Editors' Suggestion

Sound waves propagating in a slightly rarefied gas over a smooth solid boundary

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A time-evolution of a slightly rarefied gas from a uniform equilibrium state at rest is investigated on the basis of the linearized Boltzmann equation under the acoustic time scaling. By a systematic asymptotic analysis, linearized Euler sets of equations and acoustic-boundary-layer equations are derived, together with their slip and jump boundary conditions, as well as the correction formula in the Knudsen layer. Analysis is done up to the first order of the Knudsen number (Kn), with $\mathrm{Kn}^{1/2}$ being the small parameter. Several rarefaction effects, which are known as the effects of the second order in Kn in the diffusion scaling, are enhanced to be of the first order in Kn. This is because the variation of the macroscopic quantities along the normal direction is steep in the boundary layer and the compressibility of the gas is comparatively strong. The occurrence of secular terms associated with the Hilbert expansion is pointed out and a remedy for it is also given. Finally, as an application example, a sound propagation in a half space caused by a sinusoidal oscillation of flat boundary is examined on the basis of the Bhatnagar–Gross–Krook equation. The asymptotic solution agrees well with the direct numerical solution.

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I. INTRODUCTION

Propagation of sound waves in a gas is a classical problem in fluid dynamics. The application of the conventional fluid dynamics implicitly requires that the wavelength (a characteristic length) is long enough compared to the mean free path of gas molecules. This assumption breaks down in low-pressure circumstances. It also breaks down when the oscillation frequency is extremely high. Here, we generically call the gas in these situations a rarefied gas. To describe their behavior correctly, the kinetic theory of gases is required. Sound propagation in a rarefied gas has been studied extensively for more than half a century [1–14]. In the present paper, we shall focus on the acoustic behavior of gases in the slip-flow regime.

When the Knudsen number (i.e., the ratio of the mean free path to the characteristic length) is small, the overall behavior of the gas is well described by fluid-dynamic-type equations under their appropriate slip and jump boundary conditions. The regime of small Knudsen numbers is thus called the slip-flow regime. A correction due to the kinetic effect is required within a thin layer (the Knudsen layer) with a few mean-free-path thickness adjacent to the body surface. A systematic asymptotic theory has been established for steady boundary-value problems [15–18]. As to the unsteady problems under the acoustic scaling, the behavior of the gas can be described by the linearized Euler set of equations in the bulk region; see Ref. [17]. In Ref. [12], a sound propagation

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in a polyatomic rarefied gas in contact with its condensed phase is considered. For a vapor flow, the kinetic effect appears from the leading order near the vapor-liquid interfaces. Accordingly, the primary interest in this reference is to derive the appropriate fluid-dynamic-type system and Knudsen-layer corrections near the interface at the level of the leading order. The linearized Euler set of equations and acoustic-boundary-layer equations have been obtained, together with the appropriate boundary conditions at the interface and the associated corrections in the Knudsen layer, at the level of the leading order [12].

In the present work, we consider a monatomic rarefied gas over a solid boundary whose position and temperature change in time under the acoustic scaling. Under this scaling, steep variation of fluid-dynamic quantities is expected to enhance the non-Navier–Stokes effect, compared with the situation under the diffusion scaling. We try to clarify what kind of rarefaction effects appear in the slip-flow regime under the acoustic scaling.

The paper is organized as follows. The problem is stated in Sec. II and formulated in Sec. III. A preliminary analysis is given in Sec. IV for the clarity of motivation to the main analyses to be developed in Sec. V. In Sec. V, by a systematic asymptotic analysis, linearized Euler sets of equations and boundary-layer equations are derived, together with their slip and jump boundary conditions. The correction formulas in the Knudsen layer are also presented. They are obtained up to the order of the Knudsen number. Section VI discusses the features of the system, where the occurrence of secular terms in the Hilbert solution is pointed out and a remedy to it is given. In Sec. VII, as an illustrative example, a sound propagation in a half space caused by a sinusoidal oscillation of flat boundary is investigated on the basis of the Bhatnagar–Gross–Krook (BGK) equation. The asymptotic solution agrees well with the numerical solution. In particular, the proposed remedy for the secular terms is confirmed to be effective. Concluding remarks are given in Sec. VIII.

II. PROBLEM AND ASSUMPTION

Let us consider a monatomic rarefied gas around smooth rigid bodies of arbitrary shapes, which is initially at a uniform equilibrium state at rest with density ρ_0 and temperature T_0. The initial state will be taken as the reference equilibrium state. There is no external force. We will investigate the time-dependent behavior of the gas under the following assumptions: (i) the behavior of the gas is described by the Boltzmann equation for monatomic molecules; (ii) the gas molecules are reflected locally isotropically on the surface of the solid bodies by a rule of gas-surface interaction prescribed later in details; (iii) the deviation from the reference equilibrium state is so small that the equation and the initial and boundary conditions can be linearized around that equilibrium state; (iv) the mean free path ℓ_0 (or time) of the gas molecules at the reference equilibrium state is much smaller than the characteristic length (or time) of the problem; (v) the time evolution is initiated by a change of the surroundings from the reference equilibrium state with the characteristic time scale.

With a reference timescale t_0 and length scale L, we denote by $t_0 t$ the time, by $L\boldsymbol{x}$ the position, by $(2RT_0)^{1/2}\boldsymbol{\zeta}$ the molecular velocity, and by $\rho_0(2RT_0)^{-3/2}[1+\phi(t,\boldsymbol{x},\boldsymbol{\zeta})]E(\zeta)$ the velocity distribution function of molecules, respectively. Here, R is the specific gas constant (the Boltzmann constant k_B divided by the mass of a molecule m), $E(\zeta)=\pi^{-3/2}e^{-\zeta^2}$, and $\zeta=|\boldsymbol{\zeta}|$. We denote by $\rho_0(1+\omega)$ the density of the gas, by $(2RT_0)^{1/2}u_i$ the flow velocity, by $T_0(1+\tau)$ the temperature, by $p_0(1+P)$ the pressure with $p_0=\rho_0 RT_0$, by $p_0(\delta_{ij}+P_{ij})$ the stress tensor, and by $p_0(2RT_0)^{1/2}Q_i$ the heat-flow vector, respectively. Here δ_{ij} is the Kronecker delta. They are defined in terms of ϕ as

$$\omega=\int \phi E\, d\boldsymbol{\zeta}, \tag{1}$$

$$u_i=\int \zeta_i \phi E\, d\boldsymbol{\zeta}, \tag{2}$$

$$\tau=\frac{2}{3}\int \left(\zeta^2-\frac{3}{2}\right)\phi E\, d\boldsymbol{\zeta}, \tag{3}$$

$$P = \omega + \tau, \tag{4}$$

$$P_{ij} = 2\int \zeta_i \zeta_j \phi E d\boldsymbol{\zeta}, \tag{5}$$

$$Q_i = \int \zeta_i \left(\zeta^2 - \frac{5}{2}\right)\phi E d\boldsymbol{\zeta}. \tag{6}$$

The range of integration is the whole space of $\boldsymbol{\zeta}$. Equation (4) is the linearized equation of state.

From the assumption (iv), the Knudsen number $\mathrm{Kn} = \ell_0/L$ is sufficiently small.

The linearization assumption (iii) implies that the perturbation ϕ is much smaller than Kn, because the square and higher-order terms are discarded for the former but retained for the latter. Accordingly, the Mach number Ma is much smaller than Kn.

In the present paper, we are interested in the acoustic phenomena induced by slow motion or small variation of surface temperature of the bodies. Thus we set the appropriate reference time as $t_0 = L/(2RT_0)^{1/2}$ for their descriptions. Since one of the primary sources of the sound wave is an oscillation of the bodies, the surface of the bodies may have the velocity component normal to itself in the present paper.

III. FORMULATION OF THE PROBLEM

From assumptions (i)–(v), the behavior of the gas is described by the following initial and boundary-value problem for ϕ:

$$\partial_t \phi(t, \boldsymbol{x}, \boldsymbol{\zeta}) + \zeta_i \partial_i \phi(t, \boldsymbol{x}, \boldsymbol{\zeta}) = \frac{1}{k}\mathcal{L}[\phi](t, \boldsymbol{x}, \boldsymbol{\zeta}), \tag{7}$$

$$\begin{aligned}\phi(t, \boldsymbol{x}, \boldsymbol{\zeta}) = &\int_{(\zeta_{i*}-u_{\mathrm{w}i})n_i<0} \frac{|\mathring{\zeta}_{k*}n_k|}{|\mathring{\zeta}_j n_j|}\mathcal{R}(\mathring{\boldsymbol{\zeta}}_*, \mathring{\boldsymbol{\zeta}})[\phi(t, \boldsymbol{x}, \boldsymbol{\zeta}_*) - \phi_{\mathrm{ew}}(t, \boldsymbol{x}, \boldsymbol{\zeta}_*)]\frac{E(\zeta_*)}{E(\zeta)}d\boldsymbol{\zeta}_* \\ &+ \phi_{\mathrm{ew}}(t, \boldsymbol{x}, \boldsymbol{\zeta}), \quad [x_i = x_{\mathrm{w}i},\ (\zeta_i - u_{\mathrm{w}i})n_i > 0],\end{aligned} \tag{8}$$

$$\phi(0, \boldsymbol{x}, \boldsymbol{\zeta}) = 0. \tag{9}$$

Here $\partial_t = \partial/\partial t,\ \partial_i = \partial/\partial x_i,\ k = (\sqrt{\pi}/2)\mathrm{Kn},\ \zeta_* = |\boldsymbol{\zeta}_*|,\ \mathring{\boldsymbol{\zeta}} = \boldsymbol{\zeta} - (\boldsymbol{u}_{\mathrm{w}} \cdot \boldsymbol{n})\boldsymbol{n},\ \mathring{\boldsymbol{\zeta}}_* = \boldsymbol{\zeta}_* - (\boldsymbol{u}_{\mathrm{w}} \cdot \boldsymbol{n})\boldsymbol{n}$, $\phi_{\mathrm{ew}} = 2\zeta_i u_{\mathrm{w}i} + (\zeta^2 - \frac{3}{2})\tau_{\mathrm{w}}$, $\mathcal{L}$ is the linearized collision operator, $\mathcal{R}$ is the linearized reflection kernel, $Lx_{\mathrm{w}i}$ is the surface position, n_i is the unit vector normal to the surface pointed to the gas, and $(2RT_0)^{1/2}u_{\mathrm{w}i}$ and $T_0(1 + \tau_{\mathrm{w}})$ are the velocity and temperature of the body surface, respectively. For $\mathcal{R}$, the following fundamental properties are assumed:

(a) $\mathcal{R}(\mathring{\boldsymbol{\zeta}}_*, \mathring{\boldsymbol{\zeta}}) \geqslant 0,\ \ (\mathring{\zeta}_i n_i > 0, \mathring{\zeta}_{i*}n_i < 0)$.

(b) Impermeability: $\int_{\mathring{\zeta}_i n_i>0} \mathcal{R}(\mathring{\boldsymbol{\zeta}}_*, \mathring{\boldsymbol{\zeta}})d\mathring{\boldsymbol{\zeta}} = 1,\ \ (\mathring{\zeta}_{i*}n_i < 0)$.

(c) Let φ be $\varphi(\mathring{\boldsymbol{\zeta}}) = c_0 + c_i\mathring{\zeta}_i + c_4\mathring{\zeta}^2$, where c_0, c_i, and c_4 are independent of $\mathring{\boldsymbol{\zeta}}$. Among such φ, only $\varphi = c_0$ satisfies the relation $E(\mathring{\zeta})\varphi(\mathring{\boldsymbol{\zeta}}) = \int_{\mathring{\zeta}_{i*}n_i<0} \frac{|\mathring{\zeta}_{k*}n_k|}{|\mathring{\zeta}_j n_j|}\mathcal{R}(\mathring{\boldsymbol{\zeta}}_*, \mathring{\boldsymbol{\zeta}})\varphi(\mathring{\boldsymbol{\zeta}}_*)E(\mathring{\zeta}_*)d\mathring{\boldsymbol{\zeta}}_*\ (\mathring{\zeta}_i n_i > 0)$.

(d) Detailed balance:

$$|\mathring{\zeta}_{i*}n_i|\mathcal{R}(\mathring{\boldsymbol{\zeta}}_*, \mathring{\boldsymbol{\zeta}})E(\mathring{\zeta}_*) = |\mathring{\zeta}_i n_i|\mathcal{R}(-\mathring{\boldsymbol{\zeta}}, -\mathring{\boldsymbol{\zeta}}_*)E(\mathring{\zeta}), \quad (\mathring{\zeta}_i n_i > 0,\ \mathring{\zeta}_{i*}n_i < 0).$$

(e) Local isotropy: for any orthogonal transformation matrices l_{ij} (thus, $l_{ik}l_{jk} = \delta_{ij}$) that meet the condition $n_i = l_{ij}n_j$,

$$\int_{\mathring{\zeta}_{i*}n_i<0} \frac{|\mathring{\zeta}_{k*}n_k|}{|\mathring{\zeta}_j n_j|}\mathcal{R}(\mathring{\boldsymbol{\zeta}}_*, \mathring{\boldsymbol{\zeta}})f(l_{ij}\mathring{\zeta}_{j*})E(\mathring{\zeta}_*)d\mathring{\boldsymbol{\zeta}}_* = \int_{\mathring{\zeta}_{i*}n_i<0} \frac{|\mathring{\zeta}_{k*}n_k|}{|l_{ji}\mathring{\zeta}_i n_j|}\mathcal{R}(\mathring{\boldsymbol{\zeta}}_*, l_{ij}\mathring{\zeta}_j)f(\mathring{\boldsymbol{\zeta}}_*)E(\mathring{\zeta}_*)d\mathring{\boldsymbol{\zeta}}_*.$$

Here, $\mathring{\zeta} = |\mathring{\boldsymbol{\zeta}}|$ and $\mathring{\zeta}_* = |\mathring{\boldsymbol{\zeta}}_*|$. For the diffuse reflection boundary condition, $\mathcal{R}(\mathring{\boldsymbol{\zeta}}_*, \mathring{\boldsymbol{\zeta}}) = 2\sqrt{\pi}\mathring{\zeta}_j n_j E(\mathring{\zeta})$. The collision operator $\mathcal{L}$ acts on a function of $\boldsymbol{\zeta}$ and is defined by

$$\mathcal{L}[\psi(\boldsymbol{\zeta})](\boldsymbol{\zeta}) = \int [\psi(\boldsymbol{\zeta}'_*) + \psi(\boldsymbol{\zeta}') - \psi(\boldsymbol{\zeta}_*) - \psi(\boldsymbol{\zeta})] \times \hat{B}\left(\frac{|\boldsymbol{\alpha}\cdot(\boldsymbol{\zeta}_* - \boldsymbol{\zeta})|}{|\boldsymbol{\zeta}_* - \boldsymbol{\zeta}|}, |\boldsymbol{\zeta}_* - \boldsymbol{\zeta}|\right) E(\zeta_*) d\Omega(\boldsymbol{\alpha}) d\boldsymbol{\zeta}_*, \tag{10}$$

$$\boldsymbol{\zeta}'_* = \boldsymbol{\zeta}_* - [(\boldsymbol{\zeta}_* - \boldsymbol{\zeta})\cdot\boldsymbol{\alpha}]\boldsymbol{\alpha}, \quad \boldsymbol{\zeta}' = \boldsymbol{\zeta} + [(\boldsymbol{\zeta}_* - \boldsymbol{\zeta})\cdot\boldsymbol{\alpha}]\boldsymbol{\alpha}. \tag{11}$$

Here $\boldsymbol{\alpha}$ is a unit vector, $d\Omega(\boldsymbol{\alpha})$ is the solid-angle element in the direction of $\boldsymbol{\alpha}$, $\hat{B}$ is a nonnegative function which depends on the intermolecular potential. The range of integration is the whole directions of $\boldsymbol{\alpha}$ and the whole space of $\boldsymbol{\zeta}_*$. For hard-sphere molecules, $\hat{B} = |\boldsymbol{\alpha}\cdot(\boldsymbol{\zeta}_* - \boldsymbol{\zeta})|/4(2\pi)^{1/2}$.

Following Sone's method [17], we seek the solution of the problem Eqs. (7)–(9).

IV. ATTEMPT IN TERMS OF THE KNUDSEN NUMBER EXPANSION

At first, we seek a solution ϕ^*_{H} that changes moderately both in t and x_i in a power series of k: $\phi^*_{\mathrm{H}} = \phi^*_{\mathrm{H0}} + k\phi^*_{\mathrm{H1}} + \cdots$. Corresponding macroscopic quantities denoted by h^*_{H} ($h = \omega, u_i, \tau, P, P_{ij}, Q_i$) are also expanded in a power series of k: $h^*_{\mathrm{H}} = h^*_{\mathrm{H0}} + kh^*_{\mathrm{H1}} + \cdots$. The $h^*_{\mathrm{H}m}$ ($m = 0, 1, \ldots$) is expressed by Eqs. (1)–(6) with ϕ being replaced by $\phi^*_{\mathrm{H}m}$. Then, by a usual procedure as in Ref. [17], the leading-order solution ϕ^*_{H0} is found to be a local Maxwellian,

$$\phi^*_{\mathrm{H0}} = P^*_{\mathrm{H0}} + 2\zeta_i u^*_{i\mathrm{H0}} + \left(\zeta^2 - \frac{5}{2}\right)\tau^*_{\mathrm{H0}}, \tag{12}$$

such that h^*_{H0} satisfies the linearized Euler set of equations and the equation of state:

$$\partial_t \omega^*_{\mathrm{H0}} + \partial_j u^*_{j\mathrm{H0}} = 0, \tag{13}$$

$$\partial_t u^*_{i\mathrm{H0}} + \tfrac{1}{2}\partial_i P^*_{\mathrm{H0}} = 0, \tag{14}$$

$$\tfrac{3}{2}\partial_t P^*_{\mathrm{H0}} + \tfrac{5}{2}\partial_j u^*_{j\mathrm{H0}} = 0, \tag{15}$$

$$P^*_{\mathrm{H0}} = \omega^*_{\mathrm{H0}} + \tau^*_{\mathrm{H0}}. \tag{16}$$

At the initial instance, it is assumed that the gas is in the reference equilibrium state at rest and that $u_{\mathrm{w}i} = 0$ and $\tau_{\mathrm{w}} = 0$. Thus, by setting $h^*_{\mathrm{H0}} = 0$ ($t = 0$), ϕ^*_{H0} satisfies the initial condition. However, ϕ^*_{H0} cannot satisfy the boundary condition as we will see below.

