Inertial/kinetic-Alfvén wave turbulence: A twin problem in the limit of local interactions

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Inertial and kinetic-Alfvén wave turbulences have *a priori* little in common: indeed, the first one concerns rotating hydrodynamics in the limit of a small Rossby number (with $\mathbf{\Omega}_0$ the rotating rate) while the second describes high frequency plasmas in the limit of a strong uniform magnetic field $\mathbf{B_0}$. In this paper we show analytically that, in the limit of local interactions in the perpendicular direction to $\mathbf{\Omega}_0$, the inertial wave turbulence equation converges towards the same nonlinear diffusion equation as for kinetic-Alfvén waves when the same limit is taken; the only difference resides in the constants in front of the equations. Therefore, both systems share the same physical properties for the stationary phase with an energy spectrum in $k_\perp^{-5/2}$; it is preceded by a self-similar solution of the second kind during the nonstationary phase with a spectrum proportional to $k_\perp^{-8/3}$ which propagates explosively towards small scales. It is suggested that the proximity between these two problems may be used to better understand inertial or kinetic-Alfvén wave turbulence.

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I. INTRODUCTION

Turbulence under rotation is a relevant regime for many geophysical and astrophysical flows, including gaseous planets. The effect of the Coriolis force is considered to become important when the dimensionless Rossby number, R_o, the ratio of the convective to Coriolis forces, is sufficiently small ($R_o < 1$). For example, for large-scale planetary flows we find $R_o \simeq 0.05$–0.2. The Reynolds number R_e, the ratio of the convective to the viscous terms, in these systems is, in general, very large with values $R_e \gg 10^3$. Therefore, these flows can be considered in a regime of fully developed turbulence.

Many laboratory experiments have been devoted to the study of hydrodynamic turbulence under rotation. From an experimental point of view, it is not so difficult to reach a small Rossby number ($R_o \leqslant 1$) in a rapidly rotating tank but it is more difficult to achieve a Reynolds number greater than 10^5. This number is, however, sufficiently high so that the flow is in a state of fully developed turbulence. With these experiments (e.g., in a wind tunnel) it has been possible to show that, when $R_o < 1$, rotation can bidimensionalize a turbulence initially isotropic [1,2]. This leads to vortices having their axes approximately parallel to the rotation axis $\mathbf{\Omega}_0$ and to a strong correlation of the velocity in the $\mathbf{\Omega}_0$-direction [3]. Asymmetry is also found in the distribution of cyclones versus anticyclones (the first dominating the second) as well as a slowdown (compared to the case without rotation) of the freely energy decay [4]. Thanks to the particle image velocimetry (PIV) technique, it is possible to accurately measure the velocity of the particles introduced and therefore that of the fluid whose dynamics is supposed not to be influenced by the presence of these intruders. A

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restriction exists, however, because the measure is made only in a plane. It is generally the plane perpendicular to the axis of rotation that is chosen. The energy spectra measured as a function of the perpendicular wave number $k_\perp$ show a stiffening of the power law, going from an index around $-5/3$ (for $R_o = +\infty$) to a value close to -2.2 for the fastest rotation [5]. The experiment also shows that the structure functions of order p follow a self-similar law with $\xi_p = p/2$ [6] or $\xi_p = 3p/4$ [7]. Even if the origin of this self-similar behavior can be attributed to inhomogeneity, measurement, or forcing effects, it is interesting to mention that the $\xi_p = 3p/4$ law is dimensionally compatible with the exact solution of inertial wave turbulence [8] (see below). The PIV technique has also made it possible to find evidence of the inertial wave turbulence regime by measuring the frequency–wave-number spectrum. This spectrum is characterized by a signal essentially concentrated along the dispersion relation of inertial waves [9,10]. This property is the one expected when nonlinearities are weak [11]. Finally, the experiments showed the presence in the perpendicular direction only of an inverse cascade towards large scale [12] whose properties, however, seem to be different from those of a purely two-dimensional turbulence [13].

The effects of rotation on hydrodynamic turbulence were also studied through numerical simulations. A reduction of the nonlinear transfer in the direction of the rotation rate $\mathbf{\Omega}_0$ was observed as well as a slowing down of the energy decay [14,15]. A stiffening of the power law followed by the energy spectrum was measured as well as signatures of an inverse cascade [16,17]. In particular, Smith and Waleffe [18] showed with direct numerical simulations that when the flow is forced only three-dimensionally at an intermediate wave number k_i, we observe a direct cascade of energy for $k > k_i$, with a one-dimensional (1D) isotropic spectrum close to k^{-2}, and an inverse cascade for $k < k_i$, with an isotropic 1D spectrum close to k^{-3}. Their analysis shows that the energy at large scale is mainly contained into the two-dimensional (2D) state, which is defined as the fluctuations in the mode $k_\parallel = 0$ (with $\mathbf{k} \cdot \mathbf{\Omega_0} = k_\parallel \Omega_0$), while at small scale energy is mainly contained into three-dimensional (3D) modes ($k_\parallel \neq 0$). The observed 2D spectrum could be the result of nonlocal interactions between 2D and 3D modes, rather than the consequence of a 2D inverse cascade [19]. Furthermore, these simulations show that the behavior at small and large scales is strongly influenced by the aspect ratio between the vertical resolution, along $\mathbf{\Omega}_0$, and the horizontal one: a small aspect ratio, with a low resolution in the vertical direction, leads to a reduction in the number of triads and a significant alteration of the energy spectrum. Their simulations show an energy spectrum globally in $k^{-5/3}$ for a sufficiently small aspect ratio. This result suggests that the resonant triads have a fundamental role to play in rotating turbulence.

