Rapid Communications

Scaling of the production of turbulent kinetic energy and temperature variance in a differentially heated vertical channel

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(Received 21 February 2019; published 12 August 2019)

This paper investigates the scaling of turbulent kinetic energy (TKE) and temperature variance production in a differentially heated vertical channel (DHVC). In a DHVC, TKE is produced by two distinctively different mechanisms: buoyancy production and shear production. In the present work, identity equations are derived for the global integrals of shear-produced TKE and temperature variance production. The derived identity equations agree well direct numerical simulation (DNS) data. At sufficiently high Rayleigh number the global integral of the shear-produced TKE is found, based on the DNS data, to scale as $\int_0^\delta R_{\mathrm{wu}} \frac{dU}{dz}\, dz \approx 0.385 u_\tau U_{\max}^2$. Here z is the wall-normal direction, δ is the channel half-width, U is the mean streamwise velocity in the x direction, $U_{\max}$ is the maximum mean streamwise velocity, and u_τ is the friction velocity. $R_{\mathrm{wu}} = -\langle \mathrm{wu} \rangle$ is the Reynolds shear stress, where w is the velocity fluctuation in the z direction, u is the velocity fluctuation in the x direction, and angle brackets $\langle\,\rangle$ denote averaging operation. The global integral of the buoyancy-produced TKE at sufficiently high Grashof number is found to scale as $\int_0^\delta g\alpha R_{\mathrm{u}\theta}\, dz \approx u_\tau^2 U_{\max}$ where g is the gravitational acceleration, α is the thermal expansion coefficient, and $R_{\mathrm{u}\theta} = -\langle \mathrm{u}\theta \rangle$ is the covariance of the streamwise velocity fluctuation u and the temperature fluctuation θ. The global integrals of temperature variance production and temperature dissipation ϵ_θ are found to grow with the Grashof number in a logarithmic-like fashion as $\int_0^\delta R_{\mathrm{w}\theta} \frac{d\Theta}{dz}\, dz = \int_0^\delta \epsilon_\theta\, dz \approx [0.5\ln(\mathrm{Gr}) - 1.8] u_\tau \theta_\tau^2$ where Θ is the mean transformed temperature, $R_{\mathrm{w}\theta} = -\langle \mathrm{w}\theta \rangle$ is the wall-normal turbulent transport of heat, θ_τ is the friction temperature, and Gr is the Grashof number. Based on the characteristics of the flux Richardson number, a four-layer structure is proposed for the TKE budget equation in a DHVC.

DOI: 10.1103/PhysRevFluids.4.081501

I. INTRODUCTION

In pressure-driven turbulent flow through a pipe or channel, or shear-driven turbulent flow over a flat plate, the turbulent kinetic energy (TKE) is produced by Reynolds shear stress times the mean shear, commonly called shear production [1]. In the turbulent natural convection confined within a horizontal channel, i.e., turbulent Rayleigh-Bénard convection, TKE is produced by buoyancy only, since the mean shear is zero. For turbulent flow in a differentially heated vertical channel (DHVC), both buoyancy production and shear production contribute to TKE generation. Hence, DHVC is an interesting case to study the two mechanisms of TKE generation.

Given the difficulties in experimentally studying TKE production in a DHVC, an ideal tool is numerical simulation and, in particular, direct numerical simulation (DNS). There have been a number of excellent DNS studies of DHVC, for example, by Versteegh [2], Kiš [3], and Ng [4].

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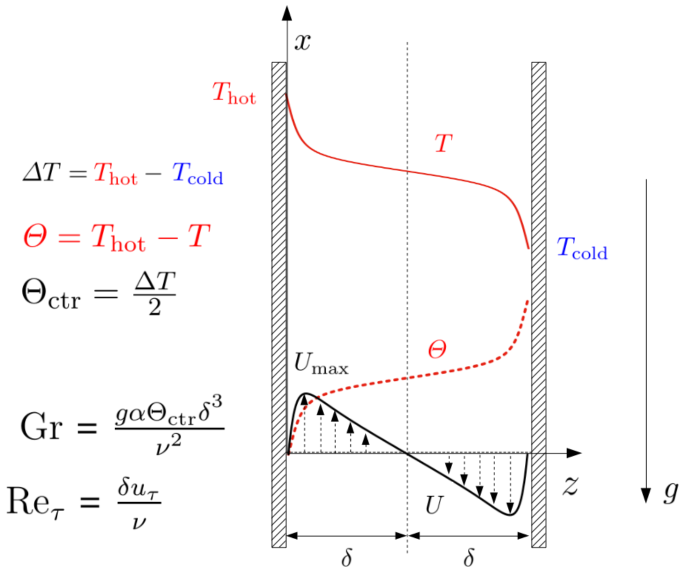

FIG. 1. Sketch of a differentially heated vertical channel (DHVC). $\Theta \stackrel{\text{def}}{=} T_{\text{hot}} - T$ is the mean transformed temperature. Under ideal conditions, the mean flow is antisymmetric about the channel center line.

