_m = \frac{q_a^m + iq_b^m}{2}, \quad \hat{q}_m = \hat{q}_{-m}^*. \quad (14)$$

Thus, while the complex exponential ansatz for the Fourier series uses a single complex amplitude to retain both the magnitude and phase information of the m th azimuthal mode, the sine-cosine ansatz of Eq. (12) introduces two real coefficients to retain the same information. For instance, the first azimuthal mode ($m = 1$) is given by the superposition of two waves, $q_a^1(x, r, t) \cos(\phi) + q_b^1(x, r, t) \sin(\phi)$. These will be individually referred to as $m = 1a$ and $1b$, respectively, in the subsequent text. Similarly, the two waves associated with $m = 2$ will be referred to as $m = 2a$ and $2b$.

The mechanisms associated with each of the above modes can be deduced from their spatial distributions, similar to the use of Mathieu functions for elliptical jets by Kinzie and McLaughlin [35]. For illustration, Fig. 7 shows the components of the first and second azimuthal modes ($m = 1a, 1b$ and $2a, 2b$, respectively), together with a high AR nozzle exit for orientation. The mode shapes and, by extension, their individual physical interpretations necessarily depend on the choice of the azimuthal origin, $\phi = 0^\circ$, with respect to the major or minor axes. Choosing one of these for $\phi = 0$ greatly simplifies the analysis because of symmetry considerations, as shown below. Following Bridges [31], $\phi = 0^\circ$ is assigned to the minor axis plane of the nozzle exit (positive y direction). Thus, mode $1a$ represents fluctuations that are out of phase on either side of the minor axis [Fig. 7(a)], and are associated with flapping motion in the minor axis direction. Correspondingly, mode $1b$ [Fig. 7(b)] captures flapping motion along the major axis direction.

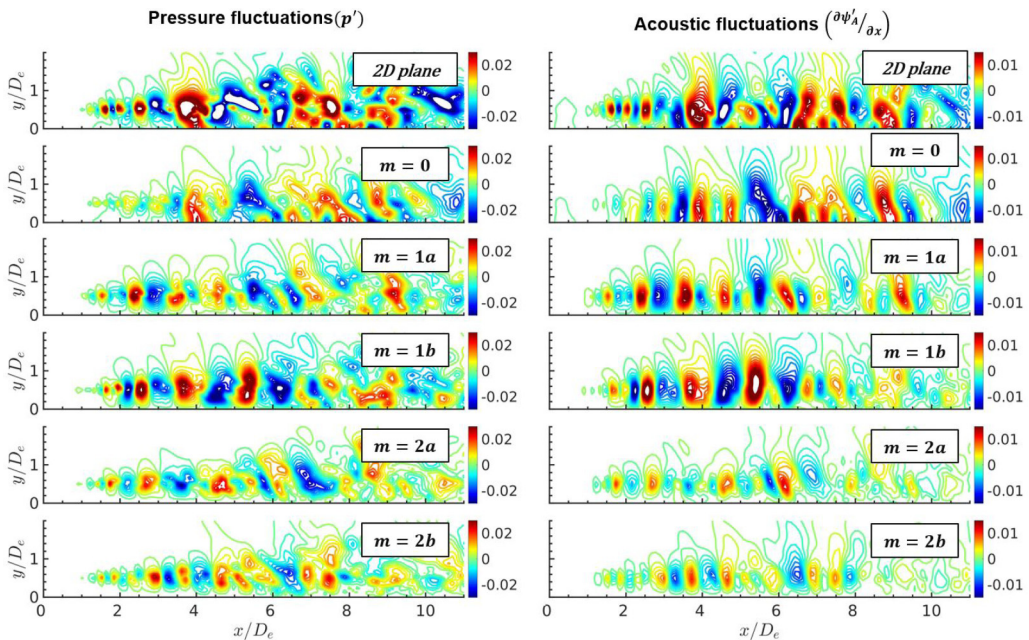


FIG. 8. Contours of pressure perturbations (p' , left column) and acoustic fluctuations ($\frac{\partial \psi_A}{\partial x}$, right column) of the circular jet on a 2D cross section along with their leading azimuthal Fourier modes.

Modes $2a$ and $2b$ are associated with more complex mechanisms. For elliptical jets, Kinzie and McLaughlin [35] associate mode $2a$ [Fig. 7(c)] to “pinched” elliptical fluctuations and mode $2b$ [Fig. 7(d)] to off-axis fluctuations.

B. Rectangular modal dynamics

The well-known azimuthal modes of the circular jet [6,14] are summarized for subsequent reference. The left column of Fig. 8 shows, starting from the top, instantaneous snapshots of p' on a 2D slice of the circular jet and the leading azimuthal Fourier modes, axisymmetric ($m = 0$), $1a$, $1b$, $2a$, and $2b$, respectively. A coherent wavelike structure related to the Kelvin-Helmholtz shear layer instability is apparent near the nozzle exit. Further downstream, however, the jet breaks down into finer scale turbulence. Mode $m = 0$ exhibits more coherence and filters out the finer scale turbulence downstream. The increased coherence illustrates the advantages afforded by the azimuthal mode decomposition in general, and the $m = 0$ mode in particular for round jets; its use in the construction of two-point wave-packet model parameters is discussed in Jaunet *et al.* [22]. In contrast, the finer scales in the raw pressure field, which are not as acoustically efficient, are retained in the higher azimuthal modes of the jets. No clear qualitative differences are evident between the higher azimuthal wave-number pairs, $m = 1a$, $1b$, or $2a$, $2b$. The similar spatial distributions and fluctuation amplitudes of each pair are consistent with the discussion of Fig. 7 and are the consequence of polar symmetry of circular jets, which is lacking in the rectangular jets examined below.

The right column of Fig. 8 displays the corresponding results with the streamwise acoustic fluctuation variable ($\frac{\partial \psi_A}{\partial x}$). Even without azimuthal decomposition, $\frac{\partial \psi_A}{\partial x}$ displays a more organized spatial distribution and greater coherence compared to the raw pressure perturbation field; this readily highlights the advantage of this FT component over p' , particularly in the region downstream of the potential core collapse. The $m = 0$ mode has the form of a spatially localized wave packet with a growth and decay envelope representing, in the aggregate, acoustic sources for downstream shallow

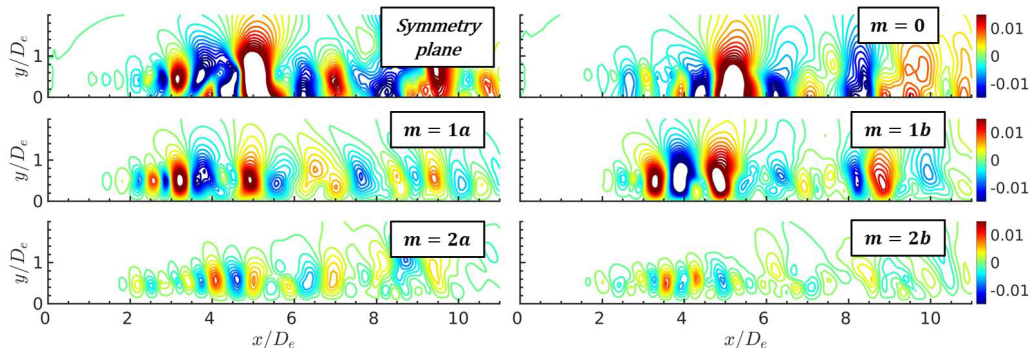


FIG. 9. The streamwise acoustic fluctuations ($\frac{\partial \psi'_A}{\partial x}$) of the AR = 1 jet on the symmetry plane and its leading azimuthal modes.

angle noise radiation. The dominant axial length scale of this wave packet is a few jet diameters larger than the potential core length, beyond which the fluctuations diminish. Higher modes of the acoustic fluctuations, $1a$, $1b$, $2a$, and $2b$, also display decay in the fluctuation magnitudes, evident from the decreasing saturation of the contours in the figure. Furthermore, these higher modes also show successively sharper cutoff with streamwise distance from the nozzle exit. Both the decreasing fluctuation magnitude and shorter streamwise lengths of the higher modes are crucial in the acoustic source modeling of circular jets [19].

Figure 9 shows $\frac{\partial \psi'_A}{\partial x}$ contours on the symmetry plane of the AR = 1 jet together with the three leading azimuthal modes (since p' fluctuations are less illustrative, they are not shown). As in the case of the mean flow, the AR = 1 fluctuation data also show similarities with round jets, including the initial wavelike structure near the nozzle exit followed by a breakdown into more chaotic finer scale structures further downstream. Likewise, the $m = 0$ shows more coherence than the raw data, and a rapid fluctuation decay downstream of the potential core. The higher azimuthal modes exhibit progressively more rapid streamwise decay. These similarities with the circular jet are not unexpected given the rapid transformation of the square jet to an axisymmetric mean cross section [Fig. 3(b)].

The higher AR nozzles (AR = 4 or 8), however, do not transition to axisymmetric cross sections in the domain of interest as discussed in the context of Fig. 3. Figure 10 shows the corresponding $\frac{\partial \psi'_A}{\partial x}$ results on the major and minor axis planes of AR = 4 and 8 jet in the left and right columns, respectively. The major (black) and minor (magenta) axis liplines are also marked with dashed lines to provide a scale for reference. On the minor axis planes of both jets (Fig. 10 first row), the structures associated with the initial breakdown along the lipline are similar to the circular and AR = 1 jets, but significantly closer to the centerline of the jet. However, on the major axis plane (second row), the fluctuation pattern is different. Two sets of structures are evident, marked A and B, respectively. The former sequence is closer to the centerline, with phase similar to the structures on the minor axis planes of the respective jets; these are the signatures of the lipline structures of the minor axis shear layer that essentially protrude on the symmetry plane. The outer structures, marked B, are those that evolve due to the major axis shear layers. This higher complexity is a consequence of the disparity between the distances of the major and minor axis liplines from the centerline of the jet, as the AR is increased. A similar effect in the context of an AR = 2 jet was discussed by Chakrabarti *et al.* [56]. By an axial distance of $x = 4D_e$, however, these separate wave trains cannot be clearly distinguished from each other.