From a theoretical point of view, rotating turbulence can be understood via the phenomenology. The first works on the subject [20,21] show that a steep spectrum in k^{-2} is expected at large scales, where the Coriolis force is important, that is, for wave numbers smaller than a critical wave number k_Ω. It can be noted, however, that this phenomenological prediction does not include anisotropy, a fundamental property of this turbulence. In the field of spectral theory, the most important advances have been realized by means of closure methods like eddy-damped quasi-normal Markovian (EDQNM) [22]. Cambon and Jacquin [23] developed a formalism based on a decomposition in eigenmodes. The ad hoc closure used leads to the spectral equations of the system. The simulation of these equations made it possible to understand more precisely some of the properties observed as the anisotropic transfer mechanism (see also [15]). More recently, inhomogeneous effects were introduced to take into account the existence of infinitely large boundaries in the direction perpendicular to the axis of rotation [24]. The confined fluid under rapid rotation then behaves differently from the free one with essentially a wall dissipation effect which dynamically emerges before the classical volume dissipation [25].

In this paper, we first recall the main properties of inertial wave turbulence with a brief derivation of the kinetic equation (Sec. II). In Sec. III, we take the limit of local interactions for wave numbers perpendicular to the rotating axis and derive the corresponding nonlinear diffusion equation. Their solutions are discussed in Sec. IV while we present a numerical simulation in Sec. V. Finally, a discussion is developed around the proximity of this problem with kinetic-Alfvén (or oblique whistler) wave turbulence, an interesting regime for understanding multiscale solar wind turbulence [26], for which the same nonlinear diffusion equation can be found [27,28].

II. INERTIAL WAVE TURBULENCE THEORY

The theory of inertial wave turbulence was developed by Galtier [8]. In this section, we summarize the main steps and properties found. The basic equation from which the theory is developed is

$$\frac{\partial \mathbf{w}}{\partial t} - 2\,(\mathbf{\Omega}_0 \cdot \nabla)\,\mathbf{u} = (\mathbf{w} \cdot \nabla)\,\mathbf{u} - (\mathbf{u} \cdot \nabla)\,\mathbf{w} + \nu\,\nabla^2\,\mathbf{w}, \tag{1}$$

with $\mathbf{u}$ the velocity, $\mathbf{w}$ the vorticity, $\mathbf{\Omega}_0 \equiv \Omega_0 \hat{\mathbf{e}}_\parallel$, $\hat{\mathbf{e}}_\parallel$ a unit vector ($|\hat{\mathbf{e}}_\parallel| = 1$), and ν the kinematic viscosity. We introduce the helicity basis

$$\mathbf{h}^s_{\mathbf{k}} \equiv \mathbf{h}^s(\mathbf{k}) = (\hat{\mathbf{e}}_k \times \hat{\mathbf{e}}_\parallel) \times \hat{\mathbf{e}}_k + is(\hat{\mathbf{e}}_k \times \hat{\mathbf{e}}_\parallel), \tag{2}$$

with $\mathbf{k} = k\hat{\mathbf{e}}_k = \mathbf{k}_\perp + k_\parallel \hat{\mathbf{e}}_\parallel$ ($k = |\mathbf{k}|$, $k_\perp = |\mathbf{k}_\perp|$, $|\hat{\mathbf{e}}_k| = 1$), and $s = \pm$ the directional polarity. This basis satisfies the following properties:

$$\mathbf{h}^{-s}_{\mathbf{k}} = \mathbf{h}^s_{-\mathbf{k}}, \tag{3}$$

$$is\big(\hat{\mathbf{e}}_k \times \mathbf{h}^s_{\mathbf{k}}\big) = \mathbf{h}^s_{\mathbf{k}}, \tag{4}$$

$$\mathbf{k} \cdot \mathbf{h}^s_{\mathbf{k}} = 0, \tag{5}$$

$$\mathbf{h}^s_{\mathbf{k}} \cdot \mathbf{h}^{s'}_{\mathbf{k}} = \frac{2k_\perp^2}{k^2}\delta(s+s'). \tag{6}$$

The projection of the Fourier transform of the velocity $\mathbf{u}_k$ on this helical basis leads to the following definition for A^s_k

$$\mathbf{u}_k \equiv \mathbf{u}(\mathbf{k}) = \sum_s A^s(\mathbf{k})\,\mathbf{h}^s_{\mathbf{k}} \equiv \sum_s A^s_k\,\mathbf{h}^s_{\mathbf{k}}. \tag{7}$$

We also find

$$\mathbf{w}_k \equiv \mathbf{w}(\mathbf{k}) = i\mathbf{k} \times \mathbf{u}_k = k \sum_s s\,A^s_k\,\mathbf{h}^s_{\mathbf{k}}. \tag{8}$$

The Fourier transform of Eq. (1) gives in the inviscid case ($\nu = 0$)

$$\frac{\partial \mathbf{w}_k}{\partial t} - 2i\Omega_0 k_\parallel \mathbf{u}_k = [(\mathbf{w} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{w}]_{\mathbf{k}}, \tag{9}$$

where the index $\mathbf{k}$ in the right-hand side (RHS) means the Fourier transform.

In the linear case, if we introduce Eqs. (7) and (8), we obtain after projection

$$sk\frac{\partial A^s_k}{\partial t} = -2i\Omega_0 k_\parallel A^s_k\,; \tag{10}$$

hence the dispersion relation

$$\omega_k = \frac{2\Omega_0 k_\parallel}{k}. \tag{11}$$

The inertial waves are transverse, dispersive, helical waves with a left polarization [29].

