Stability of a static liquid bridge knowing only its shape

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The shape and the stability of a static liquid bridge under either pressure control or volume control can be derived by knowing only the shape of a master liquid bridge. No perturbation problems need be solved.

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I. INTRODUCTION

A static liquid bridge is a fluid body spanning two solid disks, pinned at their edges; cf. Fig. 1. The equilibrium shape of a bridge is determined by its volume or its pressure, once the diameter of the disks and their distance apart are set, given the surface tension and the density difference across the bridge surface (cf. Refs. [1,2]). Figure 2 shows how a liquid bridge experiment might be run. By increasing or decreasing the pressure above reservoir (1), fluid is added or removed from the bridge submerged in reservoir (2).

The static equilibrium shape of a bridge is due to a balance between the mean curvature of its surface and the pressure difference across the surface. This equilibrium shape may become unstable as the volume of the bridge is decreased. If the density difference across the surface is zero, the instability is a consequence of an imbalance between the transverse and longitudinal curvatures. This was discovered by Rayleigh and Plateau in the case of liquid jets [3,4]. Extensions of their work [5–13] have led to several observations on static bridge stability, described in reviews [14–16]. The observations, most important to us, obtained from perturbation calculations holding the volume fixed are: (a) A liquid bridge suspended between two coaxial circular ends under volume control and having no density difference across its surface can become unstable in two ways as its volume is decreased, first to disturbances unsymmetric about the midplane, second to disturbances symmetric about the mid plane. (b) The unsymmetric instability is a bifurcation point; i.e., the bridge has not reached its theoretical least volume. (c) The symmetric instability is a turning point; i.e., the bridge has reached its least volume. (d) If the density difference is not zero, then all critical points are turning points [15].

We present a new view of the bridge stability problem: The shape and the stability of a liquid bridge ought to be understood in terms of a special pressure-controlled problem. Notwithstanding the fact that volume control is the case of most interest in the literature, it is a subset of a more inclusive master problem. This problem corresponds to the tanks in Fig. 2 having large enough cross sections that the hydrostatic head in each tank remains essentially constant.

Our plan is to present a model of a pressure-controlled experiment and then to identify the master problem where given all the details of this problem we can derive all the details of every other bridge problem. Our attention is directed to bridge shapes, then critical points and limits on the range of the pressures possible.

Three results arise from our work which set it apart from all earlier studies. First, the stability of every bridge problem, whether the density difference is zero or not and whether it is under volume or

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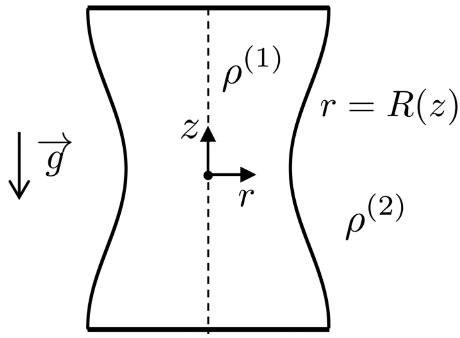

FIG. 1. A sketch of a bridge spanning two disks.

pressure control, can be determined merely by knowing the bridge shape corresponding to a master problem. Second, no perturbation problem need be solved to determine the stability of any static bridge. Finally, the distinction between the instability arising from pressure control versus volume control can be lost when the density difference is made zero.

II. MODEL: THE NONLINEAR EQUATIONS

We direct the reader's attention to the problem of predicting the shape and the stability of the surface of a liquid bridge spanning two circular disks. The surface is pinned to the edges of the disks, the disks are horizontal, they have the same diameter, denoted by D, and they lie a distance $2L$ apart. Their centers lie on a line parallel to the direction of gravity.

We denote by V, $\Delta\rho = \rho^{(1)} - \rho^{(2)}$, and γ the volume of the bridge, the density difference across the surface, and its surface tension. Ordinarily, D, L, g, $\Delta\rho$, and γ are input variables, and $R(z)$, the shape of the surface, is an output. We assume $\Delta\rho$ to be positive or zero.

Our view is that if an experiment is to be run, pressure ought to be the primary input, once the values of L, D, $\Delta\rho$, and γ are set. Thus, we sketch a possible experimental configuration to which our results pertain. Figure 2 defines the notation where $A^{(1)}$ and $A^{(2)}$ denote the cross sectional areas of the reservoirs. The volumes of fluids (1) and (2) are denoted $V^{(1)}$ and $V^{(2)}$. They are fixed. The volume of the bridge, V, is part of $V^{(1)}$.

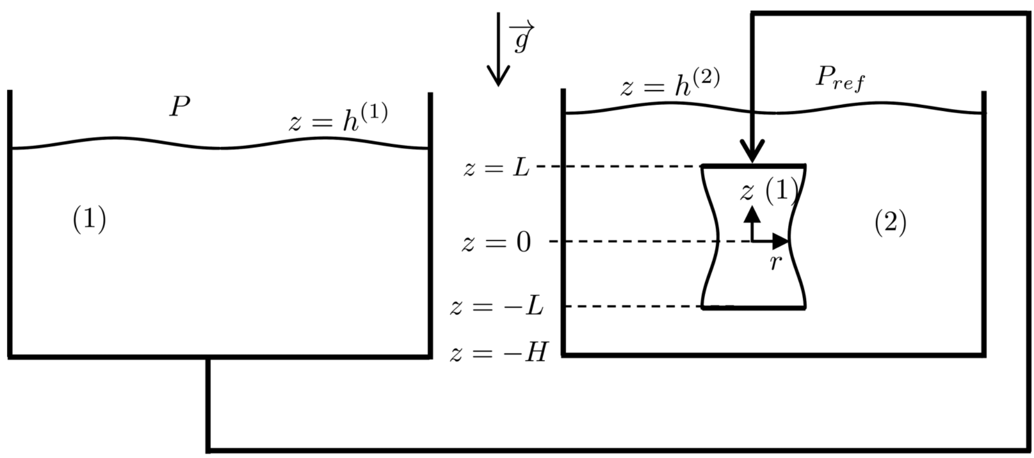

FIG. 2. A sketch of a bridge whose surface shape is being controlled by the pressure at the left-hand reservoir.

Assuming the volume in the connecting line is not important, we have

$$V^{(1)} = A^{(1)}(h^{(1)} + H) + V \tag{1}$$

and

$$V^{(2)} = A^{(2)}(h^{(2)} + H) - V, \tag{2}$$

whereupon we have

$$\rho^{(2)}gh^{(2)} - \rho^{(1)}gh^{(1)} = \rho^{(2)}g\left(\frac{V^{(2)}}{A^{(2)}} - H\right) - \rho^{(1)}g\left(\frac{V^{(1)}}{A^{(1)}} - H\right) + \left(\frac{\rho^{(2)}g}{A^{(2)}} + \frac{\rho^{(1)}g}{A^{(1)}}\right)V. \tag{3}$$

Our model is hydrostatic, hence the pressures are

$$P^{(1)} = P + \rho^{(1)}g(h^{(1)} - z) \tag{4}$$

and

$$P^{(2)} = P_{\text{ref}} + \rho^{(2)}g(h^{(2)} - z), \tag{5}$$

and the shape of the bridge, $R(z)$, is the solution to the equation

$$P^{(1)} + \gamma 2\mathcal{H} = P^{(2)}, \tag{6}$$

where $2\mathcal{H}$ denotes twice the mean curvature of the surface. Whereupon, denoting

$$P - P_{\text{ref}} - \left\{\rho^{(2)}g\left(\frac{V^{(2)}}{A^{(2)}} - H\right) - \rho^{(1)}g\left(\frac{V^{(1)}}{A^{(1)}} - H\right)\right\}$$

by P, we have in scaled variables

$$-P + B_*V + Bz = 2\mathcal{H} = \frac{d}{dz}\frac{R_z}{\left(1 + R_z^2\right)^{\frac{1}{2}}} - \frac{1}{R\left(1 + R_z^2\right)^{\frac{1}{2}}} \tag{7}$$

and

$$R = 1 \quad \text{at} \;\; z = \pm L,$$

where lengths are scaled by $\frac{1}{2}D$, and pressures are scaled by $\frac{\gamma}{\frac{1}{2}D}$, where B_* denotes

$$\left(\frac{\rho^{(1)}g}{\gamma}\frac{1}{A^{(1)}} + \frac{\rho^{(2)}g}{\gamma}\frac{1}{A^{(2)}}\right)\left(\frac{1}{2}D\right)^4,$$

and B denotes the Bond number, $\frac{\Delta\rho g}{\gamma}(\frac{1}{2}D)^2$. The volume of the bridge, V, is given by

$$V = \int_{-L}^{L} \pi R^2 dz. \tag{8}$$

Upon solving Eq. (7) we obtain the shape of the bridge. Setting the values of the inputs L, B, and B_*, we wish to know how the shape depends upon P. We observe that neither B nor B_* is negative and that either may be the larger. If $\Delta\rho = 0$, we have $B = 0$. If $A^{(1)}$ and $A^{(2)}$ are large, then B_* is small, but g remains present unless both B and B_* vanish. Thus, if $\Delta\rho = 0$, g remains in B_*. We set the values of B, B_*, and L, then to each P the outputs are $R(z)$ and V. We can solve Eqs. (7) and (8) for $R(z)$ and V whether or not the resulting shape is stable to small perturbations and we begin by asking: are there limitations on the range of P?