The spatially localized wave-packet structure remains evident in the $m = 0$ modes. The shear layer structures growing along the major axis lipline (designated B earlier) do not display a discernible signature on the axisymmetric mode. Being further away from the axis, they subtend a

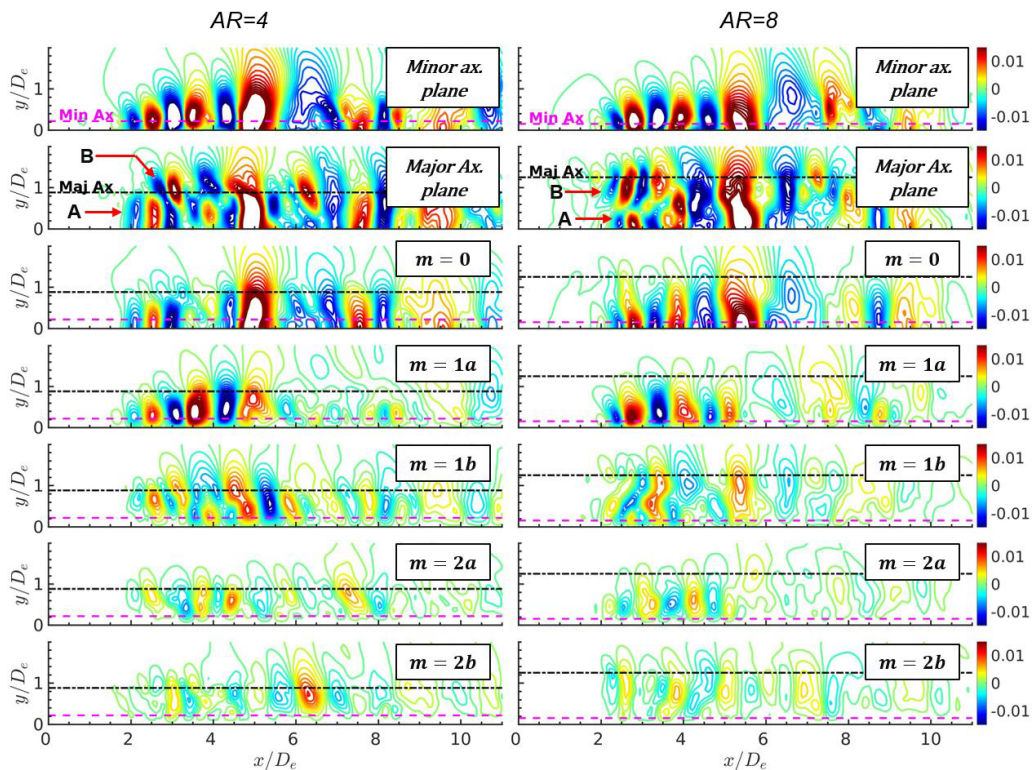


FIG. 10. Contours of streamwise acoustic fluctuations ($\frac{\partial \psi_A}{\partial x}$) of the AR = 4 (left column) and AR = 8 (right column) jets shown on the major and minor axis planes along with the leading azimuthal Fourier modes.

relatively small azimuthal angle on the centerline and thus have a small contribution to the azimuthal averaging operation used to obtain the $m = 0$ mode [Eq. (13)]. Although qualitatively similar to the $m = 0$ mode in the circular and square jets, the decay with streamwise distance is more rapid, and the lobes become more radially extended. The former effect is related to the decrease in the potential core length with increasing AR, as discussed previously, while the latter effect is due to the effective averaging of the shear layer around the azimuth and is evaluated quantitatively in a subsequent section.

Higher azimuthal modes exhibit significant differences from their circular and square counterparts. The two components of $m = 1$ and $m = 1a$ and $1b$, respectively, in Fig. 10, are distinct from each other, and emphasize fluctuation distributions from shear layers on the minor and major axis planes, respectively. This is consistent with the directivity inherent in the two coefficients as discussed in the context of Fig. 7. The chosen ansatz [Eq. (12)] thus retains the dynamics of the two symmetry planes of the rectangular jet for $m = 1$. As a consequence, $m = 1b$ (the sine coefficient) is chaotic and dominated by finer spatial scales. In contrast, $m = 1a$ (the cosine coefficient) has a more organized structure, with greater fluctuation magnitudes, which reflect the dominance of flapping motions along the minor axis plane. This is the primary instability mechanism of rectangular and elliptical jets as shown by theoretical [69] and experimental [27,28] studies. The manifestations of these differences between the two coefficients of the first mode on the near-acoustic field asymmetry are discussed in a subsequent section. The coefficients of the second mode $m = 2a$ and $2b$, however, exhibit more chaotic structures that do not resemble the fluctuation pattern on either of the symmetry planes of the jets. This is consistent with the higher complexity of the $m = 2$ mode discussed in the context of Fig. 7.

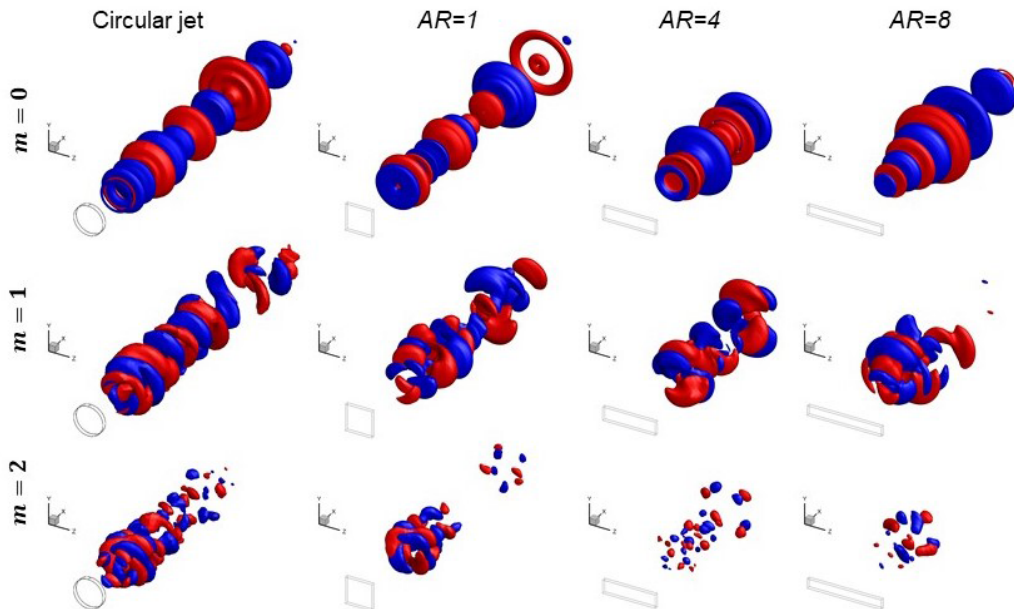


FIG. 11. Isosurfaces ($\frac{\partial \psi'_A}{\partial x} = \pm 0.007$) showing the reconstructed 3D azimuthal mode shapes of an instantaneous snapshot of the acoustic fluctuations in the near field of the various AR jets.

C. Three-dimensional reconstruction of modes

Overall, both circular and rectangular jets display similar decreases in the fluctuation magnitudes of the higher azimuthal modes. The greater influence of lower modes suggests that procedures developed for circular jets, that consider only a few dominant Fourier modes, may also apply to high AR rectangular jets. This convergence of instantaneous statistics in the azimuthal Fourier space, as well as the dynamics of the individual modes, is now clarified by considering three-dimensional (3D) reconstructions of the leading azimuthal modes. Qualitative considerations are presented first, followed by more quantitative evaluations.

Three-dimensional mode shapes are reconstructed using the 2D Fourier coefficients [$p_a^m(r, x, t)$ and $p_b^m(r, x, t)$] and their corresponding azimuthal variations ($\cos m\phi$ and $\sin m\phi$) according to Eq. (12). Figure 11 shows the instantaneous 3D snapshots of the three leading rectangular jet azimuthal modes using isosurfaces of the same value ($\frac{\partial \psi'_A}{\partial x} = \pm 0.007$) and compares them to those of the circular jet. For the $AR = 1$ jet, a steep drop in higher azimuthal mode content is evidenced by the reduced spatial support of the isosurfaces and the higher azimuthal modes become further restricted to the near nozzle regions. These observations are quite similar to those by Cavalieri *et al.* [70] in the context of the higher azimuthal modes of circular jets. This reiterates a crucial observation, since the streamwise noncompactness of the acoustic source field is a necessary criterion for superdirective acoustic radiation from a jet [19,21].

More interestingly, despite the persistent mean asymmetry of the higher AR jets, their higher azimuthal modes (Fig. 11) also show similar characteristics and, indeed, an even steeper drop off in amplitudes. This observation is associated with the relative simplicity of the acoustic fluctuations in the high AR rectangular jets (Fig. 6 second row), compared to the hydrodynamic content of the turbulent eddies (Fig. 6 third row). While the former is spatially localized with distributions that are relatively independent of the AR, the latter clearly highlights the greater spreading rate and the more chaotic nature of the higher AR jets. The 3D mode shapes at higher AR also further highlight the progressively diminishing streamwise extent of the azimuthal modes of the rectangular jets compared to the circular jet as discussed previously in the context of the 2D Fourier coefficients.

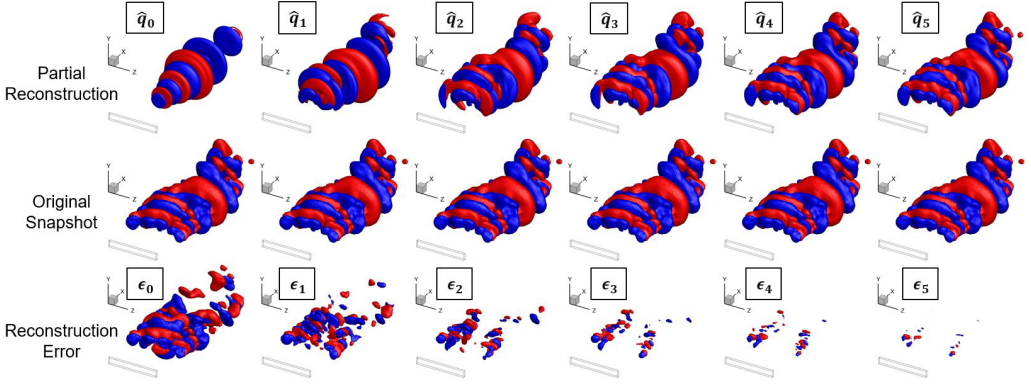


FIG. 12. A comparison of partial reconstructions using a few leading azimuthal modes to the original instantaneous snapshot of the acoustic fluctuations in the $AR = 8$ rectangular jet. The reconstruction error for each of the partial reconstructions is also shown.

These results show promise regarding the potential suitability of the cylindrical coordinate system, and by extension the Fourier basis in the azimuthal direction, in describing the instantaneous dynamics of rectangular jets.