In the nonlinear case, the introduction of Eqs. (7) and (8) leads to the following expression (after some manipulations):

$$\frac{\partial A_k^s}{\partial t} + is\omega_k A_k^s = i \sum_{s_p s_q} \int L_{-\mathbf{k}\,\mathbf{p}\,\mathbf{q}}^{s s_p s_q} A_p^{s_p} A_q^{s_q}\, \delta(\mathbf{k}-\mathbf{p}-\mathbf{q})\, d\mathbf{p} d\mathbf{q}, \tag{12}$$

with the interacting coefficient

$$L_{\mathbf{k}\,\mathbf{p}\,\mathbf{q}}^{s s_p s_q} = \frac{sk}{4k_\perp^2}(s_p p - s_q q)\big[\big(\mathbf{q}\cdot\mathbf{h}_\mathbf{p}^{\mathbf{s_p}}\big)\big(\mathbf{h}_\mathbf{q}^{\mathbf{s_q}}\cdot\mathbf{h}_\mathbf{k}^{\mathbf{s}}\big) - \big(\mathbf{p}\cdot\mathbf{h}_\mathbf{q}^{\mathbf{s_q}}\big)\big(\mathbf{h}_\mathbf{p}^{\mathbf{s_p}}\cdot\mathbf{h}_\mathbf{k}^{\mathbf{s}}\big)\big]. \tag{13}$$

The wave amplitude being assumed small, the dynamics over a short timescale—of the order of the wave period $1/\omega$—will be given by the linear terms. Over a large timescale τ—such that $\tau \gg 1/\omega$—the nonlinear terms will be nonnegligible and will modify the wave amplitude. Therefore, it is relevant to separate the amplitude from the phase and to introduce a small parameter $0 < \epsilon \ll 1$ such that by definition

$$A_k^s \equiv \epsilon a_k^s e^{-is\omega_k t}. \tag{14}$$

Hence the wave amplitude equation

$$\frac{\partial a_k^s}{\partial t} = i\epsilon \sum_{s_p s_q} \int L_{-\mathbf{k}\,\mathbf{p}\,\mathbf{q}}^{s s_p s_q} a_\mathbf{p}^{s_p} a_\mathbf{q}^{s_q} e^{i(s\omega_k - s_p\omega_p - s_q\omega_q)t}\, \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) d\mathbf{p} d\mathbf{q}. \tag{15}$$

We find a classical form for three-wave interactions with a term in the RHS of weak amplitude, a quadratic nonlinearity, and an exponential function which will give a nonzero contribution only when its coefficient cancels (resonance condition). After some manipulation, we can write the resonance condition in the following way:

$$\frac{s_q q - s_p p}{s\omega_k} = \frac{sk - s_q q}{s_p\omega_p} = \frac{s_p p - sk}{s_q\omega_q}. \tag{16}$$

It is interesting to discuss about the particular case of strongly local interactions which often give the main contribution to the nonlinear dynamics. In this case, we have $k \simeq p \simeq q$ and the previous expression simplifies to give at the main order

$$\frac{s_q - s_p}{sk_\parallel} \simeq \frac{s - s_q}{s_p p_\parallel} \simeq \frac{s_p - s}{s_q q_\parallel}. \tag{17}$$

If $k_\parallel$ is non null, the term in the left will give a nonnegligible contribution only when $s_p = -s_q$. [We do not consider the case $s_p = s_q$ which is not relevant at the main order in the case of local interactions as we can see on expression (13) that becomes negligible.] The immediate consequence is that either the term in the middle or the term in the right cancels its numerator (at main order), which implies that the corresponding denominator must also cancel (at main order): for example, if $s = s_p$ then $q_\parallel \simeq 0$. This condition means that the transfer in the parallel direction is negligible: indeed, the integration in the parallel direction of Eq. (15) reduces to only a few modes which limits strongly the transfer between parallel modes. The cascade along the parallel direction is then possible but relatively weak compared to the one in the perpendicular direction. This situation is close to magnetohydrodynamic turbulence where the parallel transfer along the external uniform magnetic field $\mathbf{B_0}$ is completely frozen [30]. It is even completely similar to the case of kinetic-Alfven (or whistler) wave turbulence [31,32] where a weak cascade is allowed along $\mathbf{B_0}$ (see discussion below).

The kinetic equations of inertial wave turbulence can be derived in a classical way. They simplify the party when the limit $k_\perp \gg k_\parallel$ is taken: this limit is justified by the arguments given above about the asymmetry in terms of cascade. Then it is possible to show that [8]

$$\partial_t \begin{Bmatrix} E_k \\ H_k \end{Bmatrix} = \frac{\epsilon^2 \Omega_0^2}{4} \sum_{s s_p s_q} \int_{\Delta_\perp} \frac{s k_\parallel s_p p_\parallel}{k_\perp^2 p_\perp^2 q_\perp^2} \left(\frac{s_q q_\perp - s_p p_\perp}{\omega_k} \right)^2$$

$$\times (s k_\perp + s_p p_\perp + s_q q_\perp)^2 \sin\theta \, \delta(s\omega_k + s_p \omega_p + s_q \omega_q)$$

$$\begin{Bmatrix} E_q(p_\perp E_k - k_\perp E_p) + (p_\perp s H_k / k_\perp - k_\perp s_p H_p / p_\perp) s_q H_q / q_\perp \\ s k_\perp [E_q(p_\perp s H_k / k_\perp - k_\perp s_p H_p / p_\perp) + (p_\perp E_k - k_\perp E_p) s_q H_q / q_\perp] \end{Bmatrix} \delta(k_\parallel + p_\parallel + q_\parallel) dp_\perp dq_\perp dp_\parallel dq_\parallel \tag{18}$$

with $E_k \equiv E(k_\perp, k_\parallel) = 2\pi k_\perp E(\mathbf{k}_\perp, k_\parallel)$ and $H_k \equiv H(k_\perp, k_\parallel) = 2\pi k_\perp H(\mathbf{k}_\perp, k_\parallel)$ the asymmetric spectra of energy and kinetic helicity, respectively. In these expressions, θ is the angle between $\mathbf{k}_\perp$ and $\mathbf{p}_\perp$ in the triangle $\mathbf{k}_\perp + \mathbf{p}_\perp + \mathbf{q}_\perp = \mathbf{0}$ where $\Delta_\perp$ is the domain of integration (corresponding to this triangle).

