Rapid Communications

Non-intermittent turbulence: Lagrangian chaos and irreversibility

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Turbulent flows are special examples of extended dynamical systems distinguished by their *intermittent*, *chaotic*, and *irreversible* behavior. However, the exact nature of the effect of intermittency on the chaotic nature of turbulence, and vice versa, is still not known. By using a recent discovery [U. Frisch, A. Pomyalov, I. Procaccia, and S. S. Ray, Phys. Rev. Lett. **108**, 074501 (2012)] of Fourier decimation, we manipulate the nonlinearity to try and isolate the origins of intermittency, chaos, and irreversibility in homogeneous, isotropic turbulence. In particular, we show that within the Lagrangian framework it is possible to have nonintermittent, yet chaotic, turbulent flows, with an emergent time reversibility as the effective degrees of freedom are reduced through decimation. These results suggest a new microscopic way, starting from the equations of motion, of understanding turbulence beyond what is possible through phenomenological models.

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A detailed understanding of incompressible turbulent flows and the mathematical structure of the underlying equations of hydrodynamics, such as the Navier-Stokes and Euler equations, has proved to be amongst the toughest problems in the natural sciences. The work of Frisch *et al.* [1], which built on the idea originally proposed by L'vov *et al.* [2], has opened a new window to understanding turbulent flows by discovering a surprising connection between ideas of equilibrium statistical mechanics and turbulence in special dimensions. This idea—now popularly known as *Fourier decimation*—allows us, in a self-consistent way, to reduce the effective degrees of freedom (in Fourier space) of the integer d-dimensional Navier-Stokes equation to obtain solutions in noninteger D dimensions. In particular, it was shown that [1], starting from the two-dimensional Navier-Stokes equation, we can obtain Kolmogorov-like, yet equilibrium with a vanishing flux, solutions at a critical dimension $D_c = 4/3$.

Since the works of Hopf [3] and Lee [4], the journey to find precise connections between thermalized solutions of the inviscid equations of hydrodynamics [5–11] has been an arduous one. It is fair to say that the most significant success in this journey has been in two-dimensional turbulence where ideas of equilibrium statistical mechanics were used, with great success, by Kraichnan [12] to explain the dual cascades. Therefore, until recently one had to use phenomenological tools to explain the non-Gaussian, intermittent signatures in turbulence [13]. With the discovery of the decimation trick (described below), however, a new approach based entirely on isolating the role of triadic interactions without further assumptions became possible.

The importance of the work of Frisch *et al.* [1] lies in providing a framework to study the nonlinearity of the Navier-Stokes (or Euler) equations mathematically and numerically. This is because the generalized Galerkin projector (defined precisely later) allows us to tinker with the nonlinear, triadic interactions and retain only a subset of them on a *fractal* or *homogeneous* Fourier set without changing the invariants of the original d-dimensional Navier-Stokes equation. Indeed, it is worth pointing out that the method is general enough to study systems beyond fluid turbulence.

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On the back of this discovery, several important studies have been made by using the Fourier decimation technique to obtain a much better understanding of the role of nonlinearity in equations of hydrodynamics [14–22] (see, also Ref. [10] for a review on this subject).

As a result of this, a critical and intriguing result has emerged in the last couple of years [16,17,20,21]: Decimation results in the reduction of intermittency—both Eulerian and Lagrangian—accompanied by a tendency toward Gaussian statistics. This, in itself, is striking because it is well known that apart from the chaotic nature of turbulent flows, the ubiquitous fingerprint of turbulence is intermittency. Decimation allows us to get rid of Eulerian and Lagrangian intermittency in a nontrivial way—as shown before [16,17,20,21]—but its effect on the chaotic and reversible properties has not been studied yet.

In this Rapid Communication we answer the Lagrangian aspect of this question by showing the existence of Lagrangian chaos with an emergent time reversibility in a nonintermittent turbulent flow. It is worth recalling that in recent papers [23,24] the irreversibility of turbulent flows and its dependence on the intensity of turbulence was extensively studied and understood within the context of rare *flight-crash* events by using Lagrangian methods; our work extends this and shows that microsurgeries on the nonlinear term allows us to recover the time reversibility.

