Spectrum of shallow water gravity waves generated by confined two-dimensional turbulence

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We apply Lighthill's theory of aeroacoustic sound generation to shallow water gravity waves generated by spatially confined two-dimensional turbulence. We show that the frequency spectrum of surface waves at large distances from the source of turbulence is, under suitable conditions, proportional to the spatiotemporal spectrum of the energy momentum tensor associated with the turbulent fields acting as the wave source and hence that it follows a power-law behavior. We compute the exponent for shallow water waves generated by isotropic two-dimensional turbulence and show that the integrated power radiated scales as ω^{-3} when the turbulent fluctuations arise from an inverse energy cascade and as ω^{-7} when they arise from the enstrophy cascade.

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I. INTRODUCTION

Waves interact with fluid structures through different processes [1,2], both linear and nonlinear. Through these interactions, the fluid structure imprints on the wave some of its properties. This idea forms the basis of much of our current noninvasive acoustic [3] and optical [4] measurement techniques. In the case of dynamical structures, such as those generated by turbulent flows, the connection between measurable quantities imprinted on the wave and the flow properties relies on a theoretical understanding of turbulence.

The simplest turbulence-wave interaction setup arises when the turbulence is confined in space, forming a turbulent patch. Such a patch can either scatter existing waves or spontaneously generate them. In his seminal paper on aerodynamic sound generation Lighthill [2,5] showed that the acoustic power of waves generated by confined three-dimensional compressible turbulence scales as M^8, where M is the Mach number, a conclusion subsequently elaborated by Proudman [6] and Lighthill himself [7]. Since then, this idea has been applied to wave generation from localized turbulent sources in rotating flows supporting both surface gravity waves [8,9] and inertial-gravity waves [10], as well as to systems supporting magnetohydrodynamic [11], Alfvénic [12], and even gravitational waves [13]. In this paper, we apply Lighthill's original idea to calculate the frequency spectrum of surface gravity waves generated by a confined patch of two-dimensional turbulence in the shallow

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water regime. Measurements of this spectrum can be used to infer the properties of the turbulent flow, such as typical energy scales, and to predict transitions between different regimes.

II. THEORETICAL FRAMEWORK

We consider an incompressible fluid of undisturbed depth h_o and velocity $\mathbf{v}(\mathbf{r}, t)$ in the presence of uniform gravitational acceleration $\mathbf{g} = -g\hat{\mathbf{z}}$. The free surface elevation is described by $h(\mathbf{r}_\perp, t) = h_o + \xi(\mathbf{r}_\perp, t)$, where $\xi(\mathbf{r}_\perp, t)$ is the local surface displacement and $\mathbf{r}_\perp = (x_1, x_2) = (x, y)$ refers to the in-plane coordinates. The typical length scales of $\xi(\mathbf{r}_\perp, t)$ are supposed to be much larger than h_o and the capillary length $l_c = 2\pi\sqrt{\sigma/\rho g}$ of the fluid. Here, ρ is the mass density and σ is the surface tension. Under these conditions we can neglect both the vertical velocity and the variation of the velocity $\mathbf{v}_\perp$ with height, allowing a two-dimensional representation of the flow in terms of $h(\mathbf{r}_\perp, t)$ and $\mathbf{v}_\perp(\mathbf{r}_\perp, t)$, referred to as shallow water theory [1]. In the following we omit the subscript $\perp$.

The equations for h and $\mathbf{v}$ are obtained by integrating over the fluid depth and read

$$\partial_t h + \nabla \cdot (h\mathbf{v}) = \mathbf{0}, \tag{1}$$

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -g\nabla h, \tag{2}$$

where $\nabla \equiv (\partial/\partial x, \partial/\partial y)$ is the horizontal gradient. We rewrite these equations as mass and momentum conservation equations, obtaining

$$\partial_t h + \partial_i (h v^i) = 0, \tag{3}$$

$$\partial_t (h v^i) + \partial_j (h v^i v^j) = -\tfrac{1}{2} g \partial_i (h^2), \tag{4}$$

where $i, j = 1, 2$. Note that the divergence of $\mathbf{v}$ is nonzero. A nondispersive wave equation for the surface displacement ξ can be obtained by combining both equations,

$$\partial_{tt}\xi - c^2 \nabla^2 \xi = \partial_{ij} T^{ij}, \tag{5}$$

where $c \equiv \sqrt{g h_o}$ is the speed of nondispersive linear shallow water gravity waves and

