Second-order velocity structure functions in direct numerical simulations of turbulence with R_λ up to 2250

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We report results of a series of high-resolution direct numerical simulations (DNSs) of forced incompressible isotropic turbulence with the number of grid points and the Taylor scale Reynolds number R_λ up to $12\,288^3$ and $\sim$2250, respectively. The DNSs show that there exists a scaling range (approximately $100 < r/\eta < 400$), at which the second-order two-point velocity structure functions $S_2(r)$ fit well with a simple power-law, $S_2(r)/(\langle\epsilon\rangle r)^{2/3} = C_2(r/L_0)^\zeta$, where r is the distance between the two points, η is the Kolmogorov length scale, $\langle\epsilon\rangle$ is the mean rate of energy dissipation per unit mass, and L_0 is the integral length scale. The exponent ζ is constant independent from R_λ. However, the coefficient C_2 is dependent of R_λ or the viscosity. This implies that the power-law scaling range of $100 < r/\eta < 400$ for R_λ up to $\sim$2250 is not the so-called "inertial subrange" in the sense that the statistics in the range are independent from the viscosity, as assumed in various turbulence theories. This suggests that the constancy of the scaling exponent of a structure function within a certain range does not necessarily mean that the exponent is the scaling exponent in "the inertial subrange."

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I. INTRODUCTION

Underlying the celebrated work by Kolmogorov (1941) [1] (called K41) is the idea of the existence of a certain kind of universality in the statistics of turbulent flows at high Reynolds numbers (Re). Although the assumptions used in K41 are not all good, the idea of the existence of universality itself is still at the heart of many modern theories of turbulence. The K41 theory gives among others

$$S_n^L(r) = C_n^L(r\langle\epsilon\rangle)^{n/3} \tag{1}$$

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within the inertial subrange (ISR) $L_0 \gg r \gg \eta$, where $S_n^L(r)$ is the nth order longitudinal structure function defined as

$$S_n^L(r) \equiv \langle (\Delta u^L)^n \rangle, \quad \Delta u^L \equiv [\mathbf{u}(\mathbf{x} + r\mathbf{e}) - \mathbf{u}(\mathbf{x})] \cdot \mathbf{e}, \tag{2}$$

$\mathbf{u}(\mathbf{x})$ is the fluid velocity at position $\mathbf{x}$; C_n^L is a universal constant independent of r, ν, and Re; $\mathbf{e}$ is a unit vector in an arbitrary direction; L_0 is the characteristic length of scale of energy containing eddies; η is the Kolmogorov length scale defined by $\eta = (\nu^3/\langle \epsilon \rangle)^{1/4}$; ν is the kinematic viscosity; and $\langle \epsilon \rangle$ is the mean rate of the energy dissipation per unit mass.

As is well known, high Re turbulence exhibits strong intermittency at small scales. It is a prototype of a wide class of intermittent phenomena in nature and technology, and has stimulated extensive studies and ideas as seen in the development of the idea of fractals (e.g., Ref. [2]). Kolmogorov himself later revised his theory to take into account the intermittency, and proposed a refined similarity hypothesis (RSH) [3], which gives

$$S_n^L(r) = C_n^L (r\langle \epsilon \rangle)^{n/3} (r/L_0)^{\zeta_n^L} \tag{3}$$

instead of Eq. (1), within the ISR, where $\zeta_3^L = 0$, and C_n^L and ζ_n^L are constants independent of r, ν, and Re. If intermittency effect is negligible, then ζ_n^L must be 0 for any n in accordance with Eq. (1).

If the ISR is sufficiently wide (see NOTE shown below), then Eq. (3) for $n = 2$ gives

$$E(k) = K_2 \langle \epsilon \rangle^{2/3} k^{-5/3} (L_0 k)^{-\mu_2} \tag{4}$$

in the wave-number space, where $E(k)$ is the energy spectrum, k is the wave number, and K_2 and μ_2 are appropriate constants independent of k and Re or ν. The K41 theory gives $\mu_2 = 0$.

Since the development of RSH, various theories on intermittency including multi-fractal models have thus far been proposed (e.g., Ref. [4]). As with RSH, in these models, $S_n^L(r)$ takes the form of Eq. (3). The difference between the models appears not to be in the form, but in the value of the exponent ζ_n^L. With these theories, the exponent ζ_n^L is assumed to be universal in the sense that it is determined by the intrinsic nature of the dynamics, but is not affected by the boundary or the initial conditions under the appropriate conditions. This idea is supported by analytically solvable models including Kraichnan's passive scalar model [5].

Results of experiments and DNSs so far reported have suggested that ζ_n^L is in fact not 0, except for $n = 3$ (e.g., Refs. [6–9] for experiments and Refs. [10–14] for DNSs). However, it is to be recalled that in any real turbulence, experiment and DNS, Re as well as L_0/r and r/η are finite however large they may be.

If ζ_n^L is estimated to be nonzero in a certain range of r in experiment or DNS, then one might think that it is due to the intermittency effect. However, this is not necessarily correct. To see this, consider, e.g., $S_3^L(r)$. It is shown rigorously under weak assumptions that $\zeta_3^L \to 0$ as r/L_0, η/r, $1/\mathrm{Re} \to 0$, i.e., the estimate $\zeta_3^L \neq 0$ by any experiment or DNS is simply due to the finiteness of Re. One should not confuse the corrections due to the finiteness of r/L_0, η/r and/or $1/\mathrm{Re}$ with the intrinsic corrections due to the intermittency, which remains nonzero in the limit r/L_0, η/r, $1/\mathrm{Re} \to 0$. (e.g., Refs. [15,16]).

If the exponent ζ_n^L or μ_2 fits well to the value predicted by a certain theory, then one might think that it gives a support for the theory. However, this is not necessarily correct. To see this, consider, e.g., the energy spectrum $E(k)$. It is known that $\mu_2 \sim 0$ in a certain range of k such that $k\eta \sim 0.1 - 0.2$ (e.g., Refs. [17,18]). But this does not give the verification of the K41 theory. A close inspection shows that the prefactor K_2 in Eq. (4) estimated by fitting $E(k)$ in the range $k\eta \sim 0.1 - 0.2$ to Eq. (4) with $\mu_2 = 0$ depends on Re, i.e., ν (e.g., Refs. [19,20]). If data with only R_λ at most up to $\sim$200 are available, then such a fitting results in an overestimate of the so-called Kolmogorov constant K_2. The range should not be misidentified as the ISR assumed in the K41 theory.

