Rapid Communications

Derivation of Zagarola-Smits scaling in zero-pressure-gradient turbulent boundary layers

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This Rapid Communication derives the Zagarola-Smits scaling directly from the governing equations for zero-pressure-gradient turbulent boundary layers (ZPG TBLs). It has long been observed that the scaling of the mean streamwise velocity in turbulent boundary layer flows differs in the near surface region and in the outer layer. In the inner region of small-velocity-defect boundary layers, it is generally accepted that the proper velocity scale is the friction velocity, u_τ, and the proper length scale is the viscous length scale, ν/u_τ. In the outer region, the most generally used length scale is the boundary layer thickness, δ. However, there is no consensus on velocity scales in the outer layer. Zagarola and Smits [ASME Paper No. FEDSM98-4950 (1998)] proposed a velocity scale, $U_{ZS} = (\delta_1/\delta)U_\infty$, where δ_1 is the displacement thickness and U_∞ is the freestream velocity. However, there are some concerns about Zagarola-Smits scaling due to the lack of a theoretical base. In this paper, the Zagarola-Smits scaling is derived directly from a combination of integral, similarity, and order-of-magnitude analysis of the mean continuity equation. The analysis also reveals that V_∞, the mean wall-normal velocity at the edge of the boundary layer, is a proper scale for the mean wall-normal velocity V. Extending the analysis to the streamwise mean momentum equation, we find that the Reynolds shear stress in ZPG TBLs scales as $U_\infty V_\infty$ in the outer region. This paper also provides a detailed analysis of the mass and mean momentum balance in the outer region of ZPG TBLs.

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I. INTRODUCTION

A zero-pressure-gradient turbulent boundary layer (ZPG TBL) occurs when a uniform stream flows parallel over a flat plate. Despite its simple geometry and intensive research over a century, the statistical behavior of ZPG TBLs is still not fully understood. The most widely accepted description of a turbulent boundary layer is the classical one in terms of an inner region and an outer region. In the inner region, it is generally accepted that the proper velocity scale is the friction velocity, u_τ, which is defined by the wall shear stress as $u_\tau \equiv \sqrt{\tau_{wall}/\rho}$, and the proper length scale is the viscous length scale, ν/u_τ, where ν is the kinematic viscosity of the fluid.

In the outer region, a proper length scale is the boundary layer thickness, δ, while integral and displacement thicknesses have also been used. However, there is no consensus on the velocity scales for the outer region. Some researchers use u_τ as the outer velocity scale [1], and some have used U_∞ [2]. Mixed scaling has also been proposed for the Reynolds stresses in the outer region of ZPG TBLs to account for the dependence on the Reynolds number. For instance, based on empirical evidence,

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DeGraaff and Eaton [3] proposed $u_\tau U_\infty$ as a proper scale for the streamwise Reynolds normal stress $\langle uu \rangle$ in both the inner and outer regions.

In their study of turbulent pipe flows, Zagarola and Smits [4] proposed a new velocity scale for the mean streamwise velocity deficit in the outer layer as

$$U_{\text{zs}} \equiv U_c - U_b, \tag{1}$$

where U_c is the mean centerline velocity and U_b is the bulk mean velocity. Zagarola and Smits [5] further extended this scaling to ZPG TBLs as

$$U_{\text{zs}} \equiv \frac{\delta_1}{\delta} U_\infty, \tag{2}$$

where U_∞ is the freestream velocity, $\delta_1 \equiv \int_0^\infty (1 - U/U_\infty)\, dy$ is the displacement thickness, and δ is the boundary layer thickness.

Using ZPG TBLs data over a wide range of Reynolds numbers, Zagarola and Smits [5] have demonstrated that U_{zs} collapses the mean streamwise velocity deficit, $(U_\infty - U)$, in the outer region better than U_∞ or U_τ. Panton [6] showed that scaling with U_{zs} is equivalent to using a higher order theory for the dependency on the Reynolds number, therefore reducing the influence of the Reynolds number on the scaled velocity deficit. The astounding success of U_{zs} in scaling the mean streamwise velocity deficit of various turbulent wall-bounded flows has been confirmed by other researchers using a wide variety of data sets from different experiments, e.g., Refs. [6–17]. The flow cases include TBLs with pressure gradient (small to large velocity defect cases) and TBLs with rough walls and turbulent pipe flows.

Despite the fact that it can scale the mean streamwise velocity deficit of the outer layer of general TBLs as opposed to u_τ, U_{zs} is regarded by some researchers as empirical. There have been a few theoretical studies to derive or justify the ZS scaling from self-similarity analysis, e.g., Refs. [9,14,18] or with composite expansions [6]. In this work, we will derive the Zagarola-Smits scaling directly from the governing equations, through a combination of integral, similarity, and order-of-magnitude analysis of the continuity equation. The analysis also reveals that V_∞, the mean wall-normal velocity at the edge of the boundary layer, is the companion velocity scale for the mean wall-normal velocity V. The analysis is extended to the mean momentum balance equation and reveals that a valid scale for the Reynolds shear stress is $R_{12,s} = U_\infty V_\infty = H u_\tau^2$, where H is the shape factor of TBLs.

It is important to stress that there may exist multiple possible sets of outer scales for the ZPG TBLs. We do not advocate that the set found here is the only possible one or that it is superior to other ones. Furthermore, the goal of this work is not to find scales that can "collapse profiles" at moderate Reynolds number. The present analysis is strictly valid only at infinite Reynolds number, like almost all the other theoretical analyses of the ZPG TBL.