The inertial wave turbulence equations were derived for the first time by Galtier [8]. They were studied numerically by Bellet *et al.* [33], and rederived with the Hamiltonian formalism by Gelash *et al.* [34]. After applying the generalized Zakharov transformation [35], the constant flux solutions found are

$$E(k_\perp, k_\parallel) \sim k_\perp^{-5/2} k_\parallel^{-1/2}, \tag{19}$$

$$H(k_\perp, k_\parallel) \sim k_\perp^{-3/2} k_\parallel^{-1/2}. \tag{20}$$

These solutions correspond to a direct energy cascade [8]. In particular, that means the dynamics of the kinetic helicity is driven by the energy. The numerical simulation of these equations have shown a relatively good agreement with the predictions [33]. Several other physical properties of inertial wave turbulence can be derived from expression (18). First, we see that an initial state with zero helicity will not produce helicity, whatever the scale considered. The energy is thus the main driven of this turbulence. Second, we observe that there is no nonlinear coupling when the wave vectors $\mathbf{p}_\perp$ and $\mathbf{q}_\perp$ are collinear (because then $\sin\theta = 0$). Third, there is no nonlinear coupling when $p_\perp$ and $q_\perp$ are equal if in the mean time their polarities s_p and s_q are also equal. It is a property that seems quite general since it is common to other types of helical waves [31,36–38]. Finally, we recall that these equations cannot describe the 2D modes ($k_\parallel = 0$) and also the 3D modes if there are too large (in particular in $k_\perp$) for which turbulence becomes strong. The kinetic equations of inertial wave turbulence are therefore limited to a finite domain in the Fourier space.

III. LOCAL TRIADIC INTERACTIONS LIMIT

In this section we shall study the local (in the perpendicular direction) triadic interactions limit for which the inertial wave turbulence equation simplifies. For the sake of simplicity, we will only consider the energy. Kinetic helicity turns out to be more difficult to manipulate analytically. Furthermore, the meaning and the behavior of the kinetic helicity is quite different from the magnetic helicity that we find in plasmas with a direct cascade for the kinetic helicity and an inverse cascade for the magnetic helicity. In the strongly anisotropic limit $k_\perp \gg k_\parallel$, Eq. (18) writes

$$\partial_t E_k = \sum_{s s_p s_q} \int T_{\mathbf{kpq}}^{s s_p s_q} dp_\perp dq_\perp dp_\parallel dq_\parallel .$$

By definition

$$T_{\mathbf{kpq}}^{s s_p s_q} \equiv \frac{\Omega_0^2}{4} \frac{s k_\parallel s_p p_\parallel}{k_\perp^2 p_\perp^2 q_\perp^2} \left(\frac{s_q q_\perp - s_p p_\perp}{\omega_k} \right)^2 (s k_\perp + s_p p_\perp + s_q q_\perp)^2 E_q (p_\perp E_k - k_\perp E_p) \sin\theta \, \delta(g_{kpq}) \delta_{k_\parallel p_\parallel q_\parallel}, \tag{21}$$

with $\omega_k = 2\Omega_0 k_\parallel / k_\perp$. Note that the small parameter ϵ is now absorbed in the time variable. The resonance condition leads to the remarkable identity

$$\frac{s_p p_\perp - s k_\perp}{s_q \omega_q} = \frac{s_q q_\perp - s_p p_\perp}{s \omega_k} = \frac{s k_\perp - s_q q_\perp}{s_p \omega_p}, \tag{22}$$

which can be used to demonstrate the symmetrical relation

$$T^{s_p s s_q}_{\mathbf{pkq}} = -T^{s s_p s_q}_{\mathbf{kpq}}. \tag{23}$$

In the limit of strongly local interactions we can write

$$p_\perp = k_\perp(1+\epsilon_p) \quad \text{and} \quad q_\perp = k_\perp(1+\epsilon_q), \tag{24}$$

with $\epsilon_p \ll 1$ and $\epsilon_q \ll 1$. We can introduce an arbitrary function $f(k_\perp, k_\parallel)$ and integrate the kinetic equation to find

$$\begin{aligned}\partial_t \int f(k_\perp, k_\parallel) E_k dk_\perp dk_\parallel &= \sum_{s s_p s_q} \int f(k_\perp, k_\parallel) T^{s s_p s_q}_{\mathbf{kpq}} dk_\perp dk_\parallel dp_\perp dq_\perp dp_\parallel dq_\parallel \\ &= \frac{1}{2} \sum_{s s_p s_q} \int (f(k_\perp, k_\parallel) - f(p_\perp, p_\parallel)) T^{s s_p s_q}_{\mathbf{kpq}} dk_\perp dk_\parallel dp_\perp dq_\perp dp_\parallel dq_\parallel. \end{aligned} \tag{25}$$

For local interactions we have

$$f(p_\perp, p_\parallel) \simeq f(k_\perp, k_\parallel) + (p_\perp - k_\perp) \frac{\partial f(k_\perp, k_\parallel)}{\partial k_\perp} = f(k_\perp, k_\parallel) + \epsilon_p k_\perp \frac{\partial f(k_\perp, k_\parallel)}{\partial k_\perp}, \tag{26}$$

if we neglect the contribution of the parallel wave number; this assumption is fully compatible with the weak cascade along the parallel direction. We obtain

$$\partial_t \int f(k_\perp, k_\parallel) E_k dk_\perp dk_\parallel = -\frac{1}{2} \sum_{s s_p s_q} \int \epsilon_p k_\perp \frac{\partial f(k_\perp, k_\parallel)}{\partial k_\perp} T^{s s_p s_q}_{\mathbf{kpq}} dk_\perp dk_\parallel dp_\perp dq_\perp dp_\parallel dq_\parallel. \tag{27}$$

Using an integration by part, we find the relation

$$\partial_t E_k = \frac{1}{2} \frac{\partial}{\partial k_\perp} \left(\sum_{s s_p s_q} \int \epsilon_p k_\perp T^{s s_p s_q}_{\mathbf{kpq}} dp_\perp dq_\perp dp_\parallel dq_\parallel \right). \tag{28}$$