In this respect, it is worthwhile to recall the results of a series of DNSs of forced incompressible isotropic turbulence conducted on the K-computer with a number of grid points of up to 12288^3

TABLE I. DNS parameters and turbulence characteristics at the final time step $t = t_F$. The numbers such as 2048 in the run-name are those of the grid points in each direction of the Cartesian coordinates. L_0 is the integral length scale, $T(= L_0/U)$ is eddy turnover time, and U is the rms value of the fluctuating velocity.

Run	R_λ	$k_{max}\eta$	$10^5\nu$	$10^2\epsilon$	L_0	$10^4\eta$	t_F/T
2048-1	732	1.01	4.40	7.07	1.23	10.5	4.7
4096-2	675	1.95	4.40	8.31	1.05	10.1	2.1
4096-1	1131	0.99	1.73	7.52	1.09	5.12	2.4
6144-1	1423	0.98	1.02	8.06	1.12	3.39	1.7
8192-1	1747	0.99	0.70	7.97	1.10	2.56	1.2
12288-1	2250	1.00	0.41	8.02	1.10	1.71	0.7

and a Taylor scale Reynolds number R_λ of approximately 2300 [21]. In the DNSs, the forcing is at only a low k range. The DNSs showed that there is a wave-number range within which the energy spectrum $E(k)$ fits approximately well to the form Eq. (4) with nonzero μ_2 in accordance with previous studies [12,20,22]. In this paper, this series of DNSs is called K-DNS, and the range is called a T-range, as in Ref. [21]. In the K-DNS, $\mu_2 \approx 0.12$, and the T-range is approximately $5 \times 10^{-3} < k\eta < 2 \times 10^{-2}$. The coefficient K_2 is independent of k in agreement with the theory. However, a close inspection showed that it has a ν-dependence. Although the k-independence of K_2 is in accordance with Eq. (4), the dependence of K_2 on ν is in conflict with the theories and/or models in which K_2 must be ν-independent. This suggests that even if there is a range showing a simple power-law scaling in k, the range may be not the "ISR" supposed in the theories giving Eq. (3).

The similar may also be the case in the r-space, and one may ask questions such as "Are the Re, L_0/r, and r/η in the experiments and DNSs so far made are sufficiently large to allow the examination or assessment of the theories that give Eq. (3)?" "Are the estimates of the exponents (ζ_n^L)'s so far reported acceptable as those in the 'ISR' for which the theories are supposed to be applicable?" The purpose of the present study is to obtain some idea regarding these questions on the basis of the data by K-DNS.

NOTE: $S_2^L(r)$ is uniquely determined by the spectrum $E(k)$ in homogeneous isotropic turbulence, and if $E(k)$ is given by Eq. (3) for the entire range of k, then Eq. (3) for $n = 2$ is equivalent to Eq. (4) for appropriate sets of constants K_2, C_2^L, μ_2, and ζ_2^L. From these facts, one might think that the rate of convergence of $S_2^L(r)$ to Eq. (3) with increasing Re must be similar to the rate of the convergence of $E(k)$ to Eq. (4). However, this is not necessarily correct, because the range in which $E(k)$ is given by Eq. (3), even if it in fact exists, must be finite, i.e., the range cannot be the entire k-range in any real turbulence, and also because the rate of convergence of a function to a certain asymptotic form with increasing Re may be in general different from that of the spectrum, as seen, e.g., in Ref. [23], which shows that the rate of convergence of the Lagrangian structure function to scaling behavior with increasing Re may be different from that of the spectrum.

II. NUMERICAL RESULTS

In this paper, we use the data by K-DNS and focus mostly on the second-order moment $S_2^L(r)$. For convenience, the DNS parameters are reproduced in Table I from Ref. [21]. (As regards Run 12288-1, the simulation time has been extended up to $t = 0.7T$. Table I shows updated data.) It is much easier to obtain reliable statistics for $S_2^L(r)$ than for $S_n^L(r)$ with $n \geqslant 4$ within the given limitation of the computational resources provided.

In the analysis presented below, we estimate $S_n^L(r)$ by taking the average of $S_n^L(r)$ over three different directions of $\mathbf{e}$ in Eq. (2) perpendicular to each other. As for $S_2^L(r)$, we also use the

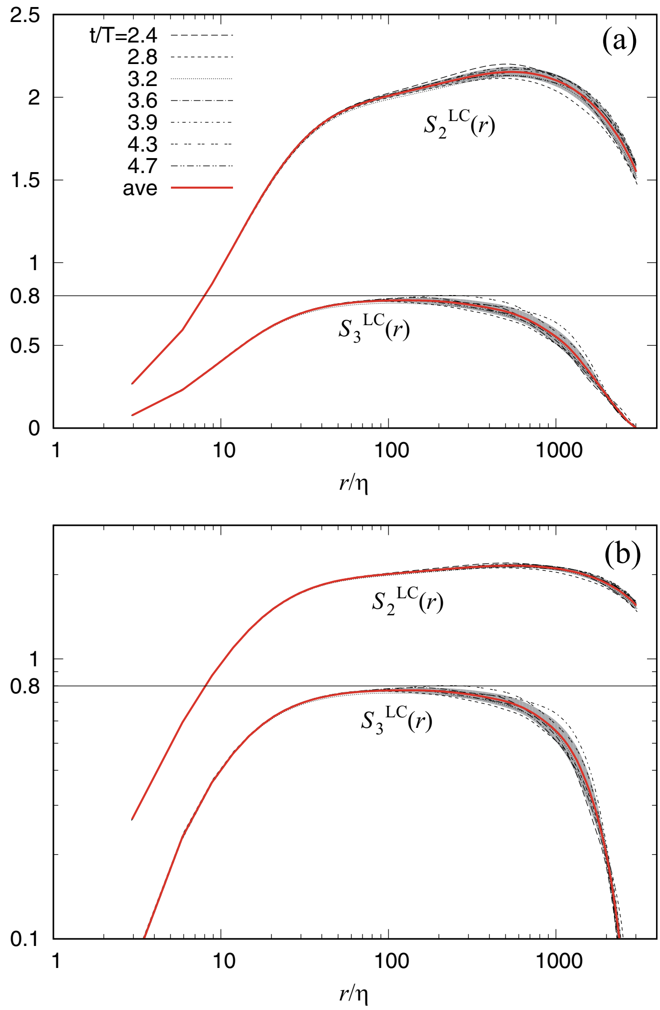

FIG. 1. $S_2^{\mathrm{LC}}(r)$ and $S_3^{\mathrm{LC}}(r)$ at $R_\lambda \sim 730$ in Run2048-1. The solid lines and shadow regions show the average over several snapshots, and the standard deviations, respectively. The broken lines show several instances of snapshot data. (a) Linear-log plot and (b) log-log plot.