Thus, having a solution $R(z)$ and V at P, we can increase or decrease P and find a nearby solution if, at P, we can find $\dot{R} = \frac{dR}{dP}$, where $\dot{R}$, obtained by differentiating Eq. (7), is the solution to

Eq. (9),viz.,

$$-1 + B_* \int_{-L}^{L} 2\pi R\dot{R}dz = \frac{1}{R}\frac{d}{dz}\frac{\dot{R}_z R}{\left(1+R_z^2\right)^{\frac{3}{2}}} + \frac{1}{R^2\left(1+R_z^2\right)^{\frac{1}{2}}}\dot{R} \equiv \mathcal{L}(\dot{R}) \tag{9}$$

and

$$\dot{R} = 0 \quad \text{at} \quad z = \pm L.$$

This is a linear inhomogeneous problem which may or may not have a solution. If it does not, then we have reached a limiting value of P.

Along the way as P is increasing or decreasing toward its limiting values, the bridge shape may become unstable and we may not be able to reach the limits of P. To find out, we denote a small perturbation to the shape by $R_1(\theta, z)$ and, holding B, B_*, L, and P fixed, R_1 must satisfy Eq. (10), viz.,

$$B_* \int_{-L}^{L} R \int_0^{2\pi} R_1 d\theta dz = \frac{1}{R}\frac{d}{dz}\frac{RR_{1z}}{\left(1+R_1^2\right)^{\frac{3}{2}}} + \frac{1}{R^2\left(1+R_1^2\right)^{\frac{1}{2}}} \tag{10}$$

and

$$R_1 = 0 \quad \text{at} \quad z = \pm L.$$

This homogeneous problem has the solution $R_1 = 0$. If $R_1 = 0$ is the only solution, then the bridge is stable to a small perturbation; otherwise, if there are solutions not zero, then the bridge shape has reached a critical condition. The bridge-shape problem is hydrostatic, so too the perturbation problem. A static bridge shape signals that the bridge has reached a least potential energy, whereupon the bridge is stable if every perturbation raises the potential energy (cf. Appendix A).

To find out whether or not $\dot{R}$ exists and whether or not R_1 is zero, we solve the eigenvalue problem, Eq. (11), for the eigenvalues, λ^2, and the eigenfunctions, ψ, viz.,

$$B_* \int_{-L}^{L} R \int_0^{2\pi} \psi d\theta dz = \frac{1}{R}\frac{d}{dz}\frac{R\psi_z}{\left(1+R_z^2\right)^{\frac{3}{2}}} + \frac{1}{R^2\left(1+R_z^2\right)^{\frac{1}{2}}}(\psi + \psi_{\theta\theta}) + \lambda^2\psi \tag{11}$$

and

$$\psi = 0 \quad \text{at} \quad z = \pm L.$$

If there is an eigenvalue equal to zero, then the homogenous part of Eq. (9) has a solution other than zero and Eq. (10) has a solution that is not zero. The bridge is at a critical pressure and the pressure may be at its limiting value.

We can restrict the perturbations of interest. Thus, the eigenfunctions can be written

$$\psi = \psi(z), \quad m = 0 \tag{12}$$

and

$$\psi = \hat{\psi}(z)\cos m\theta, \quad m = 1, 2, ... \tag{13}$$

If $m = 1, 2, ...$, we have

$$0 = \frac{1}{R}\frac{d}{dz}\frac{R\hat{\psi}_z}{\left(1+R_z^2\right)^{\frac{3}{2}}} + \frac{1}{R^2}\frac{1}{\left(1+R_z^2\right)^{\frac{1}{2}}}(1-m^2)\hat{\psi} + \lambda^2\hat{\psi} \tag{14}$$

and

$$\hat{\psi} = 0 \quad \text{at} \quad z = \pm L,$$

and upon multiplying Eq. (14) by $R\hat{\psi}$ and integrating over $(-L, L)$, we have

$$0 = -\int_{-L}^{L} \frac{R\hat{\psi}_z^2}{\left(1+R_z^2\right)^{\frac{3}{2}}} dz + (1-m^2)\int_{-L}^{L} \frac{\hat{\psi}^2}{R\left(1+R_z^2\right)^{\frac{1}{2}}} dz + \lambda^2 \int_{-L}^{L} \hat{\psi}^2 dz. \tag{15}$$

Hence, all λ^2's must be positive if $m = 1, 2, \ldots$.

Thus, only perturbations corresponding to $m = 0$ may lead to critical shapes of the bridge and our eigenvalue problem is

$$2\pi B_* \int_{-L}^{L} R\psi dz = \mathcal{L}(\psi) + \lambda^2 \psi \tag{16}$$

and

$$\psi = 0 \quad \text{at } z = \pm L.$$

Multiplying Eq. (16) by $R\psi$ and integrating, we have, at $m = 0$,

$$2\pi B_* \left(\int_{-L}^{L} R\psi dz\right)^2 = -\int_{-L}^{L} \frac{R\psi_z^2}{\left(1+R_z^2\right)^{\frac{3}{2}}} dz + \int_{-L}^{L} \frac{\psi^2}{R\left(1+R_z^2\right)^{\frac{1}{2}}} dz + \lambda^2 \int_{-L}^{L} \psi^2 dz. \tag{17}$$

The second term on the right-hand side, accounting for transverse curvature, presents the possibility of negative eigenvalues and hence critical points.

Thus, having set the values of B, B_*, and L, we may be increasing or decreasing P in search of a critical point. And assuming all the eigenvalues begin positive corresponding to a stable bridge (cf. Appendix A), we may come upon a value of P where the least eigenvalue reaches zero. We have arrived at a critical point, i.e., $R_1 \neq 0$, and we observe that the homogeneous $\dot{R}$ problem has the nonzero solution $\dot{R} = \psi$. To learn if the inhomogeneous $\dot{R}$ problem has a solution, we multiply Eq. (9) by $R\psi$, Eq. (16) by $R\dot{R}$, subtract and integrate over $(-L, L)$. Doing this, we have the solvability condition. It is

$$\int_{-L}^{L} (1) R\psi dz = 0, \tag{18}$$

and ordinarily it is not satisfied, i.e., the critical point is a turning point, $\dot{R}$ does not exist, and we have reached a limiting value of P. In fact, Eq. (17) at $\lambda^2 = 0$ is

$$2\pi B_* \left(\int_{-L}^{L} R\psi dz\right)^2 = -\int_{-L}^{L} \frac{R}{\left(1+R_z^2\right)^{\frac{3}{2}}} \psi_z^2 dz + \int_{-L}^{L} \frac{1}{R\left(1+R_z^2\right)^{\frac{1}{2}}} \psi^2 dz, \tag{19}$$

where the right-hand side is ordinarily not zero and so $\int_{-L}^{L} R\psi dz$ is not zero.

We plan to present our results as curves of V versus P, where P implies $R(z)$ and $R(z)$ implies V. Each curve corresponds to set values of B, B_*, and L. Now we need to do this for only one value of B_*, viz., $B_* = 0$, to obtain curves for any value of B_*. And $B_* = 0$ is the easiest problem to solve, it is the master problem, given L and B.

At $B_* = 0$ our R, $\dot{R}$ and ψ, λ^2 problems are,
first,

$$P + Bz = \frac{d}{dz} \frac{R_z}{\left(1+R_z^2\right)^{\frac{1}{2}}} - \frac{1}{R} \frac{1}{\left(1+R_z^2\right)^{\frac{1}{2}}} \tag{20}$$

and

$$R = 1 \quad \text{at } z = \pm L,$$

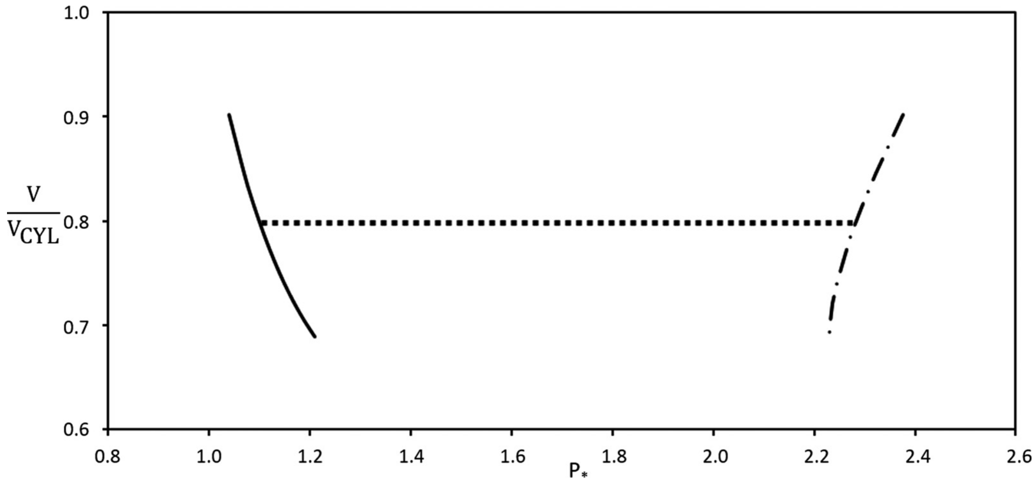

FIG. 3. The V vs P_* curve on the right at $B_* > 0$ is obtained from the V vs P curve on the left at $B_* = 0$ without solving Eq. (7). $L = \frac{3}{4}\pi$, $B = \frac{1}{100}$, $B_* = \frac{1}{20}$.

second,

$$-1 = \frac{1}{R}\frac{d}{dz}\frac{R\dot{R}_z}{\left(1+R_z^2\right)^{\frac{3}{2}}} + \frac{1}{R^2}\frac{1}{\left(1+R_z^2\right)^{\frac{1}{2}}}\dot{R} \tag{21}$$

and

$$\dot{R} = 0 \quad \text{at} \;\; z = \pm L,$$

and third,

$$0 = \frac{1}{R}\frac{d}{dz}\frac{R\phi_z}{\left(1+R_z^2\right)^{\frac{3}{2}}} + \frac{1}{R^2}\frac{1}{\left(1+R_z^2\right)^{\frac{1}{2}}}\phi + \mu^2\phi = \mathcal{L}(\phi) + \mu^2\phi \tag{22}$$

and

$$\phi = 0 \quad \text{at} \;\; z = \pm L,$$

where we have renamed ψ and λ^2, ϕ and μ^2 at $B_* = 0$. Equation (22) is a Sturm-Liouville problem, whereupon the fundamental eigenfunction, ϕ_1, is singly signed (cf. Ref. [17]).