Using DNS data, Versteegh and Nieuwstadt [5] investigated the TKE budget in a DHVC. They interpreted the budgets in terms of physical processes and found that the shear production is negative in a near-wall region. They also compared the turbulence budgets with the first linear instability mode. The turbulence transport equations, including Reynolds stresses and heat fluxes, were also comprehensively investigated by Kiš and Herwig [6] using DNS data. The budget equation was further studied by Ng *et al.* [7].

Applying the Grossmann-Lohse (GL) theory, Ng *et al.* [8] divided a DHVC into two regions: a boundary layer and a bulk region. Different scaling for the kinetic and thermal dissipations are developed for the two regions [8].

Despite decades of research, however, the underlying physics of the turbulent flow and heat transfer in a DHVC is still not well understood. A better understanding of TKE and temperature variance production will contribute to the development of turbulence models for momentum and heat transfer. Proper scaling is crucial in the understanding of a turbulent flow and heat transfer [9]. The major motivation of the present work is to identify the scaling properties of the TKE and temperature variance productions.

The rest of the paper is organized as follows. In Sec. II identity equations are derived for the global integrals of shear-produced TKE and temperature variance production. The global integral equations are then compared with DNS data. In Sec. III scaling for the buoyancy-produced TKE is determined. Based on the characteristics of buoyancy and shear production of TKE, a four-layer structure is proposed for a DHVC. Section IV summarizes the work.

II. IDENTITY EQUATIONS FOR SHEAR-PRODUCED TKE AND TEMPERATURE VARIANCE PRODUCTION

As illustrated in Fig. 1, natural convection within a DHVC occurs between two vertical walls infinite in the x and y directions. Gravity is in the vertical direction, pointing downwards. The wall-normal z direction originates at the left wall, which is kept at a constant temperature T_{hot}, hotter than the right wall temperature T_{cold}. The temperature difference between the two walls is denoted as $\Delta T = T_{\text{hot}} - T_{\text{cold}}$. Under ideal conditions, the fluid viscosity ν, thermal diffusivity k, and thermal expansion coefficient α are assumed to be constant. The Boussinesq approximation is employed to relate buoyancy force to the temperature variation.

In the present work, a tilde denotes the instantaneous flow variable, an upper case letter denotes the mean flow variable, and a lower case letter denotes the fluctuation [1]. For example, in $\tilde{u} = U + u$, $\tilde{u}$ is the instantaneous velocity in the x direction, U is its mean, and u is its fluctuation. In $\tilde{t} = T + t$, $\tilde{t}$ is the instantaneous temperature, T is the mean temperature, and t is the temperature fluctuation.

The no-slip boundary condition dictates $U_{z=0} = 0$ and $T_{z=0} = T_{\text{hot}}$. To formally match the no-slip velocity boundary condition, a transformed temperature is introduced as $\tilde{\theta} \overset{\text{def}}{=} T_{\text{hot}} - \tilde{t}$ [10]. The mean transformed temperature is $\Theta \overset{\text{def}}{=} T_{\text{hot}} - T$, and the transformed temperature fluctuation is $\theta = -t$. The no-slip thermal boundary condition using the transformed mean temperature is $\Theta_{z=0} = 0$.

Readers are referred to Versteegh [2] and Wei [10] for details on the derivation of the governing equations. The mean momentum balance (MMB) equation in the x direction and the mean thermal energy balance (MHB) equation are

$$0 = \nu \frac{d^2 U}{dz^2} + \frac{dR_{\text{wu}}}{dz} + g\alpha(\Theta_{\text{ctr}} - \Theta), \quad \text{MMB}, \tag{1a}$$

$$0 = k \frac{d^2 \Theta}{dz^2} + \frac{dR_{\text{w}\theta}}{dz}, \quad \text{MHB}, \tag{1b}$$

where $R_{\text{wu}} = -\langle \text{wu} \rangle$ is the wall-normal turbulent transport of streamwise momentum, commonly called kinematic Reynolds shear stress. $R_{\text{w}\theta} = -\langle \text{w}\theta \rangle$ is the wall-normal turbulent transport of heat. Angle brackets $\langle\ \rangle$ denote Reynolds averaging. $\Theta_{\text{ctr}} = 0.5\Delta T$ is the mean transformed temperature at the channel center line.