$$T^{ij}(\mathbf{r}, t) \equiv h v^i(\mathbf{r}, t) v^j(\mathbf{r}, t) + g\delta^{ij}\xi(\mathbf{r}, t)^2/2 \tag{6}$$

is the energy-momentum tensor. Implicit in this description is the exclusion of waves with $O(h_o)$ wavelength or less. These waves are in any case damped more rapidly by viscosity, in a distance of order ch_o^2/ν or less [2], where ν is the kinematic viscosity of the fluid. Our detection point $\mathbf{r}$ must therefore lie farther out than this distance. In the following we assume the waves are generated by a turbulent source with a Reynolds number large enough for the development of an inertial range in its spectrum [14] and assume that this source is maintained against decay. As a result, our formulation differs from setups leading to spontaneous gravity wave emission by stable or unstable flows [15]. Viscous effects outside the source region will be neglected even though these can be added to the shallow water description [16].

Although Eq. (5) is nonlinear, we proceed in the spirit of Lighthill's theory for aeroacoustic sound generation [2,5] and treat the right-hand side of the equation as a known quadrupolar source term $\partial_{ij} T^{ij}(\mathbf{r}, t)$ localized in a two-dimensional domain $\mathcal{A}$. This is certainly appropriate in the present study. In this case, waves outside $\mathcal{A}$ obey a linear nondispersive wave equation and we can calculate the radiated wave power, as schematically depicted in Fig. 1.

To do so, we solve Eq. (5) in Fourier space, obtaining

$$\hat{\xi}(\mathbf{r}, \omega) = \int_{\mathcal{A}} \frac{1}{c^2} \partial_{ij} \hat{T}^{ij}(\mathbf{r}', \omega) G(\mathbf{r} - \mathbf{r}', \omega) \mathbf{d}\mathbf{r}', \tag{7}$$

where $\hat{(\cdot)}$ denotes the temporal Fourier transform and $(\cdot)'$ runs over the whole two-dimensional domain $\mathcal{A}$ where the source of the gravity waves is located. We assume $\mathcal{A}$ is spatially bounded.

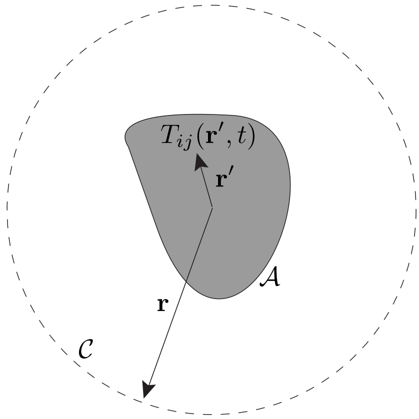

FIG. 1. A bounded two-dimensional domain $\mathcal{A}$ displays a turbulent energy-momentum tensor $T^{ij}(\mathbf{r}', t)$ which is the source of shallow gravity water waves measured at a point $\mathbf{r}$ within the circumference $\mathcal{C}$, with $|\mathbf{r}| = r \gg L = \mathcal{A}^{1/2}$.

The Fourier-transformed Green's function of the Helmholtz equation in two dimensions is given by $G(\mathbf{r} - \mathbf{r}', \omega) = \frac{i}{4} H_0^{(+)}(\frac{\omega|\mathbf{r}-\mathbf{r}'|}{c})$, where $H_0^{(+)}$ is the zero-order Hankel function of the first kind. As we are concerned with the power output of this bounded source at large distances, we use the asymptotic expansion of the Hankel function far away from the source region (and a recurrence equation for its derivatives). Integrating Eq. (7) by parts and using the fact that T^{ij} is spatially bounded, we find that at leading order in $1/r$, $r = |\mathbf{r}|$,

$$\hat{\xi}(\mathbf{r}, \omega) = \frac{i}{4} \frac{\omega^2}{c^4} \sqrt{\frac{2c}{\pi\omega}} \frac{n_i n_j}{r^{1/2}} e^{i\frac{\omega r}{c}} e^{-i\frac{3\pi}{4}} \int_{\mathcal{A}} \hat{T}^{ij}(\mathbf{r}', \omega) e^{-i\frac{\omega \mathbf{r}' \cdot \mathbf{r}}{rc}} \mathbf{dr}'.$$