Assume we set the values of B and L and we solve our bridge-shape problem at $B_* = 0$ and then obtain a curve of V versus P. At each point along the curve we have the bridge shape $R(z)$ and the eigenvalues $\mu_1^2 < \mu_2^2 < \cdots$.

Now at any value of B_* not zero, we can derive the volume versus pressure curve from the curve at $B_* = 0$. Thus, at any B_* not zero, we rename our variables P_*, $R_*(z)$, and V_*. Then, at the point (P, V) on the curve V versus P at $B_* = 0$, we set $P_* = P + B_*V$, $V_* = V$ and at the point (P_*, V_*) we have $R_*(z)$ at B_*, equal to $R(z)$ at $B_* = 0$ (cf. Fig. 3), where the scaling factor, V_{CYL}, $V_{\text{CYL}} = 2\pi L$, denotes the volume of a cylindrical bridge at the same value of L.

At the value of B_* of interest, at set values of L and B, we can obtain the eigenvalues without solving the eigenvalue problem at B_*. Thus, at a point (P, V) at $B_* = 0$ we have $R(z)$ and we can obtain the eigenvalues $\mu_1^2 < \mu_2^2 < \cdots$, which are the λ^2's at $B_* = 0$. Then at L, B, V, $B_* \neq 0$ and $P_* = P + B_*V$ we have the same bridge shape and we can obtain the eigenvalues $\lambda_1^2 < \lambda_2^2 < \cdots$.

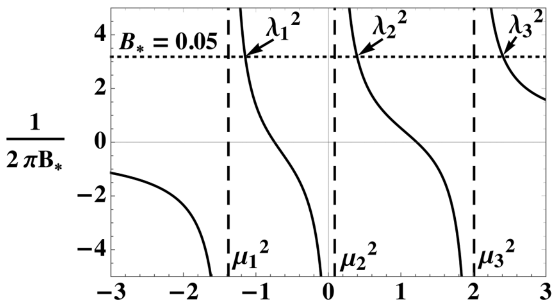

FIG. 4. Graphs of the solutions to Eq. (27), where the eigenvalues, $\lambda_1^2, \lambda_2^2, \cdots$, at $B_* > 0$ are obtained from the eigenvalues, $\mu_1^2, \mu_2^2, \cdots$, at $B_* = 0$. The values of L, B, and V are $L = \frac{3}{4}\pi$, $B = \frac{1}{10}$, $V = 0.976$.

To do this we expand ψ in the set of ϕ_j's and we have

$$\psi = \sum c_j \phi_j, \quad c_j = \int_{-L}^{L} \psi R \phi_j dz, \quad \text{where} \quad \int_{-L}^{L} R\phi_j^2 dz = 1, \tag{23}$$

whereupon we obtain from Eqs. (17) and (23)

$$2\pi B_* \left(\int_{-L}^{L} R\psi dz \right) \int_{-L}^{L} R\phi_j dz = \left(\lambda^2 - \mu_j^2\right) \int_{-L}^{L} \psi R \phi_j dz, \tag{24}$$

and hence we have

$$2\pi B_* \int_{-L}^{L} R\psi dz \, I_j = \left(\lambda^2 - \mu_j^2\right) c_j, \text{ where } I_j = \int_{-L}^{L} R\phi_j dz \tag{25}$$

and

$$2\pi B_* \int_{-L}^{L} R\psi dz \sum \frac{I_j^2}{\left(\lambda^2 - \mu_j^2\right)} = \int_{-L}^{L} R\psi dz. \tag{26}$$

Thus, we derive the eigenvalues at B_* from those at $B_* = 0$ by solving

$$\sum \frac{I_j^2}{\left(\lambda^2 - \mu_j^2\right)} = \frac{1}{2\pi B_*}. \tag{27}$$

The solution is sketched in Fig. 4, and at any B, L, and V we not only have $\mu_1^2 < \lambda_1^2 < \mu_2^2 < \lambda_2^2 < \cdots$ for all B_* but for any B_* we have the values of $\lambda_1^2, \lambda_2^2 \ldots$ corresponding to the point $(P + B_* V, V)$ on the V versus P_* curve at B_*.

We can make some predictions. At fixed values of B and L and at each point along the V versus P curve at $B_* = 0$ we can find the μ^2's. Likewise at $B_* > 0$, at each point along the V versus P_* curve, we can derive the λ^2's from the μ^2's. The stable ranges, $\mu_1^2 > 0$, $\lambda_1^2 > 0$, appear where the slopes of the V, P or V, P_* curves are positive. The stable range of P_* increases as B_* increases. Wherever $\mu_1^2 = 0$ or $\lambda_1^2 = 0$ we have a turning point. Thus, in Fig. 5, at $B_* = 0$ we have two turning points. And if $\mu_1^2 = 0$ at (P, V), λ_1^2 is not zero at $(P + B_* V, V)$. It is positive there. The slope of the V versus P_* curve at $(P + B_* V, V)$ must be positive. In fact, at $(P + B_* V, V)$ we have

$$\dot{R} = A\phi_1 \tag{28}$$

and

$$-1 + 2\pi B_* \int_{-L}^{L} RA\phi_1 dz = 0, \tag{29}$$

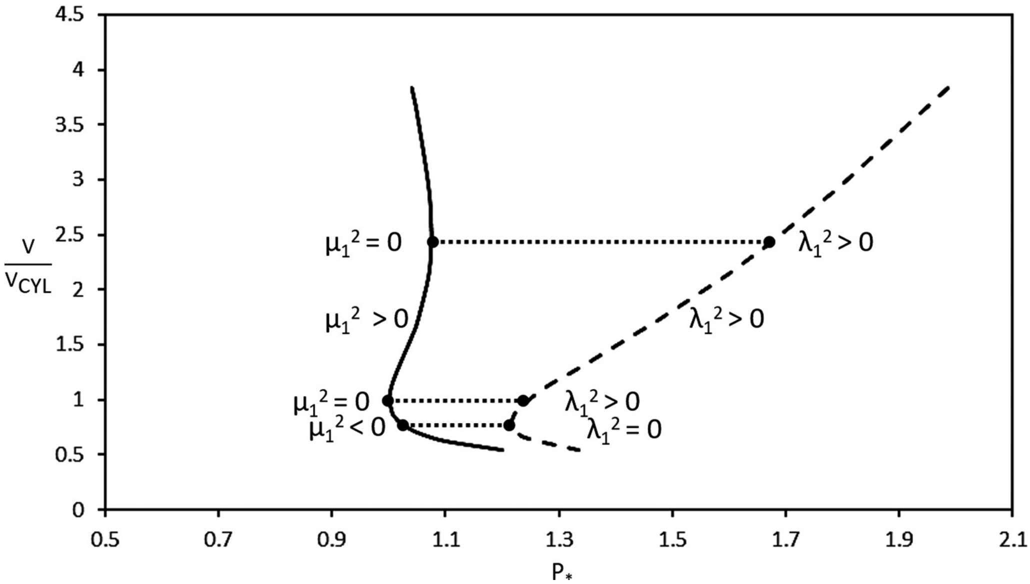

FIG. 5. A sketch of the least eigenvalues along the V vs P curve at $B_* = 0$, solid line, and along the V vs P_* curve at $B_* > 0$, dashed line. The eigenvalues, λ_1^2, derive from the eigenvalues, μ_1^2, as indicated in Fig. 4. Where μ_1^2 or λ_1^2 is zero, turning points appear. The values of L, B, and B_* are $L = \frac{1}{2}\pi$, $B = \frac{1}{100}$, $B_* = \frac{1}{100}$.

whereupon

$$\frac{dV}{dP_*} = 2\pi \int_{-L}^{L} R\dot{R}dz = \frac{1}{B_*} \tag{30}$$

and the slope of the curve must be positive.

In place of pressure control, it may be that a bridge shape is under the control of the bridge volume and that stability would then depend on perturbations at constant volume. There are many experiments thought to be run this way (cf. Refs. [18,19]) and many stability predictions made holding volume constant.

In this case where $R(z)$ derives from V, Eq. (7) must be replaced by

$$-P + Bz = \frac{d}{dz}\frac{R_z}{\left(1+R_z^2\right)^{\frac{1}{2}}} - \frac{1}{R\left(1+R_z^2\right)^{\frac{1}{2}}}, \qquad R = 1 \quad \text{at } z = \pm L, \tag{31}$$

and

$$\int_{-L}^{L} \pi R^2 dz = V,$$

where $(-P + B_* V)$ in Eq. (7) is replaced by $-P$ in Eq. (31), a constant to be determined.