Suppose that ϕ^*_{H} satisfies the boundary condition Eq. (8). Then, ϕ^*_{H0} ought to satisfy Eq. (8) with ϕ_{ew} being replaced by ϕ^*_{ew0}, where $\phi^*_{\mathrm{ew0}} = 2\zeta_i u^*_{\mathrm{w}i0} + (\zeta^2 - \frac{3}{2})\tau^*_{\mathrm{w0}}$ and $u^*_{\mathrm{w}i0}$ and τ^*_{w0} are the leading-order term of $u_{\mathrm{w}i}$ and τ_{w} in their expansion in k. Because ϕ^*_{H0}, ϕ^*_{ew0}, and E are smooth in $\boldsymbol{\zeta}_*$ and $\boldsymbol{\zeta}$, the part $[\phi^*_{\mathrm{H0}}(t, \boldsymbol{x}, \boldsymbol{\zeta}_*) - \phi^*_{\mathrm{ew0}}(t, \boldsymbol{x}, \boldsymbol{\zeta}_*)]E(\zeta_*)/E(\zeta)$ on the right-hand side can be replaced by $[\phi^*_{\mathrm{H0}}(t, \boldsymbol{x}, \mathring{\boldsymbol{\zeta}}_*) - \phi^*_{\mathrm{ew0}}(t, \boldsymbol{x}, \mathring{\boldsymbol{\zeta}}_*)]E(\mathring{\zeta}_*)/E(\mathring{\zeta})$ within the negligible error in the linearized framework. Therefore, from the property (c) in Sec. III, ϕ^*_{H0} satisfies the boundary condition if and only if

$$u^*_{i\mathrm{H0}} = u^*_{\mathrm{w}i0}, \quad \tau^*_{\mathrm{H0}} = \tau^*_{\mathrm{w0}}, \tag{17}$$

on the boundary. However, these conditions are too many as the conditions for the linearized Euler set of equations. To illustrate it, let us first combine Eqs. (14) and (15) to derive the wave equation $\partial_t^2 P^*_{\mathrm{H0}} - \frac{5}{6}\partial_j^2 P^*_{\mathrm{H0}} = 0$. Then, multiply Eq. (14) by n_i to have $n_i\partial_i P^*_{\mathrm{H0}} = -2\partial_t(n_i u^*_{i\mathrm{H0}})$ within the negligible error on the surface, because $\partial_t n_i = O(u_{\mathrm{w}i})$. When Eq. (17) holds, its partial information $n_i u^*_{i\mathrm{H0}} = n_i u^*_{\mathrm{w}i0}$ gives the Neumann boundary condition $n_i\partial_i P^*_{\mathrm{H0}} = -2\partial_t(n_i u^*_{\mathrm{w}i0})$ for the wave equation, which is enough to determine P^*_{H0} under a given initial data. Once $P^*_{\rm H0}$ is determined, the other quantities $u^*_{i\rm H0}$, $\omega^*_{\rm H0}$, and $\tau^*_{\rm H0}$ are immediately obtained by integrating Eqs. (14) and (13) with respect to t under the initial conditions and with the aid of Eq. (16). In this process, the remaining part of the condition Eq. (17), i.e., $(\delta_{ij} - n_i n_j)u^*_{j\rm H0} = (\delta_{ij} - n_i n_j)u^*_{{\rm w}j0}$ and $\tau^*_{\rm H0} = \tau^*_{\rm w0}$ are not used at all. Hence, the system Eqs. (13)–(17) is overdetermined.

When the moderate solution does not satisfy the boundary condition, it is a usual procedure to introduce the Knudsen layer adjacent to the boundary. In this procedure, we express the solution ϕ as $\phi = \phi^*_{\rm H} + \phi^*_{\rm K}$ in the layer with the thickness of the order of k (the Knudsen layer), where $\phi^*_{\rm K}$ varies with the distance from the boundary in the scale of $O(k)$ $[n_i\partial_i\phi^*_{\rm K} = k^{-1}O(\phi^*_{\rm K})]$ and vanishes sufficiently fast away from the boundary. However, $\phi^*_{\rm K}$ is not capable of working as a correction at the leading order. This is a consequence of the existence and uniqueness theorem for the half-space problem of the linearized Boltzmann equation [19] and the fact that $\phi^*_{\rm H0}$ is a local Maxwellian. The procedure constructing the entire solution with the aid of the Knudsen layer does not work.

Judging from the above facts and as discussed in Refs. [12,17], a boundary layer of diffusive nature is expected to appear in addition to the Knudsen layer. Denoting by L_d the characteristic length of this layer, $L_d \sim \sqrt{(\mu_0/\rho_0)t_0}$, where μ_0 is the reference viscosity. In view of the relation of μ_0 to ℓ_0, $L_d \sim \sqrt{k}L$, so that L_d is sufficiently small compared to L [20]. In the rest of the present paper, we seek the solution by taking account of this layer and expand the velocity distribution function and macroscopic quantities in a power series of

$$\varepsilon = \sqrt{k}. \tag{18}$$

We will start the analysis again from seeking the moderate solution.

V. ASYMPTOTIC ANALYSIS

A. Hilbert solution

Putting aside the initial and boundary conditions, we seek the Hilbert solution $\phi_{\rm H}$ that changes moderately both in t and x_i. Corresponding macroscopic quantities are denoted by $h_{\rm H}$ ($h = \omega$, u_i, τ, P, P_{ij}, Q_i). The $\phi_{\rm H}$ and $h_{\rm H}$ are expanded in a power series of ε:

$$\phi_{\rm H} = \phi_{\rm H0} + \varepsilon\phi_{\rm H1} + \varepsilon^2\phi_{\rm H2} + \varepsilon^3\phi_{\rm H3} + \cdots, \tag{19}$$

$$h_{\rm H} = h_{\rm H0} + \varepsilon h_{\rm H1} + \varepsilon^2 h_{\rm H2} + \varepsilon^3 h_{\rm H3} + \cdots. \tag{20}$$

Here $h_{{\rm H}m}$ $(m = 0, 1, \ldots)$ is defined by Eqs. (1)–(6) with ϕ being replaced by $\phi_{{\rm H}m}$. Substituting the expansion Eq. (19) and $k = \varepsilon^2$ into Eq. (7) and arranging the terms by the order of ε, a series of linear integral equations for $\phi_{{\rm H}m}$ is obtained as follows:

$$\mathcal{L}[\phi_{\rm H0}] = 0, \tag{21}$$

$$\mathcal{L}[\phi_{\rm H1}] = 0, \tag{22}$$

$$\mathcal{L}[\phi_{{\rm H}m}] = \partial_t\phi_{{\rm H}m-2} + \zeta_i\partial_i\phi_{{\rm H}m-2}, \quad (m \geqslant 2). \tag{23}$$

The solutions of Eqs. (21) and (22) are given as

$$\phi_{\rm H0} = \phi_{\rm eH0}, \quad \phi_{\rm H1} = \phi_{\rm eH1}, \tag{24}$$

where

$$\phi_{{\rm eH}m} = P_{{\rm H}m} + 2\zeta_i u_{i{\rm H}m} + \left(\zeta^2 - \tfrac{5}{2}\right)\tau_{{\rm H}m}, \quad (m = 0, 1, \ldots). \tag{25}$$

Corresponding to Eq. (24), $P_{ij\rm H0}$, $Q_{i\rm H0}$, $P_{ij\rm H1}$, and $Q_{i\rm H1}$ are given by

$$P_{ij\rm H0} = P_{\rm H0}\delta_{ij}, \quad Q_{i\rm H0} = 0, \quad P_{ij\rm H1} = P_{\rm H1}\delta_{ij}, \quad Q_{i\rm H1} = 0. \tag{26}$$

Equation (23) for $\phi_{\mathrm{H}m}$ $(m \geqslant 2)$ is inhomogeneous and it can be solved provided that the solvability condition

$$\int \psi_J(\partial_t\phi_{\mathrm{H}m-2} + \zeta_i\partial_i\phi_{\mathrm{H}m-2})E d\boldsymbol{\zeta} = 0, \quad (J = 0, 1, 2, 3, 4), \tag{27}$$

or equivalently

$$\partial_t\omega_{\mathrm{H}m-2} + \partial_j u_{j\mathrm{H}m-2} = 0, \tag{28}$$

$$\partial_t u_{i\mathrm{H}m-2} + \tfrac{1}{2}\partial_j P_{ij\mathrm{H}m-2} = 0, \tag{29}$$

$$\partial_t P_{\mathrm{H}m-2} + \tfrac{5}{3}\partial_j u_{j\mathrm{H}m-2} + \tfrac{2}{3}\partial_j Q_{j\mathrm{H}m-2} = 0, \tag{30}$$

are satisfied. Here ψ_J is the collision invariant, i.e., $\psi_0 = 1$, $\psi_i = \zeta_i$ $(i = 1, 2, 3)$, and $\psi_4 = \zeta^2$.

Substituting Eq. (26) into the solvability condition Eqs. (28)–(30) for $m = 2, 3$, the following sets of equations are obtained:

$$\begin{cases} \partial_t\omega_{\mathrm{H}0} + \partial_j u_{j\mathrm{H}0} = 0, & (31) \\ \partial_t u_{i\mathrm{H}0} + \tfrac{1}{2}\partial_i P_{\mathrm{H}0} = 0, & (32) \\ \tfrac{3}{2}\partial_t P_{\mathrm{H}0} + \tfrac{5}{2}\partial_j u_{j\mathrm{H}0} = 0, & (33) \end{cases}$$

$$\begin{cases} \partial_t\omega_{\mathrm{H}1} + \partial_j u_{j\mathrm{H}1} = 0, & (34) \\ \partial_t u_{i\mathrm{H}1} + \tfrac{1}{2}\partial_i P_{\mathrm{H}1} = 0, & (35) \\ \tfrac{3}{2}\partial_t P_{\mathrm{H}1} + \tfrac{5}{2}\partial_j u_{j\mathrm{H}1} = 0. & (36) \end{cases}$$

Both sets of Eqs. (31)–(33) and (34)–(36) are linearized Euler sets of equations. No attenuation effects appear up to this level.

With the aid of Eqs. (24) and (31)–(33), the equation for $\phi_{\mathrm{H}2}$ can be rewritten as

$$\mathcal{L}[\phi_{\mathrm{H}2}] = \zeta_{ij}\overline{\partial_i u_{j\mathrm{H}0}} + \zeta_i\big(\zeta^2 - \tfrac{5}{2}\big)\partial_i\tau_{\mathrm{H}0}, \tag{37}$$

where $\zeta_{ij} = \zeta_i\zeta_j - (1/3)\zeta^2\delta_{ij}$ and $\overline{f_{ij}} = f_{ij} + f_{ji} - (2/3)f_{kk}\delta_{ij}$. This equation can be solved to yield

$$\phi_{\mathrm{H}2} = \phi_{\mathrm{eH}2} - \tfrac{1}{2}\zeta_{ij}B(\zeta)\overline{\partial_i u_{j\mathrm{H}0}} - \zeta_i A(\zeta)\partial_i\tau_{\mathrm{H}0}, \tag{38}$$

where A and B are functions defined in Appendix A. Equation (38) leads to the following expression of $P_{ij\mathrm{H}2}$ and $Q_{i\mathrm{H}2}$:

$$P_{ij\mathrm{H}2} = P_{\mathrm{H}2}\delta_{ij} - \gamma_1\overline{\partial_i u_{j\mathrm{H}0}}, \quad Q_{i\mathrm{H}2} = -\tfrac{5}{4}\gamma_2\partial_i\tau_{\mathrm{H}0}, \tag{39}$$

where γ_1 and γ_2 are constants defined in Appendix A. Substituting Eq. (39) into the solvability condition Eqs. (28)–(30) for $m = 4$, the following is obtained:

$$\partial_t\omega_{\mathrm{H}2} + \partial_j u_{j\mathrm{H}2} = 0, \tag{40}$$

$$\partial_t u_{i\mathrm{H}2} + \tfrac{1}{2}\partial_i P_{\mathrm{H}2} = \tfrac{1}{2}\gamma_1\partial_j\overline{\partial_i u_{j\mathrm{H}0}}, \tag{41}$$

$$\tfrac{3}{2}\partial_t P_{\mathrm{H}2} + \tfrac{5}{2}\partial_j u_{j\mathrm{H}2} = \tfrac{5}{4}\gamma_2\partial_j^2\tau_{\mathrm{H}0}. \tag{42}$$

This is again a linearized Euler set of equations. The main difference from Eqs. (31)–(33) and (34)–(36) is the occurrence of the attenuation effects.

Equations (31)–(33), (34)–(36), and (40)–(42) are the linearized Euler sets of equations for $h_{\mathrm{H}m}$ $(m = 0, 1, 2;\ h = \omega, u_i, \tau, P)$. They are supplemented by the equation of state:

$$P_{\mathrm{H}m} = \omega_{\mathrm{H}m} + \tau_{\mathrm{H}m}, \quad (m = 0, 1, 2). \tag{43}$$

B. Acoustic boundary layer

Obviously, ϕ_{H0} in Sec. V A is identical to ϕ^*_{H0} in Sec. IV. Hence, the same arguments as those in Sec. IV about the latter apply, and ϕ_{H0} is found not to match the boundary condition. The boundary-layer correction of diffusive nature needs to be introduced. Let us express the solution ϕ as $\phi = \phi_{\mathrm{H}} + \phi_{\mathrm{B}}$ in the layer with the thickness of the order of ε, where ϕ_{B} varies with the distance from the boundary in the scale of $O(\varepsilon)$ $[n_i \partial_i \phi_{\mathrm{B}} = \varepsilon^{-1} O(\phi_{\mathrm{B}})]$ and vanishes sufficiently fast away from the boundary [21]. Corresponding macroscopic quantities are denoted by h_{B} ($h = \omega, u_i, \tau, P, P_{ij}, Q_i$).

The ϕ_{B} satisfies the same equation as ϕ_{H} or ϕ. We shall, however, rewrite it by a new coordinate system which is more suitable in describing the boundary layer. Let us first introduce coordinates χ_1 and χ_2 fixed on the surface and time t to express the position $x_{\mathrm{w}i}$ on the boundary by

$$x_{\mathrm{w}i} = x_{\mathrm{w}i}(t, \chi_1, \chi_2). \tag{44}$$

Here, χ_1 and χ_2 are orthogonal to each other. We denote by $t_i^{(1)}$ and $t_i^{(2)}$ the tangential unit vectors along the coordinates χ_1 and χ_2, respectively. The χ_1 and χ_2 are taken so that n_i, $t_i^{(1)}$, and $t_i^{(2)}$ form the right-handed system with this order. The velocity of the boundary $u_{\mathrm{w}i}$ and the unit normal vector to the boundary n_i are also the functions of t, χ_1, and χ_2:

$$u_{\mathrm{w}i}(t, \chi_1, \chi_2) = \partial_t x_{\mathrm{w}i}, \tag{45}$$

$$n_i(t, \chi_1, \chi_2) = \frac{(\partial_{\chi_1}\boldsymbol{x}_{\mathrm{w}} \times \partial_{\chi_2}\boldsymbol{x}_{\mathrm{w}})_i}{|\partial_{\chi_1}\boldsymbol{x}_{\mathrm{w}} \times \partial_{\chi_2}\boldsymbol{x}_{\mathrm{w}}|}, \tag{46}$$

where $\partial_{\chi_{1,2}} = \partial/\partial\chi_{1,2}$. Then we introduce a stretched coordinate y normal to the boundary as

$$x_i = \varepsilon y n_i(t, \chi_1, \chi_2) + x_{\mathrm{w}i}(t, \chi_1, \chi_2). \tag{47}$$

Then, within the negligible error in the linearized framework, the original equation for ϕ_{B} is transformed into

$$\partial_t \phi_{\mathrm{B}} + \frac{\zeta_n}{\varepsilon}\partial_y \phi_{\mathrm{B}} + \zeta_i\big(\partial_i\chi_1\partial_{\chi_1} + \partial_i\chi_2\partial_{\chi_2}\big)\phi_{\mathrm{B}} = \frac{1}{\varepsilon^2}\mathcal{L}[\phi_{\mathrm{B}}], \tag{48}$$

where $\partial_y = \partial/\partial y$ and $\zeta_n = \zeta_i n_i$. By assumption, ϕ_{B} and h_{B} vanish rapidly as $y \to \infty$:

$$\phi_{\mathrm{B}} \to 0 \ \text{ as } \ y \to \infty, \tag{49}$$

$$h_{\mathrm{B}} \to 0 \ \text{ as } \ y \to \infty. \tag{50}$$

By expanding ϕ_{B} and h_{B} in a power series of ε as [22]

$$\phi_{\mathrm{B}} = \phi_{\mathrm{B0}} + \varepsilon\phi_{\mathrm{B1}} + \varepsilon^2\phi_{\mathrm{B2}} + \cdots, \tag{51}$$

$$h_{\mathrm{B}} = h_{\mathrm{B0}} + \varepsilon h_{\mathrm{B1}} + \varepsilon^2 h_{\mathrm{B2}} + \cdots, \tag{52}$$

we have a series of equations for $\phi_{\mathrm{B}m}$ ($m = 0, 1, 2, 3, 4$):

$$\mathcal{L}[\phi_{\mathrm{B0}}] = 0, \tag{53}$$

$$\mathcal{L}[\phi_{\mathrm{B1}}] = \zeta_n \partial_y \phi_{\mathrm{B0}}, \tag{54}$$

$$\mathcal{L}[\phi_{\mathrm{B2}}] = \zeta_n \partial_y \phi_{\mathrm{B1}} + \partial_t \phi_{\mathrm{B0}} + \zeta_i \mathcal{D}_i[\phi_{\mathrm{B0}}], \tag{55}$$

$$\mathcal{L}[\phi_{\mathrm{B3}}] = \zeta_n \partial_y \phi_{\mathrm{B2}} + \partial_t \phi_{\mathrm{B1}} + \zeta_i \mathcal{D}_i[\phi_{\mathrm{B1}}] + \zeta_i \mathcal{D}_i^{(1)}[\phi_{\mathrm{B0}}], \tag{56}$$

$$\mathcal{L}[\phi_{\mathrm{B4}}] = \zeta_n \partial_y \phi_{\mathrm{B3}} + \partial_t \phi_{\mathrm{B2}} + \zeta_i \mathcal{D}_i[\phi_{\mathrm{B2}}] + \zeta_i \mathcal{D}_i^{(1)}[\phi_{\mathrm{B1}}] + \zeta_i \mathcal{D}_i^{(2)}[\phi_{\mathrm{B0}}]. \tag{57}$$