The first line contains terms that depend only on the distance from the source, while the second line is a truncated spatial Fourier transform of $\hat{T}^{ij}(\mathbf{r}', \omega)$. Here, we have used $\mathbf{r} = r\mathbf{n}$ so that the wave vector is $\mathbf{k} = (\omega/c)\mathbf{n} = k\mathbf{n}$, which is nothing but the dispersion relation for the wave equation. From the above expression we can compute the spectrum of the waves generated by the turbulent source. The frequency spectrum arises from the product

$$\begin{aligned} &\langle \hat{\xi}(\mathbf{r}, \omega) \hat{\xi}(\mathbf{r}, \omega')^* \rangle \\ &= \frac{1}{8\pi} \frac{(\omega\omega')^{3/2} e^{i\frac{(\omega-\omega')r}{c}}}{c^7} \frac{n_i n_j n_l n_m}{r} \int_{\mathcal{A}} \int_{\mathcal{A}} \langle \hat{T}^{ij}(\mathbf{r}', \omega) \hat{T}^{lm}(\mathbf{r}'', \omega')^* \rangle e^{-i\mathbf{k}\cdot\mathbf{r}'} e^{i\mathbf{k}'\cdot\mathbf{r}''} \mathbf{dr}' \mathbf{dr}'', \end{aligned} \tag{8}$$

where $\langle \cdot \rangle$ stands for ensemble averaging over the realizations of the turbulent source, the asterisk indicates complex conjugation, and, by isotropy,

$$n_i n_j n_l n_m = \tfrac{1}{8}(\delta_{ij}\delta_{lm} + \delta_{il}\delta_{jm} + \delta_{im}\delta_{jl}).$$

The output power is obtained by integrating Eq. (8) over a circumference $\mathcal{C}$ of length $2\pi r$ as $r \to \infty$. We obtain

$$\begin{aligned} &\oint_{\mathcal{C}} \langle \hat{\xi}(\mathbf{r}, \omega) \hat{\xi}(\mathbf{r}, \omega')^* \rangle \, dl \\ &= \frac{\pi^4}{2c^7} (\omega\omega')^{3/2} e^{i\frac{(\omega-\omega')r}{c}} \left(\left\langle \hat{T}_{\mathcal{A}}^{ii}(\mathbf{k}, \omega) \hat{T}_{\mathcal{A}}^{jj}(\mathbf{k}', \omega')^* \right\rangle + 2\left\langle \hat{T}_{\mathcal{A}}^{ij}(\mathbf{k}, \omega) \hat{T}_{\mathcal{A}}^{ij}(\mathbf{k}', \omega')^* \right\rangle \right), \end{aligned} \tag{9}$$

where

$$\hat{T}^{ij}_{\mathcal{A}}(\mathbf{k},\omega) = \left(\frac{1}{2\pi}\right)^2 \int_{\mathcal{A}} T^{ij}(\mathbf{r}',\omega)e^{-i\mathbf{k}\cdot\mathbf{r}'}\mathbf{dr}' \tag{10}$$

is the truncated spatiotemporal Fourier transform over the domain $\mathcal{A}$. In the following we use these expressions to compute the wave spectrum and output power of the source when turbulent fluctuations are present in $\mathcal{A}$. To simplify the calculation, we examine the terms in T^{ij} that serve as the source of two-dimensional gravity waves.

III. TWO-DIMENSIONAL TURBULENCE AS A WAVE SOURCE

In water waves the velocity field and surface displacement are strongly coupled. However, in regions of bounded bulk vorticity, this coupling can break down and the contribution from the velocity fluctuations to T^{ij} may be much larger than that arising from surface height deformations. In this case, the dominant term in T^{ij} within $\mathcal{A}$ will be $h_o v^i v^j$ [see Eq. (6)]. This observation is quantified by the ratio $|g\xi^2|/|h_o v^i v^j| = \Lambda^2/M^2$, where Λ is the surface fluctuation to mean depth ratio and $M \equiv |\mathbf{v}|/c$ is the Mach number, and this ratio characterizes the wave source. One can also reformulate this ratio as $|(\xi/h_o)(g\xi/v^i v^j)| = \Lambda/\mathrm{Fr}$, where Fr is the Froude number of the problem, a number used extensively in water wave theory as a control parameter for flow bifurcations [1]. Thus when $\Lambda \ll M$ or Fr we only need to be concerned with the first term in T^{ij} within $\mathcal{A}$.