The $\dot{R}$ problem, where now $\dot{R} = \frac{dR}{dV}$, is

$$-\dot{P} = \frac{1}{R}\frac{d}{dz}\frac{R\dot{R}_z}{\left(1+R_z^2\right)^{\frac{3}{2}}} + \frac{1}{R^2\left(1+R_z^2\right)^{\frac{1}{2}}}\dot{R} = \mathcal{L}(\dot{R}), \qquad \dot{R} = 0 \quad \text{at } z = \pm L, \tag{32}$$

and

$$\int_{-L}^{L} 2\pi\, \dot{R}R dz = 1,$$

and the perturbation problem at constant V is

$$-P_1 = \frac{1}{R}\frac{d}{dz}\frac{RR_{1_z}}{\left(1+R_z^2\right)^{\frac{3}{2}}} + \frac{1}{R^2\left(1+R_z^2\right)^{\frac{1}{2}}}R_1 = \mathcal{L}(R_1), \quad R_1 = 0 \quad \text{at } z = \pm L, \tag{33}$$

and

$$\int_{-L}^{L} RR_1 dz = 0,$$

where again only $m = 0$ perturbations need to be taken into account. Critical points correspond to solutions R_1 not zero, turning points to $\dot{R}$ failing to exist. The variable B_* plays no role.

The eigenvalue problem is

$$-C = \mathcal{L}(\psi) + \lambda^2\psi, \quad \psi = 0 \text{ at } z = \pm L, \tag{34}$$

and

$$\int_{-L}^{L} R\psi dz = 0,$$

and we denote its solutions ψ, C, and λ^2.

If, at any point along our V versus P curve, at input values of L and B, we have $\lambda^2 = 0$, then $R_1 = \psi$, $P_1 = C$ and that point is a critical point. And it is also a turning point, now in V, i.e., V has reached a limiting value, because the solvability condition that must be satisfied for $\dot{R}$ to exist, viz.,

$$-C\int_{-L}^{L} R\dot{R}dz = \frac{-C}{2\pi} = 0, \tag{35}$$

is not satisfied because C is ordinarily not zero. In fact, if C were zero at $\lambda^2 = 0$, then we would have $\psi = \phi$, $\lambda^2 = \mu^2$, and $\int_{-L}^{L} R\phi dz = 0$, which is ordinarily not possible and certainly not possible at $\mu_1^2 = 0$ because ϕ_1 is singly signed.

Because the points on the V versus P curve at V control are the points on the V versus P curve at P control, at $B_* = 0$, at any point we can again expand the ψ's in the ϕ_j's, whereupon we have

$$C\int_{-L}^{L} R\phi_j dz = \left(\lambda^2 - \mu_j^2\right)\int_{-L}^{L} \psi R\phi_j dz, \tag{36}$$

$$C\frac{I_j}{\left(\lambda^2 - \mu_j^2\right)} = c_j, \tag{37}$$

and

$$C\sum \frac{I_j^2}{\left(\lambda^2 - \mu_j^2\right)} = 0, \tag{38}$$

and thus, due to C not zero, we have

$$\sum \frac{I_j^2}{\left(\lambda^2 - \mu_j^2\right)} = 0, \tag{39}$$

where the curves, left-hand side versus λ^2, are the same as before and the λ^2's here are the λ^2's earlier at $B_* \to \infty$. These λ^2's are known in terms of the μ^2's and lie to the right of the earlier λ^2's, but in the same intervals.

Setting B and L and increasing or decreasing V in search of a critical point we may have all λ^2's positive before arriving at a critical value of V where $\lambda_1^2 = 0$. At this point the homogeneous $\dot{R}$

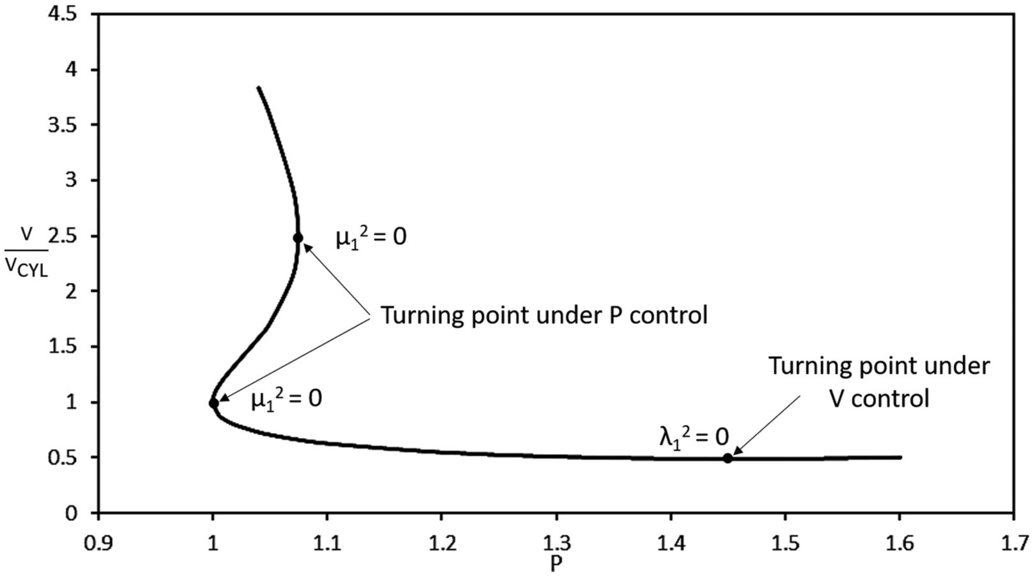

FIG. 6. Turning points and therefore points where the least eigenvalue is zero can be read directly off base curves, $B_* = 0, L = \frac{1}{2}\pi, B = \frac{1}{100}$.

problem has the solution $\dot{R} = \psi_1$, $\dot{P} = C_1$, and a solvability condition must be satisfied. It is

$$-\dot{P}\underbrace{\int_{-L}^{L} R\psi_1 dz}_{\text{zero}} + C_1 \underbrace{\int_{-L}^{L} R\dot{R} dz}_{\frac{1}{2\pi}} = 0, \tag{40}$$

and it is not satisfied. The critical point is a turning point and we have reached a limiting value of V.

We observe that turning points can be read directly off the base curve (cf. Fig. 6).

We have come to the point where, if B is not zero, no matter whether V or P is the input and no matter the value of B_* we have:

First, given B and L: (1) All the V versus P_* curves at B_* derive from the V versus P curve at $B_* = 0$; (2) All critical points are turning points and can be read off the base curves. No eigenvalue problems need to be solved to obtain critical points; (3) The bridge shapes depend on V, independent of B_*.

Second: (1) Only $m = 0$ perturbations need to be taken into account (2) The λ^2's at B_* can be obtained from the μ^2's at $B_* = 0$. (3) The λ^2's at B_* and the μ^2's at $B_* = 0$ are nested, viz., $\mu_1^2 < \lambda_1^2 < \mu_2^2 < \ldots$

To this point we have assumed B is not zero. But $B = 0$ may be achieved by setting $\Delta\rho = 0$. And many experiments have been run where $\Delta\rho$ is made as near to zero as possible [18]. This is an important limit to which we now direct our attention and we can say much more about the problem. In fact if B is not zero, critical points correspond to λ_1^2 equal to zero. But if B is zero μ_2^2 must also be taken into the account.

III. THE CASE $B = 0$

At $B = 0$ there is a special case that can be worked out by hand. At any L and B_* it is

$$R = 1 \text{ for all } z, \quad V = \pi 2L,$$

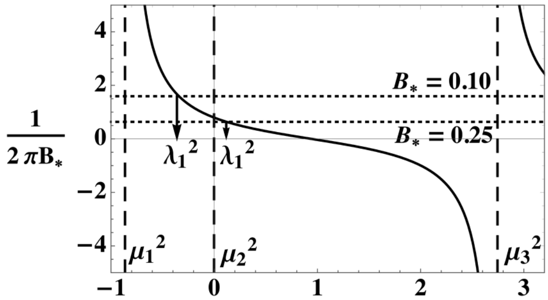

FIG. 7. Graphs of the solutions to Eq. (41), showing the first two eigenvalues, (λ_1^2, μ_2^2), at a small value at B_* and (μ_2^2, λ_1^2) at a large value of B_*. $L = \frac{3}{4}\pi$, $V = 0.975$.

and

$$P_* = 1 + B_* V.$$

The bridge shape is a cylinder and we may increase or decrease the pressure from the cylindrical bridge value causing the surface to move inward or outward. The cylindrical bridge problem is worked out in Appendix B.

At $B = 0$ our equations are again: Eqs. (7), (9), (10), and (11), but with B now set to zero. Eq. (7) has solutions $R(z)$ even in z. And, assuming $R(z)$ is even, the eigenvalue problems have solutions either even or odd in z, e.g., at $B_* = 0$, ϕ_1 is even, ϕ_2 is odd, etc. Due to this symmetry, at $B = 0$ critical points need not be turning points.