The total shear stress and total heat flux can be obtained by integrating the MMB Eq. (1a) and the MHB Eq. (1b) along the wall-normal direction and applying boundary conditions as

$$\nu \frac{dU}{dz} + R_{\text{wu}} = u_\tau^2 - g\alpha \int_0^z (\Theta_{\text{ctr}} - \Theta)\, dz, \tag{2a}$$

$$k \frac{d\Theta}{dz} + R_{\text{w}\theta} = u_\tau \theta_\tau, \tag{2b}$$

where $u_\tau \overset{\text{def}}{=} \sqrt{\frac{\tau_{\text{w}}}{\rho}} = \sqrt{\nu \frac{dU}{dz}|_{z=0}}$ is the friction velocity defined by the wall shear stress τ_{w}. Equation (2a) was presented by Shiri and George [11]. θ_τ is the friction temperature defined as $\theta_\tau \overset{\text{def}}{=} \frac{|q_{\text{w}}|}{\rho c_p u_\tau} = \frac{k}{u_\tau} \frac{d\Theta}{dz}|_{z=0}$. Here q_{w} is the wall heat flux, ρ is fluid density, and c_p is heat capacity. Equation (2b) indicates that the total heat flux in a DHVC is a constant in the wall-normal direction.

Identity equations for the integrals of TKE production and temperature variance production can be obtained in two steps [12]: first, multiplying the MMB Eq. (1a) by U and multiplying the MHB Eq. (1b) by Θ to produce

$$0 = \nu \frac{d^2 U}{dz^2} U + \frac{dR_{\text{wu}}}{dz} U + g\alpha(\Theta_{\text{ctr}} - \Theta) U, \tag{3a}$$

$$0 = k \frac{d^2 \Theta}{dz^2} \Theta + \frac{dR_{\text{w}\theta}}{dz} \Theta. \tag{3b}$$

Second, integrating Eqs. (3a) and (3b) along the wall-normal direction and applying boundary conditions yields

$$0 = \left\{ \nu \frac{dU}{dz} U - \nu \int_0^z \left(\frac{dU}{dz} \right)^2 dz \right\} + \left\{ R_{\text{wu}} U - \int_0^z R_{\text{wu}} \frac{dU}{dz}\, dz \right\} + g\alpha \int_0^z (\Theta_{\text{ctr}} - \Theta) U\, dz, \tag{4a}$$

$$0 = \left\{ k \frac{d\Theta}{dz} \Theta - k \int_0^z \left(\frac{d\Theta}{dz} \right)^2 dz \right\} + \left\{ R_{\text{w}\theta} \Theta - \int_0^z R_{\text{w}\theta} \frac{d\Theta}{dz}\, dz \right\}. \tag{4b}$$

Note that integration by parts is applied to simplify the integrals.

Substituting the total shear stress and total heat flux relations in Eqs. (2a) and (2b) into Eqs. (4a) and (4b) yields

$$\int_0^z R_{\mathrm{wu}} \frac{dU}{dz} dz + \nu \int_0^z \left(\frac{dU}{dz}\right)^2 dz = g\alpha \int_0^z (\Theta_{\mathrm{ctr}} - \Theta) U\, dz + \left[u_\tau^2 - g\alpha \int_0^z (\Theta_{\mathrm{ctr}} - \Theta) dz\right] U, \tag{5a}$$

$$\int_0^z R_{\mathrm{w}\theta} \frac{d\Theta}{dz} dz + k \int_0^z \left(\frac{d\Theta}{dz}\right)^2 dy = u_\tau \theta_\tau \Theta. \tag{5b}$$

As shown in Fig. 1, the mean velocity U is antisymmetric about the channel center line with $U_{z=\delta} = 0$. As a result, the global integral, from $z = 0$ to channel center line $z = \delta$, can be expressed as

$$\int_0^\delta R_{\mathrm{wu}} \frac{dU}{dz}\, dz + \nu \int_0^\delta \left(\frac{dU}{dz}\right)^2 dz = g\alpha \int_0^\delta (\Theta_{\mathrm{ctr}} - \Theta) U\, dz, \tag{6a}$$

$$\int_0^\delta R_{\mathrm{w}\theta} \frac{d\Theta}{dz}\, dz + k \int_0^\delta \left(\frac{d\Theta}{dz}\right)^2 dz = u_\tau \theta_\tau \Theta_{\mathrm{ctr}}. \tag{6b}$$