The asymptotic form of $T^{s s_p s_q}_{\mathbf{kpq}}$ can be found by using the locality in the perpendicular direction. In particular, we find the relations

$$k_\perp^2 p_\perp^2 q_\perp^2 = k_\perp^6, \tag{29}$$

$$\left(\frac{s_q q_\perp - s_p p_\perp}{\omega_k} \right)^2 = \left(\frac{s_q - s_p + s_q \epsilon_q - s_p \epsilon_p}{2\Omega_0 k_\parallel} \right)^2 k_\perp^4, \tag{30}$$

$$(s k_\perp + s_p p_\perp + s_q q_\perp)^2 = (s + s_p + s_q)^2 k_\perp^2, \tag{31}$$

$$E_q(p_\perp E_k - k_\perp E_p) = -\epsilon_p k_\perp^3 E_k \frac{\partial (E_k / k_\perp)}{\partial k_\perp}, \tag{32}$$

$$\sin\theta = \sin(\pi/3) = \frac{\sqrt{3}}{2}, \tag{33}$$

$$\delta(g_{kpq}) = \frac{k_\perp}{2\Omega_0} \delta(s k_\parallel + s_p p_\parallel + s_q q_\parallel). \tag{34}$$

After simplification, we arrive at

$$T^{ss_ps_q}_{\mathbf{kpq}} = -\frac{\sqrt{3}}{64\Omega_0}\frac{ss_pp_\parallel}{k_\parallel}(s_q - s_p + s_q\epsilon_q - s_p\epsilon_p)^2(s + s_p + s_q)^2\epsilon_p k_\perp^4 E_k \frac{\partial(E_k/k_\perp)}{\partial k_\perp}\delta(sk_\parallel + s_pp_\parallel + s_qq_\parallel)\delta(k_\parallel + p_\parallel + q_\parallel). \tag{35}$$

With this form we see that the transfer will be significantly higher when $s_ps_q = -1$, therefore we will only consider this type of interaction. Then the expression of the transfer reduces to

$$T^{ss_p-s_p}_{\mathbf{kpq}} = -\frac{\sqrt{3}}{16\Omega_0}\frac{ss_pp_\parallel}{k_\parallel}\epsilon_p k_\perp^4 E_k \frac{\partial(E_k/k_\perp)}{\partial k_\perp}\delta(sk_\parallel + s_pp_\parallel - s_pq_\parallel)\delta(k_\parallel + p_\parallel + q_\parallel). \tag{36}$$

The resonance condition leads to two possible combinations for the parallel wave numbers, namely,

$$k_\parallel + p_\parallel - q_\parallel = 0 \quad \text{and} \quad k_\parallel + p_\parallel + q_\parallel = 0, \tag{37}$$

$$k_\parallel - p_\parallel + q_\parallel = 0 \quad \text{and} \quad k_\parallel + p_\parallel + q_\parallel = 0. \tag{38}$$

The solution corresponds either to $q_\parallel = 0$ or $p_\parallel = 0$, which means in particular that the strong locality assumption is not allowed for the parallel direction. The second solution cancels the transfer, therefore we will only consider the first solution for which we obtain (with $p_\parallel = -k_\parallel$)

$$\partial_t E_k = \frac{\sqrt{3}}{16\Omega_0}\frac{\partial}{\partial k_\perp}\left(k_\perp^7 E_k \frac{\partial(E_k/k_\perp)}{\partial k_\perp}\right)\int_{-\epsilon}^{+\epsilon}\epsilon_p^2 d\epsilon_p \int_{-\epsilon}^{+\epsilon} d\epsilon_q = C\frac{\partial}{\partial k_\perp}\left(k_\perp^7 E_k \frac{\partial(E_k/k_\perp)}{\partial k_\perp}\right), \tag{39}$$

with $C = \epsilon^4/(4\sqrt{3}\Omega_0)$. The nonlinear diffusion equation (39) is the main result of the paper. It describes weak inertial wave turbulence for strongly local interactions in the perpendicular direction. This equation is derived rigorously from the kinetic equations of rotating turbulence under the assumptions of locality (in the perpendicular direction) and of strong anisotropy for which $k_\perp \gg k_\parallel$. Expression (39) has the same form as for whistler wave turbulence [27,28,39]. The only one difference resides in the constant in front of the equation: for whistler waves this constant, called $\tilde{C}$, is related to C via the relation $\tilde{C} = C\Omega_0 d_i/B_0$, where d_i is the ion inertial length.

Equation (39) can also be derived using phenomenological arguments, but in this case the coefficient C is unknown. Below, we shall derive a general nonlinear diffusion equation to find the minimum requirement leading to Eq. (39). Let us consider the diffusion equation

$$\partial_t E(\mathbf{k}) = -\nabla \cdot \mathbf{F} = -\frac{1}{k_\perp}\frac{\partial(F_\perp k_\perp)}{\partial k_\perp}, \tag{40}$$

where $\mathbf{F}$ is a flux. Turbulence is assumed to be axisymmetric and we neglect the parallel cascade ($F_\parallel = 0$). We model the perpendicular flux as follows

$$F_\perp = -D\frac{\partial E(\mathbf{k})}{\partial k_\perp}, \tag{41}$$

where D is a diffusion coefficient. Dimensionally, we find

$$D = \frac{k_\perp^2}{\tau}, \tag{42}$$

with τ the typical time of energy transfer. To find its phenomenological expression we introduce the dispersion relation

$$\omega_k \sim k_\perp^\xi, \tag{43}$$

from which we define the wave-time $\tau_W \sim k_\perp^{-\xi}$. Only the perpendicular component of the wave vector is retained in the dispersion relation since the dynamics is assumed to be in that direction. In wave turbulence, for three-wave interactions we have [40]

$$\tau \sim \frac{\tau_{NL}^2}{\tau_W}, \tag{44}$$

where τ_{NL} is given by the fluid equation. For rotating turbulence it is Navier-Stokes and $\tau_{NL} \sim 1/(k_\perp u)$. After some manipulation we arrive at

$$\partial_t E_k \sim \frac{\partial}{\partial k_\perp}\left(\tau_{NL}^{-2} k_\perp^{3-\xi} \frac{\partial (E_k/k_\perp)}{\partial k_\perp}\right), \tag{45}$$

where we introduced the axisymmetric energy spectrum $E_k \sim k_\perp E(\mathbf{k})$. In principal, this diffusion equation is valid for any problem of anisotropic wave turbulence if three-wave interactions dominate. Let us introduce

$$\tau_{NL} \sim k_\perp^\rho E_k^\sigma, \tag{46}$$

then we obtain the nonlinear diffusion equation

$$\partial_t E_k \sim \frac{\partial}{\partial k_\perp}\left(k_\perp^{3-\xi-2\rho} E_k^{-2\sigma} \frac{\partial (E_k/k_\perp)}{\partial k_\perp}\right). \tag{47}$$