The calculation of the surface gravity wave spectrum is done by averaging over the turbulent source within $\mathcal{A}$. To do so, we need the scaling of $\langle \hat{T}^{ij}_{\mathcal{A}}(\mathbf{k},\omega)\hat{T}^{lm}_{\mathcal{A}}(\mathbf{k}',\omega')\rangle$ as a function of the different length scales of the problem. If we adopt Kolmogorov's self-similar scaling argument for isotropic fully developed turbulence [17], the important parameters of the turbulent flow will be the mean dissipation rate per unit of mass ϵ, the fluid viscosity ν, the typical scale l at which velocity (eddy) fluctuations develop, and the corresponding velocity fluctuation scale δv. Then, the typical spatial scales of the problem are l, the size of the bounded turbulent region $L \simeq \mathcal{A}^{1/2}$, and the point where the output power is measured r. When $l \ll L \ll r$, the dynamical processes occurring at these length scales separate. Accordingly, the typical timescales are the eddy turnover time $l/\delta v$, the wave propagation time L/c across $\mathcal{A}$, and the time r/c it takes to arrive at the measurement point. These timescales must satisfy certain conditions to ensure that our statistical approach based on stationary isotropic turbulence is valid, specifically that, once generated, the time for the wave to exit the turbulent region is much shorter than the eddy turnover time. Coupling these conditions with the condition $\Lambda \ll M$ on T^{ij}, we require that

$$\Lambda \ll M \ll l/L \ll 1. \tag{11}$$

Condition (11) implies that we can assume the inertial window extends all the way down to the smallest wave number $k_o \equiv 2\pi/L$. This means that we may drop the spatial truncation implied by the subscript $\mathcal{A}$ in Eq. (9), while remembering that $\mathcal{A}$ is finite.

We now estimate the scaling of $\langle \hat{T}^{ij}(\mathbf{k},\omega)\hat{T}^{lm}(\mathbf{k}',\omega')\rangle$. First, we simplify the calculation by assuming that the velocity fluctuations are quasinormal following Millionshchikov's hypothesis [18]. This is an approximation, as odd cumulants are nonzero in fully developed turbulence [19]. Second, for the wave-vector part of the spectrum, we use the turbulent scaling of stationary velocity fluctuations for isotropic two-dimensional turbulence in k space, with $k = |\mathbf{k}| = 2\pi/l$. In two dimensions, the isotropic turbulent kinetic energy spectrum for fluctuations with wave number k consists of two power laws in k, with exponents depending on the conserved flux cascading across scales. On large scales ($k_o < k < k_c$) the turbulence is characterized by an inverse energy cascade with a Kolmogorov spectrum, namely, $E(k) \simeq C_\epsilon \epsilon^{2/3}k^{-5/3}$, where ϵ is the energy flux which scales as (distance^2 $\times$ time^{-3}). On smaller scales ($k_c < k < k_d$) the spectrum is dominated by the enstrophy spectrum $E(k) \simeq C_\eta \eta^{2/3}k^{-3}$, where η is the enstrophy flux which scales as (time^{-3}). Here, C_ϵ and C_η are numerical constants of order 1 which depend on the structure of the flow [14]; for simplicity, the logarithmic correction to the spectrum, $\log{(k/k_o)}^{-1/3}$, found by Kraichnan [20], is not taken into account, although it can be added to the calculations below. The crossover occurs at wave number $k_c = \sqrt{\eta/\epsilon}$. At scales smaller than the Kraichnan dissipation scale $k_d = (\eta/\nu^3)^{1/6}$ the flow is dominated by viscous dissipation. These isotropic turbulent scalings can be used when the entire inertial window is spanned, i.e., when condition (11) is fulfilled. Lastly, we assume that the frequency part of the spectrum can be described by a decorrelation function $g(\omega, \theta_{\mathbf{k}})$ that decays much faster than a power law in ω with a correlation frequency $\theta_{\mathbf{k}} = \epsilon^{1/3}k^{2/3}/\sqrt{2\pi}$ [18,21], which is simply the inverse of the eddy turnover time. This approximation provides a reasonable estimate of the energy-momentum spectrum as it assumes that its correlation time is much longer than L/c and thus much longer that $2\pi/\omega$ (the wave period) [13], which is again consistent with condition (11). Specific temporal autocorrelation functions, such as those proposed by Kraichnan [21], can also be used, but as discussed by Proudman [6] and Lighthill [7], this will have no measurable effect on the above estimate.