Recently Wei [10,13] found that a proper velocity scale in the outer layer of a DHVC is the maximum streamwise velocity $U_{\max}$, a proper scale for the kinematic Reynolds shear stress is $u_\tau U_{\max}$, and a proper scale for the $R_{\mathrm{w}\theta}$ and $R_{\mathrm{u}\theta}$ is $u_\tau \theta_\tau$. The mean temperature Θ is scaled by the friction temperature θ_τ [14]. The scaled variables are denoted as

$$\eta \equiv \frac{z}{\delta};\quad U^* \equiv \frac{U}{U_{\max}};\quad R_{\mathrm{wu}}^* \equiv \frac{R_{\mathrm{wu}}}{u_\tau U_{\max}};\quad \Theta^+ \equiv \frac{\Theta}{\theta_\tau};\quad R_{\mathrm{w}\theta}^+ \equiv \frac{R_{\mathrm{w}\theta}}{u_\tau \theta_\tau};\quad R_{\mathrm{u}\theta}^+ \equiv \frac{R_{\mathrm{u}\theta}}{u_\tau \theta_\tau}. \tag{7}$$

The symbol η denotes outer-scaled distance, and the superscript $^+$ denotes variables scaled by friction temperature and/or friction velocity, as in studies of turbulent wall-bounded flows. Using the scaled variables, the global integrals can be written as

$$\int_0^1 R_{\mathrm{wu}}^* \frac{dU^*}{d\eta}\, d\eta + \frac{1}{\mathrm{Re}_\tau} \int_0^1 \left[\frac{dU^*}{d\eta}\right]^2 d\eta = \frac{g\alpha\theta_\tau\delta}{u_\tau U_{\max}} \int_0^1 (\Theta_{\mathrm{ctr}}^+ - \Theta^+) U^*\, d\eta, \tag{8a}$$

$$\int_0^1 R_{\mathrm{w}\theta}^+ \frac{d\Theta^+}{d\eta}\, d\eta + \frac{1}{\mathrm{Pe}_\tau} \int_0^1 \left(\frac{d\Theta^+}{d\eta}\right)^2 d\eta = \Theta_{\mathrm{ctr}}^+. \tag{8b}$$

Here $\mathrm{Re}_\tau = \delta u_\tau/\nu$ is a Reynolds number based on the friction velocity and channel half-width. Pe_τ is a Péclet number defined as $\mathrm{Pe}_\tau = \delta u_\tau/k = \mathrm{Pr}\,\mathrm{Re}_\tau$ where $\mathrm{Pr} = \nu/k$ is the Prandtl number. Next we will evaluate the identity Eqs. (8a) and (8b) using data from three independent DNS studies by Versteegh [2], Kiš [3], and Ng [4].

Figure 2(a) shows that the DNS data agree well with the identity Eq. (8a). There is small but noticeable scatter among the three DNS studies, in the shear-production term in particular. One possible cause for the scatter is the domain size used in different simulations. The domain size of Versteegh [2] was $12 \times 6 \times 1$, that of Kiš [3] was $24 \times 12 \times 1$, and that of Ng *et al.* [4,8] was $8 \times 4 \times 1$ (more discussions on the effect of domain size can be found in Kiš and Herwig [6] and Ng *et al.* [8]).

DNS data indicate that, at sufficiently high Grashof number $\mathrm{Gr} \gtrsim 10^5$, the global integral of shear-produced TKE can be approximated as

$$\int_0^1 R_{\mathrm{wu}}^* \frac{dU^*}{d\eta}\, d\eta \approx 0.385 \quad \text{or} \quad \int_0^\delta R_{\mathrm{wu}} \frac{dU}{dz}\, dz \approx 0.385\, u_\tau U_{\max}^2. \tag{9}$$

The constant value of the global integral of shear-produced TKE in the normalized variables can be understood as follows: at high Rayleigh number, R_{wu}^* and U^* become "self-similar" in the outer layer of a turbulent DHVC, i.e., independent of the Rayleigh number [10]. Moreover, the near-wall region in which a different scaling is required occupies an ever-smaller fraction of the channel