At input values of L and B_* and at any V we have two series of eigenvalues, viz., $\mu_2^2 < \mu_4^2 < \cdots$ corresponding to odd eigenfunctions. These are independent of B_* and are called odd eigenvalues for short. The other series, $\lambda_1^2 < \lambda_3^2 < \cdots$, corresponding to even eigenfunctions (called even eigenvalues for short) satisfy

$$\sum_{j=1,3,5,\cdots} \frac{I_j^2}{\left(\lambda^2 - \mu_j^2\right)} = \frac{1}{2\pi B_*} \tag{41}$$

or

$$\sum_{j=1,3,5,\cdots} \frac{I_j^2}{\left(\lambda^2 - \mu_j^2\right)} = 0, \tag{42}$$

according to whether pressure or volume is held fixed. A sketch is presented in Fig. 7 to illustrate that, due to symmetry, λ_1^2 may be on either side of μ_2^2.

The sketch in Fig. 7 is at pressure control for set values of L and V. It depicts the least eigenvalue corresponding to an even eigenfunction at two values of B_*. The least eigenvalue corresponding to an odd eigenfunction lies at μ_2^2 and is independent of B_*. The volume control case is obtained as $B_* \to \infty$ and is marked on the figure.

We see that at small values of B_* the least eigenvalue corresponds to an even eigenfunction, at large values of B_* it corresponds to an odd eigenfunction.

What is now new at B equal to zero, is, at $B_* > 0$, we do not know whether λ_1^2 or μ_2^2 is the least eigenvalue.

We observe that there is a value of B_* where $\lambda_1^2 = \mu_2^2$ at our given L and V. What is more, at a given L and another V we ought to find another B_* where $\lambda_1^2 = \mu_2^2$. For a given L, we expect to find a special (B_*, V) pair such that $\lambda_1^2 = 0 = \mu_2^2$. For greater B_*'s holding V fixed only odd eigenvalues are the first to reach zero. The special B_* is denoted $B_*^\dagger$. It has experimental significance (cf. Appendix C).

IV. OBTAINING BIFURCATION POINTS AT B = 0

At input values of L and B_* we suppose we have a stable bridge at a certain value of P_*. We may then increase or decrease P_* until we reach a critical value, a value of P_* where the least eigenvalue reaches zero. If the eigenvalue is odd, i.e., $\lambda^2 = \mu_2^2$, $\psi = \phi_2$, then we have, because R is even and ϕ_2 is odd,

$$\int_{-L}^{L} R\psi\, dz = \int_{-L}^{L} R\phi_2 dz = 0, \tag{43}$$

whereupon $\dot{R}$ exists because the solvability condition Eq. (19) is satisfied. Thus, we can increase P_* through its critical value and continue to find even solutions, $R(z)$, to the bridge-shape problem, though these solutions ought to be unstable. This critical point is not a turning point. It is called a bifurcation point and we ought to expect new bridge shapes branching from this point, though not shapes even in z. Bifurcation points do not appear if B is not zero. They require the symmetry of $R(z)$ attending B equal to zero.

If, however, the least eigenvalue is even, then R and ψ being even, we ordinarily have

$$\int_{-L}^{L} R\psi\, dz \neq 0. \tag{44}$$

The solvability condition is not satisfied, $\dot{R}$ does not exist and we have reached a limiting value of P_*, i.e., a turning point.

Having the V versus P curve at $B_* = 0$ we can obtain the V versus P_* curve at any $B_* \neq 0$ and turning points can be read off these curves. We can also obtain bifurcation points by knowing only bridge shapes along the V versus P or V versus P_* curve. To do this we need to know that at bifurcation points the least eigenvalue reaching zero corresponds to an odd eigenfunction and we need to know that $R_z(\pm L) = 0$. At input values of L and B_* and at any P_* we have $R(z)$ and V and we obtain an equation for R_z by differentiating Eq. (7) with respect to z, viz.,

$$0 = \frac{1}{R}\frac{d}{dz}\frac{R}{\left(1+R_z^2\right)^{\frac{3}{2}}}(R_z)_z + \frac{1}{R^2}\frac{1}{\left(1+R_z^2\right)^{\frac{1}{2}}}R_z. \tag{45}$$

Now at a bifurcation point where $\int_{-L}^{L} R\psi\, dz = 0$ we have

$$0 = \frac{1}{R}\frac{d}{dz}\frac{R}{\left(1+R_z^2\right)^{\frac{3}{2}}}\,\psi_z + \frac{1}{R^2}\frac{1}{\left(1+R_z^2\right)^{\frac{1}{2}}}\,\psi. \tag{46}$$

Multiplying Eq. (46) by RR_z, Eq. (45) by $R\psi$, subtracting and integrating over $(-L, L)$ we have

$$0 = \left.\frac{R_z R\psi_z}{\left(1+R_z^2\right)^{\frac{3}{2}}}\right|_{-L}^{L},$$

where $R(L) = 1 = R(-L)$ and where R is even, R_z is odd, ψ is odd and ψ_z is even. And because $\psi_z(L)$ cannot be zero we conclude

$$R_z(L) = 0 = R_z(-L).$$

Thus, at a bifurcation point the bridge shape must be such that $R_z = 0$ at $z = \pm L$.

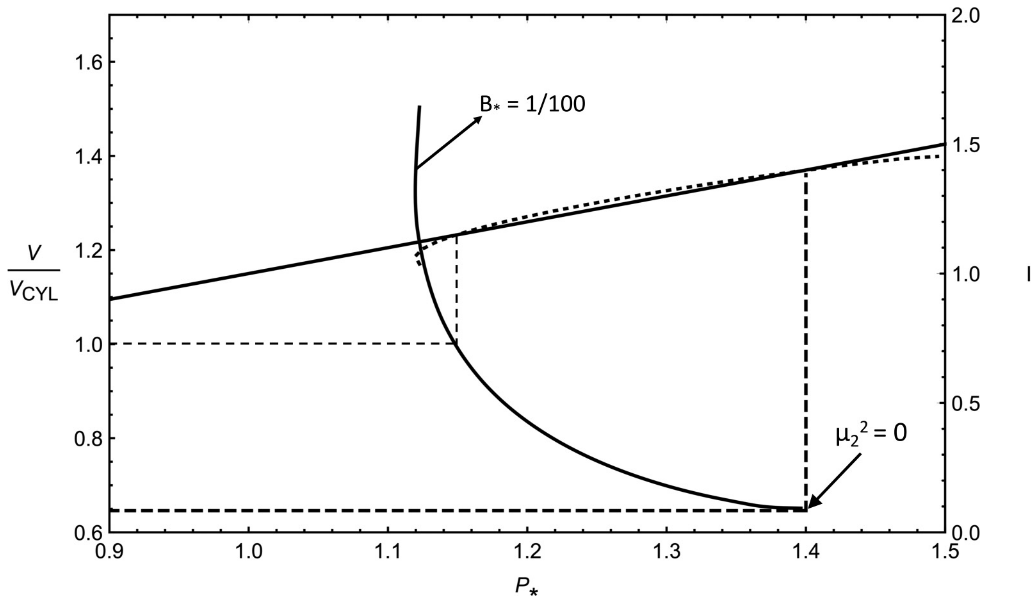

FIG. 8. The solid curve is the base curve V vs P_* at $B_* > 0$. The dot-dash curve is $\frac{1}{2L}\int_{-L}^{L}\frac{1}{R}\frac{1}{(1+R_z^2)^{\frac{1}{2}}}dz + B_*V$ vs P_* and the dot-dot curve is Eq. (49). Their intersection identifies the point (P_*, V) where $\mu_2^2 = 0$, i.e., a bifurcation point. The values of L and B_* are $L = \frac{3}{4}\pi$ and $B_* = \frac{1}{100}$.

Now integrating Eq. (7) over $(-L, L)$ we have

$$(-P_* + B_*V)2L = -\left.\frac{R_z}{\left(1+R_z^2\right)^{\frac{3}{2}}}\right|_{-L}^{L} - \int_{-L}^{L}\frac{1}{R}\frac{1}{\left(1+R_z^2\right)^{\frac{1}{2}}}dz, \tag{47}$$

whereupon at a bifurcation point we must have

$$\frac{1}{2L}\int_{-L}^{L}\frac{1}{R}\frac{1}{\left(1+R_z^2\right)^{\frac{1}{2}}}dz + B_*V = P_*. \tag{48}$$

Hence along with a plot of V versus P_* we can plot $\frac{1}{2L}\int_{-L}^{L}\frac{1}{R}\frac{1}{(1+R_z^2)^{\frac{1}{2}}}dz + B_*V$ versus P_*. Then if we also plot the straight line,

$$\frac{1}{2L}\int_{-L}^{L}\frac{1}{R}\frac{1}{\left(1+R_z^2\right)^{\frac{1}{2}}}dz + B_*V = P_*, \tag{49}$$

which is a locus of bifurcation points, the intersection will be a bifurcation point, i.e., a point along the V versus P_* curve where $\mu_2^2 = 0$. The two curves must cross if $R(z) = 1$ for all z. But this is the case of a cylindrical bridge where $R_z(\pm L)$ is always zero; cf. Appendix B.

Because μ_2^2 is independent of B_* we now have, given L, the volume at which a bifurcation point appears for all B_* and we obtain this knowing only the base shape, $R(z)$, at $B_* = 0$.

From the base shape at any B_* we derive both the turning points and the bifurcation points. At $B_* = 0$ and at the value of V where $\mu_2^2 = 0$, μ_1^2 is already negative. Holding this V fixed and increasing B_*, λ_1^2 is negative at first and increases to zero whereupon $\lambda_1^2 = 0 = \mu_2^2$ at $B_*^\dagger$. At larger B_*'s, the instability is always a bifurcation point.

The construction is illustrated in Fig. 8, where $\frac{1}{2L}\int_{-L}^{L}\frac{1}{R}\frac{1}{(1+R_z^2)^{\frac{1}{2}}}dz + B_*V$ is denoted I.