Here

$$\mathcal{D}_i[f] = (\partial_i \chi_1)_{\mathrm{w}} \partial_{\chi_1} f + (\partial_i \chi_2)_{\mathrm{w}} \partial_{\chi_2} f,$$
$$\mathcal{D}_i^{(1)}[f] = y n_j [(\partial_j \partial_i \chi_1)_{\mathrm{w}} \partial_{\chi_1} f + \left(\partial_j \partial_i \chi_2\right)_{\mathrm{w}} \partial_{\chi_2} f],$$
$$\mathcal{D}_i^{(2)}[f] = \tfrac{1}{2} y^2 n_j n_k [(\partial_k \partial_j \partial_i \chi_1)_{\mathrm{w}} \partial_{\chi_1} f + (\partial_k \partial_j \partial_i \chi_2)_{\mathrm{w}} \partial_{\chi_2} f],$$

and $(\cdot)_{\mathrm{w}}$ denotes the value on the boundary ($y = 0$). From Eqs. (49) and (50), $\phi_{\mathrm{B}m}$ and $h_{\mathrm{B}m}$ ($h = \omega$, u_i, τ, P, P_{ij}, Q_i) also vanish sufficiently fast as $y \to \infty$:

$$\phi_{\mathrm{B}m} \to 0 \ \text{ as } \ y \to \infty, \tag{58}$$

$$h_{\mathrm{B}m} \to 0 \ \text{ as } \ y \to \infty. \tag{59}$$

For Eqs. (54)–(57) to be solvable, the condition

$$\int \psi_J \big(\zeta_n \partial_y \phi_{\mathrm{B}m-1} + \partial_t \phi_{\mathrm{B}m-2} + \zeta_i \mathcal{D}_i[\phi_{\mathrm{B}m-2}] + \zeta_i \mathcal{D}_i^{(1)}[\phi_{\mathrm{B}m-3}]$$
$$+ \zeta_i \mathcal{D}_i^{(2)}[\phi_{\mathrm{B}m-4}]\big) E d\boldsymbol{\zeta} = 0, \quad (J = 0, 1, 2, 3, 4), \tag{60}$$

or equivalently

$$\partial_y (n_j u_{j\mathrm{B}m-1}) + \partial_t \omega_{\mathrm{B}m-2} + \mathcal{D}_j[u_{j\mathrm{B}m-2}] + \mathcal{D}_j^{(1)}[u_{j\mathrm{B}m-3}] + \mathcal{D}_j^{(2)}[u_{j\mathrm{B}m-4}] = 0, \tag{61}$$

$$\tfrac{1}{2}\partial_y (n_j P_{ij\mathrm{B}m-1}) + \partial_t u_{i\mathrm{B}m-2} + \tfrac{1}{2}\mathcal{D}_j[P_{ij\mathrm{B}m-2}] + \tfrac{1}{2}\mathcal{D}_j^{(1)}[P_{ij\mathrm{B}m-3}] + \tfrac{1}{2}\mathcal{D}_j^{(2)}[P_{ij\mathrm{B}m-4}] = 0, \tag{62}$$

$$\tfrac{2}{5}\partial_y (n_j Q_{j\mathrm{B}m-1}) + \partial_t \tau_{\mathrm{B}m-2} - \tfrac{2}{5}\partial_t P_{\mathrm{B}m-2} + \tfrac{2}{5}\big(\mathcal{D}_j[Q_{j\mathrm{B}m-2}] + \mathcal{D}_j^{(1)}[Q_{j\mathrm{B}m-3}] + \mathcal{D}_j^{(2)}[Q_{j\mathrm{B}m-4}]\big) = 0, \tag{63}$$

must be satisfied for $m = 1, 2, 3, 4$. Here, the quantities $\phi_{\mathrm{B}n}$ and $h_{\mathrm{B}n}$ ($n \leqslant -1$) should be read as zero.

The solution process of Eqs. (53)–(57) is similar to that in Sec. V A. Here, we omit the details and show the resulting set of equations for the boundary-layer correction up to the order of the Knudsen number:

$$\partial_y (n_i u_{i\mathrm{B}0}) = 0, \tag{64}$$

$$\partial_y P_{\mathrm{B}0} = 0, \tag{65}$$

$$\partial_t \big(t_i^{(\alpha)} u_{i\mathrm{B}0}\big) - \frac{1}{2}\gamma_1 \partial_y^2 \big(t_i^{(\alpha)} u_{i\mathrm{B}0}\big) = 0, \tag{66}$$

$$\partial_t \tau_{\mathrm{B}0} - \frac{1}{2}\gamma_2 \partial_y^2 \tau_{\mathrm{B}0} = 0, \tag{67}$$

$$\partial_y (n_i u_{i\mathrm{B}1}) = -\partial_t \omega_{\mathrm{B}0} + \sum_{\beta=1}^{2} \big[(-1)^\beta g_{3-\beta} t_i^{(\beta)} u_{i\mathrm{B}0} - \chi_{\beta,\beta} \partial_{\chi_\beta} \big(t_i^{(\beta)} u_{i\mathrm{B}0}\big)\big], \tag{68}$$

$$\partial_y P_{\mathrm{B}1} = 0, \tag{69}$$

$$\partial_t \big(t_i^{(\alpha)} u_{i\mathrm{B}1}\big) - \frac{1}{2}\gamma_1 \partial_y^2 \big(t_i^{(\alpha)} u_{i\mathrm{B}1}\big) = \gamma_1 \bar{\kappa} \partial_y \big(t_i^{(\alpha)} u_{i\mathrm{B}0}\big), \tag{70}$$

$$\partial_t \tau_{\mathrm{B}1} - \frac{1}{2}\gamma_2 \partial_y^2 \tau_{\mathrm{B}1} = \gamma_2 \bar{\kappa} \partial_y \tau_{\mathrm{B}0}, \tag{71}$$

$$\partial_y(n_i u_{iB2}) = -\partial_t \omega_{B1} - 2\bar{\kappa} n_i u_{iB1} + \sum_{\beta=1}^{2} \big[-\chi_{\beta,\beta}\partial_{\chi_\beta}\big(t_i^{(\beta)} u_{iB1}\big) + (-1)^{\beta} g_{3-\beta} t_i^{(\beta)} u_{iB1} + y\kappa_\beta \chi_{\beta,\beta}\partial_{\chi_\beta}\big(t_i^{(\beta)} u_{iB0}\big) - y\vartheta\chi_{3-\beta,3-\beta}\partial_{\chi_{3-\beta}}\big(t_i^{(\beta)} u_{iB0}\big) - (-1)^{\beta} y(\kappa_{3-\beta} g_{3-\beta} - \vartheta g_\beta) t_i^{(\beta)} u_{iB0} \big], \tag{72}$$

$$\partial_y\left[P_{B2} - \frac{1}{6}(\gamma_1\gamma_2 - 4\gamma_3)\partial_y^2 \tau_{B0}\right] = -2\partial_t(n_i u_{iB1}) + \gamma_1 \partial_y^2 (n_i u_{iB1}), \tag{73}$$

$$\begin{aligned}
&\partial_t(t_i^{(\alpha)} u_{iB2}) - \frac{1}{2}\gamma_1 \partial_y^2 (t_i^{(\alpha)} u_{iB2}) \\
&= \gamma_1 \bar{\kappa}\,\partial_y\big(t_i^{(\alpha)} u_{iB1}\big) + \frac{1}{2}\gamma_1 \Bigg\{ -\sum_{\beta=1}^{2}(-1)^{\beta} g_{3-\beta}\chi_{\beta,\beta}\partial_{\chi_\beta}\big(t_i^{(\alpha)} u_{iB0}\big) \\
&\quad + 3(-1)^{\alpha}\sum_{\beta=1}^{2} g_\beta \chi_{\beta,\beta}\partial_{\chi_\beta}(t_i^{(3-\alpha)} u_{iB0}) + \sum_{\beta=1}^{2}\chi_{\beta,\beta}\partial_{\chi_\beta}\big[\chi_{\beta,\beta}\partial_{\chi_\beta}\big(t_i^{(\alpha)} u_{iB0}\big)\big] \\
&\quad + \sum_{\beta=1}^{2}(-1)^{\alpha+\beta+1}\chi_{\beta,\beta}\partial_{\chi_\beta}\big[\chi_{3-\beta,3-\beta}\partial_{\chi_{3-\beta}}\big(t_i^{(3-\alpha)} u_{iB0}\big)\big] - y\big(\kappa_1^2 + \kappa_2^2 + 2\vartheta^2\big)\partial_y\big(t_i^{(\alpha)} u_{iB0}\big) \\
&\quad + \big(t_i^{(\alpha)} u_{iB0}\big)\Big[-2\kappa_\alpha\bar{\kappa} - 2\big(g_1^2 + g_2^2\big) + \sum_{\beta=1}^{2}(-1)^{\beta}\chi_{\beta,\beta}\partial_{\chi_\beta} g_{3-\beta}\Big] \\
&\quad + \big(t_i^{(3-\alpha)} u_{iB0}\big)\Big[2\vartheta\bar{\kappa} + (-1)^{\alpha}\sum_{\beta=1}^{2}\chi_{\beta,\beta}\partial_{\chi_\beta} g_\beta\Big]\Bigg\} - \frac{1}{2}\chi_{\alpha,\alpha}\partial_{\chi_\alpha}\left[P_{B2} - \frac{1}{6}(\gamma_1\gamma_2 - 4\gamma_3)\partial_y^2\tau_{B0}\right] \\
&\quad - \frac{1}{4}(\gamma_1\gamma_{10} - 2\gamma_6)\partial_y^4\big(t_i^{(\alpha)} u_{iB0}\big),
\end{aligned} \tag{74}$$

$$\partial_t \tau_{B2} - \frac{1}{2}\gamma_2\partial_y^2\tau_{B2} = \frac{2}{5}\partial_t P_{B2} + \frac{1}{2}\gamma_2\Bigg\{2\bar{\kappa}\,\partial_y\tau_{B1} - y\big(\kappa_1^2 + \kappa_2^2 + 2\vartheta^2\big)\partial_y\tau_{B0} + \sum_{\beta=1}^{2}\big[\chi_{\beta,\beta}\partial_{\chi_\beta}(\chi_{\beta,\beta}\partial_{\chi_\beta}\tau_{B0}) - (-1)^{\beta} g_{3-\beta}\chi_{\beta,\beta}\partial_{\chi_\beta}\tau_{B0}\big]\Bigg\} - \frac{1}{10}\left(\gamma_2\gamma_3 - \frac{13}{2}\gamma_{11}\right)\partial_y^4\tau_{B0}. \tag{75}$$

Equations (64)–(75) are supplemented by the equation of state:

$$P_{Bm} = \omega_{Bm} + \tau_{Bm}, \quad (m = 0, 1, 2). \tag{76}$$

In Eqs. (66), (70), and (74), $\alpha = 1, 2$. The γ_3, γ_6, γ_{10}, and γ_{11} are constants defined in Appendix A. The κ_α, g_α, and ϑ_α are determined by the geometric properties of the boundary and the χ_α-coordinate curve on it [18]:

$$\chi_{\alpha,\alpha}\partial_{\chi_\alpha} t_i^{(\alpha)} = -\kappa_\alpha n_i - (-1)^{\alpha} g_\alpha t_i^{(3-\alpha)}, \tag{77}$$

$$\chi_{\alpha,\alpha}\partial_{\chi_\alpha} t_i^{(3-\alpha)} = (-1)^{\alpha}\big(-\vartheta_\alpha n_i + g_\alpha t_i^{(\alpha)}\big), \tag{78}$$

$$\chi_{\alpha,\alpha}\partial_{\chi_\alpha} n_i = \kappa_\alpha t_i^{(\alpha)} + (-1)^{\alpha}\vartheta_\alpha t_i^{(3-\alpha)}, \tag{79}$$

where κ_α/L, g_α/L, and ϑ_α/L are the normal curvature, geodesic curvature, and geodesic torsion [23] on the χ_α-coordinate curve at the point $x_{wi}(t,\chi_1,\chi_2)$, respectively. In addition, $\vartheta_1 = -\vartheta_2 \equiv \vartheta$, $\bar{\kappa} = (\kappa_1+\kappa_2)/2$, and $\chi_{\alpha,\alpha} = |\partial_{\chi_\alpha}\boldsymbol{x}_{\rm w}|^{-1}$. The $\bar{\kappa}/L$ is the mean curvature. The κ_α is taken negative when the center of curvature of the χ_α-coordinate curve lies on the side of the gas. The g_α is taken positive when the center of curvature of the curve, which is generated by projecting the χ_α-coordinate curve to the tangential surface at the point $x_{wi}(t,\chi_1,\chi_2)$, lies in the direction of $\boldsymbol{n}\times\boldsymbol{t}^{(\alpha)}$ when it is looked at from x_{wi}. The ϑ_α is taken positive when the geodesic tangent to the χ_α-coordinate curve at the point $x_{wi}(t,\chi_1,\chi_2)$ departs from its osculating plane in the direction of the binormal vector. The κ_α, g_α, and ϑ are functions only of χ_1 and χ_2. Although the sign of κ_α in Eqs. (77)–(79) is opposite to that in Eq. (6.61) in Ref. [18], it is due to the difference between the two definitions.

As for the expressions of $\phi_{{\rm B}m}$ $(m=0,1,2)$, $P_{ij{\rm B}m}$, and $Q_{i{\rm B}m}$ $(m=0,1,2,3)$, the reader is referred to Appendix B.

C. Knudsen layer and slip/jump boundary conditions

In Secs. V A and V B, we obtain the sets of equations which describe the overall behavior of the gas and its correction in the boundary layer up to the order of the Knudsen number. However, we have not yet taken into account the initial and boundary conditions, i.e., Eqs. (9) and (8).

At the initial instance, it is assumed that the gas is in the reference equilibrium state at rest and that $u_{wi}=0$ and $\tau_{\rm w}=0$. Thus, if we put

$$h_{{\rm H}m}=0,\quad h_{{\rm B}m}=0,\quad (t=0;\ m=0,1,2), \tag{80}$$

then the sum of the Hilbert and boundary-layer solutions matches the initial and boundary conditions at the initial instance. However, as the time goes on, it departs from a Maxwellian, except for the leading order, and thus ceases to match the boundary condition. Hence, we need to introduce the so-called Knudsen-layer correction in the thin layer with the thickness of a few mean free paths in the vicinity of the boundary.

Since the sum of the Hilbert and boundary-layer solutions is a Maxwellian at the leading order, from the property (c) in Sec. III, it matches the boundary condition Eq. (8) at this order provided that

$$n_i u_{i{\rm H}0} + n_i u_{i{\rm B}0} = n_i u_{{\rm w}i0},\quad (x_i = x_{{\rm w}i}), \tag{81}$$

$$t_i^{(\alpha)} u_{i{\rm H}0} + t_i^{(\alpha)} u_{i{\rm B}0} = t_i^{(\alpha)} u_{{\rm w}i0},\quad (x_i = x_{{\rm w}i};\ \alpha=1,2), \tag{82}$$

$$\tau_{{\rm H}0}+\tau_{{\rm B}0}=\tau_{{\rm w}0},\quad (x_i=x_{{\rm w}i}), \tag{83}$$

are satisfied on the boundary. Here, $u_{{\rm w}im}$ and $\tau_{{\rm w}m}$ are the coefficients of the expansion of the boundary data, i.e., $u_{wi}=u_{{\rm w}i0}+\varepsilon u_{{\rm w}i1}+\varepsilon^2 u_{{\rm w}i2}+\cdots$ and $\tau_{\rm w}=\tau_{{\rm w}0}+\varepsilon\tau_{{\rm w}1}+\varepsilon^2\tau_{{\rm w}2}+\cdots$. In contrast to the case in Sec. IV, the condition Eqs. (81)–(83) is compatible with Eqs. (31)–(33) and (64)–(67), thanks to the diffusive nature of Eqs. (64)–(67). The Knudsen-layer correction is required from the first order of ε. Let us express the solution ϕ as $\phi=\phi_{\rm H}+\phi_{\rm B}+\phi_{\rm K}$ in the layer with the thickness of the order of ε^2, where $\phi_{\rm K}$ varies with the distance from the boundary in the scale of $O(\varepsilon^2)$ $[n_i\partial_i\phi_{\rm K}=\varepsilon^{-2}O(\phi_{\rm K})]$ and vanishes sufficiently fast away from the boundary. Corresponding macroscopic quantities are denoted by $h_{\rm K}$ $(h=\omega, u_i, \tau, P, P_{ij}, Q_i)$.