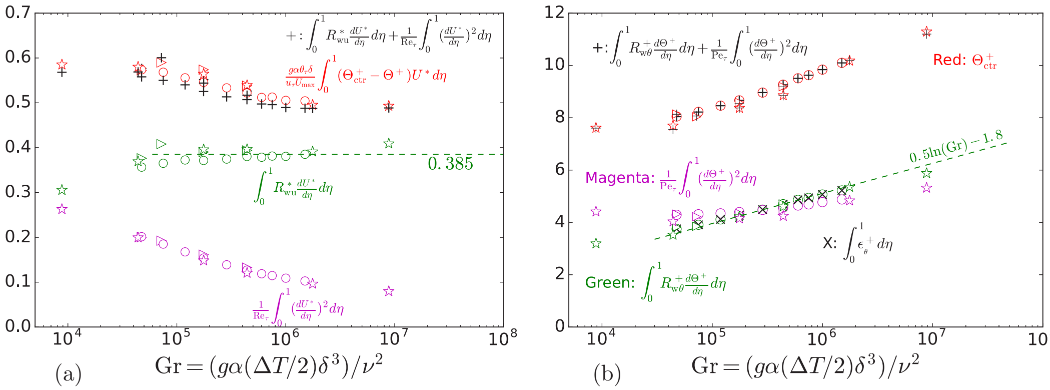

FIG. 2. (a) DNS data to evaluate TKE identity Eq. (8a). DNS data of Versteegh are triangles ▷, DNS data of Kiš are circles ○, and DNS data of Ng are stars ☆. (b) DNS data to evaluate the temperature variance production identity Eq. (8b). Black × symbols denote the the magnitude of $\int_0^1 \epsilon_\theta^+ d\eta$. The dissipation data are from DNS of Kiš [3] only.

as the Rayleigh number increases [10]. Hence, the global integral of shear-produced TKE will be dominated by the outer layer and will approach a constant value in the normalized variables, or $O(u_\tau U_{\max}^2)$ in dimensional form. More DNS, over a wider range of Rayleigh number and Prandtl number, is required to check the validity and more precisely determine the numerical factor in Eq. (9).

Figure 2(b) shows that the identity Eq. (8b) for the temperature variance production is supported very well by the DNS data. At low Grashof number, $\mathrm{Gr} \lesssim 4 \times 10^5$, the dissipation of temperature variance by mean temperature gradient is larger than the production term. At sufficiently high Grashof number, the global integral of the temperature variance production increases with the Grashof number in a logarithmic-like fashion:

$$\int_0^1 R^+_{\mathrm{w}\theta} \frac{d\Theta^+}{d\eta}\, d\eta \approx 0.5\ln(\mathrm{Gr}) - 1.8 \quad \text{or} \quad \int_0^\delta R_{\mathrm{w}\theta} \frac{d\Theta}{dz}\, dz \approx [0.5\ln(\mathrm{Gr}) - 1.8] u_\tau \theta_\tau^2. \tag{10}$$

It is known that in a turbulent channel flow, the passive scalar variance production is strongly influenced by the Prandtl number [12]. All DNS data used in the present work are for $\mathrm{Pr} \approx 0.7$; Prandtl number effects on the global integral of the temperature variance production in a DHVC remain to be studied.

Assuming a statistically steady state, the temperature variance budget equation [1,6,15,16] is

$$\frac{\partial}{\partial t}\left(\frac{\langle\theta^2\rangle}{2}\right) = k\frac{\partial^2}{\partial z^2}\left(\frac{1}{2}\langle\theta^2\rangle\right) - \frac{1}{2}\frac{\partial\langle \mathrm{w}\theta^2\rangle}{\partial z} + R_{\mathrm{w}\theta}\frac{\partial\Theta}{\partial z} - k\left\langle\frac{\partial\theta}{\partial x_j}\frac{\partial\theta}{\partial x_j}\right\rangle, \tag{11}$$

where the terms on the right side are the diffusional transport, turbulent transport, production, and dissipation term, respectively. For brevity the dissipation of temperature variance is denoted hereafter as $\epsilon_\theta = k\langle\frac{\partial\theta}{\partial x_j}\frac{\partial\theta}{\partial x_j}\rangle$. The diffusional and turbulent transport terms only redistribute the temperature variance, and their integrals from 0 to δ are both zero. Hence, the global integral of dissipation $\int_0^\delta \epsilon_\theta\, dz$ equals the global integral of production $\int_0^\delta R_{\mathrm{w}\theta}\frac{d\Theta}{dz}\, dz$. As a result, at sufficiently high Grashof number, the global integral of temperature variance dissipation can also be approximated:

$$\int_0^\delta \epsilon_\theta\, dz \approx [0.5\ln(\mathrm{Gr}) - 1.8] u_\tau \theta_\tau^2. \tag{12}$$