As an additional qualitative assessment, the compact support in the chosen Fourier basis is evaluated by reconstructing the flow using only a few leading modes. A partial reconstruction considering only the leading M azimuthal modes is obtained from

$$\hat{q}_M(x, r, \phi, t) = \sum_{m=0}^M q_a^m(x, r, t) \cos(m\phi) + q_b^m(x, r, t) \sin(m\phi). \quad (15)$$

These partial reconstructions are compared with the instantaneous fluctuation snapshot of the $AR = 8$ rectangular jet in Fig. 12. The first row of Fig. 12 shows the partial reconstructions of $\frac{\partial \psi_A}{\partial x}$ using progressively larger numbers of azimuthal modes up to $M = 5$. The original 3D snapshot of acoustic fluctuations is repeated below each partial reconstruction for comparison purposes in the second row. The reconstruction error is obtained as the difference between the partial reconstruction and the LES snapshot:

$$\epsilon_M(x, r, \phi, t) = q(x, r, \phi, t) - \hat{q}_M(x, r, \phi, t) \quad (16)$$

is plotted in the third row of Fig. 12. The reconstruction error decreases rapidly with M , evident from the close correspondence between the five mode reconstruction (\hat{q}_5) and the original snapshot. Similar results for the $AR = 4$ jet are shown in Chakrabarti *et al.* [37]. Interestingly, with the consideration of additional modes, the streamwise extent of the reconstruction error also reduces rapidly and the error is restricted to only the near nozzle regions for reconstructions with four or five azimuthal modes.

D. RMS reconstruction accuracy

A quantitative assessment of the reconstruction is performed to compare the convergence rates of the azimuthal Fourier series for different AR , as well as to differentiate between the rates of convergence of p' and $\frac{\partial \psi_A}{\partial x}$ for a given jet. A meaningful comparison is predicated on an appropriate error norm; here we choose the volume integrated mean square error given as

$$\|\epsilon_M\| = \frac{\int_V \frac{1}{T} \int_0^T \epsilon_M(x, r, \phi, t)^2 dt dV}{\int_V \frac{1}{T} \int_0^T q(x, r, \phi, t)^2 dt dV}. \quad (17)$$

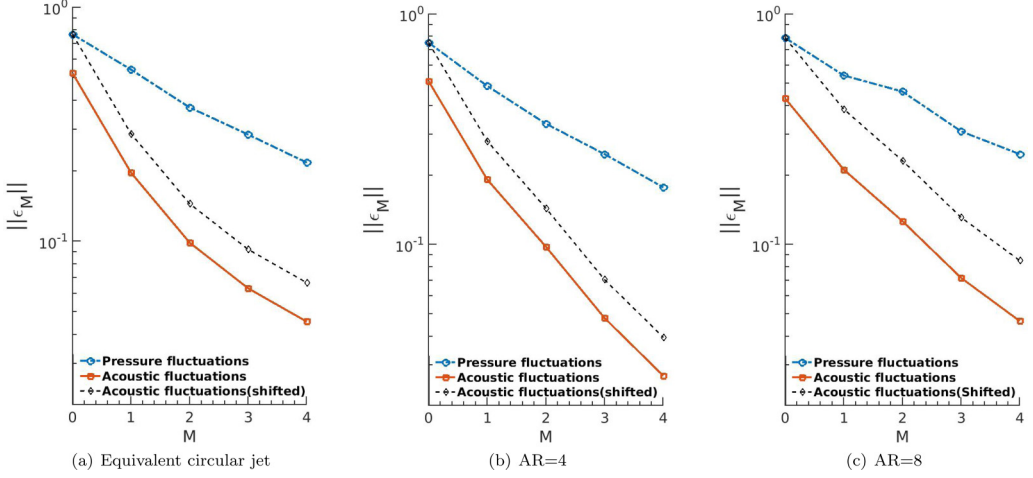


FIG. 13. Comparison of the rates of convergence for the circular jet (a) and the AR = 4 (b) and AR = 8 (c) rectangular jets.

The denominator is the RMS of the fluctuation field being analyzed; it normalizes the error to yield the fraction of the total fluctuation energy of the field. The volume of integration is a cylinder containing all acoustically significant regions within the jet. Thus, the streamwise and radial extents are $x/D_e = 25$ and $3D_e$, respectively, which coincides with the radial location of the FWH surface.

Figure 13 compares the reconstruction error norm as a function of number of modes used for reconstruction for the AR = 4 and 8 rectangular jets with the reference results for the circular jet. The error norm decreases with number of modes as anticipated. In each case, the axisymmetric mode of $\frac{\partial \psi'_A}{\partial x}$ exhibits a lower reconstruction error than p' and the error drops at a sharper rate, indicating a more rapid convergence in the azimuthal Fourier space. To highlight this, the figures also include a plot of the $\frac{\partial \psi'_A}{\partial x}$ error, shifted to match the pressure fluctuation reconstruction errors for $m = 0$ alone. Thus, the acoustic fluctuations exhibit lower-rank behavior in azimuthal Fourier space with lower reconstruction errors compared to the pressure fluctuations. This confirms that the advantage of employing the acoustic fluctuations instead of the raw pressure fluctuation field in circular jets carries over to rectangular jets.

Second-order statistics provide further insights into the behavior of the individual Fourier modes in establishing the acoustic field. The modal energies (mean squared magnitudes) of the leading azimuthal Fourier modes are defined as

$$q_m^{MS}(x, r) = \begin{cases} \overline{q_{a0}^2} & \text{if } m = 0 \\ \frac{\overline{q_{am}^2 + q_{bm}^2}}{2} & \text{if } m > 0 \end{cases} \quad (18)$$

For completeness, a derivation of the above expression is given in the Appendix [Eq. (A6)]. A similar definition for the modal energies has been adopted by Faranosov *et al.* [71].

Figure 14 compares the spatial energy distribution of the leading azimuthal modes of $\frac{\partial \psi'_A}{\partial x}$ in the different jets. The circular jet (Fig. 14 first row) clearly exhibits the two major trends of decreasing modal energy and diminishing spatial support with increasing mode numbers, which have facilitated reduced order modeling. Results for rectangular jets (third and fourth rows) also confirm the progressively smaller energies of the higher azimuthal modes, at rates that are generally commensurate with those for the circular jet. Thus, the property of rapid convergence of statistics with number of modes holds even for high AR rectangular jets which display features such as axis switching. Two trends of interest may be identified. First, modal energies exhibit a greater radial spread with increasing AR; this is most evident from a comparison of the $m = 0$ energy distribution

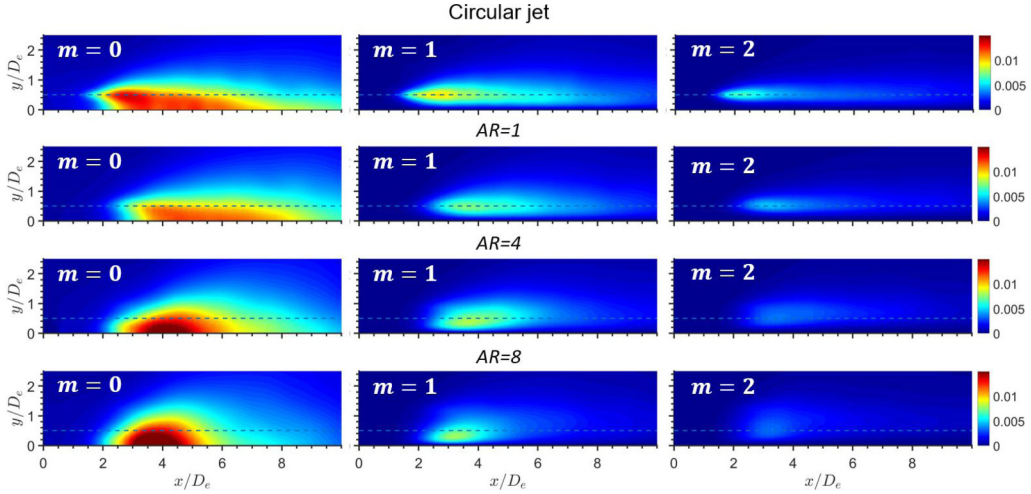


FIG. 14. Modal energies of the three leading azimuthal modes of the acoustic fluctuations in the nonaxisymmetric jets. The dashed line in the figure represents the lipline of the circular jet ($y/D_e = 0.5$) for scale.

at the lipline (marked with a dashed line) of the circular jet ($y/D_e = 0.5$). This is consistent with the azimuthal averaging effect; the major axis shear layers are geometrically further away from the axis, similar to the effect in Fig. 10. The second trend concerns the decrease in the streamwise extent of all azimuthal modes with increase in the AR, again most clearly evidenced by the marked decrease in the streamwise extension of the $m = 0$ mode of the $AR = 8$ jet (see the first column of Fig. 14). It has been previously shown in the context of circular jets [11] that the streamwise extent of the acoustic sources is correlated with the length of the potential core. This result holds for rectangular jets as well, since the decrease in the streamwise length scale of the acoustic sources [39] of the high AR jets is correlated with a progressive reduction in their potential core lengths of the higher AR rectangular jets (Fig. 4).

A suitable measure for a direct quantitative comparison of modal energies may be obtained as follows. Since the radial distributions of modal energies vary with AR, comparing the energy magnitudes along a specific streamwise line (as is common in circular jets) is not illuminating. This is evident from Fig. 14, where the dashed lines in the figures mark the radial distance corresponding to the lipline of the circular jet ($r = 0.5D_e$). In contrast to the circular jet, where most of the energy for all azimuthal modes is concentrated around the $r = 0.5D_e$ line, higher AR jets exhibit a significantly greater spatial spread with only a fraction of the energy being contained along the $r = 0.5D_e$ line. An integrated measure of energy magnitude, q_m^{iMS} , with radial distance at each axial station is thus employed:

$$q_m^{iMS}(x) = \int_0^3 q_m^{MS}(x, r) r dr \quad (19)$$

where q is the flow variable of interest ($\frac{\partial \psi'_A}{\partial x}$ or p'). The radial domain of integration is again chosen from the axis to $r = 3D_e$, which is the location of the FWH surface and is sufficiently distant from the turbulent core for all modal energies to have decreased by two orders of magnitude.