Let us now turn to a precise definition of the decimation protocol [1]. In particular, we begin with the three-dimensional ($d = 3$), (unit-density) Navier-Stokes equation for the velocity field $\boldsymbol{u}(\boldsymbol{x},t)$:

$$\partial_t \boldsymbol{u} = -\nabla p - (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nu \nabla^2 \boldsymbol{u} + \mathbf{F} \tag{1}$$

augmented with the incompressibility condition $\nabla \cdot \boldsymbol{u} = 0$, where p is the pressure, ν is the kinematic viscosity, and the force $\mathbf{F}$ drives the flow to a nonequilibrium statistically steady state. The velocity field in Fourier space $\hat{\boldsymbol{u}}(\boldsymbol{k},t)$ allows us to define the generalized Galerkin projection via

$$\mathbf{v}(\boldsymbol{x},t) = \mathcal{P}\,\mathbf{u}(\boldsymbol{x},t) = \sum_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\, \gamma_{\boldsymbol{k}} \hat{\boldsymbol{u}}(\boldsymbol{k},t), \tag{2}$$

where $\mathcal{P}$ is the generalized Galerkin projector and $\mathbf{v}(\boldsymbol{x},t)$ is the decimated velocity field. The factors $\gamma_{\boldsymbol{k}}$, chosen via

$$\gamma_{\boldsymbol{k}} = \begin{cases} 1 & \text{with probability } h_k \\ 0 & \text{with probability } 1 - h_k, \qquad k \equiv |\boldsymbol{k}| \end{cases} \tag{3}$$

introduce a quenched disorder in the Fourier lattice by eliminating a randomly, but prechosen subset of Fourier modes from the dynamics.

By making use of the definitions above, we can write the decimated Navier-Stokes equations:

$$\partial_t \mathbf{v} = \mathcal{P}[-\nabla p - (\mathbf{v} \cdot \nabla \mathbf{v})] + \nu \nabla^{\mathbf{2}} \mathbf{v} + \mathbf{F}. \tag{4}$$

It is easy to see that the projection of the nonlinear term, the initial conditions, and the forcing on the reduced Fourier lattice ensures that the dynamics exclude at all times the Fourier modes eliminated via the projector $\mathcal{P}$. Furthermore, it is essential to preserve the Hermitian symmetry $\gamma_{\boldsymbol{k}} = \gamma_{-\boldsymbol{k}}$ which ensures that $\mathcal{P}$ is a self-adjoint operator. This is because such a self-adjoint operator can commute with the Laplacian and gradient operators and hence, by making use of $\mathcal{P}\mathbf{v} = \mathbf{v}$, the inviscid invariants (energy and helicity) of the three-dimensional equation are preserved even under decimation.

The definition of the decimation projector allows us some freedom in choosing the nature of the Fourier lattice on which the dynamics is constrained. One way, as in Frisch *et al.* [1] is to choose

$$h_k \propto (k/k_0)^{D-3}, \quad \text{with} \qquad 0 < D \leqslant 3,$$

where k_0 is a reference wave number, taken to be 1. This strategy of *fractal decimation* restricts all dynamics to a D-dimensional Fourier subspace in an embedding three-dimensional space. A second choice, introduced by Buzzicotti *et al.* [21],

$$h_k = \alpha, \quad \forall k; \quad \text{with} \qquad 0 \leqslant \alpha \leqslant 1$$