IV. ENERGY-MOMENTUM AND WAVE SPECTRA

With the above assumptions, we use the velocity field correlator [13,18]

$$\langle \hat{v}^i(\mathbf{k}, \omega)\hat{v}^j(\mathbf{k}', \omega')^*\rangle = P^{ij}(\mathbf{k})g(\omega, \theta_{\mathbf{k}})\delta(\mathbf{k} - \mathbf{k}')\delta(\omega - \omega') \tag{12}$$

to compute $\langle \hat{T}^{ij}(\mathbf{k}, \omega)\hat{T}^{lm}(\mathbf{k}', \omega')\rangle$. Here, $g(\omega, \theta_{\mathbf{k}})$ is the Fourier transform of the temporal decorrelation function of velocity fluctuations proposed by Kraichan [21], while $P^{ij}(\mathbf{k})$ is the power spectrum of the velocity fluctuations, assumed to be isotropic and homogeneous, so that $P^{ij}(\mathbf{k}) = (\delta^{ij} - k^i k^j/k^2)P^t(k)$. The latter is not in fact completely correct as the two-dimensional velocity field is not divergenceless, and thus another term, of the form $P^l(k)k^i k^j/k^2$, should be added to the correlator [18]. However, as $\Lambda \ll M$, we may neglect the compressible part of the correlator. The functional form of $P^t(k)$ can be found by integrating the correlator in (12) to compute the steady state kinetic energy fluctuations $\langle |\mathbf{v}|^2\rangle$ in real space and relating the result to the one-dimensional energy spectrum $E(k)$. Thus $P^t(k) = 4\pi E(k)/k = \gamma_\beta k^{-\beta}$, with $\beta = 8/3$ and $\gamma_{8/3} = 4\pi C_\epsilon \epsilon^{2/3}$ for the inverse energy cascade, and $\beta = 4$ and $\gamma_4 = 4\pi C_\eta \eta^{2/3}$ for the direct enstrophy cascade.

It follows that $\langle \hat{T}^{ij}(\mathbf{k}, \omega)\hat{T}^{lm}(\mathbf{k}', \omega')^*\rangle$ is proportional to the convolution of $P^{ij}(\mathbf{k})$ with itself over the space of two-dimensional wave vectors involved in the turbulent cascade and also to the convolution of $g(\omega, \theta_{\mathbf{k}})$ with itself in frequency space. The final expression for the energy-momentum spectrum is

$$\begin{aligned}\langle \hat{T}^{ij}(\mathbf{k}, \omega)\hat{T}^{lm}(\mathbf{k}', \omega')^*\rangle &= h_o^2\delta(\mathbf{k} - \mathbf{k}')\delta(\omega - \omega')\iint [P^{il}(\mathbf{q})P^{jm}(\mathbf{k} - \mathbf{q})^* + P^{im}(\mathbf{q})P^{jl}(\mathbf{k} - \mathbf{q})^*]\\ &\quad \times g(\Omega, \theta_{\mathbf{q}})g(\omega - \Omega, \theta_{\mathbf{k}-\mathbf{q}})d\mathbf{q}d\Omega,\end{aligned} \tag{13}$$

where the integral extends over all wave vectors and frequencies within the inertial window for either cascade. There is no contribution from terms of the form $P^{ij}(\mathbf{q})P^{lm}(\mathbf{q})^*$, because these contain the factor $\delta(\mathbf{k})$. Choosing Kraichnan's squared exponential decorrelation function in frequency space $g(\omega, \theta_{\mathbf{k}}) = \frac{1}{\theta_{\mathbf{k}}}\exp{(-\omega^2/\pi\theta_{\mathbf{k}}^2)}$ [13] to describe the frequency part of the turbulent spectrum, the convolution in Eq. (9) reads

$$\begin{aligned}&\langle \hat{T}^{ii}(\mathbf{k}, \omega)\hat{T}^{jj}(\mathbf{k}', \omega')^*\rangle + 2\langle \hat{T}^{ij}(\mathbf{k}, \omega)\hat{T}^{ij}(\mathbf{k}', \omega')^*\rangle\\ &\quad = h_o^2\delta(\mathbf{k} - \mathbf{k}')\delta(\omega - \omega')\iint P^t(|\mathbf{q}|)P^t(|\mathbf{k} - \mathbf{q}|)\frac{\exp\left(-\Omega^2/\pi\theta_q^2\right)}{\theta_{\mathbf{q}}}\frac{\exp\left[-(\omega - \Omega)^2/\pi\theta_{\mathbf{k}-\mathbf{q}}^2\right]}{\theta_{\mathbf{k}-\mathbf{q}}}\\ &\quad\quad \times \left(4\frac{[(\mathbf{k} - \mathbf{q})\cdot \mathbf{q}]^2}{|\mathbf{k} - \mathbf{q}|^2\mathbf{q}^2} + 2\right)d\mathbf{q}d\Omega,\end{aligned} \tag{14}$$