Figure 15 shows the integrated energies of the leading azimuthal modes of $\frac{\partial \psi'_A}{\partial x}$ (a) together with corresponding results for p' (b) of the circular jet for reference. Near the nozzle exit, the separation of the energies between the azimuthal modes is quite small for pressure, but not so for the acoustic component, the $m = 0$ content of which is already much larger than the others. Further downstream, the content of all modes increases; however, the increase is largest for the lower modes. This reiterates the low-rank nature of fluctuation statistics in the azimuthal Fourier space and, therefore, a

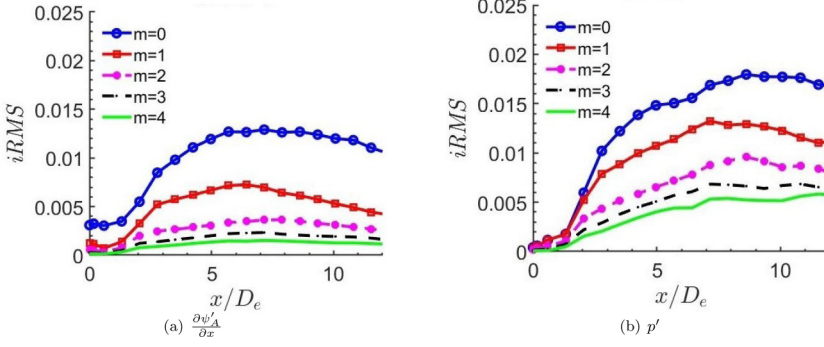


FIG. 15. Streamwise variation of the radially integrated modal energies of five leading azimuthal modes for the circular jet.

rapid convergence in the three leading azimuthal modes. Figure 15 quantifies some of the previously noted features and distinctions between pressure and acoustic fluctuations. While all azimuthal modes of $\frac{\partial \psi'_A}{\partial x}$ exhibit clear growth and subsequent decay with streamwise distance, the integrated modal energies of p' continue to increase for much longer streamwise distances. This growth and decay envelope of acoustic fluctuations reveals a characteristic streamwise acoustic source length scale, which is a critical component of jet acoustic emissions, as discussed in Unnikrishnan *et al.* [39]. Further, acoustic fluctuations exhibit a more rapid convergence in the azimuthal Fourier space as manifested by the larger differences in modal energies with increasing mode numbers, most evident around $x/D_e = 5$. Azimuthal modes of p' do not show as pronounced a drop off in energy.

The convergence properties of $\frac{\partial \psi'_A}{\partial x}$ are inherent in the defining Poisson equation (7), the solution of which may be distinguished into homogeneous and particular components. The homogeneous solution component is relatively small, since it depends on relatively distant far-field boundary conditions, where $\psi_A = 0$. The particular solution component is dependent on the nature of the source term, which connects pressure fluctuations, $\partial p'/\partial t$, to the acoustic potential, ψ_A , and may be analyzed in Fourier space. To illustrate in two dimensions ($x - \theta$) for simplicity, Eq. (7) may be written in cylindrical coordinates as

$$\frac{1}{r^2} \frac{\partial^2 \psi_A}{\partial \theta^2} + \frac{\partial^2 \psi_A}{\partial x^2} = -\frac{1}{c^2} \frac{\partial p'}{\partial t} \quad (20)$$

or, in wave-number-frequency space,

$$\left(\frac{m^2}{r^2} + k_x^2 \right) \widehat{\psi}_A = -\frac{\omega}{c^2} \widehat{p}' \quad (21)$$

where $\widehat{\psi}_A(k_r, k_x, m, \omega)$ is the Fourier transform of ψ_A . $\widehat{\psi}_A$ is thus directly dependent on \widehat{p}' by a wave-number-dependent scaling factor:

$$\widehat{\psi}_A = -\frac{\omega}{c^2} \frac{\widehat{p}'}{\left(\frac{m^2}{r^2} + k_x^2 \right)}. \quad (22)$$

In the quiescent far field, where the speed of sound is constant, this relationship is greatly simplified [72]. While the connection is not as straightforward in the near field, a key observation is that as the radial distance decreases, the sensitivity of this relationship to azimuthal wave numbers increases. Near the centerline of the jet, the higher azimuthal modes have very low weights as the quantity m^2/r^2 in the denominator becomes large. In this region, therefore, the acoustic fluctuations filter out the higher azimuthal modes that may dominate the fully turbulent dynamics, i.e., including

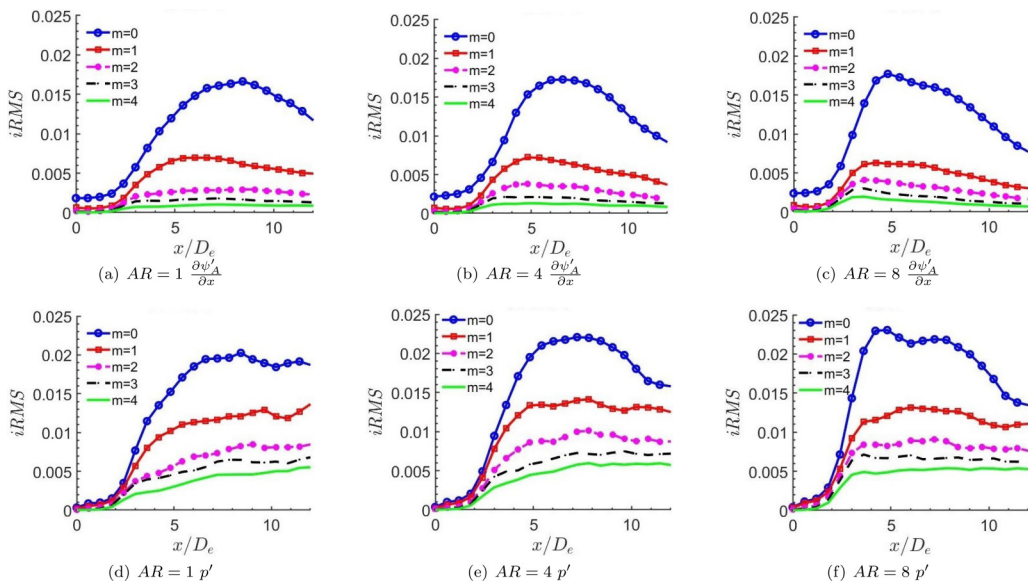


FIG. 16. Streamwise variation of the radially integrated RMS (iRMS) of the five leading azimuthal modes of the acoustic (row 1) and pressure fluctuations (row 2) for the $AR = 1$ (a, d), $AR = 4$ (b, e), and $AR = 8$ (c, f) jets.

nonacoustic components. Moving radially outward from the jet, the higher azimuthal modes are dissipated faster owing to the physics of acoustic propagation. The combined result of these two effects manifests in the superior convergence of the acoustic fluctuations in the azimuthal Fourier spectrum when compared to the pressure fluctuations.

The integrated modal energies of $\frac{\partial \psi'_A}{\partial x}$ and p' for the rectangular jets are compared in Fig. 16. The same trends as observed for the circular jets persist; in particular, the energy content decreases for higher mode numbers. As such, acoustic fluctuations of even high AR rectangular jets have appreciably low-rank representation in the azimuthal Fourier basis. This result further verifies the observation that despite the nonaxisymmetric nature of the rectangular nozzle plume significant simplifications of the rectangular jet may be achieved by considering only the leading azimuthal Fourier modes, especially if $\frac{\partial \psi'_A}{\partial x}$ is employed instead of p' as discussed previously in the context of Fig. 13. Additionally, with increasing AR, (i) the $m = 0$ mode becomes relatively more energetic compared to $m = 1$ and (ii) for all azimuthal mode numbers the streamwise location corresponding to the peak value of the RMS shifts progressively upstream and the postpeak reduction becomes steeper, especially when compared to the circular jet, consistent with the decrease in potential core length with AR.

VI. AZIMUTHAL HOMOGENEITY CONSIDERATIONS

The fluctuations in an azimuthal Fourier basis retain the rapid convergence property even for rectangular jets, implying that despite their higher complexity, the cylindrical coordinates are well suited to represent their acoustic dynamics. However, as noted earlier in the context of the mean flow description, rectangular jets exhibit several differences from circular jets, including axis switching and nonaxisymmetric far-field radiation. Given the relatively negligible contribution of the higher azimuthal Fourier modes in all of the jets, these differences between the dynamics of high AR rectangular and circular jets can only be associated with key differences in the dynamics of the leading azimuthal modes. Indeed, a major point of distinction between circular and rectangular jets

is that the former exhibit azimuthal statistical homogeneity that greatly simplifies their modeling [14,15,19]. This section examines the advantages afforded by the azimuthal homogeneity property in the context of the chosen Fourier decomposition ansatz, and evaluates its applicability to rectangular jets.

The implication of azimuthal homogeneity in the context of Eq. (12) may be examined in the simpler notation that suppresses explicit x and r dependence:

$$q(\phi, t) = a_0(t) + \sum_{m=1}^{\infty} a_m(t) \cos m\phi + b_m(t) \sin m\phi. \quad (23)$$

Homogeneity in the ϕ direction requires that the two point cross-correlation function depend only on the angular separation, $\Delta\phi = \phi_1 - \phi_2$, and not on ϕ itself [14]:

$$\overline{q(\phi_1, t)q(\phi_2, t)} = g(\phi_1 - \phi_2). \quad (24)$$

Here, $\overline{(\cdot)}$ refers to an averaging operation in time. A direct consequence of this, as shown in the Appendix, is that the individual Fourier modes obey the following constraints:

$$\overline{a_m^2} = \overline{b_m^2}, \quad (25)$$

$$\overline{a_m a_n} = \overline{b_m b_n} = \overline{a_m b_n} = 0. \quad (26)$$

These may be interpreted by considering the sense of direction associated with the azimuthal modes, based on the locations of the extrema of the sine and cosine functions, as discussed in the context of Fig. 7. The first constraint [Eq. (25)], therefore, reflects the polar symmetry property of circular jets; specifically, both cosine and sine coefficients (a_m and b_m , respectively) have identical statistics due to azimuthal statistical invariance.

The second constraint [Eq. (26)] reflects the property that in the homogeneous and periodic direction, the Fourier modes diagonalize the cross correlation matrix and are the spatial POD modes [15]. As a result, the decomposition Eq. (23) has correspondence to the space-time variable separation achieved by POD, where the sine and cosine functions represent the spatial POD modes and are mutually orthogonal in space by construct. By extension, the Fourier coefficients (a_m and b_m) correspond to the temporal POD modes and must be mutually orthogonal in time, as indicated by the above noted constraint.

These properties have simplified circular jet noise modeling efforts. For instance, POD analyses of the individual azimuthal modes (instead of the full 3D dataset) are routinely used to develop reduced-order models [6,20]. The calculation of OASPL levels using azimuthal modes is also simplified because the mutual incoherence of the individual azimuthal modes dictated by the statistical homogeneity [Eq. (26)] permits fluctuation RMS value to be represented as the sum of squares of the RMS values of the individual azimuthal Fourier modes [15,19]. Thus, by considering only the three leading modes for a circular jet, the energy of the overall fluctuation field is given as (see the Appendix)

$$\overline{q^2(\phi, t)} = \overline{a_0^2} + \frac{\overline{a_1^2} + \overline{b_1^2}}{2} + \frac{\overline{a_2^2} + \overline{b_2^2}}{2}. \quad (27)$$

The azimuthal invariance of the fluctuation RMS value, as evidenced by the absence of the azimuthal angle ϕ on the right-hand side of the above expression, implies that even if infinitely many azimuthal modes are considered, the overall RMSs of the fluctuations retain their axisymmetric form for a circular jet [15].