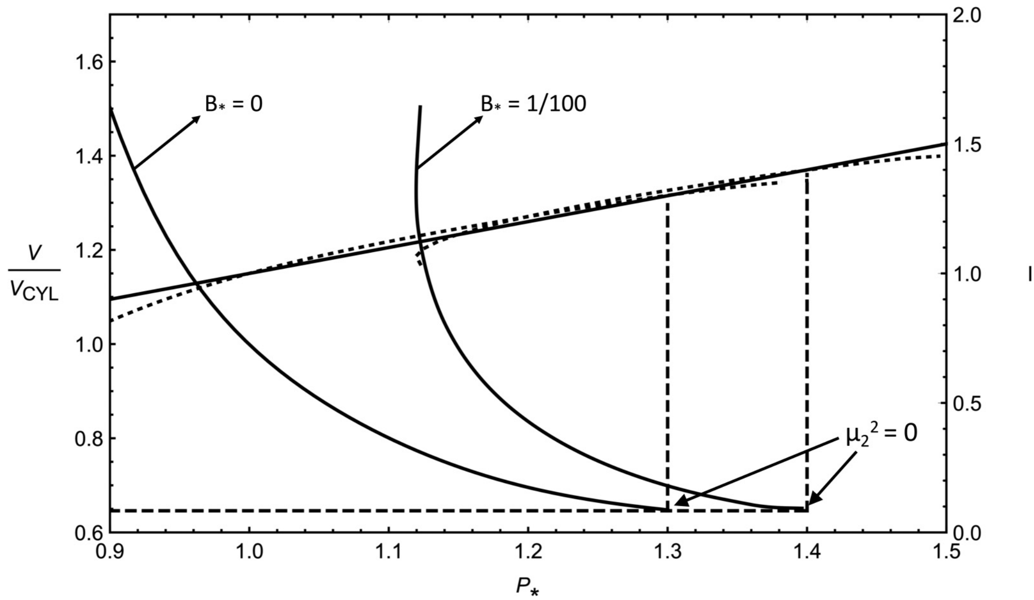

FIG. 9. At two values of B_* viz., $B_* = 0$ and $B_* = .01$, we obtain bifurcation points, $\mu_2^2 = 0$ at the same value of V. The value of L is $\frac{3}{4}\pi$.

At the same value of L we can carry out the construction at any other value of B_* and obtain the bifurcation point on the corresponding V versus P_* curve at the same value of V.

In particular we can do this at $B_* = 0$. Thus, at $B_* = 0$ we plot V versus P and $\frac{1}{2L}\int_{-L}^{L}\frac{1}{R}\frac{1}{(1+R_z^2)^{\frac{1}{2}}}$ versus P obtaining the bifurcation point on the V versus P curve and hence the bifurcation point on the V versus P_* curve for any B_* not zero; cf. Fig. 9.

The idea is, at L and V and for any B_* we have the same $R(z)$ and the same μ_2^2 as we do at $B_* = 0$, and the point where $\mu_2^2 = 0$ on the V versus P curve at $B_* = 0$ is also a bifurcation point for a bridge under volume control.

At $B = 0$ we now have the possibility of bifurcation points. Like turning points they can be derived from the base shape. Only the base shape at $B_* = 0$ is needed.

We present selected results in Appendix C. Upon setting the value of L we can obtain the V versus P curve at $B_* = 0$. At each V along the curve we have the eigenvalues $\mu_1^2 < \mu_2^2 < \mu_3^2 \cdots$. Then upon increasing B_* at V, at each value of B_* we have the eigenvalues $\lambda_1^2 < \lambda_3^2 < \cdots$ and $\mu_2^2 < \mu_4^2 \cdots$. Of most interest is whether the least eigenvalue corresponding to an even eigenfunction, viz., λ_1^2, or to an odd eigenfunction, viz., μ_2^2 reaches zero first. As B_* increases, λ_1^2 increases, starting at μ_1^2 at $B_* = 0$. However, μ_2^2 is obtained at $B_* = 0$ and does not change as B_* increases. If μ_1^2 is positive, then all μ^2's and λ^2's are positive. If μ_1^2 is zero, then we have a turning point at $B_* = 0$, but all other μ^2's and all λ^2's are positive. Hence, the results presented in Appendix C all correspond to $\mu_1^2 < 0$ and different possibilities for μ_2^2, positive, zero, or negative.

V. CONCLUSIONS

There is one and only one liquid bridge problem of interest. It is obtained by controlling the pressure and building reservoirs of large cross sections. It is the simplest problem to solve being completely constraint-free. Thus, from the solution to a completely constraint-free problem we can derive the solution to all other bridge problems, necessarily constrained, without further calculation.

The fundamental classification is in terms of the Bond number. If the Bond number is not zero, then all critical points are turning points corresponding to limiting values of pressure. If the Bond number is zero, then bifurcation points may arise. No eigenvalues need to be obtained, not even the unconstrained μ^2's at $B_* = 0$.

We can now see how to design an experiment to measure the surface tension between two liquids having nearly the same density.

APPENDIX A: POTENTIAL ENERGY CHANGE ON PERTURBATION

Our aim is to establish the result that: If a base solution is given a perturbation which is a multiple of an eigenfunction, then the sign of the change in the potential energy is the sign of the corresponding eigenvalue. Thus, to have a stable base solution, all the eigenvalues must be positive.

Assume we have set the values of L and B and we have a base solution $R = R_0(z)$. Now the base problem and the perturbation problem are static problems and we imagine the bridge is under volume control, hence the value of V is set, and we know that the only perturbations that can lead to a critical condition are axisymmetric.

The base solution is such that the potential energy of the bridge is least among all the possible bridge shapes satisfying the constraint, viz., the values of L, B, and V, and the solution is stable if all allowable perturbations cause the potential energy to increase.

The potential energy of a bridge having a shape $R(z)$ is

$$PE = \int_{-L}^{L} dz\left\{2\pi R\left(1 + R_z^2\right)^{\frac{1}{2}} + Bz\pi R^2\right\} \tag{A1}$$

and its volume is

$$V = \int_{-L}^{L} \pi R^2 dz. \tag{A2}$$

We impose a perturbation on a base shape $R_0(z)$, viz., $R(z) = R_0(z) + \epsilon R_1(z) + \frac{1}{2}\epsilon^2 R_2(z)$ and we calculate the change in the potential energy,viz., $PE(\epsilon) - PE(\epsilon = 0)$. The result, divided by 2π, is

$$\epsilon \int_{-L}^{L} dz\left\{\frac{R_0 R_{0z} R_{1z}}{\left(1 + R_{0z}^2\right)^{\frac{1}{2}}} + \left(1 + R_{0z}^2\right)^{\frac{1}{2}} R_1 + BzR_0R_1\right\}$$
$$+ \frac{1}{2}\epsilon^2 \int_{-L}^{L} dz\left\{\frac{R_0 R_{1z}^2}{\left(1 + R_{0z}^2\right)^{\frac{3}{2}}} + \frac{R_0 R_{0z} R_{2z}}{\left(1 + R_{0z}^2\right)^{\frac{1}{2}}} + 2\frac{R_{0z} R_1 R_{1z}}{\left(1 + R_{0z}^2\right)^{\frac{1}{2}}} + \left(1 + R_{0z}^2\right)^{\frac{1}{2}} R_2 + Bz\left(R_1^2 + R_0R_2\right)\right\}.$$

Integrating by parts we have

$$\epsilon \int_{-L}^{L} dz\left\{-\frac{d}{dz}\left(\frac{R_0 R_{0z}}{\left(1 + R_{0z}^2\right)^{\frac{1}{2}}}\right) R_1 + \left(1 + R_{0z}^2\right)^{\frac{1}{2}} R_1 + BzR_0R_1\right\}$$
$$+ \frac{1}{2}\epsilon^2 \int_{-L}^{L} dz\left\{-\frac{d}{dz}\left(\frac{R_0 R_{1z}}{\left(1 + R_{0z}^2\right)^{\frac{3}{2}}}\right) R_1 - \frac{d}{dz}\left(\frac{R_0 R_{0z}}{\left(1 + R_{0z}^2\right)^{\frac{1}{2}}}\right) R_2 - \frac{d}{dz}\left(\frac{R_{0z}}{\left(1 + R_{0z}^2\right)^{\frac{1}{2}}}\right) R_1^2\right.$$
$$\left. + \left(1 + R_{0z}^2\right)^{\frac{1}{2}} R_2 + Bz\left(R_1^2 + R_0R_2\right)\right\},$$

where upon we have

$$\epsilon \int_{-L}^{L} dz \left\{ -\frac{d}{dz}\left(\frac{R_{0z}}{\left(1+R_{0z}^2\right)^{\frac{1}{2}}}\right) + \frac{1}{R_0\left(1+R_{oz}^2\right)^{\frac{1}{2}}} + Bz \right\} R_0 R_1$$
$$+ \frac{1}{2}\epsilon^2 \int_{-L}^{L} dz \left[\left\{ -\frac{1}{R_0}\frac{d}{dz}\left(\frac{R_0 R_{1z}}{\left(1+R_{0z}^2\right)^{\frac{3}{2}}}\right) \right\} R_0 R_1 \right.$$
$$\left. + \left\{ \frac{1}{R_0\left(1+R_{0z}^2\right)^{\frac{1}{2}}} - \frac{d}{dz}\left(\frac{R_{0z}}{\left(1+R_{oz}^2\right)^{\frac{1}{2}}}\right) + Bz \right\} R_0 R_2 + \left\{ -\frac{d}{dz}\left(\frac{R_{0z}}{\left(1+R_{0z}^2\right)^{\frac{1}{2}}}\right) + Bz \right\} R_1^2 \right].$$