following relation, which is valid for homogeneous isotropic turbulence:

$$S_2^L(r) = \frac{4}{3}\int_0^\infty E(k)H(kr)dk, \tag{5}$$

and compute $S_2^L(r)$ from the given data $E(k)$, where

$$H(\xi) = 1 + \frac{3\cos\xi}{\xi^2} - \frac{3\sin\xi}{\xi^3}.$$

In the numerical integration of Eq. (5) with respect to k for $\xi \equiv kr < 1$, $H(\xi)$ is approximated through the expansion $H(\xi) \sim \xi^2/10 - \xi^4/280 + \xi^6/15120$. The estimate by Eq. (5) was confirmed to agree well with the estimate without using $E(k)$ (see Fig. 5).

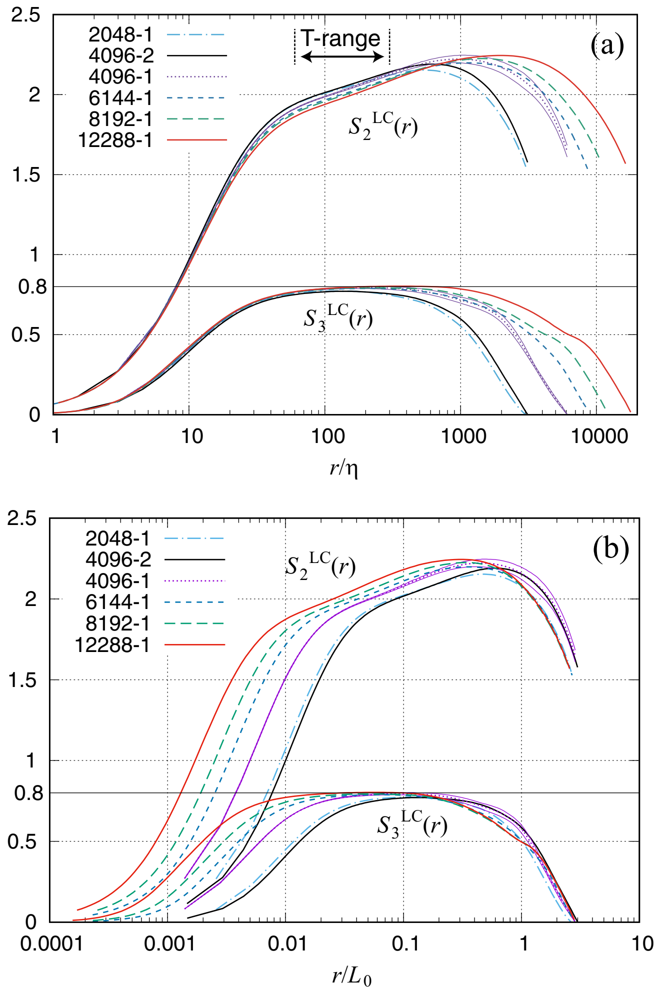

FIG. 2. (a) $S_2^{\mathrm{LC}}(r)$ and $S_3^{\mathrm{LC}}(r)$ as a function of r/η, and (b) the same as (a) but as a function of r/L_0. For "T-range" see the text.

A. Influence of simulation time and spatial resolution

Before proceeding to a detailed analysis of $S_2^L(r)$, it is worth having some idea regarding the potential influence of (i) the limitation of the simulation time and (ii) the spatial resolution in the DNS. In Ref. [21], it was shown that the influence of (i) and (ii) on $E(k)$ within the T-range is insignificant under appropriate conditions. Although it is natural to assume this is also the case for the influence on $S_2^L(r)$, because $S_2^L(r)$ is related to $E(k)$ as in Eq. (5), let us confirm this first, along the line of Ref. [21], as follows.

Figure 1 is provided to check the potential influence of the limitation of the simulation time, and shows the compensated structure functions $S_2^{\mathrm{LC}}(r)$ and $S_3^{\mathrm{LC}}(r)$ at several time steps in Run 2048-1, where $S_n^{\mathrm{LC}}(r) \equiv S_n^L(r)/(r\langle\epsilon\rangle)^{n/3}$. The functions averaged over the time steps for $t/T > 2.4$, as well as the corresponding standard deviations, are shown by the solid lines and shadowed regions, respectively. It can be seen that, although the instantaneous structure functions at different times are different to a certain extent at a large r/η range, the difference is not significant at smaller scales, for example, at $r/\eta < 500$, as can be expected from the results of Ref. [21].

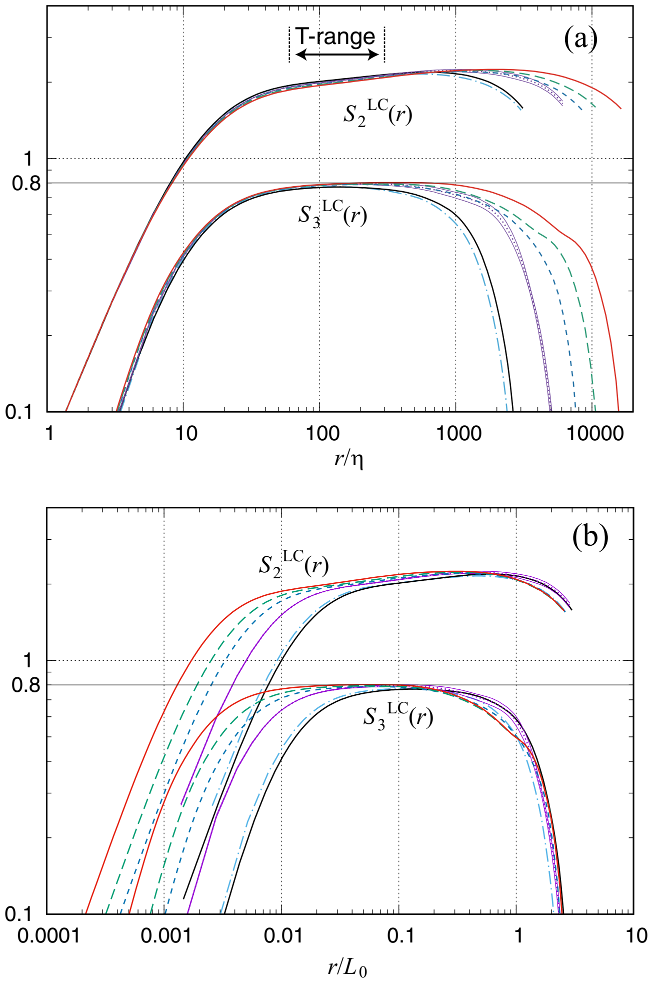

FIG. 3. The same as described in the caption of Fig. 2 but the vertical axis is in log-scale.