The $\phi_{\rm K}$ is the solution of the following problem:

$$\partial_t\phi_{\rm K}(t,\boldsymbol{x},\boldsymbol{\zeta})+\zeta_i\partial_i\phi_{\rm K}(t,\boldsymbol{x},\boldsymbol{\zeta})=\frac{1}{\varepsilon^2}\mathcal{L}[\phi_{\rm K}(t,\boldsymbol{x},\boldsymbol{\zeta})](t,\boldsymbol{x},\boldsymbol{\zeta}), \tag{84}$$

$$\begin{aligned}\phi_{\rm K}(t,\boldsymbol{x},\boldsymbol{\zeta}) = {} & \int_{(\zeta_{i*}-u_{{\rm w}i})n_i<0}\frac{|\mathring{\zeta}_{k*}n_k|}{|\mathring{\zeta}_j n_j|}\mathcal{R}(\mathring{\boldsymbol{\zeta}}_*,\mathring{\boldsymbol{\zeta}})[\phi_{\rm K}(t,\boldsymbol{x},\boldsymbol{\zeta}_*)+\phi_{\rm F}(t,\boldsymbol{x},\boldsymbol{\zeta}_*)-\phi_{\rm ew}(t,\boldsymbol{x},\boldsymbol{\zeta}_*)]\frac{E(\zeta_*)}{E(\zeta)}d\boldsymbol{\zeta}_* \\ & +\phi_{\rm ew}(t,\boldsymbol{x},\boldsymbol{\zeta})-\phi_{\rm F}(t,\boldsymbol{x},\boldsymbol{\zeta}),\quad [x_i=x_{{\rm w}i},\ (\zeta_i-u_{{\rm w}i})n_i>0],\end{aligned} \tag{85}$$

$$\phi_{\mathrm{K}}(t,\boldsymbol{x},\boldsymbol{\zeta}) \to 0 \quad \text{as} \quad n_i(x_i - x_{\mathrm{w}i}) \to \infty, \tag{86}$$

$$\phi_{\mathrm{K}}(0,\boldsymbol{x},\boldsymbol{\zeta}) = 0, \tag{87}$$

where $\phi_{\mathrm{F}}(t,\boldsymbol{x},\boldsymbol{\zeta}) = \phi_{\mathrm{H}}(t,\boldsymbol{x},\boldsymbol{\zeta}) + \phi_{\mathrm{B}}(t,\boldsymbol{x},\boldsymbol{\zeta})$ [24]. Thanks to the smoothness of ϕ_{ew}, ϕ_{F}, and E with respect to the molecular velocity, $\boldsymbol{\zeta}$ and $\boldsymbol{\zeta}_*$ in their arguments can be replaced by $\mathring{\boldsymbol{\zeta}}$ and $\mathring{\boldsymbol{\zeta}}_*$, respectively, within the negligible error in the linearized framework. Then, the boundary condition Eq. (85) can be rewritten within the negligible error as follows:

$$\begin{aligned}\phi_{\mathrm{K}}(t,\boldsymbol{x},\boldsymbol{\zeta}) = &\int_{(\zeta_{i*}-u_{\mathrm{w}i})n_i<0} \frac{|\mathring{\zeta}_{k*}n_k|}{|\mathring{\zeta}_j n_j|}\mathcal{R}(\mathring{\boldsymbol{\zeta}}_*,\mathring{\boldsymbol{\zeta}})[\phi_{\mathrm{K}}(t,\boldsymbol{x},\boldsymbol{\zeta}_*) + \phi_{\mathrm{F}}(t,\boldsymbol{x},\mathring{\boldsymbol{\zeta}}_*) - \phi_{\mathrm{ew}}(t,\boldsymbol{x},\mathring{\boldsymbol{\zeta}}_*)]\frac{E(\mathring{\zeta}_*)}{E(\mathring{\zeta})}d\boldsymbol{\zeta}_* \\ &+ \phi_{\mathrm{ew}}(t,\boldsymbol{x},\mathring{\boldsymbol{\zeta}}) - \phi_{\mathrm{F}}(t,\boldsymbol{x},\mathring{\boldsymbol{\zeta}}), \quad [x_i = x_{\mathrm{w}i},\ (\zeta_i - u_{\mathrm{w}i})n_i > 0].\end{aligned} \tag{88}$$

Putting

$$\phi_{\mathrm{K}}(t,\boldsymbol{x},\boldsymbol{\zeta}) = \phi_{\mathrm{K}}(t,\boldsymbol{x},\mathring{\boldsymbol{\zeta}} + (\boldsymbol{u}_{\mathrm{w}}\cdot\boldsymbol{n})\boldsymbol{n}) \equiv \varphi_{\mathrm{K}}(t,\boldsymbol{x},\mathring{\boldsymbol{\zeta}}), \tag{89}$$

and changing the variable of integration from $\boldsymbol{\zeta}_*$ to $\mathring{\boldsymbol{\zeta}}_*$ in Eq. (88), we have

$$\varphi_{\mathrm{K}}(t,\boldsymbol{x},\mathring{\boldsymbol{\zeta}}) = \widetilde{\mathcal{K}}[\phi_{\mathrm{ew}} - \phi_{\mathrm{F}}](t,\boldsymbol{x},\mathring{\boldsymbol{\zeta}}) + \mathcal{K}[\varphi_{\mathrm{K}}](t,\boldsymbol{x},\mathring{\boldsymbol{\zeta}}), \quad (x_i = x_{\mathrm{w}i},\ \mathring{\zeta}_i n_i > 0), \tag{90}$$

where $\mathcal{K}$ and $\widetilde{\mathcal{K}}$ are operators defined by

$$\begin{aligned}\mathcal{K}[\psi(\mathring{\boldsymbol{\zeta}})](\mathring{\boldsymbol{\zeta}}) &= \int_{\mathring{\zeta}_{i*}n_i<0} \frac{|\mathring{\zeta}_{k*}n_k|}{|\mathring{\zeta}_j n_j|}\frac{E(\mathring{\zeta}_*)}{E(\mathring{\zeta})}\mathcal{R}(\mathring{\boldsymbol{\zeta}}_*,\mathring{\boldsymbol{\zeta}})\psi(\mathring{\boldsymbol{\zeta}}_*)d\mathring{\boldsymbol{\zeta}}_*, \\ \widetilde{\mathcal{K}}[\psi(\mathring{\boldsymbol{\zeta}})](\mathring{\boldsymbol{\zeta}}) &= \psi(\mathring{\boldsymbol{\zeta}}) - \mathcal{K}[\psi(\mathring{\boldsymbol{\zeta}})](\mathring{\boldsymbol{\zeta}}).\end{aligned} \tag{91}$$

These $\mathcal{K}$ and $\widetilde{\mathcal{K}}$ are the same as those in Ref. [25], in which the domain is assumed not to deform in time. In the rest of the paper, the argument with respect to the molecular velocity is $\mathring{\boldsymbol{\zeta}}$ for ϕ_{ew} and ϕ_{F}, unless otherwise stated.

We introduce another stretched coordinate system $(\eta,\ \chi_1,\ \chi_2)$ with χ_1 and χ_2 being the same as in Sec. V B:

$$x_i = \varepsilon^2\eta n_i(t,\chi_1,\chi_2) + x_{\mathrm{w}i}(t,\chi_1,\chi_2). \tag{92}$$

The problem Eqs. (84)–(87) for ϕ_{K} can be rewritten in the following problem for φ_{K} within the negligible error in the linearized framework:

$$\partial_t\varphi_{\mathrm{K}} + \frac{\mathring{\zeta}_n}{\varepsilon^2}\partial_\eta\varphi_{\mathrm{K}} + \mathring{\zeta}_i\big(\partial_i\chi_1\partial_{\chi_1} + \partial_i\chi_2\partial_{\chi_2}\big)\varphi_{\mathrm{K}} = \frac{1}{\varepsilon^2}\mathcal{L}[\varphi_{\mathrm{K}}], \tag{93}$$

$$\varphi_{\mathrm{K}} = \widetilde{\mathcal{K}}[\phi_{\mathrm{ew}} - \phi_{\mathrm{F}}] + \mathcal{K}[\varphi_{\mathrm{K}}], \quad (\eta = 0,\ \mathring{\zeta}_n > 0), \tag{94}$$

$$\varphi_{\mathrm{K}} \to 0 \quad \text{as} \quad \eta \to \infty, \tag{95}$$

$$\varphi_{\mathrm{K}} = 0, \quad (t = 0), \tag{96}$$

where $\partial_\eta = \partial/\partial\eta$ and $\mathring{\zeta}_n = \mathring{\zeta}_i n_i = (\zeta_i - u_{\mathrm{w}i})n_i$. Note that the independent variables of each term in Eqs. (93)–(96) are now t, η, χ_1, χ_2, and $\mathring{\boldsymbol{\zeta}}$ and that $\mathcal{L}[\phi_{\mathrm{K}}(\boldsymbol{\zeta})](\boldsymbol{\zeta})$ is replaced by $\mathcal{L}[\varphi_{\mathrm{K}}(\mathring{\boldsymbol{\zeta}})](\mathring{\boldsymbol{\zeta}})$ within the negligible error. Taking into account that the correction is not required at the leading order, the expansion of φ_{K} and h_{K} in a power series of ε starts from the first order as

$$\varphi_{\mathrm{K}} = \varepsilon\varphi_{\mathrm{K}1} + \varepsilon^2\varphi_{\mathrm{K}2} + \cdots, \quad h_{\mathrm{K}} = \varepsilon h_{\mathrm{K}1} + \varepsilon^2 h_{\mathrm{K}2} + \cdots. \tag{97}$$

Here $h_{\mathrm{K}m}$ can be expressed by Eqs. (1)–(6) with ϕ being replaced by $\varphi_{\mathrm{K}m}$.

The Knudsen-layer corrections $\varphi_{\mathrm{K}1}$ and $\varphi_{\mathrm{K}2}$ can be obtained by a process similar to that in Refs. [17,18,25]. The expressions of $\varphi_{\mathrm{K}1}$ and $\varphi_{\mathrm{K}2}$ are given in Appendix C. We omit the details of derivation and show the resulting slip and jump boundary conditions, together with formulas of the Knudsen-layer correction for macroscopic quantities. They are summarized as follows:

Slip/Jump boundary conditions

$$n_i u_{i\mathrm{H}1} + n_i u_{i\mathrm{B}1} = n_i u_{\mathrm{w}i1}, \tag{98}$$

$$t_i^{(\alpha)} u_{i\mathrm{H}1} + t_i^{(\alpha)} u_{i\mathrm{B}1} = t_i^{(\alpha)} u_{\mathrm{w}i1} + b_1^{(1)} \partial_y \big(t_i^{(\alpha)} u_{i\mathrm{B}0}\big), \tag{99}$$

$$\tau_{\mathrm{H}1} + \tau_{\mathrm{B}1} = \tau_{\mathrm{w}1} + c_1^{(0)} \partial_y \tau_{\mathrm{B}0}, \tag{100}$$

$$n_i u_{i\mathrm{H}2} + n_i u_{i\mathrm{B}2} = n_i u_{\mathrm{w}i2}, \tag{101}$$

$$\begin{aligned} t_i^{(\alpha)} u_{i\mathrm{H}2} + t_i^{(\alpha)} u_{i\mathrm{B}2} =& t_i^{(\alpha)} u_{\mathrm{w}i2} + b_1^{(1)} \big[\partial_y \big(t_i^{(\alpha)} u_{i\mathrm{B}1}\big) + n_i t_j^{(\alpha)} (\partial_i u_{j\mathrm{H}0} + \partial_j u_{i\mathrm{H}0}) - \kappa_\alpha \big(t_i^{(\alpha)} u_{i\mathrm{B}0}\big) \\ &+ \vartheta \big(t_i^{(3-\alpha)} u_{i\mathrm{B}0}\big)\big] + b_2^{(1)} \big(t_j^{(\alpha)} \partial_j \tau_{\mathrm{H}0} + \chi_{\alpha,\alpha} \partial_{\chi_\alpha} \tau_{\mathrm{B}0}\big) + b_4^{(1)} \partial_y^2 \big(t_i^{(\alpha)} u_{i\mathrm{B}0}\big), \end{aligned} \tag{102}$$

$$\tau_{\mathrm{H}2} + \tau_{\mathrm{B}2} = \tau_{\mathrm{w}2} + c_1^{(0)} (\partial_y \tau_{\mathrm{B}1} + n_j \partial_j \tau_{\mathrm{H}0}) + c_6^{(0)} \partial_y^2 \tau_{\mathrm{B}0} + c_5^{(0)} [\partial_y (n_i u_{i\mathrm{B}1}) + n_i n_j \partial_i u_{j\mathrm{H}0}], \tag{103}$$

Knudsen-layer corrections

$$\omega_{\mathrm{K}1} = \Omega_1^{(0)}(\eta) \partial_y \tau_{\mathrm{B}0}, \quad \tau_{\mathrm{K}1} = \Theta_1^{(0)}(\eta) \partial_y \tau_{\mathrm{B}0}, \quad P_{\mathrm{K}1} = \omega_{\mathrm{K}1} + \tau_{\mathrm{K}1}, \tag{104}$$

$$u_{i\mathrm{K}1} n_i = 0, \quad u_{i\mathrm{K}1} t_i^{(\alpha)} = Y_1^{(1)}(\eta) \partial_y (t_m^{(\alpha)} u_{m\mathrm{B}0}), \tag{105}$$

$$\omega_{\mathrm{K}2} = \Omega_1^{(0)}(\eta) (\partial_y \tau_{\mathrm{B}1} + n_i \partial_i \tau_{\mathrm{H}0}) + \Omega_6^{(0)}(\eta) \partial_y^2 \tau_{\mathrm{B}0} + \Omega_5^{(0)}(\eta) [\partial_y (n_i u_{i\mathrm{B}1}) + n_i n_j \partial_i u_{j\mathrm{H}0}], \tag{106}$$

$$\tau_{\mathrm{K}2} = \Theta_1^{(0)}(\eta) (\partial_y \tau_{\mathrm{B}1} + n_i \partial_i \tau_{\mathrm{H}0}) + \Theta_6^{(0)}(\eta) \partial_y^2 \tau_{\mathrm{B}0} + \Theta_5^{(0)}(\eta) [\partial_y (n_i u_{i\mathrm{B}1}) + n_i n_j \partial_i u_{j\mathrm{H}0}], \tag{107}$$

$$P_{\mathrm{K}2} = \omega_{\mathrm{K}2} + \tau_{\mathrm{K}2}, \quad u_{i\mathrm{K}2} n_i = 0, \tag{108}$$

$$\begin{aligned} u_{i\mathrm{K}2} t_i^{(\alpha)} =& Y_1^{(1)}(\eta) \big[\partial_y \big(t_i^{(\alpha)} u_{i\mathrm{B}1}\big) + n_i t_j^{(\alpha)} (\partial_i u_{j\mathrm{H}0} + \partial_j u_{i\mathrm{H}0}) - \kappa_\alpha \big(t_i^{(\alpha)} u_{i\mathrm{B}0}\big) + \vartheta \big(t_i^{(3-\alpha)} u_{i\mathrm{B}0}\big)\big] \\ &+ Y_2^{(1)}(\eta) \big(t_j^{(\alpha)} \partial_j \tau_{\mathrm{H}0} + \chi_{\alpha,\alpha} \partial_{\chi_\alpha} \tau_{\mathrm{B}0}\big) + Y_4^{(1)}(\eta) \partial_y^2 \big(t_i^{(\alpha)} u_{i\mathrm{B}0}\big), \end{aligned} \tag{109}$$

where $\alpha = 1, 2$. In Eqs. (98)–(109), the quantities with subscripts H and B represent their values on the boundary. The $b_p^{(1)}$ $(p = 1, 2, 4)$ and $c_q^{(0)}$ $(q = 1, 5, 6)$ are constants and are called the slip and jump coefficients, while $Y_p^{(1)}(\eta)$, $\Omega_q^{(0)}(\eta)$, $\Theta_q^{(0)}(\eta)$, and $H_p^{(1)}(\eta)$ are called the Knudsen-layer functions (see Appendix C for their definitions). They depend on both of the gas and the surface-scattering models. Their numerical data are available for the BGK model, hard-sphere molecules, Ellipsoidal Statistical (ES) model, and Shakhov model under the diffuse reflection condition [17,25–29] from Kyoto University Research Information Repository [30]. The expressions of the stress tensor $P_{ij\mathrm{K}m}$ and heat-flow vector $Q_{i\mathrm{K}m}$ $(m = 1, 2)$ can be found in Appendix C.

D. Summary

By the analyses in previous subsections, we have obtained the fluid-dynamic-type system for the description of acoustic phenomena in slightly rarefied gases. The system is composed of (i) linearized Euler sets of Eqs. (31)–(33), (34)–(36), and (40)–(42) for the Hilbert part, (ii) a set of Eqs. (64)–(75) for the acoustic boundary-layer corrections, (iii) a set of their boundary conditions Eqs. (81)–(83), (98)–(100), and (101)–(103), and (iv) the formulas of the Knudsen-layer corrections Eqs. (104), (105) and (106)–(109). The Hilbert part and the boundary-layer corrections respectively satisfy the equations of state [Eq. (43) or (76)] and the initial condition Eq. (80) as well. The condition Eq. (59) at infinity is also imposed on the boundary-layer corrections. Although the two parts occur simultaneously in the boundary conditions, they are not actually coupled. We can focus on either part in solving them from the lowest order [31]. If desired, the boundary-layer corrections can be expressed explicitly in terms of the fundamental solution, because they are the solution of the one-dimensional half-space initial boundary-value problem of the diffusion equation. The Knudsen-layer corrections are given by the formulas once the Hilbert part and the boundary-layer corrections are obtained. The obtained system describes the phenomena within the error of $o(\varepsilon^2)$, i.e., $o(\mathrm{Kn})$.

Incidentally, we have recently become aware of Ref. [32], which studies a similar problem based on the BGK model equation with the diffuse reflection boundary condition under the time-harmonic assumption. In this reference, the solution in the acoustic boundary layer is not separated into the Hilbert and its correction parts; accordingly matching issue to the bulk region arises. The secular-term problem, which can be a potential drawback in practical application and will be the primary issue in Secs. VI B and VII, is not discussed.

VI. DISCUSSIONS

In the present section, we first summarize main features of the obtained system, and then discuss a possible practical drawback in application and a remedy for it.

A. Features

1. Leading order

At the leading order, the boundary-layer correction is not required for the normal component of flow velocity and pressure [$n_i u_{i\mathrm{B}0}(y) = 0$, $P_{\mathrm{B}0}(y) = 0$; see Eqs. (64) and (65) and condition Eq. (59)]. Thus, the Hilbert part is described by the linearized Euler set of equations under the initial condition and the boundary condition $n_i u_{i\mathrm{H}0} = n_i u_{\mathrm{w}i0}$ [see Eq. (81)] [33,34]. This agrees with the usual description in aeroacoustics [35].

Generally, however, the Hilbert part determined as above does not match the no-slip/jump condition for tangential components of flow velocity and temperature. The boundary layer emerges to cancel this mismatch. The structure of the boundary layer is determined by solving Eqs. (66) and (67) under the conditions (82), (83), (80), and (59), because the Hilbert part has already been at hands. This also agrees with the acoustic boundary-layer theory in aeroacoustics [35].

2. 1/2th *order of the Knudsen number*

In contrast to the leading order, the boundary-layer correction for the normal component of flow velocity is required in general [see Eq. (68)]. Hence its presence affects the Hilbert part or the behavior in the bulk region at this order through the boundary condition Eq. (98).

The effects of shear slip and the temperature jump are known to be at the first order of Kn in the literature [17,25]. In the present case, however, they are enhanced to appear at the order of $\sqrt{\mathrm{Kn}}$, as seen in conditions Eqs. (99) and (100). This is because of the steep variation of the macroscopic quantities in the boundary layer. Up to this order, except for the Knudsen layer, the system is equivalent to the Navier–Stokes system with corresponding slip/jump conditions.

3. First order of the Knudsen number

At the order of Kn, effects enhanced by the same mechanism as above occur not only in the boundary condition but also in the boundary-layer equations. In the boundary conditions, slip and jump effects related to the second-order normal derivative of macroscopic quantities are enhanced. In the boundary-layer equations, the enhancement induces the term $-\frac{1}{6}(\gamma_1\gamma_2 - 4\gamma_3)\partial_y^2\tau_{\mathrm{B}0}$ in Eqs. (73) and (74). The part $-\frac{1}{6}(-4\gamma_3)\partial_y^2\tau_{\mathrm{B}0}$ represents the contribution of thermal stress, which was also pointed out in Ref. [12]. The other part $-\frac{1}{6}(\gamma_1\gamma_2)\partial_y^2\tau_{\mathrm{B}0}$ is found to be the leading-order term of $-\frac{1}{3}\varepsilon^2\gamma_1\partial_j u_{j\mathrm{B}}$, a part of the Newtonian stress, from Eqs. (67) and (68) and $\tau_{\mathrm{B}0} = -\omega_{\mathrm{B}0}$. The enhancement also induces the double-Laplacian terms in y of the leading-order flow velocity and temperature in Eqs. (74) and (75) [25]. The thermal-stress and the double-Laplacian terms are not inherent in the Navier–Stokes system. Hence, the boundary layer departs from the Navier–Stokes description at the present order even if proper slip/jump conditions are applied. However, secondary effects of the non-Navier–Stokes terms in the boundary layer do not appear in the boundary conditions Eqs. (81)–(83), (98)–(100), and (101)–(103) nor in the Knudsen-layer corrections (104), (105) and (106)–(109), because any derivatives of $h_{\mathrm{B}2}$ do not appear there.