These considerations do not hold when two point correlations depend on the azimuthal angle; for example, in the case of a rectangular jet, points on the y ($\phi = 0^\circ$) and z ($\phi = 90^\circ$) axes [Fig. 5(a)], respectively, display different correlations with each other than those along $\phi = 45^\circ$ and 135° . This aspect manifests in the near- and far-field asymmetry of the rectangular jet dynamics, regardless of the rapid convergence in the azimuthal Fourier basis shown previously. A generalized expression for

the fluctuation RMS, applicable for azimuthally nonhomogeneous data, may be written as a function of the individual modal contributions as follows [derivation in the Appendix, Eq. (A5)]:

$$\begin{aligned}
 \overline{q^2(\phi, t)} = & \overline{a_0^2} + \frac{\overline{a_1^2} + \overline{b_1^2}}{2} + \frac{\overline{a_2^2} + \overline{b_2^2}}{2} \\
 & + \frac{\overline{a_1^2} - \overline{b_1^2}}{2} \cos 2\phi + \frac{\overline{a_2^2} - \overline{b_2^2}}{2} \cos 4\phi \\
 & + 2\overline{a_0 a_1} \cos \phi + 2\overline{a_0 b_1} \sin \phi + 2\overline{a_0 a_2} \cos 2\phi + 2\overline{a_0 b_2} \sin 2\phi \\
 & + 2\overline{a_1 b_1} \sin \phi \cos \phi + 2\overline{a_1 a_2} \cos \phi \cos 2\phi + 2\overline{a_1 b_2} \cos \phi \sin 2\phi \\
 & + 2\overline{b_1 a_2} \sin \phi \cos 2\phi + 2\overline{b_1 b_2} \cos \phi \sin 2\phi + 2\overline{a_2 b_2} \cos 2\phi \sin 2\phi. \quad (28)
 \end{aligned}$$

The complications associated with lack of homogeneity require consideration of additional terms to account for the inequality of sine and cosine coefficients ($\frac{\overline{a_m^2} - \overline{b_m^2}}{2}$) as well as modal interactions ($\overline{a_m a_n}$), that must be modeled.

The generalized RMS expression [Eq. (A5)] aids in identifying physical mechanisms related to azimuthal inhomogeneities and asymmetry. For example, the term $\frac{\overline{a_1^2} - \overline{b_1^2}}{2}$ is related to the dominance of flapping motions along the major or minor axes planes. This is linked to the directivity given by $\cos 2\phi$ and is visualized in Fig. 7(c). Thus, the dominance of flapping motions along one axis results in the increase of the RMS in that direction and a reduction on the other axis.

Equation (28) is generalized and does not take into account the specifics of the problem. Some simplification is afforded by exploiting horizontal and vertical symmetry of rectangular jets about the minor and major axes, respectively. The directivity of their radiation can then only assume azimuthal harmonics of the form $2n\phi$. To elucidate this point, the coupling between the axisymmetric and first azimuthal mode is reflected in the term $2\overline{a_0 a_1} \cos \phi$, the associated directivity of which, given by $\cos \phi$, is shown graphically in Fig. 7(a). This yields an increase in acoustic intensity on one side of the minor axis plane of the jet along with a reduction on the other, violating the underlying symmetry of the problem. Further, among the even harmonics, the sine function ($\sin 2n\phi$) results in a spatial variation that is antisymmetric about the major and minor axis planes of symmetry. Thus, by exploiting the symmetry of the rectangular jet problem, all terms resulting in directivities given by odd harmonics in ϕ as well as those containing even harmonics of the sine function may be removed from the expression. This simplification is a direct result of the choice of the particular ansatz for the Fourier decomposition used in this paper [Eq. (12)]. Thus, the simplified expression for the fluctuation RMS is given by

$$\begin{aligned}
 \overline{q^2(\phi, t)} = & \overline{a_0^2} + \frac{\overline{a_1^2} + \overline{b_1^2}}{2} + \frac{\overline{a_2^2} + \overline{b_2^2}}{2} \\
 & + \frac{\overline{a_1^2} - \overline{b_1^2}}{2} \cos 2\phi + \frac{\overline{a_2^2} - \overline{b_2^2}}{2} \cos 4\phi + 2\overline{a_0 a_2} \cos 2\phi. \quad (29)
 \end{aligned}$$

The terms related to azimuthal inhomogeneity are evaluated for the three nonaxisymmetric jets in Fig. 17. The modal energy of the leading $m = 0$ mode is also shown (first row) to highlight the importance of inhomogeneous processes relative to the dominant acoustic sources. Clearly, one of the most important contributors to the asymmetry are the differences between the sine and cosine coefficients of the first azimuthal mode (modes 1a and 1b). The preferential flapping in the minor axis direction noted in the context of Fig. 10 manifests in larger values for $\overline{a_1^2}$ compared to $\overline{b_1^2}$. Indeed, in rectangular jets, the cosine and sine coefficients (a_1 and b_1 , respectively) represent distinct physical processes on the major and minor axis planes. The preferential growth of instabilities and flapping motions along the minor axis plane has been shown to be a dominant dynamic feature of higher AR rectangular jets, and hypothesized as a cause of asymmetries in the far-field acoustic

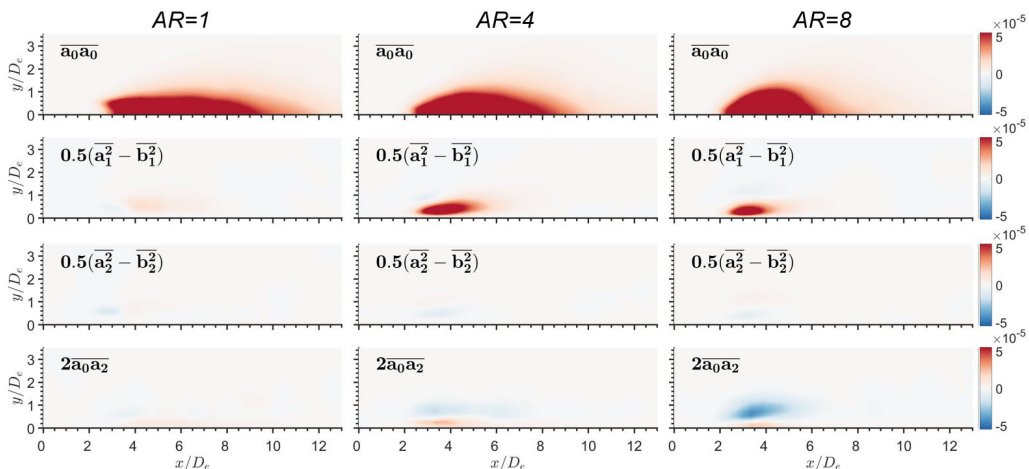


FIG. 17. Contours showing the leading $m = 0$ azimuthal mode and spatial distributions of the dominant terms associated with azimuthal inhomogeneity and acoustic radiation asymmetry for all simulated rectangular jets.

radiation of jets [62]. In addition to the above, the $AR = 8$ jet also exhibits a coupling between the axisymmetric and second azimuthal mode ($\overline{a_0 a_2}$) although it is much weaker than the flapping mechanism.

Other important features evident in Fig. 17 may be summarized as follows. The preferential flapping motion manifested in the disparity between the sine and cosine coefficients of $m = 1$ ($\overline{a_1^2} - \overline{b_1^2}$) occurs over a larger spatial spread for the $AR = 4$ jet compared to the $AR = 8$ jet. This is similar to the reduction in the streamwise length scale of the higher AR jets due to a reduction in the potential core length. However, in both these rectangular jets, the relative magnitudes and spatial spread of the flapping motions are much smaller compared to the $m = 0$ azimuthal mode ($\overline{a_0 a_0}$). Furthermore, the square jet shows a very small difference between the two coefficients of the first mode. This is due to the added symmetry of the problem; i.e., in addition to symmetry about the major and minor axes, the square ($AR = 1$) jet is also symmetric along its diagonals (fourfold mirror symmetry). Thus, any departure from axisymmetry for the square jet must be of the form $\cos 4n\phi$. Preferential flapping along either of the planes bisecting the sides in a square jet would result in an unacceptable asymmetry given by $\cos 2\phi$. Interestingly, several terms that are not identically zero in square jets are found to be negligible, for example, $\overline{a_2^2} - \overline{b_2^2}$, which would result in a fourth harmonic in the azimuthal direction. This is consistent with the axisymmetric far-field acoustic radiation obtained for the case of the square jet.

VII. REDUCED NEAR-FIELD MODEL

The above findings may be used to inform a reduced order model for the nonaxisymmetric near field of rectangular jets; for completeness the corresponding results for circular jets are also presented. Figure 18 displays the polar plot distribution of azimuthal acoustic fluctuation RMS on the FWH surface for the circular (a) and the $AR = 8$ (b) rectangular jets at the streamwise location of $x/D_e = 5$, which corresponds to the peak acoustic radiation (in the downstream shallow angle direction). As anticipated, the circular jet exhibits no variation in acoustic intensity with respect to the azimuthal angle. In this case, due to the azimuthal homogeneity, the overall RMS obtained from the LES can be faithfully reproduced by directly adding the modal energies of the individual azimuthal modes per Eq. (27). The cumulative sum of the modal energies of the four leading azimuthal modes is shown using dashed black lines. The $m = 0$ energy, shown by the dotted line,

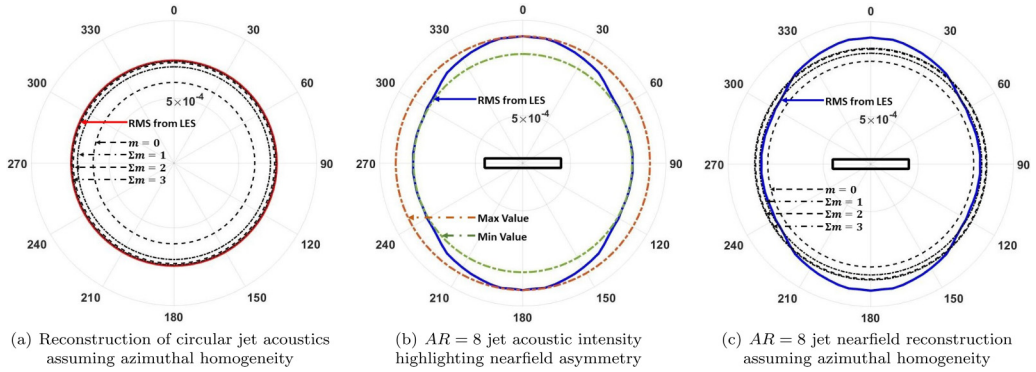


FIG. 18. Azimuthal distributions of the acoustic intensity on the FWH surface at the streamwise location of the maximal jet noise shown using polar plots for the circular and $AR = 8$ rectangular jets. Reconstructions of the acoustic intensity using the cumulative sum [Eq. (27)] of the m leading modal energies (indicated by the $\sum m$ value) are shown for both jets.

closest to the center of the plot accounts for 78% of the observed acoustic intensity. The sum of the two leading modes, namely, $m = 0$ and 1, constitutes 94% of the total acoustic intensity and is quite close to overall RMS observed from the LES. Adding higher azimuthal modes progressively recovers the LES results; for example, the sum of the first three azimuthal modes, i.e., $\Sigma m = 2$, yields 97% of the observed acoustic intensity. However, higher modal contributions diminish in consistency with the above noted rapid convergence of fluctuation statistics.