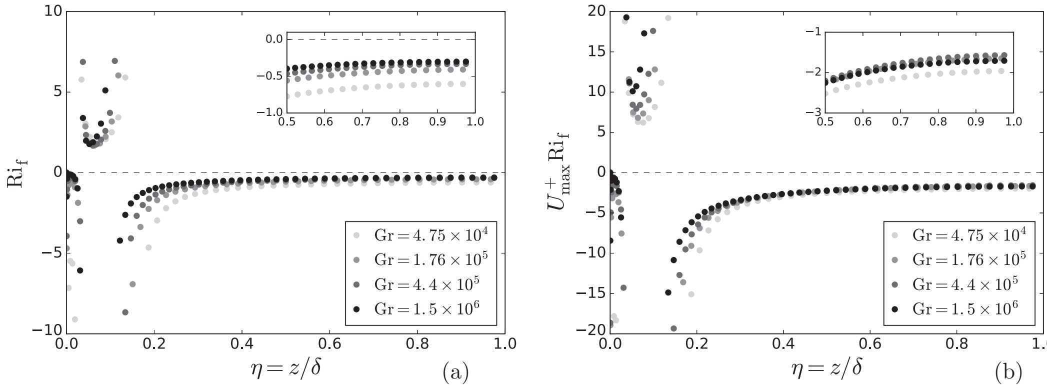

FIG. 3. (a) Flux Richardson number versus $\eta = z/\delta$. (b) Flux Richardson number multiplied by $U^+_{\max} = \frac{U_{\max}}{u_\tau}$. Insets on the top right show the region away from the wall. Data are from DNS of Kiš [3].

In other words, the global integral of temperature variance dissipation when normalized by $u_\tau\theta_\tau^2$ also increases with Grashof number in a logarithmic-like fashion, as shown in Fig. 2(b).

More DNS data, over a wider range of Raleigh numbers and Prandtl numbers, will be required to scrutinize the identity equations and better determine the numerical "constants" in Eq. (12). Next we will determine the scaling of the buoyancy-produced TKE.

III. SCALING OF THE BUOYANCY-PRODUCED TKE

Assuming homogeneity in the x-y plane, the TKE budget equation for a DHVC is [15,16]

$$\frac{\partial}{\partial t}\left(\frac{\langle \mathrm{u_j u_j}\rangle}{2}\right) = \nu\frac{\partial^2}{\partial z^2}\left(\frac{\langle \mathrm{u_j u_j}\rangle}{2}\right) - \frac{\partial}{\partial z}\left\langle \mathrm{w}\left(\frac{\mathrm{u_j u_j}}{2} + \frac{p}{\rho}\right)\right\rangle + R_{\mathrm{wu}}\frac{\partial U}{\partial z} + g\alpha R_{\mathrm{u}\theta} - \nu\left\langle \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}\right\rangle. \quad (13)$$

The first and second terms on the right side are the viscous and turbulent transport term, respectively. $R_{\mathrm{wu}}\frac{dU}{dz}$ is commonly called shear production of TKE, and $g\alpha R_{\mathrm{u}\theta}$ is commonly called buoyancy production of TKE. The last term on the right side is viscous dissipation of TKE, for brevity denoted as $\epsilon_k = \nu\langle \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}\rangle$.

In studies of a stratified atmospheric boundary layer (ABL), a flux Richardson number is often used to characterize the role of buoyancy and shear in TKE production. Following the convention commonly used in ABL studies [9,16], a flux Richardson number for a DHVC is defined as

$$\mathrm{Ri_f} \stackrel{\mathrm{def}}{=} \frac{g\alpha\langle \mathrm{ut}\rangle}{\langle \mathrm{wu}\rangle\frac{dU}{dz}} = -\frac{g\alpha(-\langle \mathrm{u}\theta\rangle)}{-\langle \mathrm{wu}\rangle\frac{dU}{dz}} = -\frac{g\alpha R_{\mathrm{u}\theta}}{R_{\mathrm{wu}}\frac{dU}{dz}}. \quad (14)$$

In a DHVC, the buoyancy-produced TKE is $g\alpha R_{\mathrm{u}\theta}$ because gravity is in the x direction, but the turbulent wall-normal heat flux term in the MHB Eq. (1b) is $R_{\mathrm{w}\theta}$ [8]. In contrast, in a stratified ABL, the buoyancy production term is $g\alpha R_{\mathrm{w}\theta}$ because gravity is in the z direction, and $R_{\mathrm{w}\theta}$ is also the turbulent wall-normal heat flux term in its MHB equation. An ABL is considered stably stratified if $\mathrm{Ri_f} > 0$, neutral if $\mathrm{Ri_f} = 0$, and unstably stratified if $\mathrm{Ri_f} < 0$.