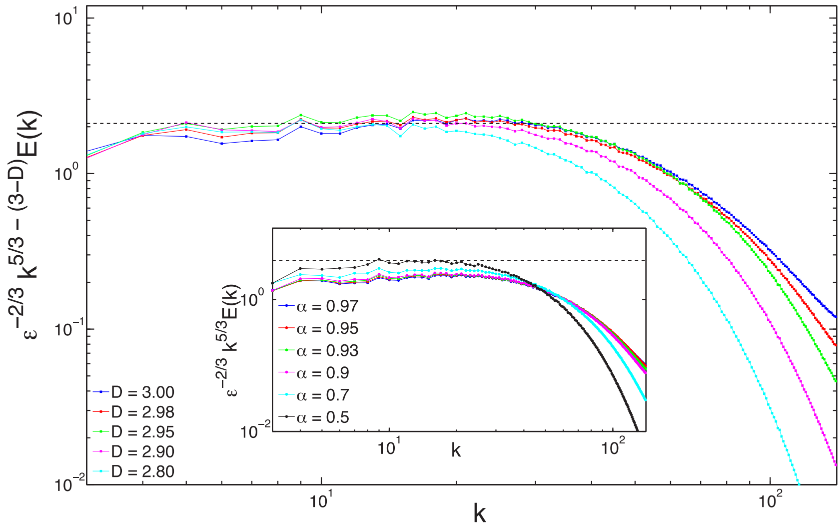

FIG. 1. Log-log plot of the compensated energy spectrum $k^{D-3}k^{5/3}E(k)$ versus the wave number k for fractal decimation. In the inset, we show the compensated spectrum $k^{5/3}E(k)$ for homogeneous decimation. The horizontal, black dashed line is a guide to the eye to suggest the extent of the inertial range. As discussed in the text, fractal decimation picks up an additional factor k^{3-D}, hence the different compensation factors for the two different decimation protocols.

allows, unlike the previous strategy, a *homogeneous* decimation in Fourier space. In this work we use both protocols because it is important to keep in mind that the two strategies are qualitatively different: For fractal decimation the effective degrees of freedom, which scales as $\sim k^D$ ensures that the Fourier modes are decimated with a larger probability for larger k; for homogeneous decimation the probability of decimation is independent of the magnitude of k.

A second distinction between the two protocols is that because of the self-similarity in fractal decimation approach, the energy spectrum picks up an additional factor k^{3-D} in addition to the Kolmogorov scaling $E(k) \sim k^{-5/3}$. We illustrate this in Fig. 1 by showing the compensated plots of the energy spectra $k^{D-3}k^{5/3}E(k)$ for fractal, and $k^{5/3}E(k)$, in the inset, for homogeneous protocols for various degrees of decimation. In addition, by using both these qualitatively different strategies, we find that our results reported here remain unchanged. Hence we conjecture that these effects depend solely on the reduction of the effective degrees of freedom and the introduction of nonlocality through the decimation operator.

The Fourier decimation approach is mathematically precise and easy to implement numerically. Hence we begin by performing direct numerical simulations by using the standard, dealiased pseudospectral method, in three dimensions, 2π-periodic boundary conditions, and a second-order Adams-Bashforth scheme for integration in time. We use $N^3 = 512^3$ collocation points and a constant energy-injection rate on the first two shells, which drives the system to a nonequilibrium statistically steady state to obtain a Taylor-scale Reynolds number $\mathrm{Re}_\lambda \simeq 100$. We then decimate, by using the strategies discussed above, by introducing a quenched disorder to yield (a) fractal decimation with $D = 2.98,\ 2.95,\ 2.90,$ and 2.80 and (b) homogeneous decimation with $\alpha = 0.97,\ 0.95,\ 0.93,\ 0.90,\ 0.70,$ and 0.50. Practically, this involves setting up a disordered Fourier lattice, as defined above, and ensuring that the equations of motion are solved at all times on this decimated lattice. Thence, the Fourier modes of the velocity field are always 0 (or decimated) at the modes which have been preselected through the fractal or homogeneous scheme. In what follows, we redefine a measure of the degree of decimation $\%_D$ as equal to the percentage of modes removed up to the wave number where the dissipation energy spectrum $k^2E(k)$ peaks in the fractal case and as