This suggests that the instantaneous structure functions $S_2^L(r)$ and $S_3^L(r)$ within the scale range are not significantly different from the time average over a certain time range. We use the value of the structure functions at the final time step in each run in the following discussion.

The influence of the resolution of the DNS was also confirmed to be insignificant within the scale range of $r/\eta < 500$, by comparing $S_2^{\mathrm{LC}}(r)$ in Run2048-1 and Run4096-2 (Figs. 2 and 3). Both runs are set at $R_\lambda \sim 700$, but with different $k_{\max}$, i.e., $k_{\max}\eta \sim 1$ in Run 2048-1 and ~ 2 in Run 4096-2. This suggests that the structure function within this range is insensitive to the difference between $k_{\max}\eta \sim 1$ and $k_{\max}\eta \sim 2$, as can be expected from the results of Ref. [21]. We use the DNS data with $k_{\max}\eta \sim 1$ in the following discussion of the range, unless otherwise stated.

B. Re-dependence of second- and third-order structure functions

Figures 2 and 3 show the compensated structure functions $S_2^{\mathrm{LC}}(r)$ and $S_3^{\mathrm{LC}}(r)$ for the six runs listed in Table I. It can be seen that, within a certain range, at $r/\eta \sim 100$ in Fig. 3(a), and $r/L_0 \sim 0.1$ in Fig. 3(b), $S_2^{\mathrm{LC}}(r)$ fits well the simple power-law scaling, and that, within the range, the curves are not horizontal but slightly tilted. The existence of such a tilt can be expected from Ref. [21],

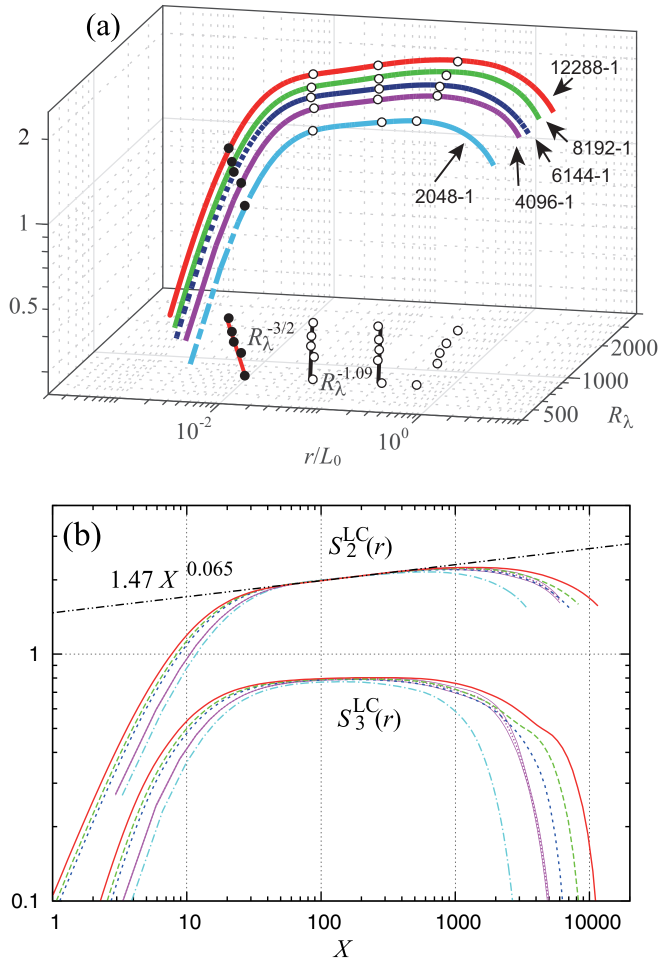

FIG. 4. (a) Overhead view of $S_2^{\rm LC}(r)$ as a function of r/L_0 and R_λ. (b) The same as described in the caption of Fig. 3 but as a function of $X = r/(L_0 R_\lambda^{-1.09})$.

which shows that the compensated spectrum $E(k)/[\langle\epsilon\rangle^{2/3}k^{-5/3}]$ is slightly tilted within the T-range. The T-range in the k-space is converted using $2r = \pi/k$ to approximately $60 < r/\eta < 300$ in the r-space, the result of which is indicated in Figs. 2(a) and 3(a). It can also be seen in Fig. 2 that the overlap of the curves within the tilted range (T-range) is not as good in Fig. 2(b) as in Fig. 2(a). Figure 2(b) suggests that $S_2^{\rm LC}(r)$ within the T-range depends not only on r/L_0 but also on R_λ.

Figure 4 a shows the dependence of $S_2^{\rm LC}(r)$ on R_λ and r/L_0. It suggests that the curves within the T-range are approximately on a single plane in the three-dimensional $[\log_{10} R_\lambda, \log_{10}(r/L_0), \log_{10} S_2^{\rm LC}(r)]$ space. The plane may be identified by following the method in Ref. [21], as follows. Let the cross section of this plane and the plane given by $S_2^{\rm LC}(r) = \gamma$ be expressed as

$$\log_{10}(r/L_0) = \alpha \log_{10} R_\lambda + \beta, \tag{6}$$

where α, β, and γ are constants. We can obtain the estimates of the constants α and β by first setting γ as $\gamma = 1.9, 1.92, 1.94, \ldots, 2.1$, such that we can identify five points (for the five runs in Table I) for each of the 11 values of γ, through which the five curves in Fig. 4(a) cross the plane $S_2^{\rm LC}(r) = \gamma$. We then obtain a set of estimates for α and β through the least squares fitting of Eq. (6)