Then using

$$Bz - \frac{d}{dz}\left(\frac{R_{0z}}{\left(1+R_{0z}^2\right)^{\frac{1}{2}}}\right) = -\frac{1}{R_0\left(1+R_{0z}^2\right)^{\frac{1}{2}}} + P_0, \tag{A3}$$

we obtain

$$\epsilon \int_{-L}^{L} dz (P_0) R_0 R_1 + \frac{1}{2}\epsilon^2 \int_{-L}^{L} dz \left[\left\{ -\frac{1}{R_0}\frac{d}{dz}\frac{R_0 R_{1z}}{\left(1+R_{0z}^2\right)^{\frac{3}{2}}} - \frac{1}{R_0^2}\frac{R_1}{\left(1+R_{0z}^2\right)^{\frac{1}{2}}} \right\} R_0 R_1 \right.$$
$$\left. + \left\{ (P_0)\left(R_1^2 + R_0 R_2\right) \right\} \right],$$

where the terms in P_0, a constant, vanish due to

$$\int_{-L}^{L} dz R_0 R_1 = 0 = \int_{-L}^{L} dz\left(R_1^2 + R_0 R_2\right), \tag{A4}$$

a result of holding V constant.

The change in potential energy is then

$$\frac{1}{2}\epsilon^2 \int_{-L}^{L} dz(-1) \left\{ \frac{1}{R_0}\frac{d}{dz}\frac{R_0 R_{1z}}{\left(1+R_{0z}^2\right)^{\frac{3}{2}}} + \frac{R_1}{R_0^2\left(1+R_{0z}^2\right)^{\frac{1}{2}}} \right\} R_0 R_1,$$

and we can choose R_1 to be any solution to the eigenvalue problem, say ψ, and find the change is given by

$$\frac{1}{2}\epsilon^2 \lambda^2 \int_{-L}^{L} R_0 \psi^2 dz.$$

We conclude that the bridge is stable or unstable to a perturbation ψ depending on whether λ^2 is positive or negative. If all λ^2 are positive, then the bridge is stable. If the least λ^2 is zero, then the bridge is critical.

APPENDIX B: CYLINDRICAL BRIDGE

At $B = 0$, we have a solution $R = 1$ for all z to Eq. (7). The corresponding bridge is called a cylindrical bridge and many cylindrical bridge experiments have been run [18]. The length, $2L$, is the main input whereupon $V = \pi 2L$ and $P_* = 1 + B_* V$.

Perturbations can be carried out at constant pressure where L and B_* are held fixed.

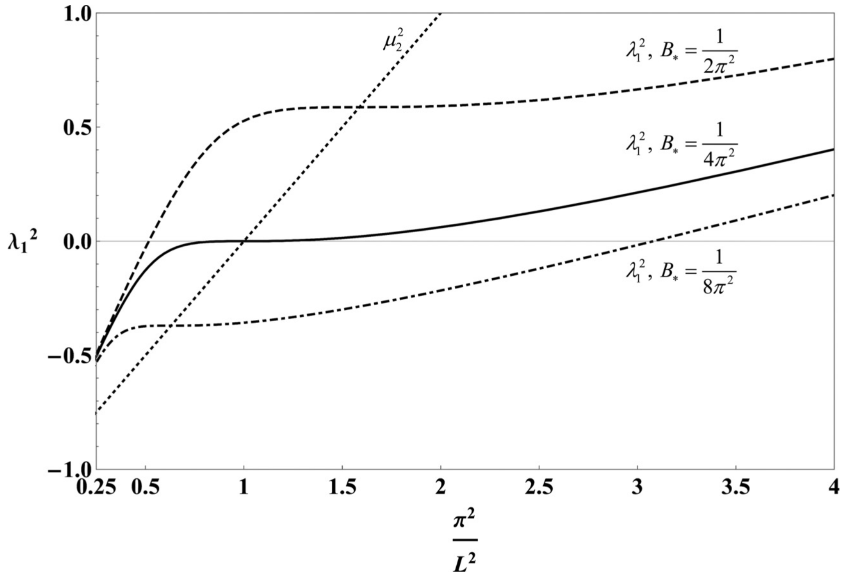

FIG. 10. The least even eigenvalues for three values of B_* and the least odd eigenvalue plotted vs $\frac{\pi^2}{L^2}$.

At constant pressure, where pressure is not perturbed, the eigenvalue problem, at B_* equal to zero, is

$$0 = \frac{d^2\phi}{dz^2} + (1+\mu^2)\phi, \phi = 0 \quad \text{at } z = \pm L. \tag{B1}$$

It has odd and even solutions, where the odd eigenvalues are $\mu_2^2, \mu_4^2, \cdots$, viz.,

$$\frac{\pi^2}{L^2} - 1, \quad \frac{4\pi^2}{L^2} - 1, \ldots,$$

and the even eigenvalues are $\mu_1^2, \mu_3^2, \cdots$, viz.,

$$\frac{1}{4}\frac{\pi^2}{L^2} - 1, \quad \frac{9}{4}\frac{\pi^2}{L^2} - 1, \ldots$$

and where $\mu_1^2 = 0$ if and only if $L = \frac{1}{2}\pi$ and $\mu_2^2 = 0$ if and only if $L = \pi$.

At a value of B_*, greater than zero, the eigenvalue problem is

$$2\pi B_* \int_{-L}^{L} \psi dz = \frac{d^2\psi}{dz^2} + (1+\lambda^2)\psi, \psi = 0 \text{ at } z = \pm L. \tag{B2}$$

The odd eigenvalues are $\mu_2^2, \mu_4^2, \cdots$ as above and the even eigenvalues, $\lambda_1^2, \lambda_3^2, \cdots$ are the solutions to the equation

$$\frac{\tan(1+\lambda^2)^{\frac{1}{2}}L}{(1+\lambda^2)^{\frac{1}{2}}L} = \frac{2\pi B_* 2L - (1+\lambda^2)}{2\pi B_* 2L}. \tag{B3}$$

In Fig. 10 we present a plot of the least even eigenvalues versus $\frac{\pi^2}{L^2}$ for three values of B_*. We have not plotted the $B_* = 0$ eigenvalue, $\mu_1^2 = \frac{1}{4}\frac{\pi^2}{L^2} - 1$. But we have plotted μ_2^2, the least odd eigenvalue. It is independent of B_*. As L increases the graphs are read from right to left.

At constant pressure short bridges become unstable to even perturbations if B_* is low enough. Upon increasing the length of the bridge, every bridge becomes unstable to an odd perturbation at

$L = \pi$ and thereafter, whether or not it is stable to an even perturbation. Thus, for large values of B_*, where the bridge is stable to even perturbations, it is already unstable to an odd perturbation and the critical length is $L = \pi$, no matter the value of B_*. Thus, no matter whether pressure or volume is being controlled the critical value of L must be π if B_* is large enough, viz., greater than $\frac{1}{4\pi^2}$.

In the case of a cylindrical bridge $\mu_2^2 = 0 = \lambda_1^2$ at $L = \pi$, $B_* = \frac{1}{4\pi^2}$. Now, a cross over point is of interest if an experiment is being planned. We denote values of B_* where $\lambda_1^2 = 0 = \mu_2^2$ by B_*^+. A graph of B_*^+ vs L is given in Appendix C for noncylindrical bridges.

APPENDIX C: NONCYLINDRICAL BRIDGE

Each of Figs. (11), (12), and (13) presents volume versus pressure curves for several values of B_*. On each figure a value of V is selected and the corresponding value of λ_1^2 is indicated at each of the several values of B_* at the selected V. In Fig. 11, V is chosen so that μ_2^2 is positive. In Fig. 12 μ_2^2 is zero, and in Fig. 13 μ_2^2 is negative, where μ_2^2 does not depend on B_*. On each figure a plot of

$$\sum_{j=1,3,5,\cdots} \frac{I_j^2}{(\lambda^2 - \mu_j^2)} \quad \text{vs} \quad \lambda^2 \tag{C1}$$

is presented for the values of L and V corresponding to that figure. The construction of λ_1^2 for a selected B_* is then shown.

Figure 14 presents the value of B_*, where $\lambda_1^2 = 0 = \mu_2^2$, as a function of L. The limiting value of L is about 1.89, below which the bridge can only break symmetrically. It corresponds to the volume-controlled bridge. We can make use of what we have found to design an experiment to measure the surface tension between two liquids of nearly equal density. By running pressure-controlled experiments we have a variable, B_*, which we do not have running volume-controlled experiments.