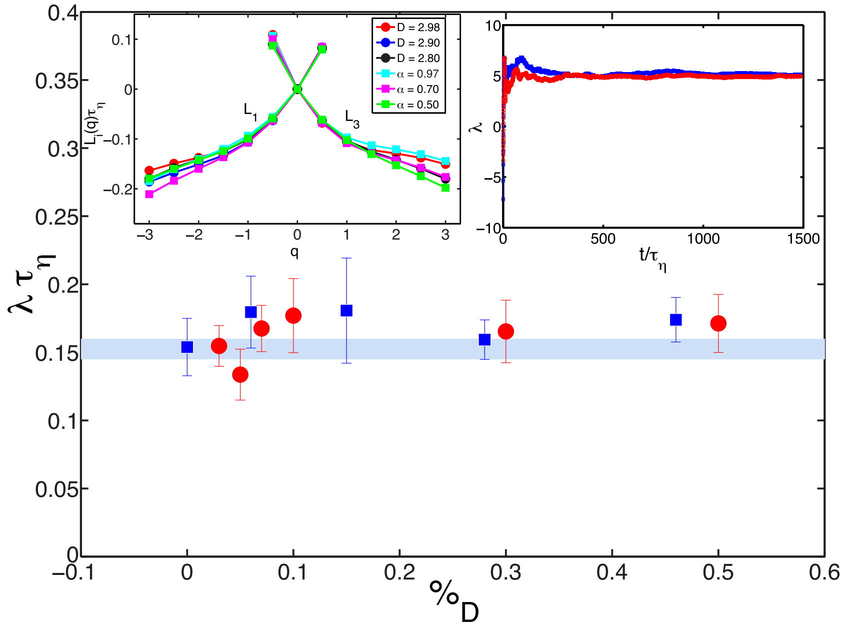

FIG. 2. Plot of the compensated (largest) Lyapunov exponent, in the Lagrangian framework, $\lambda\tau_\eta$, as a function of the degree of decimation $\%_D$ for fractal (blue, filled squares) and homogeneous (red, filled circles) decimation. The pale-blue horizontal band denotes the value obtained for the usual three-dimensional turbulent case. In the inset we show representative plots of λ_1, for the fractal ($\%_D = 0.46$, $D = 2.98$; blue squares) and the homogeneous ($\%_D = 0.1$, $\alpha = 0.9$; red circles) cases, vs the normalized time t/τ_η which shows an initial transient period followed by a saturation, at long times, to the actual exponent λ.

$1 - \alpha$ in the homogeneous case [21]. This single measure allows easier comparison of results from the two decimation strategies.

The degree of Lagrangian chaoticity of a dynamical system is characterized by the generalized Lyapunov exponents $L_i(q)$ $(i = 1,2,3)$, first proposed by Eckmann and Procaccia [25], and subsequently applied to Lagrangian turbulence [26,27]. We calculate these exponents, strictly in the Lagrangian framework, for $-3 \leqslant q \leqslant 3$, by seeding the flows with 50 000 tracers and calculating the fluid velocity gradient tensor along the particle trajectories for 100 large eddy turnover times, defined via $\frac{dX}{dt} = \mathbf{v}(\boldsymbol{X}(t),t)$. We use a trilinear interpolation scheme to calculate the fluid velocity at the typically off-grid particle position $\boldsymbol{X}$. Our measurement of the generalized exponents, shown as the top left inset in Fig. 2 for the largest L_1 and smallest L_3 exponents, are consistent with those reported by Johnson and Meneveau [26]. A useful measure of the Lagrangian chaoticity of the flow is given the Lyapunov exponent $\lambda_i = \frac{dL_i}{dq}|_{q=0}$. In the nonequilibrium statistically steady state of our flows, we calculate [27] the finite-time Lyapunov exponents which, at large enough times, converge to the true Lyapunov exponents λ_i; for a three-dimensional, incompressible flow, our labeling convention is chosen such that $\lambda_1 > \lambda_2 > \lambda_3$; also given the incompressibility $\sum_i \lambda_i = 0$. In what follows, we will call the largest Lyapunov exponent λ_1 as simply λ.