where the factors in the parentheses come from the different index contractions in the angled brackets. The result evidently depends on the functional form of $P^{ij}(\mathbf{k})$ and $g(\omega, \theta_{\mathbf{k}})$. In the present case, the integration over the frequency part is straightforward and yields

$$\int_{\infty}^{-\infty} \frac{\exp\left(-\Omega^2/\pi\theta_{\mathbf{q}}^2\right)}{\theta_{\mathbf{q}}} \frac{\exp\left[-(\omega-\Omega)^2/\pi\theta_{\mathbf{k}-\mathbf{q}}^2\right]}{\theta_{\mathbf{k}-\mathbf{q}}} d\Omega = \left(\frac{\theta_{\mathbf{q}}^2\theta_{\mathbf{k}-\mathbf{q}}^2}{\theta_{\mathbf{q}}^2+\theta_{\mathbf{k}-\mathbf{q}}^2}\right)^{1/2} \exp\left[-\omega^2/\pi\left(\theta_{\mathbf{q}}^2+\theta_{\mathbf{k}-\mathbf{q}}^2\right)\right]. \tag{15}$$

Using the functional forms of $P^t(|\mathbf{k}|)$ and $\theta_{\mathbf{k}}$, the energy-momentum spectrum now reduces to

$$\begin{aligned} &\langle \hat{T}^{ii}(\mathbf{k},\omega)\hat{T}^{jj}(\mathbf{k}',\omega')^*\rangle + 2\langle \hat{T}^{ij}(\mathbf{k},\omega)\hat{T}^{ij}(\mathbf{k}',\omega')^*\rangle \\ &= h_o^2\delta(\mathbf{k}-\mathbf{k}')\delta(\omega-\omega')\gamma_\beta^2 \int q^{-\beta}|\mathbf{k}-\mathbf{q}|^{-\beta}\left(4\frac{[(\mathbf{k}-\mathbf{q})\cdot\mathbf{q}]^2}{|\mathbf{k}-\mathbf{q}|^2\mathbf{q}^2}+2\right) \\ &\times \frac{\sqrt{2\pi}\exp\left[-2\omega^2/\epsilon^{2/3}(q^2+|\mathbf{k}-\mathbf{q}|^2)\right]}{\sqrt{\epsilon^{2/3}(q^2+|\mathbf{k}-\mathbf{q}|^2)}} d\mathbf{q}, \end{aligned} \tag{16}$$

which is a two-dimensional integral over the wave vectors within the inertial window. The angular part stems from $\mathbf{k}\cdot\mathbf{q} = kq\cos\phi$, where ϕ is the angle measured from the direction of $\mathbf{k}$. The final integral can be evaluated only numerically but such a calculation yields little insight into the nature of the spectrum. We may avoid such complications by using an aeroacoustic limit [5], which makes Eq. (16) more tractable and allows us to find analytical asymptotic forms for the spectrum [22]. We use the fact that the most important contribution of the convolution stems from the smallest wave vector of each cascade. Asymptotically, this allows us to take the limit $\mathbf{k}\to 0$ in the integrand of Eq. (16). Then, the spectra can be directly computed as

$$\frac{3\gamma_\beta^2 h_o^2\mathcal{A}}{\epsilon^{1/3}\pi^{1/2}}\left(\int_{k_-}^{k_+}\frac{\exp\left[-\xi(q)^2\right]}{q^{(2\beta-1/3)}}dq\right)\delta(\omega-\omega'), \tag{17}$$

where $\xi(q)\equiv\omega/(\epsilon^{1/3}q^{2/3})$, k_- is the lower bound of the inertial window (k_o for $\beta=8/3$ and k_c for $\beta=4$), and k_+ is its upper bound (k_c for $\beta=8/3$ and k_d for $\beta=4$). Using Eq. (10), we notice that in wave-vector space the delta function for $\mathbf{k}=\mathbf{k}'$ yields $\mathcal{A}/(2\pi)^2$. The integral in parentheses can be computed within the inertial window of each cascade using a simple change of variable. For $\beta=8/3$, we change variables to $u=(q/k_o)^{-4/3}$ and the integral in the parentheses in Eq. (17) reads