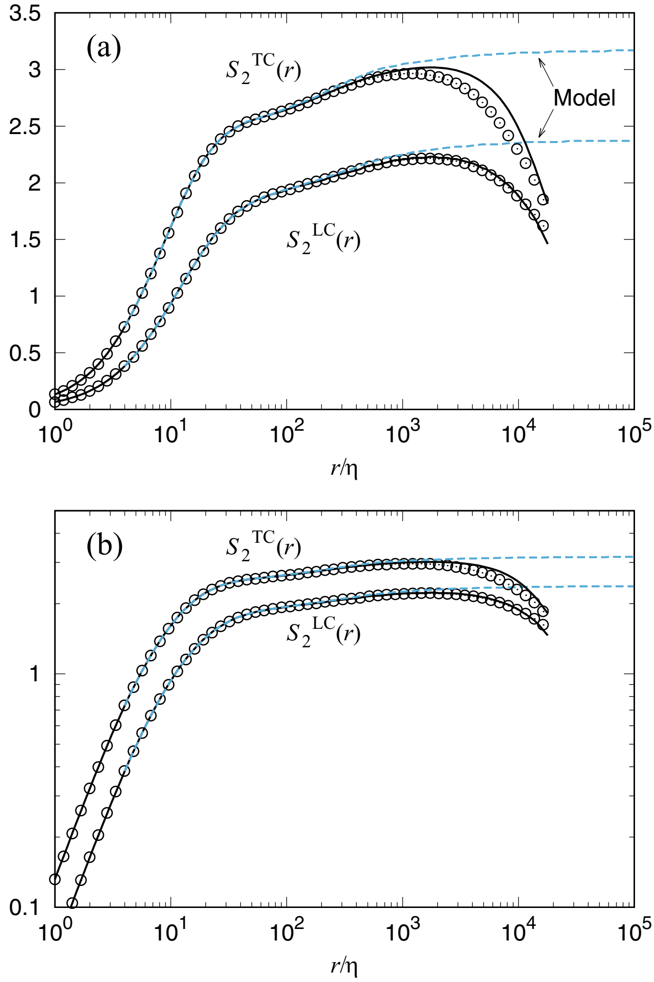

FIG. 5. K-DNS data of $S_2^{\mathrm{LC}}(r)$ and $S_2^{\mathrm{TC}}(r)$ by $E_{\mathrm{DNS}}(k)$ ($\circ$), and by taking the average over the directions of **e** (solid lines). The broken lines indicate model Eq. (10). (a) Linear-log plot and (b) log-log plot.

to the data of the five points. The 11 sets of (α, β) thus obtained give $\alpha = -1.09 \pm 0.07$. The scaling $r/L_0 \propto (R_\lambda)^\alpha$ with $\alpha = -1.09$ is shown with the solid lines on the $(R_\lambda, r/L_0)$ plane in Fig. 4(a).

The relation Eq. (6) implies that $S_2^{\mathrm{LC}}(r)$ within the T-range fits well a function of $X \equiv r/[L_0(R_\lambda)^\alpha]$. This is seen to be consistent with Fig. 4(b), where the plots $S_2^{\mathrm{LC}}(r)$ as a function of X with $\alpha = -1.09$ are seen to overlap well within a particular range, for example, $50 < X < 200$.

The least squares fitting of a straight line in Fig. 4(b) to the data of $S_2^{\mathrm{LC}}(r)$ within the range gives

$$\log_{10} S_2^{\mathrm{LC}}(r) = \zeta_2^L \log_{10} X + \beta', \tag{7}$$

where $\zeta_2^L = 0.065$ and $\beta' = 0.168$. The relation Eq. (7) is shown by the straight solid line in Fig. 4(b), where $10^{\beta'} = 1.47$ is used.

These results imply that $S_2^{\mathrm{LC}}(r)$ fits well with

$$S_2^{\mathrm{LC}}(r) = C_2^L (r/L_0)^{\zeta_2^L}, \quad C_2^L \equiv c(R_\lambda)^\delta, \tag{8}$$

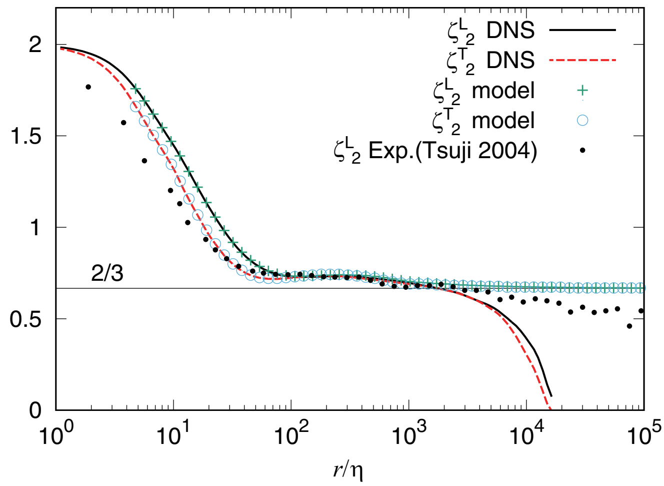

FIG. 6. ζ_2^L and ζ_2^T in Run12288-1, and by model Eq. (10). The dots show the experimental data of ζ_2^L by Tsuji [24].

within the above range, where $\zeta_2^L = 0.065$, $c = 10^{\beta'} = 1.47$, and $\delta = 1.09\zeta_2^L = 0.07$. The comparison of Eq. (8) with Eq. (3) shows that $S_2^L(r)$ within this range fits well with the form Eq. (3) for $n = 2$, with $\zeta_2^L = 0.065$. However, C_2^L is not constant, but depends on R_λ, i.e., ν.

To understand the r-dependence of $S_2^L(r)$ at a higher R_λ, it is instructive to recall the K-DNS data for $E(k)$. These data (Fig. 4(a) in Ref. [21]) suggest that, in a DNS with a high R_λ, a certain wave-number range (F-range in Ref. [21]) is observed next to the T-range (at the lower-wave-number side), in which $E^C(k) \equiv E(k)/\langle\epsilon\rangle^{2/3}k^{-5/3} \approx$ const ($\approx$1.8), i.e., $E(k)$ fits well with

$$E_F(k) \equiv K_o\langle\epsilon\rangle^{2/3}k^{-5/3}, \quad K_o \sim 1.8. \tag{9}$$

In addition, the width of the F-range extends toward the left, i.e., toward the lower wave-number range, with an increase in R_λ. In view of these observations, let us assume here a simple model of $E(k)$ at $R_\lambda \to \infty$:

$$E(k) = \begin{cases} E_{\mathrm{DNS}}(k), & \text{for } k\eta \geqslant 0.007, \\ E_F(k), & \text{for } k\eta < 0.007, \end{cases} \tag{10}$$

where $E_{\mathrm{DNS}}(k)$ is the spectrum $E(k)$ obtained by the highest resolution in K-DNS, i.e., Run12288-1 ($R_\lambda \sim 2250$). Figure 5 shows $S_2^{\mathrm{LC}}(r)$ obtained from Eqs. (10) and (5). Two kinds of K-DNS data $S_2^{\mathrm{LC}}(r)$ in Run12288-1 are also included in the figure; one is by $E_{\mathrm{DNS}}(k)$ and Eq. (5), the other is by taking the average over three directions of $\mathbf{e}$ in Eq. (2) perpendicular to each other, but without using $E_{\mathrm{DNS}}(k)$ or Eq. (5). The two curves can be seen to agree well with each other.