Figure 3(a) presents the variation of the flux Richardson number $\mathrm{Ri_f}$ in the wall-normal direction of a DHVC. The buoyancy production term $g\alpha R_{\mathrm{u}\theta}$ is positive across the whole channel, meaning that the buoyancy-produced TKE in a DHVC is always a source term. However, the shear-produced TKE in a DHVC may be positive or negative, depending on the signs of the Reynolds shear stress R_{wu} and the mean shear $\frac{dU}{dz}$. Thus, the shear-produced TKE may be a source or sink term, and $\mathrm{Ri_f}$ may be positive or negative depending on the wall-normal location.

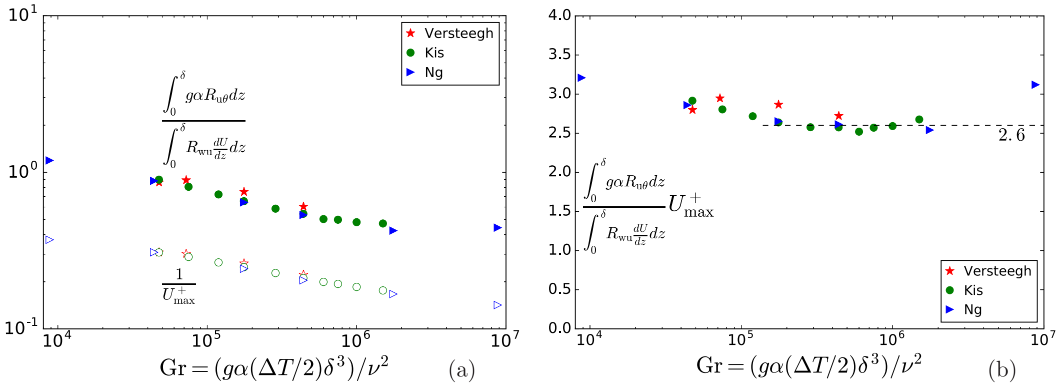

FIG. 4. (a) Ratio (filled symbols) between the global integrals of buoyancy-produced TKE and the shear-produced TKE as a function of Grashof number. Open symbols are data of $\frac{1}{U^+_{\max}}$. (b) Ratio of global integrals weighted by $U^+_{\max}$.

Based on the characteristics of $\mathrm{Ri_f}$ as shown in Fig. 3(a), the left half of a DHVC can be divided into four layers, and the layer structure is symmetric about the channel center line. In Layer I, a thin layer adjacent to the wall, $R_{\mathrm{wu}} > 0$ and $\frac{dU}{dz} > 0$, so shear-produced TKE is positive and $\mathrm{Ri_f} < 0$. In Layer II, between Layer I and the peak U location, $R_{\mathrm{wu}} < 0$ and $\frac{dU}{dz} > 0$, so shear-produced TKE is negative and $\mathrm{Ri_f} > 0$. This negative shear production layer was reported by Versteegh and Nieuwstadt [5] and Kiš and Herwig [6]. Layer III centers around the peak U location (see Fig. 1) where $\frac{dU}{dz} \approx 0$ and $\mathrm{Ri_f} \to \pm\infty$. In Layer IV, between the peak U location and the channel center line, $R_{\mathrm{wu}} < 0$ and $\frac{dU}{dz} < 0$, so shear production of TKE is positive and $\mathrm{Ri_f} < 0$. More DNS data are required to more precisely identify the width of each layer and the dependence on Rayleigh and Prandtl numbers.

Using the normalized variables in Eq. (7), the flux Richardson number can be presented as

$$\mathrm{Ri_f} = -\frac{u_\tau}{U_{\max}}\frac{g\alpha\theta_\tau\delta}{u_\tau U_{\max}}\frac{R^+_{\mathrm{u}\theta}}{R^*_{\mathrm{wu}}\frac{dU^*}{d\eta}} \quad \text{or} \quad U^+_{\max}\mathrm{Ri_f} = -\frac{g\alpha\theta_\tau\delta}{u_\tau U_{\max}}\frac{R^+_{\mathrm{u}\theta}}{R^*_{\mathrm{wu}}\frac{dU^*}{d\eta}}. \tag{15}$$

Figure 3(b) presents the flux Richardson number multiplied by $U^+_{\max}$. In Layer IV, Fig. 3(b) shows that $U^+_{\max}\mathrm{Ri_f}$ from different Grashof numbers collapse well. The deviation at $\mathrm{Gr} = 4.75 \times 10^4$ is likely caused by the low Grashof number effect, similar to the low Reynolds number effect in forced turbulent flows [10].