The conditions to obtain expression (39) are

$$\xi + 2\rho = -4, \tag{48}$$

$$\sigma = -1/2. \tag{49}$$

For rotating turbulence we have $\xi = -1$, $\rho = -3/2$, and $\sigma = -1/2$ and for kinetic-Alfvén wave turbulence we have $\xi = 1$, $\rho = -5/2$, and $\sigma = -1/2$.

IV. SOLUTIONS OF THE NONLINEAR DIFFUSION EQUATION

We shall check that Eq. (39) has the same exact power-law solutions as the original wave turbulence equation (18). Let us introduce in Eq. (18) the energy flux $\Phi_E(k_\perp)$ which is by definition

$$\frac{\partial E(k_\perp)}{\partial t} = -\frac{\partial \Phi_E(k_\perp)}{\partial k_\perp}, \tag{50}$$

and the energy spectrum $E(k_\perp) = Ak_\perp^x$ where A is by definition a positive constant. We obtain

$$\Phi_E(k_\perp) = A^2 C(1-x) k_\perp^{5+2x}. \tag{51}$$

Thus the constant flux solutions are $x = 1$ and $x = -5/2$. The first value cancels the flux and corresponds therefore to the thermodynamic solution found in [8]. The second value is the solution discussed above: it is the well-known Kolmogorov-Zakharov spectrum. In this case, we also have

$$\Phi_E(k_\perp) \equiv \Phi_0 = \frac{7}{2} A^2 C, \tag{52}$$

which is positive as expected for a direct cascade. It is also a finite capacity solution which means that we expect the formation of this solution in a finite time. Note that we find here a first reason to think that the local interactions is a good limit to describe inertial wave turbulence since the nonlinear diffusion equation is able to reproduce the exact solution derived from the kinetic equation.

The stationary spectrum is sometimes preceded by a transitory solution with different properties. In our case, since the system has a finite capacity, the nonstationary spectrum should correspond to a self-similar solution of the second kind with the form

$$E(k_\perp) = \frac{1}{\tau^\alpha} E_0\left(\frac{k_\perp}{\tau^\beta}\right), \tag{53}$$

with $\tau = t_* - t$. t_* is the time that takes the spectrum to reach the maximum wave number available. To be more precise, in principle, the spectral front will take a finite time to reach $k_\perp = +\infty$. Introducing the previous expression into Eq. (18) we find

$$\alpha = 4\beta + 1. \tag{54}$$

A second relation can be found by supposing that $E_0(\xi) \sim \xi^y$ far behind the front. Then, the stationary condition leads to

$$\alpha + y\beta = 0. \tag{55}$$

The combination of the two expressions gives eventually

$$y = -\frac{\alpha}{\beta} = -4 - \frac{1}{\beta}. \tag{56}$$

This last expression means that we have a direct relation between the power-law exponent y and the law of the front propagation. $k_f \sim \tau^\beta$. For example, if we assume that the Kolmogorov-Zakharov solution appears immediately during the nonstationary phase, then $y = -5/2$ and thus $\beta = -2/3$ (with $\alpha = -5/3$). In this case, the prediction for the front propagation is

$$k_f \sim (t_* - t)^{-2/3}. \tag{57}$$

In practice, as illustrated in the next section, the nonstationary solution is quite different: the expressions (55) and (56) are verified but the values of α, β, and y can only be found by using a numerical simulation.

V. NUMERICAL STUDY

In this section we present a numerical study of the nonlinear diffusion equation (39) with $C = 1$. A hyperviscous dissipative term is added; this term stabilizes the code around the critical time t_*. Therefore, we numerically implement the following equation:

$$\frac{\partial E(k_\perp)}{\partial t} = \frac{\partial}{\partial k_\perp}\left(k_\perp^7 E_k \frac{\partial (E(k_\perp)/k_\perp)}{\partial k_\perp}\right) - \nu k_\perp^4 E(k_\perp), \tag{58}$$

with ν the hyperviscosity. In our simulation we took $\nu = 2\times10^{-7}$. A logarithmic subdivision of the $k_\perp$-axis is used with $k_{\perp i} = 2^{i/10}$ and i an integer varying between 0 and 240. The Adams-Bashforth and Crank-Nicholson numerical schemes are implemented for the nonlinear and dissipative terms, respectively. The initial condition ($t = 0$) corresponds to a spectrum localized at large scale with $E(k_\perp) \sim k_\perp^3 \exp(-(k_\perp/k_0)^2)$ and $k_0 = 5$. No forcing is added at $t > 0$. The timestep is $\Delta t = 2\times10^{-7}$. Note that a similar numerical simulation was already performed by David and Galtier [28] for kinetic-Alfvén wave turbulence. For the consistency of the paper we perform a new simulation where an hyperviscosity of order 4 is used (instead of order 6) as well as a higher grid resolution (we use $k_{\perp i} = 2^{i/10}$ instead of $k_{\perp i} = 2^{i/8}$). As shown below, we arrive to the same conclusion, which proves that it does not depend on the type of hyperdissipation and also that the resolution is high enough for the convergence of the results. Note that we also performed a simulation with a normal viscosity (not shown) and we recover qualitatively the same results (however, the inertial range is narrower).