Figure 18(b) shows the azimuthal variation of the near-field acoustic intensity of the $AR = 8$ jet to highlight its near-field asymmetry. A schematic of the nozzle exit is also shown for orientation. A peak in the acoustic intensity is distinctly visible on the minor axis plane which was shown previously (Sec. III) to be the direction of the maximum acoustic intensity of the $AR = 8$ jet. The difference between the minimum and maximum acoustic intensity (shown in the figure using green and orange dashed lines) is about 15% of the average RMS around the azimuth. Bridges [31] and Kantola [62] also observed the minor axis plane of high AR rectangular jets to be louder. Figure 18(c) shows reconstructions of the near-field acoustic intensity of the rectangular jet following a procedure similar to that used for the circular jet in Fig. 18(a), i.e., using Eq. (27). This approach fails to capture the near-field asymmetry of the rectangular jet and results in under- and overpredictions along the major and minor axis directions, respectively.

Thus, the terms associated with nonhomogeneity, discussed in Sec. VI, are crucial to the reconstruction of the higher aspect ratio rectangular jets. This is shown in Fig. 19 for all the nonaxisymmetric jets. The near-field asymmetry of $AR = 8$ [Fig. 19(c)] is recovered by adding only two additional terms associated with inhomogeneity in Eq. (29): $0.5(\overline{a_1^2} - \overline{b_1^2}) \cos 2\phi$ and $\overline{a_0 a_2}$, incorporating preferential flapping and the coupling of the axisymmetric and second azimuthal modes, respectively. This corrected reconstruction shown using the magenta hatched line successfully reproduces the asymmetry in the near-field acoustic intensity. A similar situation is also visible for the $AR = 4$ jet [Fig. 19(b)].

Interestingly, in line with its overall similarity to the circular jet, the square jet exhibits an azimuthally uniform near-field acoustic intensity [Fig. 19(a)]. As a result, a near-field acoustic reconstruction assuming azimuthal homogeneity [following Eq. (27)] gives a close approximation of the measured acoustic field. Adding the terms related to the inhomogeneity has a negligible effect on the reconstruction. This is consistent with the relatively negligible magnitudes of the inhomogeneity terms in the square jet (as shown in Fig. 17). The near-field RMS acoustic intensity for $AR = 4$ and 8 jets displays preferential flapping along the minor axis direction [$0.5(\overline{a_1^2} - \overline{b_1^2}) \cos 2\phi$]; this is the dominant mechanism related to the near-field asymmetry as evidenced by Fig. 17. The differences

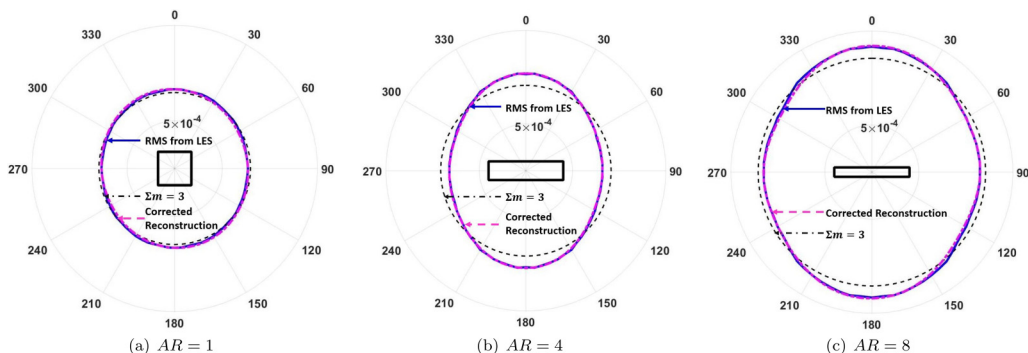


FIG. 19. Comparison of the near-field acoustic intensity reconstruction without [Eq. (27)] and with [Eq. (29)] a consideration of the terms representing the inhomogeneity of the nonaxisymmetric jets.

between the $AR = 4$ and 8 jets may be highlighted by examining the flapping term in both as a fraction of the axisymmetric modal energy. Figure 20 shows the importance of the primary term representing flapping for both the $AR = 4$ and 8 jets along a ray following the direction of maximum acoustic radiation ($\theta = 150^\circ$ from the upstream). The flapping effect term is dominant relatively close to the jet and is commensurate with the axisymmetric modal energy. Further out along the shallow angle direction, clear differences emerge between the two jets. While the relative importance of the flapping term decays rapidly for the $AR = 4$ jet, it remains a dominant term for the $AR = 8$ jet accounting for approximately 10% of the axisymmetric modal energy.

VIII. CONCLUSIONS

The advantages of azimuthal Fourier decomposition, the benefits of which are well established for circular jet analysis, are evaluated for rectangular jets using large eddy simulations and by projecting the fluctuations on a cylindrical coordinate system. A related goal is to examine the effectiveness of such a decomposition for near-field acoustic reconstruction with a reduced order model. The three ARs chosen, $AR = 1$, 4, and 8, incorporate several flow phenomena pertinent to rectangular jets, including rapid transition to axisymmetric cross section ($AR = 1$), axis switching

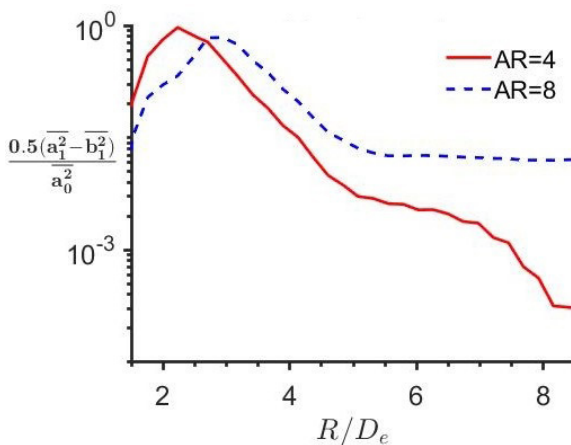


FIG. 20. Comparison of the relative importance of preferential flapping motions in rectangular jets along the direction of peak acoustic radiation.

(AR = 4), and azimuthal nonuniformity in far-field acoustic radiation (AR = 8). They also contain other important features such as potential core length reduction and jet spreading rate increase with AR.

The utility of azimuthal Fourier decomposition is examined by replacing the commonly used complex exponential ansatz with separate coefficients for sine and cosine modes; this provides a more convenient representation of features that do not display axisymmetric behavior. Fluctuations from all AR jets show rapid declines in higher azimuthal mode content; as such, the convergence observed in circular jet data remains valid for rectangular jets. Even for the AR = 8 jet, the use of the three leading azimuthal wave numbers yields relatively small error in the instantaneous fluctuation field as well as volume integrated measures. Consistent with circular jet modeling, the acoustic field from a fluid-thermodynamic decomposition exhibits lower-rank behavior than pressure.

The primary differences in plume evolution between rectangular and circular jets are evident in the first azimuthal mode, the sine and cosine coefficients of which contain the qualitatively distinct processes on the major and minor axis planes of the rectangular jets. Further, a comparison of the spatial variation of the RMS acoustic fluctuation magnitudes of the individual azimuthal modes indicates that the streamwise extent reduces with increase in the AR, consistent with a similar trend in the length of the potential core found in the corresponding mean flow fields.

Finally, the loss of azimuthal homogeneity in rectangular jets and the nonaxisymmetric RMS fluctuations is analyzed with a generalized expression for second-order statistics in terms of the azimuthal modal energies. This expression highlights, as a special case, the appreciable simplification afforded by an azimuthal homogeneity, as in circular jets. For rectangular jets, significant simplifications are also shown to be possible by incorporating their twofold mirror symmetry. By analyzing individual terms, preferential flapping along the minor axis direction is confirmed as the dominant effect in loss of axisymmetry in rectangular jets. Rectangular jet modeling is, thus, greatly aided by the rapid convergence of the fluctuation statistics in azimuthal Fourier space, as well as features of the general expression for acoustic reconstruction as a function of the individual modal energies. These findings should facilitate more advanced acoustic models for rectangular jets that leverage many of the mature techniques developed for circular jets.