The last term on the right-hand side of Eq. (103) represents the jump related to the divergence of flow velocity (in fact, $\partial_y(n_k u_{k\mathrm{B}1}) + n_i n_j \partial_i u_{j\mathrm{H}0}$ equals to the leading-order term of divergence under the rigid-body assumption [25]). Under the diffusion scaling, this jump appears at the second order of Kn [25]. Here, it is enhanced to appear at the order of Kn due to the difference of the magnitude of compressibility. The divergence vanishes at the leading order in the former, while it is not the case in general in the latter [see Eqs. (31) and (68)]. The occurrence of the corresponding jump at the order of Kn is reported in the case of a finite Mach number flow in Ref. [36].

Finally, it should be remarked that the right-hand sides of Eqs. (41) and (42) are inhomogeneous terms that represent the attenuation effect due to the viscosity and heat conductivity. Because they appear as sources in the equations, they may cause secular terms in the solution. A practical remedy for this potential drawback will be the topic of Sec. VI B.

B. Secular terms and a remedy to handle them

We have seen that at the first order of Kn the attenuation effect appears as the source terms in Eqs. (41) and (42). These terms are expressed with the solution of the homogeneous equations at the leading order. According to the perturbation theory [37–39], however, this type of sources may cause that the uniformness of the expansion (here in ε) with respect to the independent variables (here t and $\boldsymbol{x}$) no longer holds, namely, some terms in the solution become unbounded. Such terms are called the secular terms. They affect in an unfavorable way the behavior of the wave which has traveled for a long time and accordingly for a long distance. In the application example in Sec. VII, this unfavorable property will be demonstrated in Fig. 2 that appears later.

Sometimes the secularity is regarded as a weakness of the Hilbert expansion compared to the Chapman–Enskog expansion [40]. However, on the basis of the derived system, we can construct a useful system, which suppresses secular terms in the solution. The former system has been obtained by a systematic expansion up to $O(\varepsilon^2)$ [or $O(\mathrm{Kn})$] and hence its solution is correct up to $O(\varepsilon^2)$ [or $O(\mathrm{Kn})$]. The latter system is constructed from the former without dropping any component in them and accordingly the solution of the latter is expected to be correct up to $O(\varepsilon^2)$ [or $O(\mathrm{Kn})$]. We explain the way of construction below.

Summing up Eqs. (31)–(33), (34)–(36), and (40)–(42) after multiplying the corresponding power of ε, we see that h_{H} satisfies the following equations within the error of $O(\varepsilon^3)$:

$$\partial_t \omega_{\mathrm{H}} + \partial_j u_{j\mathrm{H}} = 0, \tag{110}$$

$$\partial_t u_{i\mathrm{H}} + \tfrac{1}{2}\partial_i P_{\mathrm{H}} = \tfrac{1}{2}\gamma_1 \varepsilon^2 \partial_j \overline{\partial_i u_{j\mathrm{H}}}, \tag{111}$$

$$\partial_t \tau_{\mathrm{H}} - \tfrac{2}{5}\partial_t P_{\mathrm{H}} = \tfrac{1}{2}\gamma_2 \varepsilon^2 \partial_j^2 \tau_{\mathrm{H}}, \tag{112}$$

$$P_{\mathrm{H}} = \omega_{\mathrm{H}} + \tau_{\mathrm{H}}. \tag{113}$$

Equations (110)–(113) are no other than the linearized Navier–Stokes set of equations.

As for the boundary-layer part, the derived Eqs. (64)–(75) are also identical to the same linearized Navier–Stokes set of Eqs. (110)–(113) in the curvilinear coordinates, Eq. (47), within negligible error except for the contribution of the thermal-stress term $-\frac{1}{6}(-4\gamma_3)\partial_y^2 \tau_{\mathrm{B}0}$ in Eqs. (73) and (74) and the source (inhomogeneous) terms $-\frac{1}{4}(\gamma_1\gamma_{10} - 2\gamma_6)\partial_y^4 (t_i^{(\alpha)} u_{i\mathrm{B}0})$ and $-\frac{1}{10}(\gamma_2\gamma_3 - \frac{13}{2}\gamma_{11})\partial_y^4 \tau_{\mathrm{B}0}$ in Eqs. (74) and (75) (see the last sentence in Sec. VI A 2 and the first paragraph in Sec. VI A 3). Remember that the boundary-layer correction at the $O(\varepsilon^2)$ does not affect the slip/jump conditions and Knudsen-layer corrections up to the same order (see the last sentence of the first paragraph in Sec. VI A 3). Then, we are motivated to unify the Hilbert and boundary-layer parts, say $h_\mathrm{V} = h_\mathrm{H} + h_\mathrm{B}$, and treat it as a solution of the linearized Navier–Stokes set of equations

$$\partial_t \omega_\mathrm{V} + \partial_j u_{j\mathrm{V}} = 0, \tag{114}$$

$$\partial_t u_{i\mathrm{V}} + \tfrac{1}{2}\partial_i P_\mathrm{V} = \tfrac{1}{2}\gamma_1 \varepsilon^2 \partial_j \overline{\partial_i u_{j\mathrm{V}}}, \tag{115}$$

$$\partial_t \tau_\mathrm{V} - \tfrac{2}{5}\partial_t P_\mathrm{V} = \tfrac{1}{2}\gamma_2 \varepsilon^2 \partial_j^2 \tau_\mathrm{V}, \tag{116}$$

$$P_\mathrm{V} = \omega_\mathrm{V} + \tau_\mathrm{V}, \tag{117}$$

under the following slip/jump conditions:

$$n_i u_{i\mathrm{V}} - n_i u_{\mathrm{w}i} = 0, \tag{118}$$

$$t_i^{(\alpha)} u_{i\mathrm{V}} - t_i^{(\alpha)} u_{\mathrm{w}i} = \varepsilon^2 b_1^{(1)} n_i t_j^{(\alpha)} (\partial_i u_{j\mathrm{V}} + \partial_j u_{i\mathrm{V}}) + \varepsilon^2 b_2^{(1)} t_j^{(\alpha)} \partial_j \tau_\mathrm{V} + \varepsilon^4 b_4^{(1)} t_i^{(\alpha)} n_j n_k \partial_j \partial_k u_{i\mathrm{V}}, \tag{119}$$

$$\tau_\mathrm{V} - \tau_\mathrm{w} = \varepsilon^2 c_1^{(0)} n_j \partial_j \tau_\mathrm{V} + \varepsilon^4 c_6^{(0)} n_j n_k \partial_j \partial_k \tau_\mathrm{V} + \varepsilon^2 c_5^{(0)} n_i n_j \partial_i u_{j\mathrm{V}}. \tag{120}$$

The condition Eqs. (118)–(120) has been obtained by summing up Eqs. (81)–(83), (98)–(100), and (101)–(103) after multiplied with the corresponding power of ε. In this process, the gradients in the curvilinear coordinates Eq. (47) have been rewritten in terms of the original coordinates x_i, e.g., $\partial_y = \varepsilon n_i \partial_i$, $\chi_{\alpha,\alpha}\partial_{\chi_\alpha} = t_i^{(\alpha)}\partial_i + O(\varepsilon)$. The ε in front of $n_i\partial_i$ is due to the difference of the scale of variation in the normal direction between the Hilbert and boundary-layer parts. Corresponding summation will be required later to obtain the Knudsen-layer corrections Eq. (136)–(140). By this h_V, the behavior of the gas can be described correctly up to $O(\varepsilon^2)$ in the bulk region and $O(\varepsilon)$ in the boundary layer (and outside the Knudsen layer). In Eqs. (114)–(117), the contribution of the viscosity and heat conductivity is represented not as source (inhomogeneous) terms but as high-order derivative terms (multiplied by small coefficients) expressed with the dependent variables in the equations. Thus the occurrence of the secular term is expected to be prevented.

To improve the less accuracy in the boundary layer, we can take two options. The first option (construction I) is rather straightforward: Add the following $\varepsilon^2 h_{\mathrm{B}2}^{\#}$ to h_V

$$\varepsilon^2 P_{\mathrm{B}2}^{\#} = -\frac{2}{3}\gamma_3 \varepsilon^2 \partial_y^2 \tau_{\mathrm{B}0}, \tag{121}$$

$$\varepsilon^2 \tau_{\mathrm{B}2}^{\#} = \frac{7}{30\gamma_2}\left(\gamma_2\gamma_3 - \frac{39}{14}\gamma_{11}\right)\varepsilon^2 y \partial_y^3 \tau_{\mathrm{B}0}, \tag{122}$$

$$\varepsilon^2 \omega_{\mathrm{B}2}^{\#} = \varepsilon^2 P_{\mathrm{B}2}^{\#} - \varepsilon^2 \tau_{\mathrm{B}2}^{\#}, \tag{123}$$

$$\varepsilon^2 t_i^{(\alpha)} u_{i\mathrm{B}2}^{\#} = \frac{1}{4\gamma_1}(\gamma_1\gamma_{10} - 2\gamma_6)\varepsilon^2 y \partial_y^3 \big(t_i^{(\alpha)} u_{i\mathrm{B}0}\big), \tag{124}$$

$$\varepsilon^2 n_i u_{i\mathrm{B}2}^{\#} = 0. \tag{125}$$

Equations (121)–(125) represent the correction due to the non-Navier–Stokes effect mentioned at the beginning of the previous paragraph. As for the derivation of these expressions, see Appendix D. The sum $h_\mathrm{V} + \varepsilon^2 h_{\mathrm{B}2}^{\#}$ is correct up to $O(\varepsilon^2)$ in the entire region outside of the Knudsen layer. Note that $\tau_{\mathrm{B}0}$ [or $t_i^{(\alpha)} u_{i\mathrm{B}0}$] is the solution of Eq. (67) [or (66)] under the conditions (83) [or (82)], (59), and (80), and is thus expressed in the form

$$\tau_{B0} = \tau_{w0}(t, \chi_1, \chi_2) - \tau_{H0}(t, \boldsymbol{x}_w(t, \chi_1, \chi_2)) - \int_0^t \operatorname{erf}\left(\frac{y}{\sqrt{2\gamma_2(t-s)}}\right)[\partial_s \tau_{w0}(s, \chi_1, \chi_2) - \partial_s \tau_{H0}(s, \boldsymbol{x}_w(s, \chi_1, \chi_2))]ds, \tag{126}$$

$$t_i^{(\alpha)} u_{iB0} = t_i^{(\alpha)} u_{wi0}(t, \chi_1, \chi_2) - t_i^{(\alpha)} u_{iH0}(t, \boldsymbol{x}_w(t, \chi_1, \chi_2)) - \int_0^t \operatorname{erf}\left(\frac{y}{\sqrt{2\gamma_1(t-s)}}\right) \times \left[\partial_s (t_i^{(\alpha)} u_{wi0})(s, \chi_1, \chi_2) - \partial_s (t_i^{(\alpha)} u_{iH0})(s, \boldsymbol{x}_w(s, \chi_1, \chi_2))\right]ds. \tag{127}$$

Because it is easily seen from Eqs. (31)–(33) that

$$\tau_{H0} = \frac{2}{5} P_{H0}, \tag{128}$$

$$t_i^{(\alpha)} u_{iH0}(t, \boldsymbol{x}) = -\frac{1}{2} t_i^{(\alpha)} \int_0^t \partial_i P_{H0}(s, \boldsymbol{x}) ds, \quad [n_i(x_i - x_{wi}) = O(\varepsilon)], \tag{129}$$

and from $P_{B0} = 0$ that

$$P_V = P_{H0} + O(\varepsilon), \tag{130}$$

the corrections Eqs. (121)–(125) can be recovered from the data of P_V within the tolerant error by the simple replacement of P_{H0} with P_V in Eqs. (128) and (129).

The second option (construction II) might be easier to implement: Add the following h_C to h_V:

$$P_C = -\frac{2}{3} \gamma_3 \varepsilon^4 \partial_{x_n}^2 \tau_V, \tag{131}$$

$$\tau_C = \frac{7}{30\gamma_2}\left(\gamma_2\gamma_3 - \frac{39}{14}\gamma_{11}\right)\varepsilon^4 [1 - \exp(-x_n)] \partial_{x_n}^3 \tau_V, \tag{132}$$

$$\omega_C = P_C - \tau_C, \tag{133}$$

$$t_i^{(\alpha)} u_{iC} = \frac{1}{4\gamma_1}(\gamma_1\gamma_{10} - 2\gamma_6)\varepsilon^4 [1 - \exp(-x_n)] \partial_{x_n}^3 \left(t_i^{(\alpha)} u_{iV}\right), \tag{134}$$

$$n_i u_{iC} = 0, \tag{135}$$

where $x_n \equiv (x_i - x_{wi}) n_i = \varepsilon y$ and $\partial_{x_n} = \partial/\partial x_n$. Compare the above with Eqs. (121)–(125), and recall that $\varepsilon^2 \partial_y h_V = \varepsilon^2 \partial_y h_{B0} + O(\varepsilon^3)$ ($h = \tau, t_i^{(\alpha)} u_i$) in the boundary layer. Then, it is obvious that h_C is $O(\varepsilon^2)$ and agrees with the desired non-Navier–Stokes part $\varepsilon^2 h_{B2}^{\#}$ within the error of $O(\varepsilon^3)$ in the boundary layer. It should be noted that since h_V is nonzero outside the boundary layer, εy is replaced by $1 - \exp(-x_n)$ in Eqs. (131)–(135) to suppress the secular-term problem. Outside the boundary layer, the magnitude of h_C is at most $O(\varepsilon^4)$, which is within the negligible error in the present framework. The sum $h_V + h_C$ is correct up to $O(\varepsilon^2)$ in the entire region outside of the Knudsen layer.

Finally, from Eqs. (104), (105) and (106)–(109), the following summed up Knudsen-layer corrections $\overline{h}_K$ ($h = \omega, \tau, P, u_i$) are obtained:

$$\overline{\omega}_K = \varepsilon^2 \Omega_1^{(0)}(\eta) n_j \partial_j \tau_V + \varepsilon^4 \Omega_6^{(0)}(\eta) n_j n_k \partial_j \partial_k \tau_V + \varepsilon^2 \Omega_5^{(0)}(\eta) n_i n_j \partial_i u_{jV}, \tag{136}$$

$$\overline{\tau}_K = \varepsilon^2 \Theta_1^{(0)}(\eta) n_j \partial_j \tau_V + \varepsilon^4 \Theta_6^{(0)}(\eta) n_j n_k \partial_j \partial_k \tau_V + \varepsilon^2 \Theta_5^{(0)}(\eta) n_i n_j \partial_i u_{jV}, \tag{137}$$

$$\overline{P}_K = \overline{\omega}_K + \overline{\tau}_K, \tag{138}$$

$$\overline{u}_{iK} t_i^{(\alpha)} = \varepsilon^2 Y_1^{(1)}(\eta) n_i t_j^{(\alpha)} (\partial_i u_{jV} + \partial_j u_{iV}) + \varepsilon^2 Y_2^{(1)}(\eta) t_j^{(\alpha)} \partial_j \tau_V + \varepsilon^4 Y_4^{(1)}(\eta) t_i^{(\alpha)} n_j n_k \partial_j \partial_k u_{iV}, \tag{139}$$

$$\overline{u}_{iK} n_i = 0. \tag{140}$$

We can construct the solution which is correct up to $O(\varepsilon^2)$ or $O(\mathrm{Kn})$ in the entire region by $h_\mathrm{V} + \varepsilon^2 h^\#_{\mathrm{B}2} + \overline{h}_\mathrm{K}$ (construction I) or by $h_\mathrm{V} + h_\mathrm{C} + \overline{h}_\mathrm{K}$ (construction II). The expressions for the stress tensor and heat-flow vector of each part, say $(P_{ij\mathrm{V}}, Q_{i\mathrm{V}})$, $(P_{ij\mathrm{C}}, Q_{i\mathrm{C}})$, and $(\overline{P}_{ij\mathrm{K}}, \overline{Q}_{i\mathrm{K}})$, are summarized in Appendix D.

VII. APPLICATION EXAMPLE

Consider a slightly rarefied gas in a uniform equilibrium state at rest with density ρ_0 and temperature T_0, bounded by an infinitely wide flat plate with temperature T_0. The plate location $X_\mathrm{w}(\tilde{t})$ in X_1 at time $\tilde{t}$ changes sinusoidally as $X_\mathrm{w}(\tilde{t}) = \tilde{a}\cos\tilde{\sigma}\tilde{t}$ from its initial position $X_\mathrm{w}(0) = \tilde{a}$. The mean free time of the gas molecules is much smaller than the period of oscillation of the plate: $\tilde{\sigma}\ell_0/(2RT_0)^{1/2} \ll 1$, where ℓ_0 is the mean free path in the equilibrium state at rest with density ρ_0 and temperature T_0. The amplitude of oscillation of the plate is sufficiently small so that square and higher-order terms of $\tilde{\sigma}\tilde{a}/(2RT_0)^{1/2} \equiv a$ are negligible.

Let the reference time be $t_0 = \tilde{\sigma}^{-1}$ so that $t = \tilde{\sigma}\tilde{t}$. With this t_0, we take the reference length $L = (2RT_0)^{1/2} t_0$ [41]. Because the present problem is spatially one dimensional, we simply denote the x_1 coordinate by x and the dimensionless flow velocity u_i as $(u(t,x), 0, 0)$. The same notation convention will be applied to $u_{i\mathrm{H}}$, $u_{i\mathrm{B}}$, $u_{i\mathrm{K}}$, etc.

We will investigate the harmonic behavior of the gas after a long time has passed from the initial state on the basis of the system derived above up to the second order of ε.

In seeking the solution of $h_{\mathrm{H}m}$ and $h_{\mathrm{B}m}$, the conditions Eqs. (81)–(83), (98)–(100), and (101)–(103) may be imposed at the fixed position $x = 0$ because of the oscillatory motion of the plate (see Ref. [33]).