In Fig. 1(left) we show the time evolution of the energy spectrum for $t \in [0, t_*]$. During this nonstationary phase a clear power-law spectrum in $k_\perp^{-8/3}$ is formed behind the front. As shown in Fig. 1(right), the nonstationary phase is characterized by a nonconstant energy flux $\Phi_E(k_\perp)$: we start with a flux localized at small perpendicular wave numbers which then develops towards smaller scales without reaching a plateau. The solution does not correspond to the constant flux solution derived analytically, but it is fully compatible with the power-law solution $\sim k_\perp^{-1/3}$ when we take $x = -8/3$ in Eq. (51). Note that this result could be the explanation of the recent spectrum obtained

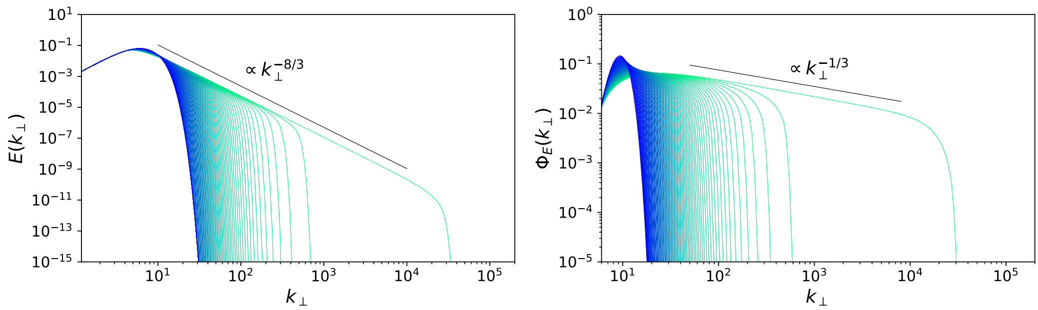

FIG. 1. Left: Time evolution (every 33 802Δt; 100 spectra are shown) of the energy spectrum $E(k_\perp)$ from $t = 0$ (blue) to t_* (green); a comparison is made to $k_\perp^{-8/3}$. Right: Time evolution of the energy flux $\Phi_E(k_\perp)$ for the same times; a comparison is made to $k_\perp^{-1/3}$.

numerically with a wave turbulence code where such anomalous scaling was found [25]. We may find here a second reason to think that the local interactions is a good limit to describe inertial wave turbulence.

To check if the energy spectrum corresponds to the self-similar solution introduced above we show in Fig. 2 the front propagation $k_f(t)$. This front is defined by taking $E(k_\perp) = 10^{-15}$ from Fig. 1(left): we then follow the point of intersection between this threshold and the spectral tail. From Fig. 2 we can define the singular time t_* at which the front can reach, in principle, $k_\perp = +\infty$. The value $t_* = 6.757\times10^{-7}$ is chosen. In Fig. 2 (inset) we display k_f as a function of $t_* - t$: a power law is observed over three decades with an index of -0.750. The negative value illustrates the explosive character of the direct cascade of energy in inertial wave turbulence. The different values measured are fully compatible with

$$\alpha = -2, \quad \beta = -3/4, \quad \text{and} \quad y = -8/3, \tag{59}$$

which therefore demonstrates the self-similar nature of the nonstationary solution.

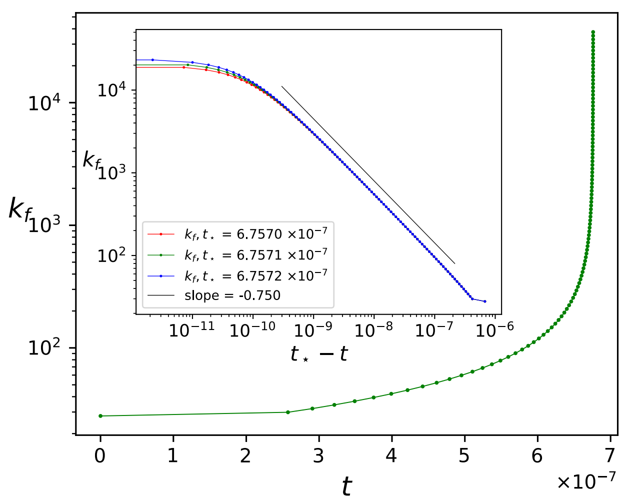

FIG. 2. Time evolution of the spectral front k_f for $t \leqslant t_*$ in linear-logarithmic coordinates (green). A sharp increase of k_f is observed from which we can define precisely the singular time $t_* = 6.757\times10^{-7}$. Inset: The temporal evolution of k_f as a function of $t_* - t$ (green) in double logarithmic coordinates. The black dashed line corresponds to $(t_* - t)^{-0.750}$. For comparison two other values of t_* are taken (red and blue).

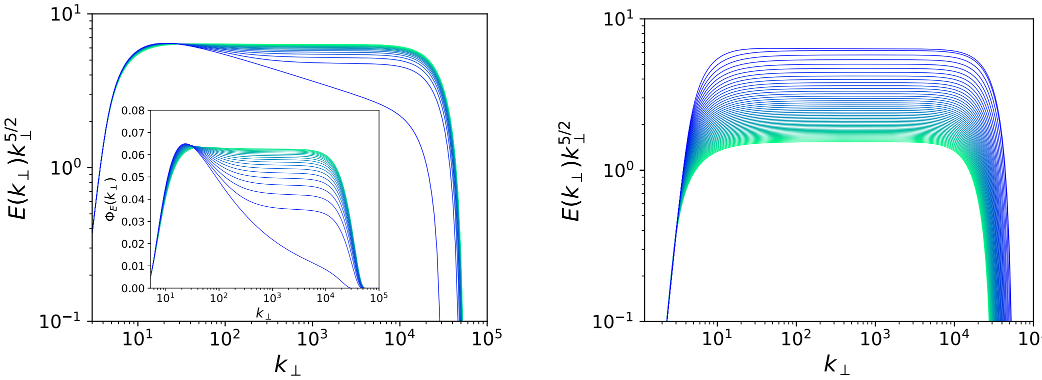

FIG. 3. Left: Time evolution (every 33 802Δt) from $t = t_*$ to $t = 4\times10^6$ (from blue to green) of the energy spectrum compensated by $k_\perp^{5/2}$; inset: time evolution of the flux for the same times. Right: Time evolution (every $10^6\Delta t$) for $t = 4\times10^6$ to $t = 5\times10^7$ of the compensated energy spectrum.