In Fig. 2 we plot the largest Lyapunov exponent compensated by the Kolmogorov timescale τ_η of the flow as a function of $\%_D$ for both decimation strategies. The fractal and homogeneous protocols are distinguished by the symbols and colors in the plot: the blue filled squares are for the fractal case and the red filled circles for the homogeneous one. We see immediately that decimation, in stark contrast to measures of intermittency, has no effect on the level of chaos in a flow. Our measurement of $\lambda\tau_\eta$ is, within error bars, consistent with the values one obtains for the three-dimensional, incompressible Navier-Stokes equation for large enough Reynolds numbers, shown by the pale-blue horizontal band

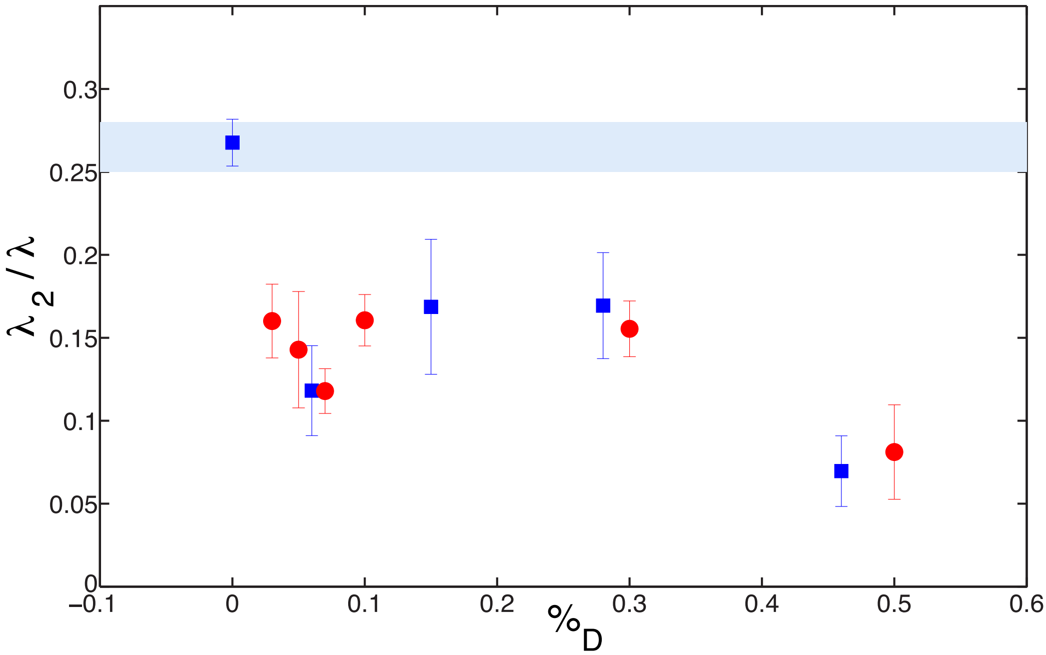

FIG. 3. Plot of the ratio λ_2/λ as a function of the degree of decimation $\%_D$ for fractal (blue filled squares) and homogeneous (red filled circles) decimation. The pale-blue horizontal band denotes the value obtained for the usual three-dimensional turbulent case.

in Fig. 2. In the inset of the same figure, we show representative plots of the time evolution of λ_1 versus the normalized time t/τ_η for the fractal ($D = 2.98$, $\%_D = 0.06$; blue squares) and the homogeneous ($\alpha = 0.90$, $\%_D = 0.1$; red circles) cases. We note the initial transients before the finite-time Lyapunov exponents asymptotically saturate to λ. We take the average and standard deviation of λ_1, by ignoring the initial transient phase, to obtain λ and its error bars as plotted in Fig. 2. It is worth reiterating that our measurements relate strictly to Lagrangian chaos, which is different from Eulerian approaches for an infinite-dimensional system which is beyond the scope of the present work.

It is known that dynamics with time reversal yield $\lambda_2 = 0$. Therefore, a useful measure of the time reversibility is the ratio λ_2/λ which is known to be between 0.25 and 0.28 [27] for three-dimensional, incompressible turbulence reflecting the fact that the Navier-Stokes equation itself is not invariant under time reversal. How does decimation affect this measure? In Fig. 3 we plot the ratio λ_2/λ as a function of the degree of decimation. The horizontal, pale-blue band indicates the value of three-dimensional turbulence and our result for $\%_D$ is in agreement with this value. Remarkably though, as the degree of decimation increases, the ratio becomes smaller and smaller. This is a surprising result and indicates that under decimation, the dynamics of the decimated Navier-Stokes equation becomes more and more time reversal as suggested by the ratio λ_2/λ reducing by a factor of more than 5 from its value for three-dimensional turbulence. This inference is especially valid as the largest exponent λ, as we have shown above, is essentially invariant under decimation and hence the decrease in the ratio is attributable only to a diminishing value of the middle exponent. Interestingly, by itself, as these observations—Figs. 2 and 3—are to characterize solutions of the Navier-Stokes equation in noninteger dimensions, it is important now to discuss what our results imply for the nature of three-dimensional, incompressible, homogeneous, isotropic turbulence.