Figure 6 shows the exponent $\zeta_2^L(r) = d[\log S_2^L(r)]/\,d(\log r)$ in Run12288-1, and ζ_2^L through model Eqs. (10) and (5). It can be seen that ζ_2^L is different from 2/3 at $r/\eta \lesssim 400$, but close to 2/3 at $r/\eta \gtrsim 1000$. This suggests that the exponent at $r/\eta \lesssim 400$ should not be confused with that at $r/\eta \gtrsim 1000$. In other words, $R_\lambda \sim 2250$ and $r/\eta \lesssim 400$ may be too small for an examination of the "ISR" scaling. Figure 6 also includes the experimental data of ζ_2^L at $R_\lambda \sim 17060$ by Tsuji [24]. The model curve of ζ_2^L is seen to be close to the experimental data at large values of r.

Model Eq. (10) may also be used to obtain an estimate of the transverse structure function $S_2^T(r) \equiv \langle(\Delta u_r^T)^2\rangle$, where Δu_r^T is given by the right-hand-side of Eq. (2), but $\mathbf{e}$ is to be understood as an arbitrary unit vector perpendicular to $\mathbf{r}$. Here, $S_2^T(r)$ can be related to $E(k)$ through a simple relation similar to Eq. (5). In addition, $S_2^{\mathrm{TC}}(r) \equiv S_2^T(r)/(r\langle\epsilon\rangle)^{2/3}$ estimated by the model along with S_2^{TC} by DNS are included in Fig. 5. Figure 6 shows, in addition to $\zeta_2^L(r)$, the scaling exponent $\zeta_2^T(r) = d[\log S_2^T(r)]/d(\log r)$ estimated using model Eq. (10), as well as an estimate for Run12288-1 (without using the model). As in the case of $\zeta_2^L(r)$, it can be seen that the exponent $\zeta_2^T(r)$ at $r/\eta \lesssim 400$ should not be confused with those at $r/\eta \gtrsim 1000$.

III. COMPARISON WITH MODELS

A. Models allowing Re-dependent C_2^L

According to K41, the longitudinal structure function $S_2^L(r)$ for $r \ll L_0$ is given by

$$S_2^L(r) = (\nu\langle\epsilon\rangle)^{1/2} B\left(\frac{r}{\eta}\right), \tag{11}$$

where $B(\xi)$ is a universal function of only $\xi \equiv r/\eta$, independent of Re, and must satisfy

$$B(\xi) \to C_2^L \xi^{2/3}, \quad \text{as} \quad \xi \to \infty, \tag{12}$$

in which C_2^L is a universal constant independent of Re. As Re $\to \infty$, $\eta \to 0$, so that Eqs. (11) and (12) yield Eq. (1) with $n = 2$, i.e.,

$$S_2^L(r) = C_2^L \langle\epsilon\rangle^{2/3} r^{\chi}, \quad \chi = 2/3, \tag{13}$$

in this limit.

Barenblatt and Chorin proposed the so-called incomplete similarity hypothesis, according to which C_2^L and χ in Eq. (13) may be Re-dependent, in contrast to K41 (cf. Refs. [25,26], and references cited therein). They argued that C_2^L and χ are given by

$$C_2^L = C_2^L(\mathrm{Re}) = A_0 + A_1 \frac{1}{\ln \mathrm{Re}} + o\left(\frac{1}{\ln \mathrm{Re}}\right), \tag{14}$$

$$\chi = \chi(\mathrm{Re}) = \frac{2}{3} + \alpha_1 \frac{1}{\ln \mathrm{Re}} + o\left(\frac{1}{\ln \mathrm{Re}}\right), \tag{15}$$

respectively, for large Re, where A_0, A_1, and α_1 are nondimensional constants independent of Re. Equation (14) implies that as Re $\to \infty$, the prefactor $C_2^L \to$ constant($=A_0$), in contrast to C_2^L in Eq. (8). Thus, Eq. (8) is not explained by the theory that gives Eqs. (14) and (15).

Barenblatt *et al.* [27] noted "the prefactor $\cdots$ is also Re-dependent, as has indeed been observed experimentally," and referred Refs. [28,29]. In Ref. [28], Praskovsky and Oncley reported that no measurable deviation from the $-5/3$ exponent in Eq. (4) with $\mu_2 = 0$ is found, while the prefactor, i.e., the Kolmogorov constant K_2 is weakly dependent on Re and the dependence is well described by $K_2 \propto R_\lambda^{-0.20}$. In Ref. [29], after surveying experimental results, Sreenivasan concluded as "the Kolmogorov constant is more or less universal, essentially independent of the flow as well as the Reynolds number." This conclusion for one-dimensional energy spectrum is also applicable to the three-dimensional one because of a simple relationship between the two spectra. These results are different from Eq. (8) in which the prefactor C_2^L increases with Re.

Grossmann [30] considered $D \equiv \langle \delta\mathbf{u} \cdot \delta\mathbf{u} \rangle$ [$\delta\mathbf{u} \equiv \mathbf{u}(\mathbf{x}+\mathbf{r}) - \mathbf{u}(\mathbf{x})$] instead of $S_2^L(r)$. In generalizing the ideas originally propose by Lohse [31], he used

$$D = D(r) = (\nu\langle\epsilon\rangle)^{1/2} B_D\left(\frac{r}{\eta}\right), \tag{16}$$

and analyzed the relationship between Eq. (16) and the expression for the normalized energy dissipation rate

$$C_\epsilon \equiv \frac{\langle\epsilon\rangle}{U^3/L_0}, \tag{17}$$

where $B_D(\xi)$ is given by

$$B_D(\xi) = \frac{1}{3}\frac{\xi^2}{[1+(\xi/a)^2]^z}, \tag{18}$$

$a = a(\mathrm{Re})$ may depend on Re, and the exponent z need not be $2/3$, in contrast to Batchelor's model [32] in which B in Eq. (11) is given by

$$B(\xi) = \frac{1}{15}\frac{\xi^2}{[1+(\xi/a)^2]^z}, \tag{19}$$

where

$$a = \left(15C_2^L\right)^{3/4}, \quad z = 2/3 \tag{20}$$

are independent of Re.