Finally, in Fig. 3(left) we show the temporal evolution for $t > t_*$ of the spectra of energy and flux (inset), respectively. The classical stationary wave turbulence predictions are finally obtained with an energy spectrum in $k_\perp^{-5/2}$—Kolmogorov-Zakharov solution—and a constant positive energy flux, as expected for a direct cascade. In Fig. 3(right) the final phase of the simulation is shown for $t \gg t_*$: it corresponds to a self-similar decay of the energy spectrum with the same power-law index.

VI. DISCUSSION

Kinetic-Alfvén (or oblique whistler) wave turbulence is a regime of plasma physics often considered [41] to describe solar wind turbulence for frequencies higher than the ion-gyro frequency (or for lengthscales smaller than the ion skin depth d_i) where standard magnetohydrodynamics (MHD) is not applicable [40]. In the presence of a strong uniform magnetic field $\mathbf{B_0}$ a weak turbulence regime is possible for which the kinetic equations can be derived [27,32,39]. These equations are greatly simplified by taking the strongly local interactions limit for wave numbers perpendicular to $\mathbf{B_0}$. The nonlinear diffusion equations found have been numerically solved to study the dynamics of the magnetic energy coupled with the magnetic helicity [27]. More recently, this regime has also been studied numerically (in the absence of helicity) to show the existence of a nonstationary solution with a spectrum proportional to $k_\perp^{-8/3}$, which is significantly different from the stationary solution in $k_\perp^{-5/2}$ [28]. The consequences of such a discovery have been discussed in the context of the turbulent collisionless solar wind plasma. It could also provide an explanation for the $-8/3$ spectrum obtained with a direct numerical simulation of electron MHD turbulence [42].

In the present paper, we show that the nonlinear diffusion equation (39) found for inertial wave turbulence is the same as the one derived for kinetic-Alfvén wave turbulence. The only difference resides in the constant in front of the equation where some physical quantities of the system appear. For rotating hydrodynamics it is the rotating rate Ω_0 while for magnetized plasmas it is the uniform magnetic field B_0. The proximity between these two problems may be used to better understand inertial or kinetic-Alfvén wave turbulence. For example, it is interesting to note that a study based on structure functions of the magnetic field, obtained with *in situ* data at one astronomical unit, reveals a non-Gaussian monoscaling in the domain where kinetic-Alfvén wave turbulence dominates [43] (see also the discussion in [44]). The statistics is compatible with $\xi_p = 0.8p$ or $0.9p$, depending on the magnetic field component, while at MHD scales classical properties of intermittency were found with a nonlinear variation of ξ_p with p. Such statistics—with a monoscaling—is rather rare in turbulence, however, for rotating hydrodynamic turbulence, such a linear scaling was also found experimentally [7]: it was for the longitudinal velocity (perpendicular components) structure function exponents. In this case the data are compatible with $\xi_p = 3p/4$, with still a non-Gaussian statistics. Scale invariance for the velocity structure function was also detected in a direct numerical simulations for a Rossby number $R_o \sim 0.06$ and in presence of helicity [45]; the measures are compatible with $\xi_p = 0.71p$. It is well-known that the law followed by ξ_p can depend on the type of structures (sheets or filaments) present [46]. In the case of rotating turbulence we find vorticity filaments mainly oriented along the rotating axis [47]. Interestingly, filaments of electric currents were also found in the regime of electron MHD under a strong mean magnetic field (whose equations can also describe kinetic-Alfvén wave turbulence) [32,42]. These structures are also oriented along the external agent ($\mathbf{B_0}$). According to these different anomalous results, it is tantalizing to think that this behavior finds its origin in the *weak* turbulence character where waves are omnipresent, probably in coexistence with fluctuations at $k_\parallel = 0$ (2D modes). The slight differences between the measurements could be attributed to inhomogeneities, always present experimentally, or to the fact that the regime of weak wave turbulence is not perfectly reached.

The impact of 2D modes is certainly nonnegligible in both problems but it has been discussed mainly in the context of rotating turbulence [8,18,24,33]. For example, it was shown numerically that a significant fraction of the energy is concentrated in modes with zero frequency and it is only for modes with the period faster than the turnover time that a significant fraction of the remaining energy is concentrated along the dispersion relation, as expected for weak turbulence [48]. In the context of the solar wind, it is difficult to remove the contribution of the 2D mode to the statistics since we have only access to one or four points (the number of satellites). However, it is known that the 2D modes have an influence on the global behavior of a plasma like it was shown numerically in incompressible MHD turbulence [49,50] (and also in Hall MHD [51]). With this scenario, the regime at kinetic scales in the solar wind would be driven by weak wave turbulence in presence of 2D modes, limiting therefore the critical balance regime [52] to MHD scales. Note, however, that this fluid description would not be complete if we did not include the kinetic effects that can modify non-trivially the overall picture [53].

In conclusion, we showed that under a particular limit of weak wave turbulence, rotating hydrodynamics, and magnetized plasmas at sub-ion scales can share the same dynamical equation which allows us to make a bridge between these two different problems. It was often stated in the past that the first problem was close to MHD and therefore to Alfvén wave turbulence, but the case of kinetic-Alfvén waves (or oblique whistler waves) is even closer since the diffusion equation is exactly the same whereas a difference exists with MHD [54]. Other differences exist with pure MHD: Alfvén waves are linearly polarized while inertial and kinetic-Alfvén waves are helical waves (with left and right polarizations, respectively). In pure MHD, the cascade along the external agent is not possible [30,55] whereas a weak transfer is always possible for the two problems discussed in this paper. Finally, it is believed that the proximity between these two problems can help to better understand them, and in particular, solar wind turbulence. However, we must keep in mind that the solar wind is a collisionless plasma where kinetic effects are present, which may limit our comparison with a fluid model [56,57].

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