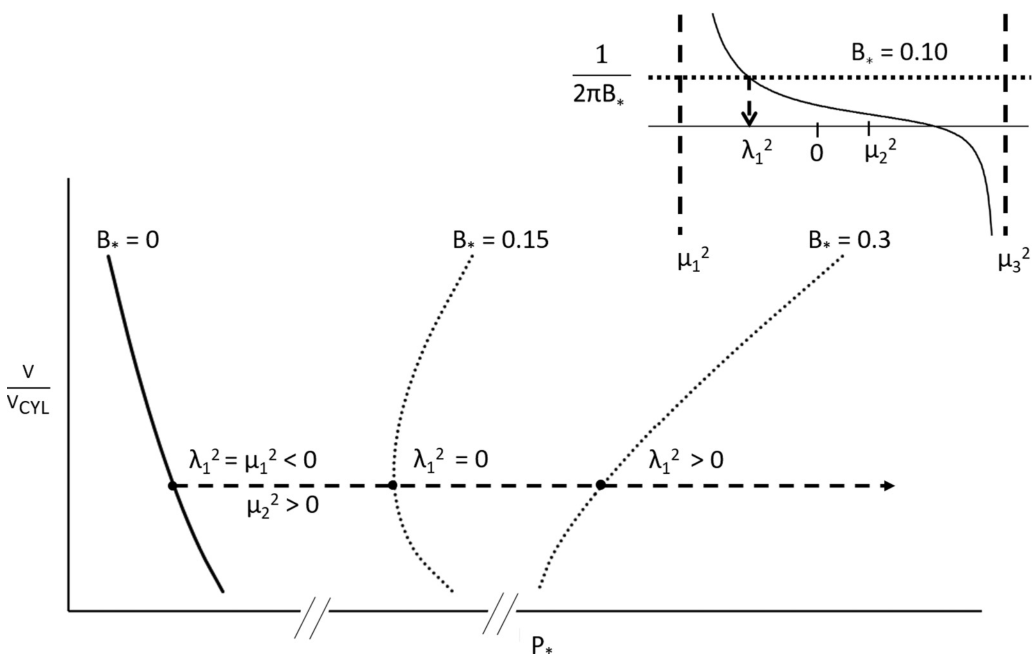

FIG. 11. Volume vs Pressure Curves at $B_* = 0, 0.15, 0.30$. A value of V is chosen so that $\mu_2^2 > 0$.

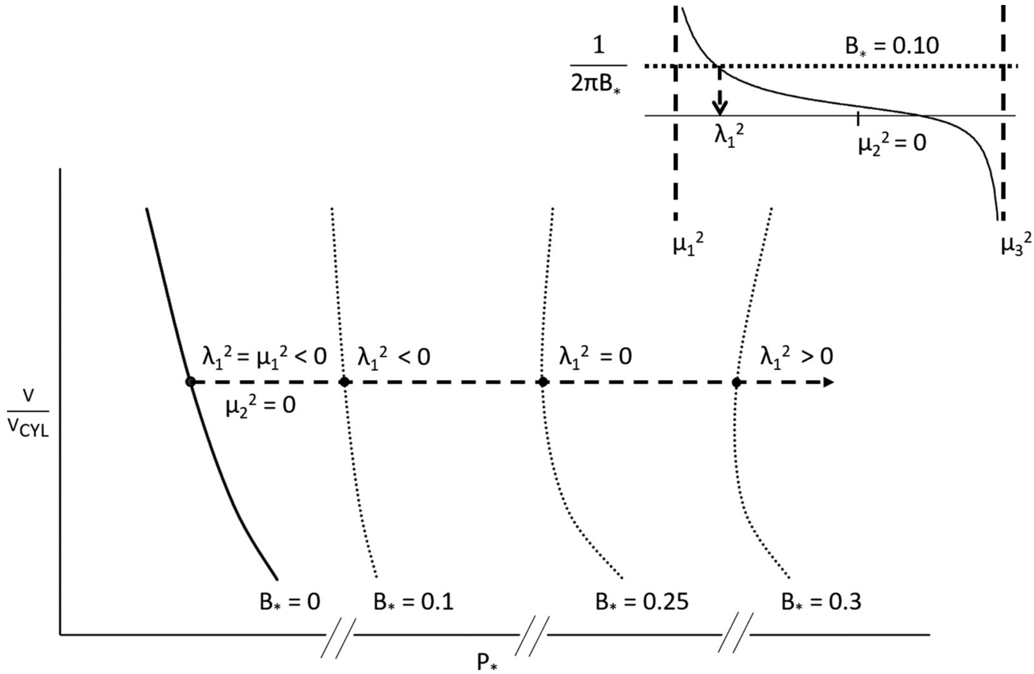

FIG. 12. Volume vs pressure curve at $B_* = 0, 0.1, 0.25, 0.30$. A value of V is chosen so that $\mu_2^2 = 0$.

To take a simple example, assume upon setting L we have $\mu_1^2 < 0$ all along the V versus P curve at $B_* = 0$ and $\mu_2^2 = 0$ at one point on the curve. Except for the value of γ, B_* is known once we have agreed on an experiment.

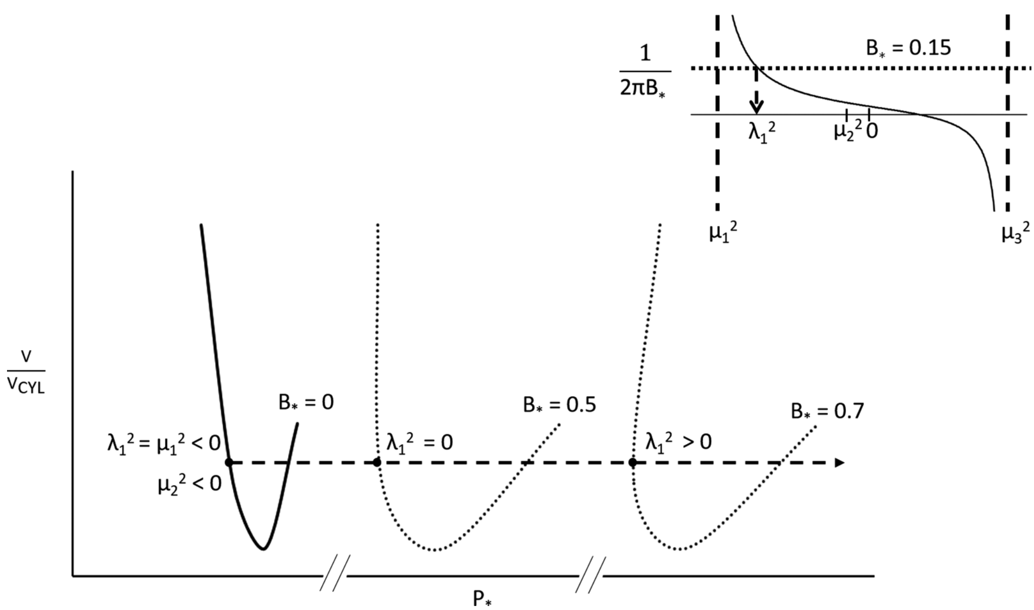

FIG. 13. Volume vs pressure curve at $B_* = 0, 0.15, 0.30$. A value of V is chosen so that $\mu_2^2 < 0$.

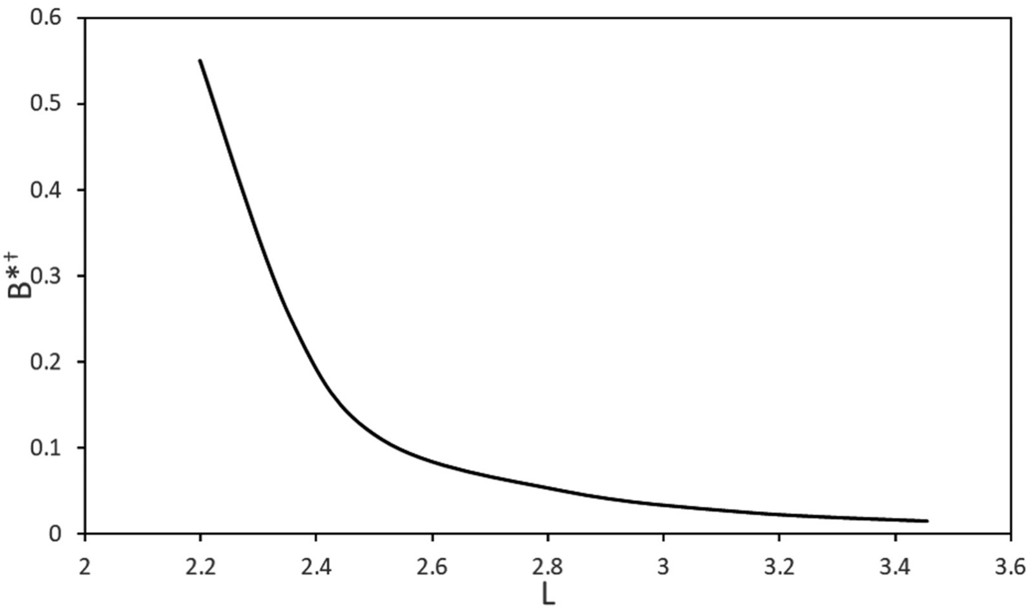

FIG. 14. A graph of $B_*^\dagger$ as a function of L. Note that $\lambda_1^2 = 0 = \mu_2^2$ at each point on the curve and that V changes along the curve.

Now what we know is that for each V there is a value of B_* such that $\lambda_1^2 = 0$ and as we decrease V the value of B_* increases until we arrive at V, say $V^\dagger$, such that $B_* = B_*^\dagger$, i.e., $\lambda_1^2 = 0 = \mu_2^2$. Thus, for any L we can plot B_* versus V at $\lambda_1^2 = 0$ i.e, the locus of turning points, where B_* is less than $B_*^\dagger$, and V is greater than $V^\dagger$, and where $B_*^\dagger$ and $V^\dagger$ depend on L.

Then the measurement is V at critical i.e., at $\lambda_1^2 = 0$, along with the shape of the critical eigenfunction, whereupon V implies B_* implies γ.

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