In homogenous isotropic turbulence D depends on $\mathbf{r}$ only through r, and given by

$$D(r) = S_2^L(r) + 2S_2^T(r) = 3S_2^L(r) + r\frac{d}{dr}S_2^L(r). \tag{21}$$

Hence, if S_2^L has a power law dependence $\propto r^\chi$, then we have from Eq. (21)

$$D(r) = (3+\chi)S_2^L(r). \tag{22}$$

Equations (16), (18), and (22) imply that

$$\frac{D(r)}{\langle\epsilon\rangle^{2/3}r^{2/3}} = (5-2z)\frac{S_2^L(r)}{\langle\epsilon\rangle^{2/3}r^{2/3}} \sim C_D\left(\frac{r}{L_0}\right)^{4/3-2z}, \quad \text{at} \quad r/\eta \gg 1, \tag{23}$$

where

$$C_D = \frac{1}{3}a^{2z}\left(\frac{L_0}{\eta}\right)^{4/3-2z} \sim \text{const.} \times (R_\lambda)^g, \quad g \equiv 2z\beta + y(4/3-2z), \tag{24}$$

and we have put

$$a \sim (R_\lambda)^\beta, \tag{25}$$

$L_0/\eta \propto (R_\lambda)^y$, and used $(\nu\langle\epsilon\rangle)^{1/2}\xi^{2/3} = \langle\epsilon\rangle^{2/3}r^{2/3}$.

Note that the model Eq. (16) with Eq. (18) contains two parameters z and a, or equivalently two parameters z and β, if we assume Eq. (25). It is therefore trivial that the scaling exponents implied by Eq. (23) agree with those of Eq. (8) for certain appropriate values of z and β, i.e., if the exponents z and β are so chosen that

$$4/3 - 2z = \zeta_2^L, \quad 2z\beta + y(4/3-2z) = \delta, \tag{26}$$

for any given set of ζ_2^L and δ, then the scaling exponents given by Eq. (23) agree with those of Eq. (8).

However, assuming the model Eq. (16) with Eq. (18) alone without any other constraint is not sufficient to determine the exponent z and the parameter a or β in Eqs. (23) and (25). The model alone allows not only the exponents that are consistent with Eq. (8), i.e., Eq. (26) but also any other arbitrary exponents. Grossmann noted "Data anaysis favors $\kappa = 0$" (where κ is the exponent given by $C_\epsilon \propto \mathrm{Re}^{-\kappa}$), and showed that if $\kappa = 0$ there are only two cases (i) $\zeta_2^L = \delta = 0$, and (ii) $\zeta_2^L \neq 0$, and a depends on Re appropriately. In either case, C_D is Re-independent (Refs. [30,31,33]). Thus, Eq. (8) is not explained by the theory.

As regards the exponent z, in view of the fact that in Grossmann's analysis, the assumptions (i) Eq. (16) with Eq. (18) is applicable at $r = L_0$, and (ii) $D_\infty \equiv D(L_0)/U^2$ is independent of Re, play a key role, it looks natural to impose the constraint that the exponent z must be so chosen that Eq. (16) with Eq. (18) fits well to the data of $D(r)$ at $r \sim L_0$. However, it is not surprising that the statistics

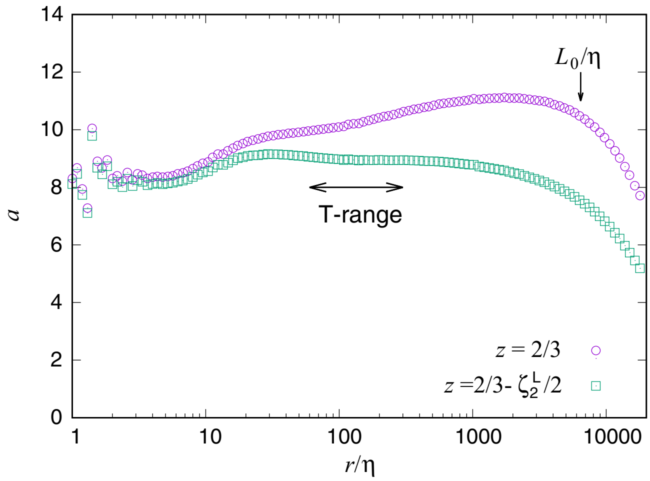

FIG. 7. DNS values of "a" given by Eq. (27) in Run 12288-1 as a function of r/η, with $4/3 - 2z = 2/3$ or $4/3 - 2z = \zeta_2^L = 0.065$. The vertical arrow shows the position of r at $r = L_0$.

in the energy containing range ($\sim L_0$) and in the ISR or T-range are different from each other. As a matter of fact, as seen in Fig. 3, the slope of $S_2^L(r)$ at $r \sim L_0$ is different from the slope in the T-range. Therefore, the exponent z thus obtained by the fitting at $r \sim L_0$ would be different from the exponent given by the T-range, i.e., $4/3 - 2z = \zeta_2^L = 0.065$ in Eq. (8).

As regards $a = a(\mathrm{Re})$, similarly to z, it looks natural to impose the constraint that the crossover a must be so chosen that Eq. (16) with Eq. (18) fits well to the data of $D(r)$ at $r \sim L_0$. To get some idea on this constraint, note that Eq. (16) under Eq. (18) is equivalent to

$$a = \frac{\xi}{[\Delta(r)\xi^2]^{1/z} - 1}, \quad \left(\Delta(r) \equiv \frac{(\nu\langle\epsilon\rangle)^{1/2}}{3D(r)}\right). \tag{27}$$

Figure 7 plots the DNS data of a given by Eq. (27) vs. $\xi = r/\eta$. While a is assumed to be independent of r in Eq. (18), it is seen in Fig. 7 that a given by Eq. (27) is not r-independent, and a at $r \sim L_0$ is different from a in the T-range. The exponent δ thus obtained by the fitting at $r \sim L_0$ would be different from the exponent given by the T-range, i.e., Eq. (26).