Unlike simpler low-dimensional dynamical systems, where the relation between intermittency and chaos is well documented [28], turbulence is rather special. Indeed, there are examples where

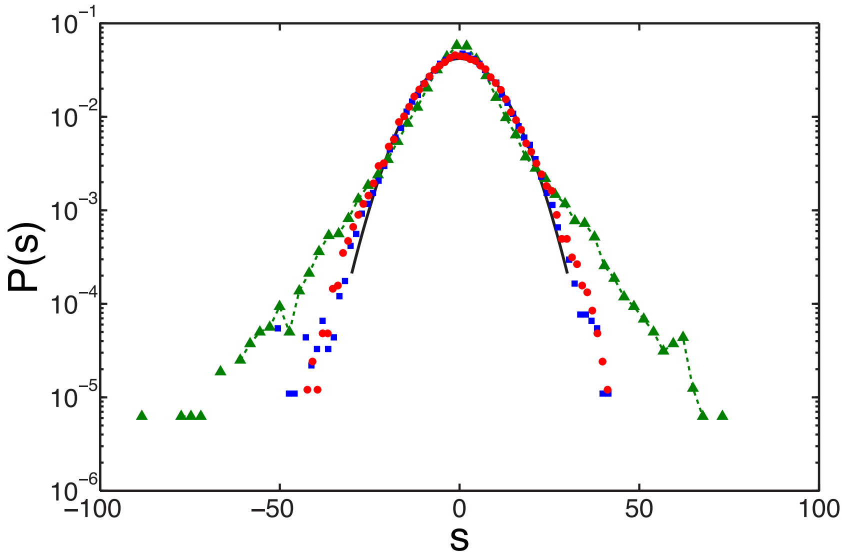

FIG. 4. Lin-log plot of the probability distribution function of the Lagrangian velocity gradients. The data from $\%_D = 0.46$ for fractal ($D = 2.80$; blue filled squares) and $\%_D = 0.5$ for homogeneous ($\alpha = 0.5$; red filled circles) decimations show a Gaussian behavior (black curve). In contrast, for the full three-dimensional case (green triangles connected by a broken line) the distribution function shows fat tails and a marked departure from Gaussianity indicating the effect of intermittency.

the dynamics of a mechanical system is chaotic but not intermittent. For turbulence, however, in the absence of a microscopic theory, it was shown by Crisanti *et al.* [29] by using the multifractal formalism, and numerically validated by shell model simulations, that the level of Eulerian chaos (as quantified by the largest Eulerian Lyapunov exponent, λ_E, not necessarily related to our Lagrangian version λ) is connected to intermittency through the long tails in the distribution of the predictability times. In other words, since the Lyapunov exponent λ_E scales with the Reynolds number as a power law (with an exponent 0.5 within the Kolmogorov theory), it can be shown, within the Parisi-Frisch multifractal formalism [30], that the Eulerian chaos in fully developed turbulence could owe its origins directly to the intermittent, non-Gaussian regime of strongly turbulent flows. As for Lagrangian chaos, intermittency influences the distribution function of the finite-time Lyapunov exponents, which should follow the large deviation principle, through its effect on the width of the associated rate function [27]. Hence, although the mean exponents are mildly affected by the tails of such distribution function, conventional wisdom suggests that the largest exponent λ ought to be influenced to some extent by the strength of intermittency in a flow.