$$\frac{3}{4k_o^4}\int_{(k_o/k_c)^{4/3}}^{1}\exp\left(-\frac{\omega^2}{\epsilon^{2/3}k_o^{4/3}}u\right)u^2du, \tag{18}$$

which can be integrated directly by parts and approximated by the dominant term, yielding $(3/2)(\epsilon^{2/3}/\omega^2)^3$ for wave frequencies within the inverse energy cascade $\epsilon^{1/3}k_o^{2/3}<\omega<\epsilon^{1/3}k_c^{2/3}$ with $k_o\ll k_c$. For $\beta=4$, we change variables to $u=(q/k_c)^{-4/3}$, and the integral in the parentheses in Eq. (17) reads

$$\frac{3}{4k_c^{20/3}}\int_{(k_c/k_d)^{4/3}}^{1}\exp\left(-\frac{\omega^2}{\epsilon^{2/3}k_c^{4/3}}u\right)u^4du, \tag{19}$$

this time yielding $18(\epsilon^{2/3}/\omega^2)^5$ for wave frequencies within the direct enstrophy cascade $\epsilon^{1/3}k_c^{2/3}<\omega<\epsilon^{1/3}k_d^{2/3}$ with $k_c\ll k_d$.

The above expressions may now be used in Eq. (9) to find the frequency dependence of the surface wave spectra, which read

$$\oint_C \langle\hat{\xi}(\mathbf{r},\omega)\hat{\xi}(\mathbf{r},\omega')^*\rangle\,dl \simeq \frac{S_\epsilon^2\epsilon^3h_o^2\mathcal{A}}{\omega^3c^7}\delta(\omega-\omega'), \tag{20}$$

for $k_o<k<k_c$ in the inverse energy cascade regime, and

$$\oint_C \langle\hat{\xi}(\mathbf{r},\omega)\hat{\xi}(\mathbf{r},\omega')^*\rangle\,dl \simeq \frac{S_\eta^2\epsilon^3\eta^{4/3}h_o^2\mathcal{A}}{\omega^7c^7}\delta(\omega-\omega'), \tag{21}$$

for $k_c < k < k_d$ within the direct enstrophy cascade regime. Here, $S_\epsilon^2 = 36\pi^{11/2}C_\epsilon^2$ and $S_\eta^2 = 432\pi^{11/2}C_\eta^2$. Thus the integrated radiated power in surface gravity waves in the shallow water regime generated by bounded two-dimensional turbulence is proportional to ω^{-3} within the inverse energy cascade and ω^{-7} within the direct enstrophy cascade.

V. DISCUSSION AND CONCLUSION

We have shown that the framework proposed by Lighthill to describe aerodynamically generated sound by three-dimensional compressible turbulence can be used to compute the frequency spectrum of shallow water gravity waves generated by confined two-dimensional turbulence subject only to the scale separation assumption (11). In the derivation we have neglected both a term of the form $\xi v^i v^j$ (assuming that $|\xi| \ll h_o$) and the potential energy term $g\xi^2$, assuming that $\Lambda \ll M$. These physically motivated assumptions allowed us to treat the right-hand side of Eq. (5) as a known source term. All our results are further limited by the requirement that $k \lesssim 2\pi/h_o$. This wave number may lie anywhere between k_o, k_c, and k_d, thereby truncating the observed inertial window. This cutoff can be tuned by choosing the scales L and h_o and the turbulent fluxes ϵ and η. Since long-range correlations are present in two dimensions, the results obtained here on the basis of scale separation are expected to hold up to logarithmic corrections [23].

In conclusion, we have predicted the spectrum of long surface gravity waves generated by spatially confined two-dimensional turbulence following Lighthill's analysis of weakly compressible turbulence. We have shown that simple but plausible hypotheses about the nature of the turbulent source lead to specific predictions for the power spectrum of the energy-momentum tensor which acts as a quadrupolar wave source. The integrated radiated power in the waves scales as ω^{-3} when the turbulent fluctuations arise from the energy cascade and as ω^{-7} when they arise from the enstrophy cascade, a fact that can be used to diagnose the source of the waves. Furthermore, experimental and numerical studies of the breakdown of the above predictions can lead to estimates of the ratio η/ϵ within the source region and its spatial scale L.

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