It may be worthwhile to note that in Grossmann's model, $D(r)$ can depend on Re only through one parameter, i.e., the crossover a, while an analysis of DNS data of the energy spectrum $E(k)$ at near dissipation range $0.5 < k\eta < 1.5$ at R_λ up to 615 suggested that $E(k)$ in the range is well characterized by three parameters, but their Re-dependencies are different from each other, and the approach of one of the parameters to a constant at $\mathrm{Re} \to \infty$ is very slow [34]. This suggests it not surprising if the Re-dependence of D in the entire range of r and/or that of C_ϵ are/is not well represented by only one parameter a, even if Eq. (16) with Eq. (18) is a good approximation in a certain sense or range.

B. Models of intermittency in the inertial subrange

There are theories of intermittency so far proposed, including Kolmogorov's RSH and various fractal models, which give estimates of the intermittency exponent $\zeta_n^{L,T}$ in Eq. (3) for $n = 1, 2, 3, \cdots$. In these theories, it is assumed that there is a certain range of r such that $L_0 \gg r \gg \eta$ and $S_n^L(r)$ $(n = 1, 2, 3, \cdots)$ fit well to the form Eq. (3) in the range, where ζ_n^L and C_n are independent of r, in the limit $r/L_0, \eta/r, 1/\mathrm{Re} \to 0$. We call here such theories as "theories of intermittency in the ISR" or shortly ISR-theories, and the range where the theories are supposed to be applicable as "inertial subrange (ISR)." In these theories, C_n may depend on the large-scale flow conditions, in contrast to K41, but C_n are determined only by the large-scale flow statistics, so that C_n are independent of R_λ, i.e., the viscosity ν.

The analysis in this study shows that $S_2^{L,T}(r)$ fits well to a simple scaling law with a constant exponent $\zeta_2^{L,T}$ in a range called T-range. The exponent at high R_λ is almost independent of R_λ. In this sense, the DNS results for the T-range appear to be consistent with the ISR theories. However, as seen in Eq. (8), the prefactor C_2 depends on R_λ, i.e., on ν in the T-range. According to Eq. (8), $C_2^L \propto (R_\lambda)^\delta$ with $\delta > 0$, which implies $C_2^L \to \infty$ as Re $\to \infty$. To the authors' knowledge, none of the ISR-theories so far proposed accounts for such a dependence of C_2 on R_λ, or takes in to account of the ν-dependence of the "ISR" statistics. It is not clear whether the theories are compatible with the ν-dependence in the "ISR," and if they are, how to account the ν-dependence.

It is questionable that the ISR-theories for the ISR are compatible with such a ν-dependence or divergence of C_2 at Re $\to \infty$. It is therefore also questionable that the T-range is the ISR to which the theories are supposed to be applied. If the ν-dependence is incompatible with the ISR-theories proposed for the ISR, then one may ask if it would make sense to assess the theories by comparison of the exponents by the ISR-theories for the ISR with those by the data for the T-range, just like one may ask if it would make sense to estimate the constants K_2 and μ_2 in Eq. (4) by the data for the range at $k\eta \sim 0.2$ (the so-called bump range).

Note that in a strict sense, the flows at large-scales are not exactly the same for different runs in the series of DNSs in the present study. It is therefore not surprising if C_2 is different for different runs. However, the boundary conditions and the method of forcing are common in the runs, and the turbulence characteristics such as $\langle \epsilon \rangle$ and L_0 listed in Table I are not so much different for different runs. It is unlikely that the simple power-law dependence of C_n on R_λ as seen in Eq. (8) can be attributed to the difference of the large-scale flow statistics for different runs in the series of our DNSs, within the framework of the ISR-theories.

IV. RESULTS AND DISCUSSIONS

It is seen in Fig. 4 that S_2^L appears to have a neat scaling range (T-range) with $r/\eta \sim 400$, and the exponent ζ_2^L at $R_\lambda \sim 732$ (in Run 2048-1) is not so much different from ζ_2^L at larger $R_\lambda \sim 2250$ (in Run 12288-1). From this, one might think that the values of $r/\eta \sim 400$ and $R_\lambda \sim 1000$ or 2000 are sufficiently large for us to have reasonable estimates of the exponents ζ_n^L in the ISR, or that the difference of the estimate at higher Re is not significant, so that it is only a matter of precision. However, the question raised by Fig. 4 is on whether the T-range (i.e., the r-range such as $r/\eta \sim 400$) is really the "ISR," the understanding of whose statistics is one of the main objectives of various ISR theories, rather than the question on the precision of the estimate of ζ_2^L.

If the T-range is not the "ISR" and there is a "true" ISR somewhere else, then in view of the fact that the influence of intermittency on $S_n(r)$ is in general stronger for larger n, it is not surprising that the scaling exponents $\zeta_n^{L,T}$ for large n in the "false" ISR are significantly different from those in the "true" ISR, even if the difference between ζ_2's in the two ranges is small for $n = 2$.

Regarding the existence of the "ISR," it may be worthwhile to note that as seen in Figs. 5 and 6, a model based on DNS-data [see the discussion in the paragraph including Eq. (9)] at high R_λ, as well as experiment suggest the possibility of the existence of a scaling range at $r/\eta > 5 \times 10^3$, where the exponents $\zeta_2^{L,T}$ are different from those in the T-range. One may ask whether there is any scaling range where C_n and the exponents $\zeta_n^{L,T}$ are ν-independent, and if such a range exists, one may also ask whether the exponents in the range are the same as or different from those in the T-range. Experiments and/or DNSs with high Re in future are hoped to give some idea on these questions.

In conclusion, the results of this study show that even if there is a certain r-range where the structure functions show simple scalings, the range is not the "ISR" in the sense that the statistics are independent of R_λ or the viscosity, and that $R_\lambda \sim 2000$ and $r/\eta \sim 400$ may be still too small to examine or to assess the theories of "ISR." In this respect, it might be worthwhile to note that $R_\lambda \sim 2250$ in Run 12288-1 is considerably larger than $R_\lambda \sim 852$ which is the largest value in experiments

[6] used in the assessment of intermittency models in the textbooks by Frisch [4] and Pope [35]. In view of Fig. 6, it looks that for the examination of the theories, r/η need be at least 1000 or so. In this sense, DNSs and/or experiments with much larger R_λ and r/η are waited.

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