We can seek the solution in the form $h_{\mathrm{H}m} = \mathrm{Re}[e^{it}\hat{h}_{\mathrm{H}m}(x)]$ and $h_{\mathrm{B}m} = \mathrm{Re}[e^{it}\hat{h}_{\mathrm{B}m}(y)]$, where i is the imaginary unit and $\hat{h}_{\mathrm{H}m}(x)$ and $\hat{h}_{\mathrm{B}m}(y)$ are complex-valued functions. The $\hat{h}_{\mathrm{H}m}(x)$ and $\hat{h}_{\mathrm{B}m}(y)$ are obtained as follows:

$$\hat{P}_{\mathrm{H}0} = \frac{5ia}{3c}\exp\left(-\frac{ix}{c}\right), \quad \hat{u}_{\mathrm{H}0} = \frac{3c}{5}\hat{P}_{\mathrm{H}0}, \quad \hat{\omega}_{\mathrm{H}0} = \frac{3}{5}\hat{P}_{\mathrm{H}0}, \quad \hat{\tau}_{\mathrm{H}0} = \frac{2}{5}\hat{P}_{\mathrm{H}0}, \tag{141}$$

$$\hat{P}_{\mathrm{H}1} = \frac{5(1-i)a\sqrt{\gamma_2}}{9c^2}\exp\left(-\frac{ix}{c}\right), \quad \hat{u}_{\mathrm{H}1} = \frac{3c}{5}\hat{P}_{\mathrm{H}1}, \quad \hat{\omega}_{\mathrm{H}1} = \frac{3}{5}\hat{P}_{\mathrm{H}1}, \quad \hat{\tau}_{\mathrm{H}1} = \frac{2}{5}\hat{P}_{\mathrm{H}1}, \tag{142}$$

$$\hat{P}_{\mathrm{H}2} = -a\left\{\frac{5i}{9c^4}\left(\gamma_1 + \frac{5\gamma_2}{12c^2}\right)x + \left[\frac{25}{27c^3}\left(1 - \frac{1}{4c^2}\right)\gamma_2 - \frac{5\gamma_1}{9c^3} + \frac{10c_1^{(0)}}{9c^2}\right]\right\}\exp\left(-\frac{ix}{c}\right),$$

$$\hat{u}_{\mathrm{H}2} = -a\left[\frac{i}{3c^3}\left(\gamma_1 + \frac{5\gamma_2}{12c^2}\right)x + \frac{2}{3c}\left(\frac{\gamma_2}{3c} + c_1^{(0)}\right)\right]\exp\left(-\frac{ix}{c}\right), \tag{143}$$

$$\hat{\omega}_{\mathrm{H}2} = \frac{3}{5}\hat{P}_{\mathrm{H}2} + \frac{\gamma_2}{2}i\frac{d^2\hat{\tau}_{\mathrm{H}0}}{dx^2}, \quad \hat{\tau}_{\mathrm{H}2} = \frac{2}{5}\hat{P}_{\mathrm{H}2} - \frac{\gamma_2}{2}i\frac{d^2\hat{\tau}_{\mathrm{H}0}}{dx^2}, \tag{144}$$

$$\hat{P}_{\mathrm{B}0} = 0, \quad \hat{u}_{\mathrm{B}0} = 0, \quad \hat{\tau}_{\mathrm{B}0} = -\hat{\omega}_{\mathrm{B}0} = -\frac{2ia}{3c}\exp\left(-\frac{(1+i)y}{\sqrt{\gamma_2}}\right), \tag{145}$$

$$\hat{P}_{\mathrm{B}1} = 0, \quad \hat{u}_{\mathrm{B}1} = -\frac{(1-i)a\sqrt{\gamma_2}}{3c}\exp\left(-\frac{(1+i)y}{\sqrt{\gamma_2}}\right), \tag{146}$$

$$\hat{\tau}_{\mathrm{B}1} = -\hat{\omega}_{\mathrm{B}1} = -(1-i)a\left(\frac{2\sqrt{\gamma_2}}{9c^2} + \frac{2}{3c\sqrt{\gamma_2}}c_1^{(0)}\right)\exp\left(-\frac{(1+i)y}{\sqrt{\gamma_2}}\right), \tag{147}$$

$$\hat{P}_{\mathrm{B}2} = \frac{2a}{3c}\left(\frac{4}{3}\gamma_1 - \gamma_2 - \frac{4\gamma_3}{3\gamma_2}\right)\exp\left(-\frac{(1+i)y}{\sqrt{\gamma_2}}\right), \tag{148}$$

$$\hat{u}_{\mathrm{B}2} = 2a\left(\frac{\gamma_2}{9c^2} + \frac{c_1^{(0)}}{3c}\right)\exp\left(-\frac{(1+i)y}{\sqrt{\gamma_2}}\right), \tag{149}$$

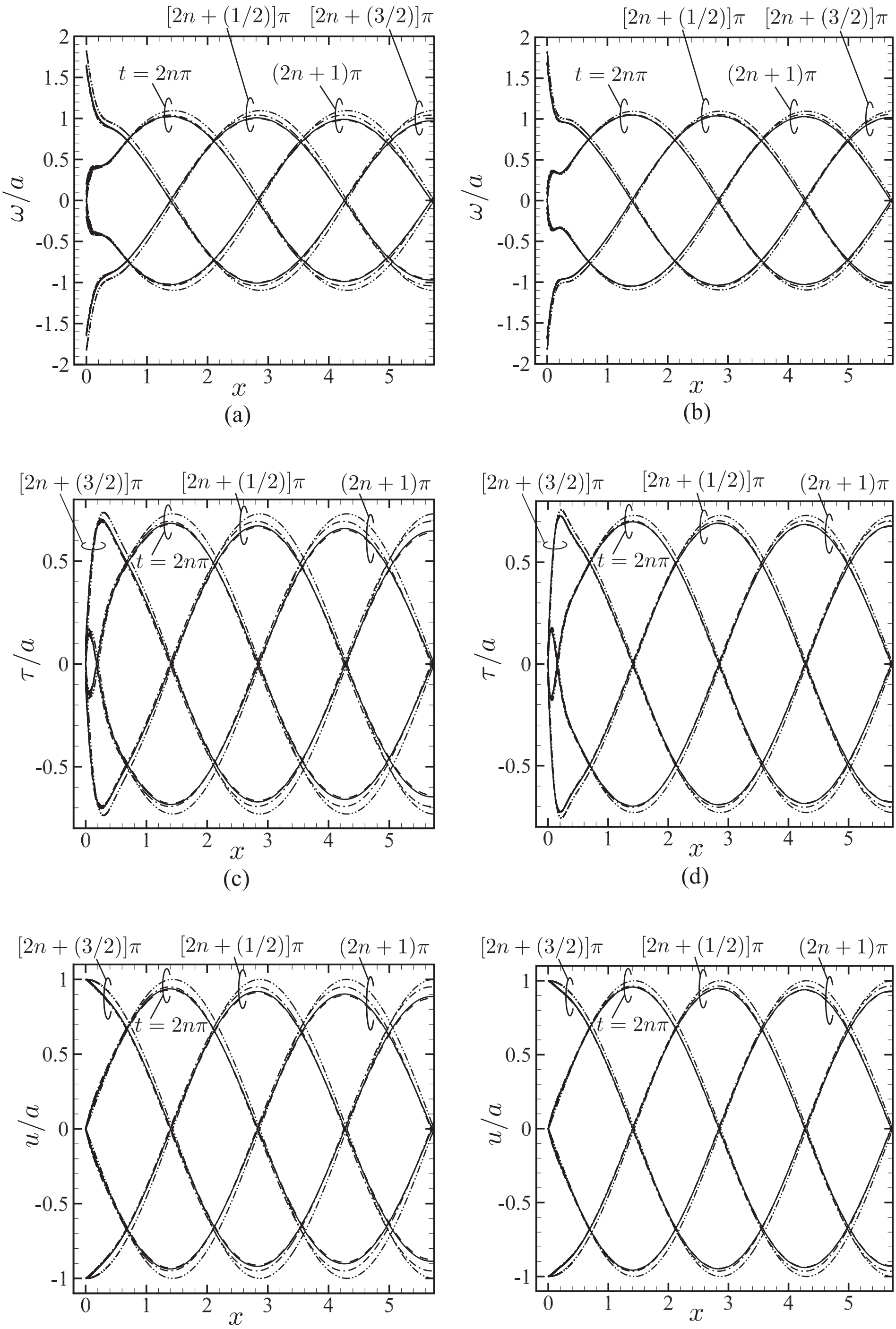

FIG. 1. The profile of macroscopic quantities. (a) ω/a for $k = 0.02$, (b) ω/a for $k = 0.01$, (c) τ/a for $k = 0.02$, (d) τ/a for $k = 0.01$, (e) u/a for $k = 0.02$, and (f) u/a for $k = 0.01$. The solid line indicates the N solution, the dashed line the HBK2 solution, the dash-dotted line the HBK1 solution, and the dash-double-dotted line the HBK0 solution.

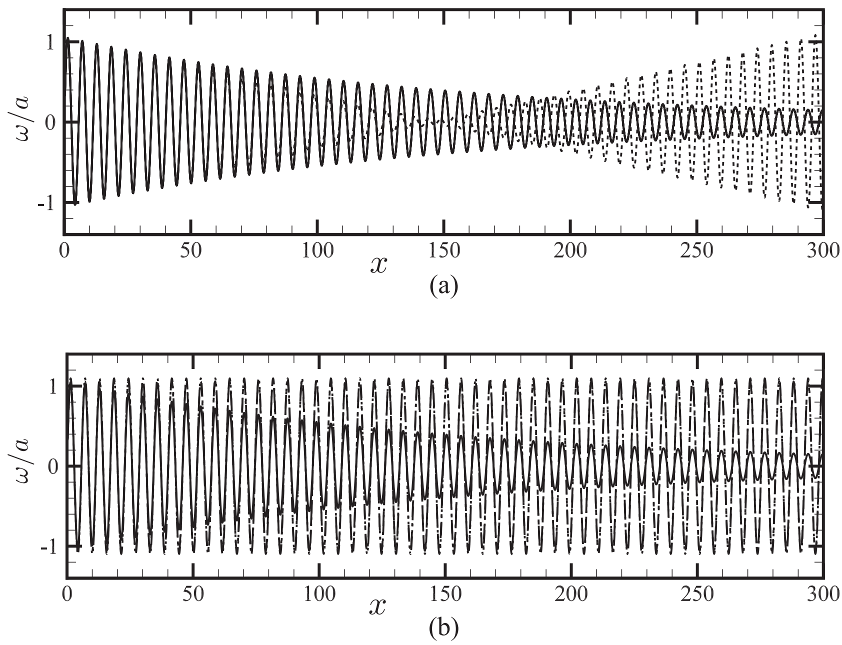

FIG. 2. The profile of ω in a far field at $t = 2n\pi$ for $k = 0.01$. (a) N, HBK2, CI, and CII solutions and (b) N, HBK1, and HBK0 solutions. The N, HBK2, CI, CII, HBK1, and HBK0 solutions are indicated by the solid, dashed, long-dashed, dotted, dash-dotted, and dash-double-dotted lines, respectively. In panel (a), the long-dashed and dotted lines are invisible because they are close to the solid line. In panel (b), the dash-dotted and dash-double-dotted lines overlap each other.

$$\hat{\tau}_{\mathrm{B2}} = a\left[\frac{1}{27c^3}\left(19 - \frac{5}{2c^2}\right)\gamma_2 - \frac{2\gamma_1}{9c^3} + c_1^{(0)}\left(\frac{14}{9c^2} + \frac{4c_1^{(0)}}{3c\gamma_2}\right) + \frac{4c_6^{(0)}}{3c\gamma_2} + \frac{5c_5^{(0)}}{3c} - \frac{2(1+i)}{15c\sqrt{\gamma_2}}\left(\gamma_2 - \frac{4}{3}\gamma_1 + \frac{7\gamma_3}{3\gamma_2} - \frac{13\gamma_{11}}{2\gamma_2^2}\right)y\right]\exp\left(-\frac{(1+i)y}{\sqrt{\gamma_2}}\right), \tag{150}$$

$$\hat{\omega}_{\mathrm{B2}} = \hat{P}_{\mathrm{B2}} - \hat{\tau}_{\mathrm{B2}}, \tag{151}$$

where $c = \sqrt{5/6}$. The Knudsen-layer correction is given as $h_{\mathrm{K}m} = \mathrm{Re}[e^{it}\hat{h}_{\mathrm{K}m}(\eta)]$, where $m = 1, 2$, $\eta = x/\varepsilon^2$, and $\hat{h}_{\mathrm{K}m}$ is defined by Eqs. (104), (105) and (106)–(109) with $(h_{\mathrm{H}m}, h_{\mathrm{B}m})$ being replaced by $(\hat{h}_{\mathrm{H}m}, \hat{h}_{\mathrm{B}m})$.

We have solved the same physical problem on the basis of the linearized BGK model under the diffuse reflection condition, again under the time-harmonic assumption, directly by the finite-difference method. In the rest of this section, we denote the numerical solution by the N solution, the asymptotic solutions which are correct up to the leading, first, and second orders of ε $[\sum_{k=0}^{m}\varepsilon^k(h_{\mathrm{H}k} + h_{\mathrm{B}k} + h_{\mathrm{K}k})\ (m = 0, 1, 2)]$ by the HBK0, HBK1, and HBK2 solutions, the solutions obtained by the constructions I and II in Sec. VI B by the CI and CII solutions, and the Navier–Stokes solution h_{V} by the V solution for short.

To confirm the validity of the asymptotic solutions, we compare the N, HBK0, HBK1, and HBK2 solutions in the region rather close to the boundary for $k = 0.02$ and 0.01 in Fig. 1. The quantities at $t = 2n\pi$, $[2n + (1/2)]\pi$, $(2n + 1)\pi$, and $[2n + (3/2)]\pi$ with n being an arbitrary large integer are shown in the figure [42]. The asymptotic solutions approach the N solution as the degree of approximation increases.

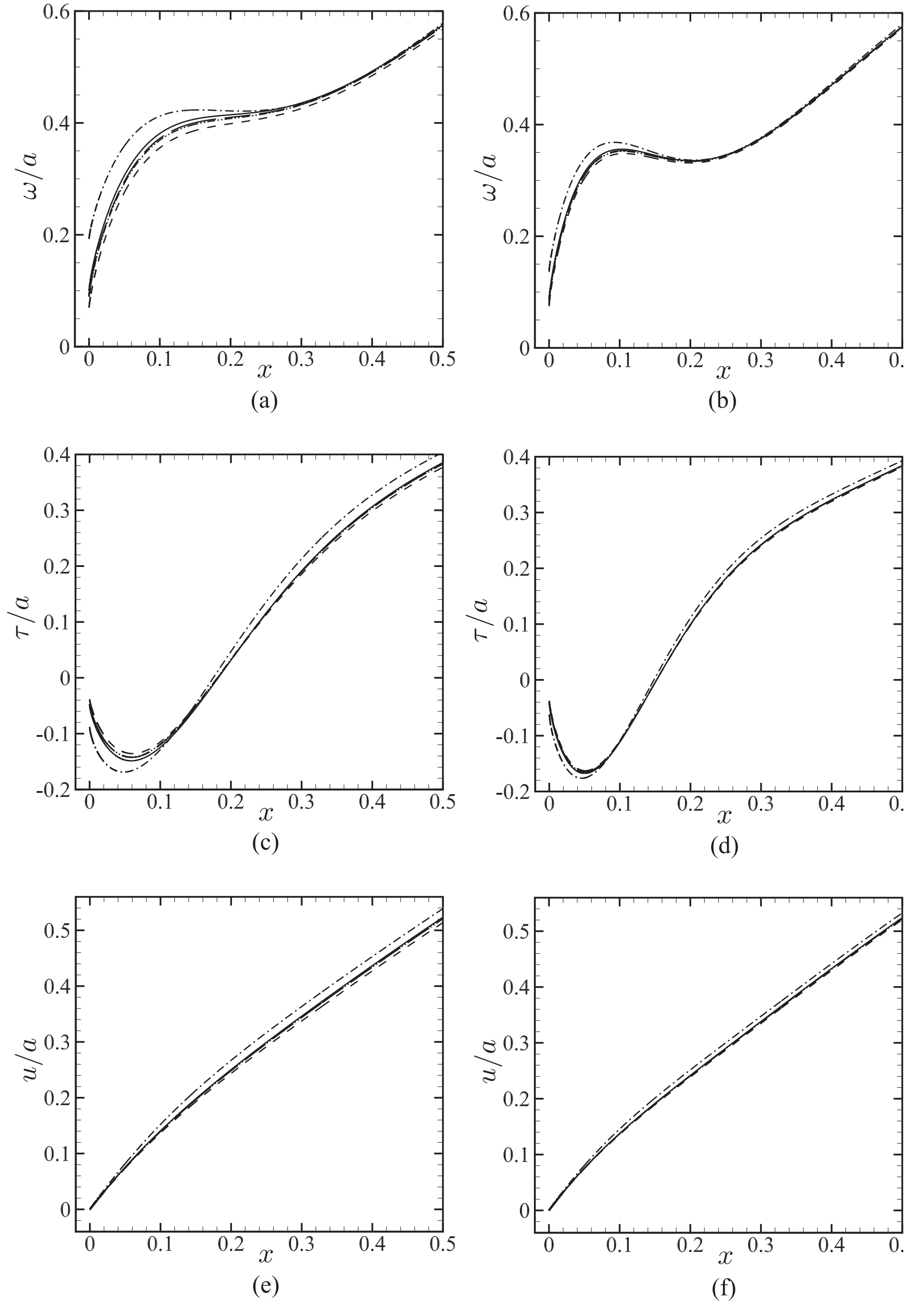

FIG. 3. The profile of macroscopic quantities near the plate at $t = 2n\pi$. (a) ω/a for $k = 0.02$, (b) ω/a for $k = 0.01$, (c) τ/a for $k = 0.02$, (d) τ/a for $k = 0.01$, (e) u/a for $k = 0.02$, and (f) u/a for $k = 0.01$. The solid line indicates the N solution, the dashed line the HBK2 solution, the dash-dotted line the HBK1 solution, the dash-double-dotted line the CI solution, and the long-dashed line the CII solution.