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APPENDIX: CONSTRAINTS ON FOURIER MODES DUE TO HOMOGENEITY

To highlight some properties associated with azimuthal homogeneity we consider a spatiotemporal function $f(\phi, t)$, with ϕ being periodic over 2π . $f(\phi, t)$ may be written as a Fourier series:

$$f(\phi, t) = a_0(t) + \sum_{m=1}^{\infty} a_m(t) \cos(m\phi) + b_m(t) \sin(m\phi). \quad (\text{A1})$$

Rapid convergence of the instantaneous statistics in the Fourier basis allows simplification by truncation of the series beyond the dominant leading terms. If only the axisymmetric ($m = 0$) and

first azimuthal ($m = 1$) modes are considered, we have

$$f(\phi, t) = a_0(t) + a_1(t) \cos(\phi) + b_1(t) \sin(\phi). \quad (\text{A2})$$

Azimuthal homogeneity imposes constraints on the Fourier coefficients a_m and b_m and may be written as

$$\overline{f(\phi_1, t)f(\phi_2, t)} = g(\phi_1 - \phi_2). \quad (\text{A3})$$

Here, $\overline{(\cdot)}$ refers to an averaging operation in time. Substituting the truncated Fourier series [Eq. (A2)] in the expression for the cross correlation leads to

$$\begin{aligned} \overline{f(\phi_1, t)f(\phi_2, t)} &= \overline{a_0^2} + \overline{a_1^2} \cos(\phi_1) \cos(\phi_2) + \overline{b_1^2} \sin(\phi_1) \sin(\phi_2) + \overline{a_0 a_1} (\cos \phi_1 + \cos \phi_2) \\ &\quad + \overline{a_0 b_1} (\sin \phi_1 + \sin \phi_2) + \overline{a_1 b_1} (\sin \phi_1 \cos \phi_2 + \sin \phi_2 \cos \phi_1). \end{aligned}$$

The resulting expression is arranged such that terms involving the product of similar Fourier modes appear on the first line and are referred to as self-terms (S). The terms on the second line represent interactions between Fourier modes and are, thus, referred to as cross terms. The self-terms may be further simplified by adding and subtracting two terms, namely, $\frac{\overline{a_1^2}}{2} \sin(\phi_1) \sin(\phi_2)$ and $\frac{\overline{b_1^2}}{2} \cos(\phi_1) \cos(\phi_2)$, to read

$$\begin{aligned} \text{S} &= \overline{a_0^2} + \overline{a_1^2} \cos(\phi_1) \cos(\phi_2) + \overline{b_1^2} \sin(\phi_1) \sin(\phi_2) \\ &= \overline{a_0^2} + \frac{\overline{a_1^2} + \overline{b_1^2}}{2} [\cos(\phi_1) \cos(\phi_2) + \sin(\phi_1) \sin(\phi_2)] \\ &\quad + \frac{\overline{a_1^2} - \overline{b_1^2}}{2} [\cos(\phi_1) \cos(\phi_2) - \sin(\phi_1) \sin(\phi_2)], \end{aligned}$$

which may be simplified to

$$\text{S} = \overline{a_0^2} + \frac{\overline{a_1^2} + \overline{b_1^2}}{2} \cos(\phi_1 - \phi_2) + \frac{\overline{a_1^2} - \overline{b_1^2}}{2} \cos(\phi_1 + \phi_2).$$

On substituting the above form of the self-terms in the expression for the cross-correlation matrix, the following expression is obtained:

$$\begin{aligned} \overline{f(\phi_1, t)f(\phi_2, t)} &= \overline{a_0^2} + \frac{\overline{a_1^2} + \overline{b_1^2}}{2} \cos(\phi_1 - \phi_2) + \frac{\overline{a_1^2} - \overline{b_1^2}}{2} \cos(\phi_1 + \phi_2) + \overline{a_0 a_1} (\cos \phi_1 + \cos \phi_2) \\ &\quad + \overline{a_0 b_1} (\sin \phi_1 + \sin \phi_2) + \overline{a_1 b_1} (\sin \phi_1 \cos \phi_2 + \sin \phi_2 \cos \phi_1). \end{aligned}$$

The right-hand side of the above expression is a function of angular separation ($\phi_1 - \phi_2$) alone; all terms on the second and third line are zero for all values of ϕ_1 and ϕ_2 . This leads to two important classes of constraints on the individual Fourier coefficients (a_0 , a_1 and b_1).

- (1) $\overline{a_m^2} = \overline{b_m^2}$: The sine and cosine coefficients of a given wave number have equal RMS.
- (2) $\overline{a_m a_n} = \overline{b_m b_n} = \overline{a_m b_n} = 0$: The individual azimuthal Fourier coefficients are mutually uncorrelated.

Under these conditions, the cross-correlation function is given by

$$\overline{f(\phi_1, t)f(\phi_2, t)} = \overline{a_0^2} + \frac{\overline{a_1^2} + \overline{b_1^2}}{2} \cos(\phi_1 - \phi_2)$$

and thus also satisfies the condition for circumferential isotropy, in that it is independent of the sign of the angular displacement [14]. Thus $\overline{f(\phi_1, t)f(\phi_2, t)} = g(\phi_1 - \phi_2) = g(\phi_2 - \phi_1)$.

The constraints imposed on individual Fourier coefficients due to the statistical homogeneity also significantly simplify the reconstruction of the RMS fluctuations using Fourier modes. In general,

the RMS of $f(\phi, t)$ in terms of the three leading Fourier coefficients is given by

$$\begin{aligned} \overline{f^2(\phi, t)} = & \overline{a_0^2} + \overline{a_1^2} \cos^2 \phi + \overline{b_1^2} \sin^2 \phi + \overline{a_2^2} \cos^2 2\phi + \overline{b_2^2} \sin^2 2\phi \\ & + 2\overline{a_0 a_1} \cos \phi + 2\overline{a_0 b_1} \sin \phi + 2\overline{a_0 a_2} \cos 2\phi + 2\overline{a_0 b_2} \sin 2\phi \\ & + \overline{a_1 b_1} \sin \phi \cos \phi + \overline{a_1 a_2} \cos \phi \cos 2\phi + \overline{a_1 b_2} \cos \phi \sin 2\phi \\ & + \overline{b_1 a_2} \sin \phi \cos 2\phi + \overline{b_1 b_2} \cos \phi \sin 2\phi + \overline{a_2 b_2} \cos 2\phi \sin 2\phi. \end{aligned} \quad (\text{A4})$$

The terms on the right-hand side of the above relation can again be split into self-terms consisting of contributions from the sine and cosine coefficients of the same azimuthal mode number (first two lines) and cross terms representing modal interactions. Using trigonometric identities, the self-terms in the above expression can be rearranged to

$$\begin{aligned} \overline{f^2(\phi, t)} = & \overline{a_0^2} + \frac{\overline{a_1^2} + \overline{b_1^2}}{2} + \frac{\overline{a_2^2} + \overline{b_2^2}}{2} \\ & + \frac{\overline{a_1^2} - \overline{b_1^2}}{2} \cos 2\phi + \frac{\overline{a_2^2} - \overline{b_2^2}}{2} \cos 4\phi \\ & + 2\overline{a_0 a_1} \cos \phi + 2\overline{a_0 b_1} \sin \phi + 2\overline{a_0 a_2} \cos 2\phi + 2\overline{a_0 b_2} \sin 2\phi \\ & + 2\overline{a_1 b_1} \sin \phi \cos \phi + 2\overline{a_1 a_2} \cos \phi \cos 2\phi + 2\overline{a_1 b_2} \cos \phi \sin 2\phi \\ & + 2\overline{b_1 a_2} \sin \phi \cos 2\phi + 2\overline{b_1 b_2} \cos \phi \sin 2\phi + 2\overline{a_2 b_2} \cos 2\phi \sin 2\phi. \end{aligned} \quad (\text{A5})$$

This form facilitates an understanding of the physical mechanisms that contribute to an asymmetry, that is, a dependence on ϕ . All the terms in this expression, apart from those on the first line, represent physical processes that are a direct result of the inhomogeneity of the statistics in the azimuthal direction.

The constraints due to homogeneity derived previously simplify the expression for the RMS to

$$\overline{f^2(\phi, t)} = \overline{a_0^2} + \frac{\overline{a_1^2} + \overline{b_1^2}}{2} + \frac{\overline{a_2^2} + \overline{b_2^2}}{2}. \quad (\text{A6})$$

Thus, the RMS of the function is given as the summation of the contributions of the individual azimuthal modes in the homogeneous direction, and is independent of the angular position ϕ . This crucial result implies that azimuthal invariance of the RMS holds no matter how many individual azimuthal modes are retained.

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- [1] C. K. W. Tam, Supersonic jet noise, *Annu. Rev. Fluid Mech.* **27**, 17 (1995).
 - [2] P. Jordan and T. Colonius, Wave packets and turbulent jet noise, *Annu. Rev. Fluid Mech.* **45**, 173 (2013).
 - [3] T. Suzuki, A review of diagnostic studies on jet-noise sources and generation mechanisms of subsonically convecting jets, *Fluid Dynamics Research* **42**, 014001 (2010).
 - [4] G. Raman, P. Panickar, and K. Chelliah, Aeroacoustics of twin supersonic jets: A review, *Int. J. Aeroacoustics* **11**, 957 (2012).
 - [5] C. Bailly and K. Fujii, High-speed jet noise, *Mechanical Engineering Reviews* **3**, 15-00496 (2016).
 - [6] J. B. Freund and T. Colonius, POD analysis of sound generation by a turbulent jet, in *Proceedings of the 40th AIAA Aerospace Sciences Meeting and Exhibit* (AIAA, Reston, VA, 2002).
 - [7] O. T. Schmidt, A. Towne, G. Rigas, T. Colonius, and G. A. Brès, Spectral analysis of jet turbulence, *J. Fluid Mech.* **855**, 953 (2018).
 - [8] K. Gudmundsson and T. Colonius, Instability wave models for the near-field fluctuations of turbulent jets, *J. Fluid Mech.* **689**, 97 (2011).