Our results, surprisingly, show this cannot be completely true. As has been shown very recently [16,17,20,21], decimation allows us to perform precise surgeries on the triadic interactions to kill intermittency. To illustrate this, we show in Fig. 4 the probability distribution function $P(s)$ of the Lagrangian velocity gradients s. The green triangles, connected with a broken line, is a plot of $P(s)$ from a three-dimensional flow exemplifying the strong non-Gaussian, intermittent character. On the other hand, the same measurement for a decimated system for $\%_D = 0.46$ ($D = 2.80$, fractal; blue filled squares) and $\%_D = 0.5$ ($\alpha = 0.5$, homogeneous; red filled circles) shows a more Gaussian behavior as indicated by the thick black Gaussian curve. Thus, despite the lack of any intermittency in the system under decimation, we do see an unchanged level of chaotic behavior as clearly seen in Fig. 2 and an emergent reversibility as is illustrated in Fig. 3.

The Fourier decimation procedure opens a new theoretical front to understand one the oldest, unsolved problems in the natural sciences. The advantage of such an approach is that being a projector it leaves the basic mathematics of the starting equations unchanged. In particular, for the Navier-Stokes, while retaining all the invariants of the system, we see a dramatic change in one hallmark of turbulent flows—intermittency—while leaving the other—chaos—unchanged. These results, along with the striking observation of the solution of the Navier-Stokes equation being more and more time reversal as an emergent phenomena with the decrease in the effective degrees of freedom lays the framework for fresh theoretical machinery to understand the nature of turbulence. It thus allows us to carefully study the effect of triadic interactions in chaos, reversibility, and intermittency in solutions of the Navier-Stokes equation. Furthermore, this projector, while destroying the Lagrangian character of the Navier-Stokes equation and making the advection term nonlocal, is an efficient numerical and theoretical tool to understand the full scope of the nonlinear term and the triadic interactions without resorting to more phenomenological modeling of the velocity or dissipation field as done in the past [13]. Our work also adds to the characterization of the decimated Navier-Stokes equation which is important for subgrid modeling like in large-eddy-simulations.

A second implication of our results is in the field of dynamical systems where the ideas of intermittency and chaos are often intrinsically linked [28]. Fully developed, strong turbulence is extremely complicated compared to lower-dimensional models. This is because turbulence is characterized by fluctuations both spatially and temporally along with increasing degrees of freedom as smaller scales emerge with the Reynolds number. Although there are some results which indicate transition to chaos through period doublings [31], we still do not have a complete grasp of the role of intermittency *inter alia* chaos as we do for low-dimensional dynamical systems [28,32]. This work forces us to take a fresh look at this issue.

Before we conclude, it is important to be cautious about how much we should infer from the Lyapunov exponents obtained within the Lagrangian framework. The largest exponent λ is a measure of the Lagrangian—and not the Eulerian—chaos in the system in that it directly measures the sensitivity of the system to initial conditions and how trajectories which start infinitesimally close to each other separate exponentially fast. In particular, this is also not a measure of the intermittency of the system. Furthermore, the middle exponent λ_2, whose behavior is a measure of irreversibility in turbulent flows, may also indicate a neutral evolution of the flow with the largest and smallest exponents balancing each other. Indeed, seperate measurements of irreversibility [23] in decimated systems should suggest how robust are the indications from Lyapunov measurements on this point. This is left for future work.

Furthermore, it is interesting to note that the inverse cascade regime in two-dimensional turbulence is a prototypical example of a nonintermittent, turbulent system. Although it is tempting to equate that absence of intermittency with the effective degrees of freedom—higher dimensionality leading to higher intermittency—the results from two-, three-, and four-dimensional turbulence [33] are not conclusive yet to make such a conjecture. What is clear though from this and earlier works, is that the full set of triads and the locality of the nonlinear interactions in the absence of the decimation operator (for the same Reynolds number), is essential for intermittent behavior of numerical solutions of the three-dimensional Navier-Stokes equation. A possible test for experiments could be to use the idea of an *a posteriori* static mask as developed in Ref. [16]. However, the decimated two-dimensional Navier-Stokes is special [1]; fractal decimation leads, eventually, to equilibrium, fluxless Gaussian statistics, and hence nonintermittent solutions. Indeed, within the shell model framework, this problem of the interplay of equilibrium and cascade solutions has been studied recently by several authors [34]. In our study of three-dimensional flows, the observed phenomena cannot be attributed to equipartition and equilibrium solutions.

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