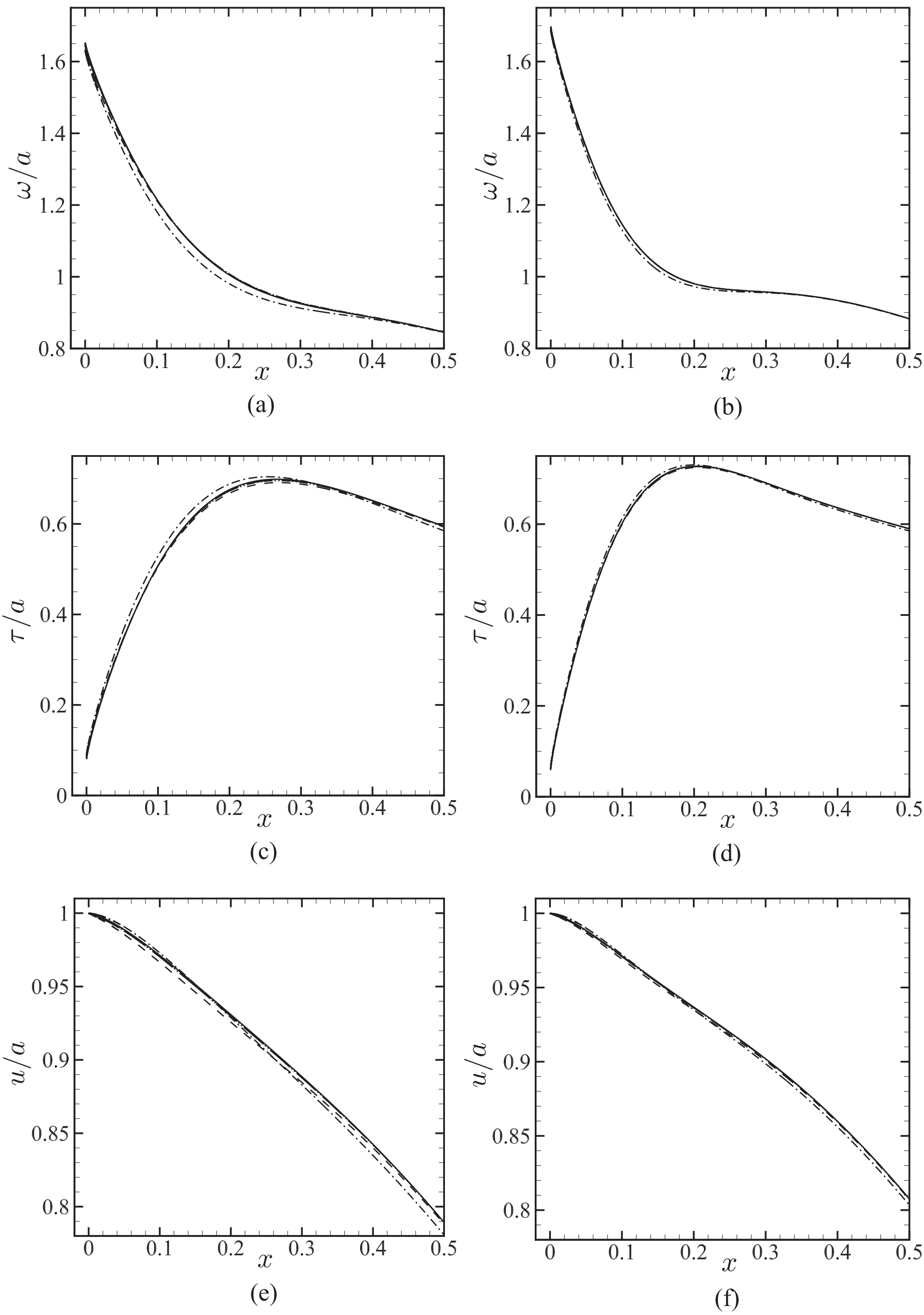

FIG. 4. The profile of macroscopic quantities near the plate at $t = [2n + (3/2)]\pi$. (a) ω/a for $k = 0.02$, (b) ω/a for $k = 0.01$, (c) τ/a for $k = 0.02$, (d) τ/a for $k = 0.01$, (e) u/a for $k = 0.02$, and (f) u/a for $k = 0.01$. See the caption of Fig. 3.

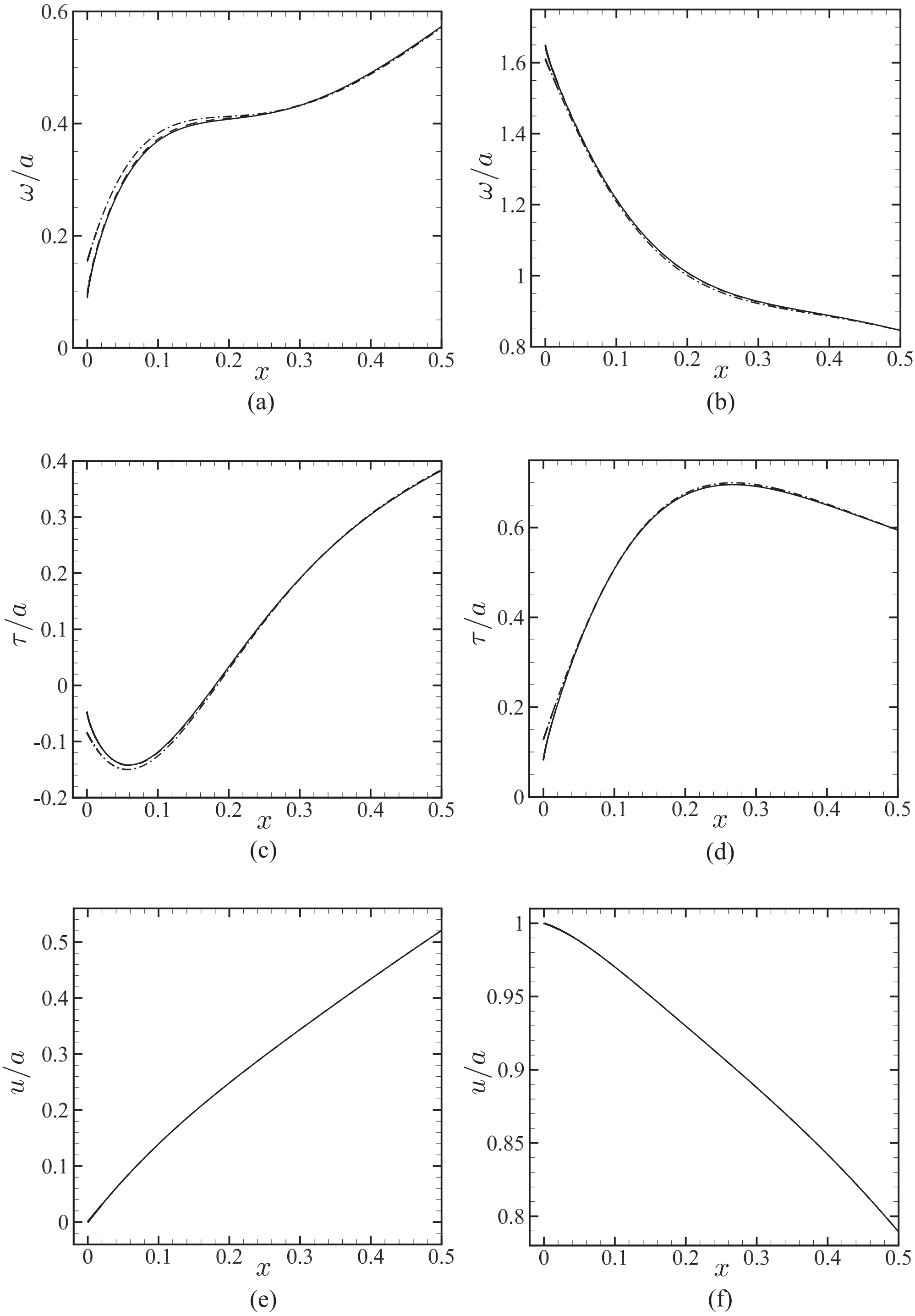

FIG. 5. The profile of macroscopic quantities near the plate for $k = 0.02$. (a) ω/a at $t = 2n\pi$, (b) ω/a at $t = [2n + (3/2)]\pi$, (c) τ/a at $t = 2n\pi$, (d) τ/a at $t = [2n + (3/2)]\pi$, (e) u/a at $t = 2n\pi$, and (f) u/a at $t = [2n + (3/2)]\pi$. The solid line indicates the CI solution, the dashed line the CII solution, and the dash-dotted line the V solution.

Although good agreement between the HBK2 and N solutions is achieved in Fig. 1, the secular-term problem is anticipated in a far field. The comparisons in much wider region are made in Fig. 2 for ω with $k = 0.01$. The HBK2 solution deviates significantly both in amplitude and phase from the N solution at a far distance. The amplitude of HBK2 solution grows unlimitedly as x increases [see also Eqs. (143) and (144); the $\hat{\omega}_{\mathrm{H2}}$ includes the term proportional to x in front of $\exp(-ix/c)$]. This clearly demonstrates the drawback of HBK2 solution due to the secular term. The HBK1 and HBK0 solutions also deviate in amplitude due to the absence of attenuation effect at those approximation levels. In marked contrast with them, the CI and CII solutions agree well with the N solution including in the far field. Similar results are obtained for τ and u, though they are omitted here. These results show that the remedy proposed in Sec. VI B is effective. The CI and CII solutions keep the same (or practically even better) level of accuracy as the HBK2 solution in the boundary layer and the Knudsen layer, which is clearly shown in Figs. 3 and 4. In these figures, comparisons are made among the N, HBK2, HBK1, CI, and CII solutions. Among them, only the HBK1 solution appreciably deviates from the others, because of its lower degree of approximation.

Finally, to examine the contributions of the non-Navier–Stokes part and of the Knudsen-layer correction, we show the CI, CII, and V solutions near the plate for $k = 0.02$ in Fig. 5. In Figs. 5(a)–5(d), the V solution slightly deviates from the CI and CII solutions, while the three solutions are identical for u in Figs. 5(e) and 5(f). This difference between u and the other quantities can be understood, by our discussions in Sec. VI B that there is no non-Navier–Stokes effect in the boundary-layer correction nor the Knudsen-layer correction for the normal component of flow velocity [see Eqs. (135) and (140)].

In conclusion, the numerical evidence presented in this section confirms that the Constructions I and II are essentially identical. Moreover, it shows that they are practically most reliable asymptotic description for acoustic phenomena in the entire region in the slip flow regime.

VIII. CONCLUDING REMARKS

In the present work, a time-evolution of a slightly rarefied monatomic gas from a reference uniform equilibrium state at rest, which is induced by the motion or variation of the temperature of the smooth solid boundary, is investigated on the basis of the linearized Boltzmann equation under the acoustic time scaling. By a systematic asymptotic analysis, linearized Euler sets of equations and boundary-layer equations are derived up to the first order of the Knudsen number, together with their slip and jump boundary conditions, as well as the correction formula in the Knudsen layer. Since the variation of the quantities in the normal direction is steep in the boundary layer and the compressibility is strong in the acoustic scaling compared to the diffusion scaling, the slips and jumps appear from an earlier stage compared with the case of diffusion scaling. Thus, attention needs to be paid to the boundary condition in investigating the acoustic phenomena in the slip-flow regime. The occurrence of secular terms associated with the Hilbert expansion is pointed out and a remedy for it is also given. As an application example, a sound propagation in a half space caused by a sinusoidal oscillation of flat boundary is examined on the basis of the Bhatnagar–Gross–Krook equation. The asymptotic solution agrees well with the direct numerical solution. In particular, the proposed remedy for the secular terms is confirmed to be effective.

ACKNOWLEDGMENTS

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APPENDIX A: ISOTROPIC FUNCTIONS AND RELATED CONSTANTS

The isotropic functions A, B, D_1, D_2, F, and F_d are defined as the solutions of the following integral equations:

$$\mathcal{L}[\zeta_i A(\zeta)] = -\zeta_i\left(\zeta^2 - \frac{5}{2}\right) \quad \text{with} \quad \int \zeta^2 A(\zeta)E(\zeta)d\boldsymbol{\zeta} = 0, \tag{A1}$$

$$\mathcal{L}[\zeta_{ij}B(\zeta)] = -2\zeta_{ij}, \tag{A2}$$

$$\mathcal{L}[(\zeta_i\delta_{jk} + \zeta_j\delta_{ki} + \zeta_k\delta_{ij})D_1(\zeta) + \zeta_i\zeta_j\zeta_k D_2(\zeta)]$$
$$= \gamma_1(\zeta_i\delta_{jk} + \zeta_j\delta_{ki} + \zeta_k\delta_{ij}) - \zeta_i\zeta_j\zeta_k B(\zeta) \quad \text{with} \quad \int [5\zeta^2 D_1(\zeta) + \zeta^4 D_2(\zeta)]E(\zeta)d\boldsymbol{\zeta} = 0, \tag{A3}$$

$$\mathcal{L}[\zeta_{ij}F(\zeta)] = \zeta_{ij}A(\zeta), \tag{A4}$$

$$\mathcal{L}[F_d(\zeta)] = -\frac{5}{6}\gamma_2\left(\zeta^2 - \frac{3}{2}\right) + \frac{1}{3}\zeta^2 A(\zeta) \quad \text{with} \quad \int F_d(\zeta)E(\zeta)d\boldsymbol{\zeta} = \int \zeta^2 F_d(\zeta)E(\zeta)d\boldsymbol{\zeta} = 0. \tag{A5}$$

The related constants γ_i ($i = 1, 2, 3, 6, 10, 11$) are defined by

$$\gamma_1 = I_6(B), \quad \gamma_2 = 2I_6(A), \quad \gamma_3 = -2I_6(F) = I_6(AB) = 5I_6(D_1) + I_8(D_2),$$
$$\gamma_6 = \frac{1}{2}I_6(BD_1) + \frac{3}{14}I_8(BD_2), \quad \gamma_{10} = \frac{1}{2}I_6(B^2),$$
$$\gamma_{11} = -\frac{4}{39}\left\{\frac{1}{2}\gamma_2\left[15I_4(A^2) + \gamma_3\right] + 4I_6(AF) + 15I_4(AF_d)\right\},$$
$$I_n(X) = \frac{8}{15\sqrt{\pi}}\int_0^\infty z^n X(z)\exp(-z^2)dz. \tag{A6}$$

Their value depends on the intermolecular potential and is available for the BGK model, hard-sphere molecules, ES model, and Shakhov model [17,25,28–30].

APPENDIX B: BOUNDARY-LAYER CORRECTION AND CORRESPONDING STRESS AND HEAT FLOW

The boundary-layer corrections $\phi_{\mathrm{B}0}$, $\phi_{\mathrm{B}1}$, and $\phi_{\mathrm{B}2}$ are given by

$$\phi_{\mathrm{B}0} = \phi_{\mathrm{eB}0}, \tag{B1}$$

$$\phi_{\mathrm{B}1} = \phi_{\mathrm{eB}1} - \zeta_i\zeta_j B(\zeta)n_i(\delta_{jk} - n_j n_k)\partial_y u_{k\mathrm{B}0} - \zeta_i A(\zeta)n_i\partial_y\tau_{\mathrm{B}0}, \tag{B2}$$

$$\phi_{\mathrm{B}2} = \phi_{\mathrm{eB}2} - \zeta_{ij}B(\zeta)[n_i(\delta_{jk} - n_j n_k)\partial_y u_{k\mathrm{B}1} + n_i n_j\partial_y(n_k u_{k\mathrm{B}1}) + \mathcal{D}_i[u_{j\mathrm{B}0}]]$$
$$- \zeta_i A(\zeta)(n_i\partial_y\tau_{\mathrm{B}1} + \mathcal{D}_i[\tau_{\mathrm{B}0}]) - [n_i n_j\zeta_{ij}F(\zeta) + F_d(\zeta)]\partial_y^2\tau_{\mathrm{B}0}$$
$$+ [(\zeta_i\delta_{jk} + \zeta_j\delta_{ki} + \zeta_k\delta_{ij})D_1(\zeta) + \zeta_i\zeta_j\zeta_k D_2(\zeta)]n_j n_k(\delta_{i\ell} - n_i n_\ell)\partial_y^2 u_{\ell\mathrm{B}0}, \tag{B3}$$

where

$$\phi_{\mathrm{eB}m} = P_{\mathrm{B}m} + 2\zeta_i u_{i\mathrm{B}m} + \left(\zeta^2 - \frac{5}{2}\right)\tau_{\mathrm{B}m}, \quad (m = 0, 1, 2), \tag{B4}$$

and A, B, D_1, D_2, F, and F_d are defined in Appendix A.

The stress tensor P_{ijBm} and heat-flow vector Q_{iBm} ($m = 0, 1, 2, 3$) are given by

$$P_{ijB0} = P_{B0}\delta_{ij}, \quad Q_{iB0} = 0, \tag{B5}$$

$$P_{ijB1}n_in_j = P_{ijB1}t_i^{(\alpha)}t_j^{(\alpha)} = P_{B1}, \tag{B6}$$

$$P_{ijB1}n_it_j^{(\alpha)} = -\gamma_1\partial_y\big(t_i^{(\alpha)}u_{iB0}\big), \quad P_{ijB1}t_i^{(\alpha)}t_j^{(3-\alpha)} = 0, \tag{B7}$$

$$Q_{iB1}n_i = -\frac{5}{4}\gamma_2\partial_y\tau_{B0}, \quad Q_{iB1}t_i^{(\alpha)} = 0, \tag{B8}$$

$$P_{ijB2}n_in_j = P_{B2} + \frac{2}{3}\gamma_3\partial_y^2\tau_{B0} - \frac{2}{3}\gamma_1\left\{2\partial_y(n_iu_{iB1}) - \sum_{\beta=1}^{2}\big[\chi_{\beta,\beta}\partial_{\chi_\beta}\big(t_i^{(\beta)}u_{iB0}\big) - (-1)^\beta g_{3-\beta}t_i^{(\beta)}u_{iB0}\big]\right\}, \tag{B9}$$

$$\begin{aligned} P_{ijB2}t_i^{(\alpha)}t_j^{(\alpha)} = {} & P_{B2} - \frac{1}{3}\gamma_3\partial_y^2\tau_{B0} + \frac{2}{3}\gamma_1\partial_y(n_iu_{iB1}) - \frac{2}{3}\gamma_1\big[2\chi_{\alpha,\alpha}\partial_{\chi_\alpha}\big(t_i^{(\alpha)}u_{iB0}\big) \\ & - \chi_{3-\alpha,3-\alpha}\partial_{\chi_{3-\alpha}}\big(t_i^{(3-\alpha)}u_{iB0}\big) + (-1)^\alpha g_{3-\alpha}\big(t_i^{(\alpha)}u_{iB0}\big) + 2(-1)^\alpha g_\alpha\big(t_i^{(3-\alpha)}u_{iB0}\big)\big], \end{aligned} \tag{B10}$$

$$P_{ijB2}n_it_j^{(\alpha)} = -\gamma_1\big[\partial_y\big(t_i^{(\alpha)}u_{iB1}\big) - \kappa_\alpha\big(t_i^{(\alpha)}u_{iB0}\big) + \vartheta\big(t_i^{(3-\alpha)}u_{iB0}\big)\big], \tag{B11}$$

$$P_{ijB2}t_i^{(\alpha)}t_j^{(3-\alpha)} = -\gamma_1\sum_{\beta=1}^{2}[\chi_{\beta,\beta}\partial_{\chi_\beta}(t_i^{(3-\beta)}u_{iB0}) - (-1)^\beta g_\beta\big(t_i^{(\beta)}u_{iB0}\big)], \tag{B12}$$