- [9] E. Pickering, G. Rigas, O. T. Schmidt, D. Sipp, and T. Colonius, Optimal eddy viscosity for resolvent-based models of coherent structures in turbulent jets, *J. Fluid Mech.* **917**, A29 (2021).
- [10] A. V. G. Cavalieri, P. Jordan, and L. Lesshafft, Wave-packet models for jet dynamics and sound radiation, *Appl. Mech. Rev.* **71**, 020802 (2019).
- [11] R. Reba, S. Narayanan, and T. Colonius, Wave-packet models for large-scale mixing noise, *Int. J. Aeroacoustics* **9**, 533 (2010).
- [12] Y. B. Baqui, A. Agarwal, A. V. G. Cavalieri, and S. Sinayoko, A coherence-matched linear source mechanism for subsonic jet noise, *J. Fluid Mech.* **776**, 235 (2015).
- [13] D. Papamoschou, Wavepacket modeling of the jet noise source, *Int. J. Aeroacoustics* **17**, 52 (2018).
- [14] A. Michalke and H. V. Fuchs, On turbulence and noise of an axisymmetric shear flow, *J. Fluid Mech.* **70**, 179 (1975).
- [15] H. Vold, P. Shah, P. Morris, Y. Du, and D. Papamoschou, Axisymmetry and azimuthal modes in jet noise, in *Proceedings of the 18th AIAA/CEAS Aeroacoustics Conference (33rd AIAA Aeroacoustics Conference)* (AIAA, Reston, VA, 2012), p. 2214.
- [16] D. G. Crighton and M. Gaster, Stability of slowly diverging jet flow, *J. Fluid Mech.* **77**, 397 (1976).
- [17] F. C. Lajús, A. Sinha, A. V. G. Cavalieri, C. J. Deschamps, and T. Colonius, Spatial stability analysis of subsonic corrugated jets, *J. Fluid Mech.* **876**, 766 (2019).
- [18] A. Michalke, A Wave Model for Sound Generation in Circular Jets, Technical Report, Deutsche Forschungs- und Versuchsanstalt für Luft- und Raumfahrt, 1970.
- [19] A. V. G. Cavalieri, P. Jordan, T. Colonius, and Y. Gervais, Axisymmetric superdirectivity in subsonic jets, *J. Fluid Mech.* **704**, 388 (2012).
- [20] A. Sinha, D. Rodríguez, G. A. Brès, and T. Colonius, Wavepacket models for supersonic jet noise, *J. Fluid Mech.* **742**, 71 (2014).
- [21] S. C. Crow, Acoustic gain of a turbulent jet, in *Proceedings of the Physical Society Meeting*, 1972 (unpublished), Vol. 6.
- [22] V. Jaunet, P. Jordan, and A. V. G. Cavalieri, Two-point coherence of wave packets in turbulent jets, *Phys. Rev. Fluids* **2**, 024604 (2017).
- [23] F. F. Grinstein, Vortex dynamics and entrainment in rectangular free jets, *J. Fluid Mech.* **437**, 69 (2001).
- [24] E. J. Gutmark and F. F. Grinstein, Flow control with noncircular jets, *Annu. Rev. Fluid Mech.* **31**, 239 (1999).
- [25] N. Chen and H. Yu, Mechanism of axis switching in low aspect-ratio rectangular jets, *Comput. Math. Appl.* **67**, 437 (2014).
- [26] K. B. M. Q. Zaman, Axis switching and spreading of an asymmetric jet: The role of coherent structure dynamics, *J. Fluid Mech.* **316**, 1 (1996).
- [27] C. Shih, A. Krothapalli, and S. Gogineni, Experimental observations of instability modes in a rectangular jet, *AIAA J.* **30**, 2388 (1992).
- [28] E. Gutmark, K. C. Schadow, and C. J. Bicker, Near acoustic field and shock structure of rectangular supersonic jets, *AIAA J.* **28**, 1163 (1990).
- [29] F. Hussain and H. S. Husain, Elliptic jets. Part 1. Characteristics of unexcited and excited jets, *J. Fluid Mech.* **208**, 257 (1989).
- [30] F. F. Grinstein, Self-induced vortex ring dynamics in subsonic rectangular jets, *Phys. Fluids* **7**, 2519 (1995).
- [31] J. Bridges, Acoustic measurements of rectangular nozzles with bevel, in *Proceedings of the 18th AIAA/CEAS Aeroacoustics Conference (33rd AIAA Aeroacoustics Conference)* (AIAA, Reston, VA, 2012), p. 2252.
- [32] J. Veltin and D. McLaughlin, Flow field and acoustic measurements of rectangular supersonic jets, in *Proceedings of the 47th AIAA Aerospace Sciences Meeting including The New Horizons Forum and Aerospace Exposition* (AIAA, Reston, VA, 2009), p. 19.
- [33] N. S. Heeb, P. M. Sanchez, E. J. Gutmark, and K. Kailasanath, Investigation of the noise from a rectangular supersonic jet, in *Proceedings of the 19th AIAA/CEAS Aeroacoustics Conference* (AIAA, Reston, VA, 2013), p. 2239.
- [34] C. K. W. Tam, Influence of nozzle geometry on the noise of high-speed jets, *AIAA J.* **36**, 1396 (1998).

- [35] K. W. Kinzie and D. K. McLaughlin, Azimuthal mode measurements of elliptic jets, *Phys. Fluids* **9**, 2000 (1997).
- [36] C. K. W. Tam and A. T. Thies, Instability of rectangular jets, *J. Fluid Mech.* **248**, 425 (1993).
- [37] S. Chakrabarti, D. V. Gaitonde, and S. Unnikrishnan, The dynamics of azimuthal modes in rectangular jets, in *Proceedings of the AIAA Aviation 2020 Forum* (AIAA, Reston, VA, 2020), p. 3000.
- [38] P. E. Doak, Momentum potential theory of energy flux carried by momentum fluctuations, *J. Sound Vib.* **131**, 67 (1989).
- [39] S. Unnikrishnan, A. V. G. Cavalieri, and D. V. Gaitonde, Acoustically informed statistics for wave-packet models, *AIAA J.* **57**, 2421 (2019).
- [40] D. Anderson, J. C. Tannehill, and R. H. Pletcher, *Computational Fluid Mechanics and Heat Transfer* (CRC, Boca Raton, FL, 2016).
- [41] M. Vinokur, An analysis of finite-difference and finite-volume formulations of conservation laws, *J. Comput. Phys.* **81**, 1 (1989).
- [42] M. Samimy, J. H. Kim, M. Kearney-Fischer, and A. Sinha, Acoustic and flow fields of an excited high Reynolds number axisymmetric supersonic jet, *J. Fluid Mech.* **656**, 507 (2010).
- [43] M. Samimy, J. H. Kim, J. Kastner, I. Adamovich, and Y. Utkin, Active control of high-speed and high-Reynolds-number jets using plasma actuators, *J. Fluid Mech.* **578**, 305 (2007).
- [44] J. Hileman and M. Samimy, Turbulence structures and the acoustic far field of a mach 1.3 jet, *AIAA J.* **39**, 1716 (2001).
- [45] D. V. Gaitonde, Analysis of the near field in a plasma-actuator-controlled supersonic jet, *J. Propul. Power* **28**, 281 (2012).
- [46] D. V. Gaitonde and M. Samimy, Coherent structures in plasma-actuator controlled supersonic jets: Axisymmetric and mixed azimuthal modes, *Phys. Fluids* **23**, 095104 (2011).
- [47] R. Speth and D. V. Gaitonde, Parametric study of a mach 1.3 cold jet excited by the flapping mode using plasma actuators, *Comput. Fluids* **84**, 16 (2013).
- [48] S. Unnikrishnan and D. V. Gaitonde, Acoustic, hydrodynamic and thermal modes in a supersonic cold jet, *J. Fluid Mech.* **800**, 387 (2016).
- [49] K. B. M. Q. Zaman, Effect of initial condition on subsonic jet noise, *AIAA J.* **23**, 1370 (1985).
- [50] G. A. Brès, P. Jordan, V. Jaunet, M. Le Rallic, A. V. G. Cavalieri, A. Towne, S. K. Lele, T. Colonius, and O. T. Schmidt, Importance of the nozzle-exit boundary-layer state in subsonic turbulent jets, *J. Fluid Mech.* **851**, 83 (2018).
- [51] A. Karon and K. Ahuja, Effect of nozzle-exit boundary layer on jet noise, in *Proceedings of the 51st AIAA Aerospace Sciences Meeting including the New Horizons Forum and Aerospace Exposition* (AIAA, Reston, VA, 2013), p. 615.
- [52] D. Whitfield, Three-dimensional unsteady euler equations solution using flux vector splitting, in *Proceedings of the 17th Fluid Dynamics, Plasma Dynamics, and Lasers Conference* (AIAA, Reston, VA, 1984), p. 1552.
- [53] P. L. Roe, Approximate Riemann solvers, parameter vectors, and difference schemes, *J. Comput. Phys.* **43**, 357 (1981).
- [54] J. H. Morrison, Flux difference split scheme for turbulent transport equations, in *2nd International Aerospace Planes Conference* (AIAA, Reston, VA, 1990).
- [55] T. H. Pulliam and D. S. Chaussee, A diagonal form of an implicit approximate-factorization algorithm, *J. Comput. Phys.* **39**, 347 (1981).
- [56] S. Chakrabarti, C. M. Stack, S. Unnikrishnan, D. V. Gaitonde, F. Baier, A. Karnam, and E. J. Gutmark, On the turbulence statistics of a hot, overexpanded rectangular jet, in *Proceedings of the AIAA Scitech 2020 Forum* (AIAA, Reston, VA, 2020), p. 0819.
- [57] F. F. Grinstein and K. Kailasanath, Three-dimensional numerical simulations of unsteady reactive square jets, *Combust. Flame* **100**, 2 (1995).
- [58] D. J. Bodony and S. K. Lele, On using large-eddy simulation for the prediction of noise from cold and heated turbulent jets, *Phys. Fluids* **17**, 085103 (2005).
- [59] A. Krothapalli, D. Baganoff, and K. Karamcheti, On the mixing of a rectangular jet, *J. Fluid Mech.* **107**, 201 (1981).

- [60] P. Di Francescantonio, A new boundary integral formulation for the prediction of sound radiation, *J. Sound Vib.* **202**, 491 (1997).
- [61] S. Mendez, M. Shoybi, S. K. Lele, and P. Moin, On the use of the Ffowcs Williams-Hawkings equation to predict far-field jet noise from large-eddy simulations, *Int. J. Aeroacoustics* **12**, 1 (2013).
- [62] R. A. Kantola, Noise characteristics of heated high velocity rectangular jets, *J. Sound Vib.* **64**, 277 (1979).
- [63] I. A. Maia, P. Jordan, A. V. G. Cavalieri, and V. Jaunet, Two-point wavepacket modelling of jet noise, *Proc. R. Soc. A* **475**, 20190199 (2019).
- [64] M. E. Goldstein, On identifying the true sources of aerodynamic sound, *J. Fluid Mech.* **526**, 337 (2005).
- [65] C. E. Tinney and P. Jordan, The near pressure field of co-axial subsonic jets, *J. Fluid Mech.* **611**, 175 (2008).
- [66] R. E. A. Arndt, D. F. Long, and M. N. Glauser, The proper orthogonal decomposition of pressure fluctuations surrounding a turbulent jet, *J. Fluid Mech.* **340**, 1 (1997).
- [67] C. Bogey, S. Barré, V. Fleury, C. Bailly, and D. Juvé, Experimental study of the spectral properties of near-field and far-field jet noise, *Int. J. Aeroacoustics* **6**, 73 (2007).
- [68] S. Unnikrishnan and D. V. Gaitonde, Interactions between vortical, acoustic and thermal components during hypersonic transition, *J. Fluid Mech.* **868**, 611 (2019).
- [69] D. G. Crighton, Instability of an elliptic jet, *J. Fluid Mech.* **59**, 665 (1973).
- [70] A. V. G. Cavalieri, D. Rodríguez, P. Jordan, T. Colonius, and Y. Gervais, Wavepackets in the velocity field of turbulent jets, *J. Fluid Mech.* **730**, 559 (2013).
- [71] G. A. Faranosov, I. V. Belyaev, V. F. Kopiev, M. Y. Zaytsev, A. A. Aleksentsev, Y. V. Bersenev, V. A. Chursin, and T. A. Viskova, Adaptation of the azimuthal decomposition technique to jet noise measurements in full-scale tests, *AIAA J.* **55**, 572 (2017).
- [72] S. Unnikrishnan, Jet noise source localization and identification, Ph.D. thesis, The Ohio State University, 2017.