At the channel center line, DNS data indicate

$$U^+_{\max}\mathrm{Ri_f}|_{\mathrm{ctr}} \approx -1.6 \quad \text{or} \quad \mathrm{Ri_f}|_{\mathrm{ctr}} \approx -\frac{1.6}{U^+_{\max}}. \tag{16}$$

Thus, the flux Richardson number $\mathrm{Ri_f}$ becomes smaller with increasing Grashof number, because $U^+_{\max}$ increases with Grashof number [see Fig. 4(a)].

A. Scaling of the global integral of buoyancy-produced TKE

The global integral of buoyancy- and shear-produced TKE can be obtained by integrating $g\alpha R_{\mathrm{u}\theta}$ and $R_{\mathrm{wu}}\frac{dU}{dz}$ from $z = 0$ to $z = \delta$. The ratio between the global integrals of buoyancy- and shear-produced TKE is

$$\frac{\int_0^\delta g\alpha R_{\mathrm{u}\theta}\, dz}{\int_0^\delta R_{\mathrm{wu}}\frac{dU}{dz}\, dz} = \frac{1}{U^+_{\max}}\frac{g\alpha\theta_\tau\delta}{u_\tau U_{\max}}\frac{\int_0^1 R^+_{\mathrm{u}\theta}\, d\eta}{\int_0^1 R^*_{\mathrm{wu}}\frac{dU^*}{d\eta}\, d\eta}. \tag{17}$$

This ratio is shown in Fig. 4(a) as a function of Grashof numbers. DNS data indicate that as $\mathrm{Gr} \gtrsim 5 \times 10^4$, buoyancy-produced TKE becomes smaller than shear production, and the ratio decreases further with Grashof number. In other words, buoyancy contributes less and less to the total TKE production as Grashof number increases.

Figure 4(b) shows that the ratio Eq. (17) at sufficiently high Grashof number can be approximated as

$$\frac{\int_0^\delta g\alpha R_{\mathrm{u}\theta}\, dz}{\int_0^\delta R_{\mathrm{wu}} \frac{dU}{dz}\, dz} \approx \frac{2.6}{U_{\max}^+}. \tag{18}$$

Applying Eq. (9), the global integral of buoyancy-produced TKE can be approximated as

$$\int_0^\delta g\alpha R_{\mathrm{u}\theta}\, dz \approx u_\tau^2 U_{\max}. \tag{19}$$

IV. SUMMARY

Turbulent kinetic energy (TKE) and temperature variance are some of the most important quantities in understanding the underlying physics of turbulent flow and heat transfer. In a turbulent DHVC, TKE is produced by two distinctively different mechanisms: buoyancy production and shear production. In this work, starting from the mean momentum equation and the mean thermal energy equation, the global integral equations (8a), (8b) are derived for the TKE production and temperature variance production and are found to agree well with three independent DNS studies. The derived equations are general, independent of Rayleigh or Prandtl numbers.

The presently available DNS data have relatively moderate Rayleigh number ($10^5 \lesssim \mathrm{Ra} \lesssim 10^9$) and are at the same Prandtl number $\mathrm{Pr} \approx 0.7$. Based on these DNS data, the global integral of the shear-produced TKE is found to scale as $\int_0^\delta R_{\mathrm{wu}} \frac{dU}{dz}\, dz \approx 0.385 u_\tau U_{\max}^2$, and the global integral of the buoyancy-produced TKE is found to scale as $\int_0^\delta g\alpha R_{\mathrm{u}\theta}\, dz \approx u_\tau^2 U_{\max}$. More DNS studies over a wider range of Rayleigh and Prandtl numbers are required to evaluate the validity of the scaling and better determine the "constants"; it will be of particular interest to see whether the findings will vary with the Prandtl number.

A flux Richardson number $\mathrm{Ri_f}$ is defined as the ratio of buoyancy-produced and shear-produced TKE. Depending on the wall-normal location, $\mathrm{Ri_f}$ can be positive or negative. Based on the characteristics of the $\mathrm{Ri_f}$, a four-layer structure is proposed for the TKE budget equation in a DHVC.

The global integrals of temperature variance production and dissipation are found, based on the DNS data, to grow with the Grashof number in a logarithmic-like fashion: $\int_0^\delta R_{\mathrm{w}\theta} \frac{d\Theta}{dz}\, dz = \int_0^\delta \epsilon_\theta\, dz \approx [0.5\ln(\mathrm{Gr}) - 1.8] u_\tau \theta_\tau^2$. These findings will be useful for future studies of the buoyancy-driven turbulent flow and heat transfer in a DHVC.

ACKNOWLEDGMENT

The author is very grateful to Dr. T. A. M. Versteegh, Dr. P. Kiš, and Dr. C. S. Ng for generously sharing their DNS data.

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