$$Q_{iB2}n_i = -\frac{5}{4}\gamma_2\partial_y\tau_{B1}, \quad Q_{iB2}t_i^{(\alpha)} = -\frac{5}{4}\gamma_2\chi_{\alpha,\alpha}\partial_{\chi_\alpha}\tau_{B0} + \frac{1}{2}\gamma_3\partial_y^2\big(t_i^{(\alpha)}u_{iB0}\big), \tag{B13}$$

$$\begin{aligned} P_{ijB3}n_in_j = {} & P_{B3} + \frac{2}{3}\gamma_3(\partial_y^2\tau_{B1} - \bar{\kappa}\partial_y\tau_{B0}) - \frac{4}{3}\gamma_1\partial_y(n_iu_{iB2}) \\ & - \frac{2}{3}\gamma_1\left\{\sum_{\beta=1}^{2}\Big[-\chi_{\beta,\beta}\partial_{\chi_\beta}\big(t_i^{(\beta)}u_{iB1}\big) + (-1)^\beta g_{3-\beta}\big(t_i^{(\beta)}u_{iB1}\big)\Big] - 2\bar{\kappa}(n_iu_{iB1})\right\} \\ & - \frac{2}{3}\gamma_1y\sum_{\beta=1}^{2}\big[\kappa_\beta\chi_{\beta,\beta}\partial_{\chi_\beta}\big(t_i^{(\beta)}u_{iB0}\big) - \vartheta\chi_{\beta,\beta}\partial_{\chi_\beta}\big(t_i^{(3-\beta)}u_{iB0}\big) \\ & - (-1)^\beta(\kappa_{3-\beta}g_{3-\beta} - \vartheta g_\beta)\big(t_i^{(\beta)}u_{iB0}\big)\big], \end{aligned} \tag{B14}$$

$$\begin{aligned} P_{ijB3}n_it_j^{(\alpha)} = {} & -\gamma_1\big[\partial_y(t_i^{(\alpha)}u_{iB2}) + \chi_{\alpha,\alpha}\partial_{\chi_\alpha}(n_iu_{iB1}) - \kappa_\alpha\big(t_i^{(\alpha)}u_{iB1}\big) + \vartheta\big(t_i^{(3-\alpha)}u_{iB1}\big)\big] \\ & - \gamma_1y\big[\big(\kappa_\alpha^2 + \vartheta^2\big)\big(t_i^{(\alpha)}u_{iB0}\big) - 2\vartheta\bar{\kappa}\big(t_i^{(3-\alpha)}u_{iB0}\big)\big] \\ & + \gamma_3\chi_{\alpha,\alpha}\partial_{\chi_\alpha}\partial_y\tau_{B0} + \left(\frac{1}{2}\gamma_1\gamma_{10} - \gamma_6\right)\partial_y^3\big(t_i^{(\alpha)}u_{iB0}\big), \end{aligned} \tag{B15}$$

$$Q_{iB3}n_i = -\frac{5}{4}\gamma_2\partial_y\tau_{B2} + \frac{1}{2}\gamma_3\partial_y^2(n_iu_{iB1}) - \frac{13}{8}\gamma_{11}\partial_y^3\tau_{B0}. \tag{B16}$$

Here $\alpha = 1, 2$ and γ's are defined in Appendix A.

APPENDIX C: KNUDSEN-LAYER CORRECTION AND CORRESPONDING STRESS AND HEAT FLOW

The Knudsen-layer corrections φ_{K1} and φ_{K2} are given by

$$\varphi_{K1}(t, \eta, \chi_1, \chi_2, \mathring{\boldsymbol{\zeta}}) = \mathring{\zeta}_i(\delta_{ij} - n_in_j)\partial_yu_{jB0}\phi_1^{(1)}(\eta, \mathring{\zeta}_n, \mathring{\zeta}) + \partial_y\tau_{B0}\phi_1^{(0)}(\eta, \mathring{\zeta}_n, \mathring{\zeta}), \tag{C1}$$

$$\begin{aligned}\varphi_{\mathrm{K2}}(t,\eta,\chi_1,\chi_2,\mathring{\boldsymbol{\zeta}}) = {} & \mathring{\zeta}_k(\delta_{kj}-n_kn_j)[\partial_y u_{j\mathrm{B1}} + n_i(\partial_i u_{j\mathrm{H0}} + \partial_j u_{i\mathrm{H0}} + \mathcal{D}_j[u_{i\mathrm{B0}}])]\phi_1^{(1)}(\eta,\mathring{\zeta}_n,\mathring{\zeta}) \\ & + (\partial_y\tau_{\mathrm{B1}} + n_j\partial_j\tau_{\mathrm{H0}})\phi_1^{(0)}(\eta,\mathring{\zeta}_n,\mathring{\zeta}) \\ & + \mathring{\zeta}_i(\delta_{ij}-n_in_j)\big(\partial_j\tau_{\mathrm{H0}} + \mathcal{D}_j[\tau_{\mathrm{B0}}]\big)\phi_2^{(1)}(\eta,\mathring{\zeta}_n,\mathring{\zeta}) \\ & + \mathring{\zeta}_i(\delta_{ij}-n_in_j)\partial_y^2 u_{j\mathrm{B0}}\phi_4^{(1)}(\eta,\mathring{\zeta}_n,\mathring{\zeta}) + \partial_y^2\tau_{\mathrm{B0}}\phi_6^{(0)}(\eta,\mathring{\zeta}_n,\mathring{\zeta}) \\ & + [\partial_y(n_iu_{i\mathrm{B1}}) + n_in_j\partial_i u_{j\mathrm{H0}}]\phi_5^{(0)}(\eta,\mathring{\zeta}_n,\mathring{\zeta}),\end{aligned} \tag{C2}$$

where the notation of functions $\phi_p^{(1)}$ $(p=1,2,4)$ and $\phi_q^{(0)}$ $(q=1,5,6)$ are the same as those in Ref. [25]. They have already been obtained for the BGK model, hard-sphere molecules, ES model, and Shakhov model under the diffuse reflection condition [17,26–30]. The slip and jump coefficients $b_p^{(1)}$ $(p=1,2,4)$ and $c_q^{(0)}$ $(q=1,5,6)$ in the main text are determined simultaneously in solving the problems for $\phi_p^{(1)}$ and $\phi_q^{(0)}$, respectively. The Knudsen-layer functions $Y_p^{(1)}(\eta)$, $H_p^{(1)}(\eta)$, $\Omega_q^{(0)}(\eta)$, and $\Theta_q^{(0)}(\eta)$ are defined as their moment:

$$\begin{aligned} Y_p^{(1)}(\eta) &= \frac{1}{2}\int(\mathring{\zeta}^2-\mathring{\zeta}_n^2)\phi_p^{(1)}(\eta,\mathring{\zeta}_n,\mathring{\zeta})E(\mathring{\zeta})d\mathring{\boldsymbol{\zeta}}, \\ H_p^{(1)}(\eta) &= \frac{1}{2}\int(\mathring{\zeta}^2-\mathring{\zeta}_n^2)\left(\mathring{\zeta}^2-\frac{5}{2}\right)\phi_p^{(1)}(\eta,\mathring{\zeta}_n,\mathring{\zeta})E(\mathring{\zeta})d\mathring{\boldsymbol{\zeta}}, \\ \Omega_q^{(0)}(\eta) &= \int\phi_q^{(0)}(\eta,\mathring{\zeta}_n,\mathring{\zeta})E(\mathring{\zeta})d\mathring{\boldsymbol{\zeta}}, \quad \Theta_q^{(0)}(\eta) = \frac{2}{3}\int\left(\mathring{\zeta}^2-\frac{3}{2}\right)\phi_q^{(0)}(\eta,\mathring{\zeta}_n,\mathring{\zeta})E(\mathring{\zeta})d\mathring{\boldsymbol{\zeta}}. \end{aligned} \tag{C3}$$

Notations $b_p^{(1)}$, $c_q^{(0)}$, $Y_p^{(1)}$, $H_p^{(1)}$, $\Omega_q^{(0)}$, and $\Theta_q^{(0)}$ are the same as those in Refs. [25,27–30]. Incidentally, φ_{K1} and φ_{K2} given above automatically satisfy the initial condition $\varphi_{\mathrm{K1,2}}|_{t=0}=0$ as far as Eq. (80) is satisfied.

The corrections for the stress tensor and heat-flow vector are given by

$$P_{ij\mathrm{K1}}t_i^{(\alpha)}t_j^{(\alpha)} = \frac{3}{2}P_{\mathrm{K1}}, \quad P_{ij\mathrm{K1}}n_in_j = P_{ij\mathrm{K1}}n_it_j^{(\alpha)} = P_{ij\mathrm{K1}}t_i^{(\alpha)}t_j^{(3-\alpha)} = Q_{i\mathrm{K1}}n_i = 0, \tag{C4}$$

$$Q_{i\mathrm{K1}}t_i^{(\alpha)} = H_1^{(1)}(\eta)\partial_y\big(t_i^{(\alpha)}u_{i\mathrm{B0}}\big), \tag{C5}$$

$$P_{ij\mathrm{K2}}t_i^{(\alpha)}t_j^{(\alpha)} = \frac{3}{2}P_{\mathrm{K2}}, \quad P_{ij\mathrm{K2}}n_in_j = P_{ij\mathrm{K2}}n_it_j^{(\alpha)} = P_{ij\mathrm{K2}}t_i^{(\alpha)}t_j^{(3-\alpha)} = Q_{i\mathrm{K2}}n_i = 0, \tag{C6}$$

$$\begin{aligned} Q_{i\mathrm{K2}}t_i^{(\alpha)} = {} & H_1^{(1)}(\eta)\big[\partial_y\big(t_i^{(\alpha)}u_{i\mathrm{B1}}\big) + n_it_j^{(\alpha)}(\partial_iu_{j\mathrm{H0}} + \partial_ju_{i\mathrm{H0}}) - \kappa_\alpha\big(t_i^{(\alpha)}u_{i\mathrm{B0}}\big) + \vartheta\big(t_i^{(3-\alpha)}u_{i\mathrm{B0}}\big)\big] \\ & + H_2^{(1)}(\eta)\Big(t_j^{(\alpha)}\partial_j\tau_{\mathrm{H0}} + \chi_{\alpha,\alpha}\partial_{\chi_\alpha}\tau_{\mathrm{B0}}\Big) + H_4^{(1)}(\eta)\partial_y^2\big(t_i^{(\alpha)}u_{i\mathrm{B0}}\big). \end{aligned} \tag{C7}$$

APPENDIX D: SUPPLEMENT TO SEC. VI B

Suppose that $h_{\mathrm{B2}}^\dagger$ is the solution of Eqs. (72)–(75) and (76) with γ_3, γ_6, γ_{10}, and γ_{11} being zero that meets the conditions Eqs. (101)–(103), (80), and (59). The $h_{\mathrm{B2}}^\dagger$ is the Navier–Stokes part of h_{B2}. Then, the remainder $h_{\mathrm{B2}}^\# \equiv h_{\mathrm{B2}} - h_{\mathrm{B2}}^\dagger$ is found to satisfy the following set of equations with the sources of the non-Navier–Stokes effects:

$$\partial_y(n_iu_{i\mathrm{B2}}^\#) = 0, \tag{D1}$$

$$\partial_y\left(P_{\mathrm{B2}}^\# + \frac{2}{3}\gamma_3\partial_y^2\tau_{\mathrm{B0}}\right) = 0, \tag{D2}$$

$$\partial_t\big(t_i^{(\alpha)}u_{i\mathrm{B2}}^\#\big) - \frac{1}{2}\gamma_1\partial_y^2\big(t_i^{(\alpha)}u_{i\mathrm{B2}}^\#\big) = -\frac{1}{4}(\gamma_1\gamma_{10}-2\gamma_6)\partial_y^4\big(t_i^{(\alpha)}u_{i\mathrm{B0}}\big), \tag{D3}$$

$$\partial_t \tau^{\#}_{\mathrm{B2}} - \frac{1}{2}\gamma_2 \partial_y^2 \tau^{\#}_{\mathrm{B2}} = \frac{2}{5}\partial_t P^{\#}_{\mathrm{B2}} - \frac{1}{10}\left(\gamma_2\gamma_3 - \frac{13}{2}\gamma_{11}\right)\partial_y^4 \tau_{\mathrm{B0}}, \tag{D4}$$

$$P^{\#}_{\mathrm{B2}} = \omega^{\#}_{\mathrm{B2}} + \tau^{\#}_{\mathrm{B2}}. \tag{D5}$$

The $h^{\#}_{\mathrm{B2}}$ is solved to give Eqs. (121)–(125). Note that $h^{\#}_{\mathrm{B2}}$ satisfies the homogeneous Dirichlet boundary condition $h^{\#}_{\mathrm{B2}}|_{y=0} = 0$, because all the terms in the original boundary conditions Eqs. (101)–(103) are incorporated into the problem of $h^{\dagger}_{\mathrm{B2}}$ [Remind that the non-Navier–Stokes effects do not affect the boundary conditions up to $O(\varepsilon^2)$]. In obtaining Eqs. (122) and (124), we have used the fact that the solution of the following problem:

$$\partial_t g - \tfrac{1}{2}\gamma\partial_y^2 g = A\partial_y^4 f, \quad g = 0\ (y=0), \quad \partial_t f - \tfrac{1}{2}\gamma\partial_y^2 f = 0, \quad (A, \gamma : \text{given constant}), \tag{D6}$$

can be expressed as

$$g = -\frac{A}{\gamma} y \partial_y^3 f. \tag{D7}$$

The $P_{ij\mathrm{V}}$, $Q_{i\mathrm{V}}$, $P_{ij\mathrm{C}}$, $Q_{i\mathrm{C}}$, $\overline{P}_{ij\mathrm{K}}$, and $\overline{Q}_{i\mathrm{K}}$ are obtained as follows:

$$P_{ij\mathrm{V}} = P_{\mathrm{V}}\delta_{ij} - \gamma_1\varepsilon^2\overline{\partial_i u_{j\mathrm{V}}}, \quad Q_{i\mathrm{V}} = -\tfrac{5}{4}\gamma_2\varepsilon^2\partial_i\tau_{\mathrm{V}}, \tag{D8}$$

$$P_{ij\mathrm{C}} n_i n_j = \tfrac{2}{3}\gamma_3\varepsilon^4\partial_{x_n}^2\tau_{\mathrm{V}}, \quad P_{ij\mathrm{C}} t_i^{(\alpha)} t_j^{(\alpha)} = -\tfrac{1}{3}\gamma_3\varepsilon^4\partial_{x_n}^2\tau_{\mathrm{V}}, \tag{D9}$$

$$P_{ij\mathrm{C}} t_i^{(\alpha)} t_j^{(3-\alpha)} = P_{ij\mathrm{C}} n_i t_j^{(\alpha)} = Q_{i\mathrm{C}} n_i = 0, \quad Q_{i\mathrm{C}} t_i^{(\alpha)} = \tfrac{1}{2}\gamma_3\varepsilon^4\partial_{x_n}^2\left(t_i^{(\alpha)} u_{i\mathrm{V}}\right), \tag{D10}$$

$$\overline{P}_{ij\mathrm{K}} t_i^{(\alpha)} t_j^{(\alpha)} = \tfrac{3}{2}\overline{P}_{\mathrm{K}}, \quad \overline{P}_{ij\mathrm{K}} n_i n_j = \overline{P}_{ij\mathrm{K}} n_i t_j^{(\alpha)} = \overline{P}_{ij\mathrm{K}} t_i^{(\alpha)} t_j^{(3-\alpha)} = \overline{Q}_{i\mathrm{K}} n_i = 0, \tag{D11}$$

$$\overline{Q}_{i\mathrm{K}} t_i^{(\alpha)} = \varepsilon^2 H_1^{(1)}(\eta) n_i t_j^{(\alpha)}(\partial_i u_{j\mathrm{V}} + \partial_j u_{i\mathrm{V}}) + \varepsilon^2 H_2^{(1)}(\eta) t_j^{(\alpha)}\partial_j\tau_{\mathrm{V}} + \varepsilon^4 H_4^{(1)}(\eta) t_i^{(\alpha)} n_j n_k \partial_j\partial_k u_{i\mathrm{V}}. \tag{D12}$$

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[21] In the case of steady flow with a finite Mach number [17], the use of viscous boundary-layer solution and its matching to the Hilbert solution is advantageous since the problem is nonlinear. Here, the solution is expressed as the sum of the Hilbert part and its correction since the problem is linear and hence the superposition is advantageous. As will be seen later, the superposition leads to the simple condition at infinity and the equations completely decoupled from the Hilbert part at the level of velocity distribution function, accordingly at the level of macroscopic quantities.
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[31] The solution is determined at each order as follows. The $n_i u_{i\mathrm{B}m}$ $(m = 0, 1, \text{or } 2)$ is determined from Eq. (64), (68), or (72) and $P_{\mathrm{B}m}$ from Eq. (65), (69), or (73), under the condition Eq. (59). Then, Eq. (81), (98), or (101) gives the boundary condition for the Hilbert part since $n_i u_{i\mathrm{B}m}$ occurring there has already

been at hands. Under this boundary condition and initial condition Eq. (80), $\omega_{\mathrm{H}m}$, $u_{i\mathrm{H}m}$, $\tau_{\mathrm{H}m}$, and $P_{\mathrm{H}m}$ are determined from Eqs. (31)–(33), (34)–(36), or (40)–(42) with Eq. (43). Now, Eq. (82), (99), or (102) gives the boundary condition for $t_i^{(\alpha)}u_{i\mathrm{B}m}$ and Eq. (83), (100), or (103) that for $\tau_{\mathrm{B}m}$ since $t_i^{(\alpha)}u_{i\mathrm{H}m}$ and $\tau_{\mathrm{H}m}$ occurring there have already been known. Under these boundary conditions, initial condition Eq. (80), and condition Eq. (59), $t_i^{(\alpha)}u_{i\mathrm{B}m}$ is determined from Eq. (66), (70), or (74) and $\tau_{\mathrm{B}m}$ from Eq. (67), (71), or (75). The $\omega_{\mathrm{B}m}$ is determined from Eq. (76).

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[33] In our analyses, the position of the boundary $x_{\mathrm{w}i}$ is allowed to change in time, so that the boundary position may change appreciably after a long time. When the boundary oscillates around a certain point, the problem is often treated as if the position of the boundary did not change in time in the linearized setting. Although we omit the proof, it can be justified, because we can find a one-to-one coordinate transformation from the domain deforming in time, to the nondeformable domain. Therefore, it is allowed to use the same simplification to our system as well in practical applications.

[34] The Hilbert solution and the boundary-layer correction are implicitly assumed to be smooth. In order that the macroscopic quantities up to the second order of ε appearing in the system are continuous, at least the first- and second-order time derivatives of the velocity and temperature of the boundary need to vanish at $t = 0$.

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