The mean flow in ZPG TBLs is two dimensional, i.e., the mean flow varies only in the streamwise direction and the wall-normal direction and is statistically homogeneous in the third direction. The mean continuity equation and the mean momentum equation in the streamwise direction, assuming steady incompressible flow, are

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0, \tag{3}$$

$$0 = -U\frac{\partial U}{\partial x} - V\frac{\partial U}{\partial y} + \nu \frac{\partial^2 U}{\partial y^2} + \frac{\partial R_{12}}{\partial y}, \tag{4}$$

where U, V are the mean velocity component in the streamwise direction x and wall-normal direction y, respectively. $R_{12} = -\langle uv \rangle$ is the kinematic Reynolds shear stress, where lowercase letters u, v are velocity fluctuation in the streamwise direction and wall-normal direction, respectively, and angle brackets denote averaging. Note higher order terms in the mean momentum equation such as $\partial(\langle uu \rangle - \langle vv \rangle)/\partial x$ have been neglected. The mean momentum equation is written in a form of 0 = sum of forces, for the convenience of illustrating data in the figures below.

II. INTEGRAL ANALYSIS RESULTS

A recent integral analysis of the mean continuity equation and the mean momentum balance equation reveals relations among the velocity scales $U_\infty, V_\infty, u_\tau$, length scales $\delta, \delta_1, \delta_2$ and the boundary layer growth rate $d\delta/dx$ [19]. The integral results from the continuity and momentum equations can be presented, respectively, as

$$-\frac{d\delta}{dx}\frac{\delta_1}{\delta} + \frac{V_\infty}{U_\infty} = 0, \tag{5}$$

$$-\frac{d\delta}{dx}\left(\frac{\delta_1}{\delta} + \frac{\delta_2}{\delta}\right) + \frac{V_\infty}{U_\infty} + \frac{u_\tau^2}{U_\infty^2} = 0, \tag{6}$$

where $\delta_2 \equiv \int_0^\infty U/U_\infty(1 - U/U_\infty)\, dy$ is the momentum thickness. Equation (5) can be rearranged to form a relation for the growth rate of boundary layer thickness as

$$\frac{d\delta}{dx} = \frac{\delta}{\delta_1}\frac{V_\infty}{U_\infty}. \tag{7}$$

Combination of Eqs. (5) and (6) yields a relation:

$$\frac{U_\infty V_\infty}{u_\tau^2} = \frac{\delta_1}{\delta_2} = H. \tag{8}$$

These integral results will be used next in the derivation of the Zagarola-Smits scaling.

III. SIMILARITY AND ORDER-OF-MAGNITUDE ANALYSES

The objective of the combined similarity and order-of-magnitude analyses is to identify the terms with leading and small order of magnitude in an equation, as well as proper scalings for the leading terms. A key in the analysis is that the prefactor of the leading terms in a properly scaled equation must have the same order of magnitude [20].

The similarity scalings in the outer layer of ZPG TBLs are defined as

$$\eta \equiv \frac{y}{\delta(x)}; \quad U^*(\eta) \equiv \frac{U_\infty - U(x,y)}{U_s(x)}; \quad V^*(\eta) \equiv \frac{V(x,y)}{V_s(x)}; \quad R_{12}^*(\eta) \equiv \frac{R_{12}(x,y)}{R_{12,s}(x)}, \tag{9}$$

where U_s, V_s, and $R_{12,s}$, respectively, are the characteristic scales for U, V, and R_{12} and will be determined in the following analysis. Note that $U_s(x), V_s(x), R_{12,s}(x)$ are allowed to vary with downstream location. Using U^*, V^*, R_{12}^*, and η, the derivatives in the governing equations become the following:

$$\begin{aligned} &\frac{\partial U}{\partial x} = \frac{U_s}{\delta}\frac{d\delta}{dx}\eta\frac{\partial U^*}{\partial \eta} - \frac{dU_s}{dx}U^*; \qquad \frac{\partial U}{\partial y} = -\frac{U_s}{\delta}\frac{\partial U^*}{\partial \eta}; \qquad \frac{\partial^2 U}{\partial y^2} = -\frac{U_s}{\delta^2}\frac{\partial^2 U^*}{\partial \eta^2}; \\ &\frac{\partial V}{\partial y} = \frac{V_s}{\delta}\frac{\partial V^*}{\partial \eta}; \qquad \frac{\partial R_{12}}{\partial y} = \frac{R_{12,s}}{\delta}\frac{\partial R_{12}^*}{\partial \eta}. \end{aligned} \tag{10}$$

A. Similarity and order-of-magnitude analysis of the continuity equation

Substituting the $\partial U/\partial x$ term and the $\partial V/\partial y$ term in Eq. (10) into Eq. (3), the continuity equation becomes

$$\frac{U_s}{\delta}\frac{d\delta}{dx}\eta\frac{\partial U^*}{\partial \eta} - \frac{dU_s}{dx}U^* + \frac{V_s}{\delta}\frac{\partial V^*}{\partial \eta} = 0. \tag{11}$$

Equation (11) is then multiplied by δ/V_s to turn it into a nondimensional form:

$$\left[\frac{U_s}{V_s}\frac{d\delta}{dx}\right]\eta\frac{\partial U^*}{\partial\eta} - \left[\frac{\delta}{V_s}\frac{dU_s}{dx}\right]U^* + \frac{\partial V^*}{\partial\eta} = 0. \tag{12}$$

The balance in Eq. (12) can be established in two ways: (1) all the three terms have the same order of magnitude and each contributes to the balance of the equation; (2) the balance is dominated by two leading terms, and the third one is much smaller. We demonstrate below that in the outer layer of ZPG TBLs (between $\eta = 0.2$ and 1), the balance of Eq. (12) is essentially between the first and the last terms.

If the first term in Eq. (12) is a leading term, then its prefactor has to be nominally $O(1)$ because the prefactor in the third term is one. Thus

$$\left[\frac{U_s}{V_s}\frac{d\delta}{dx}\right] = O(1). \tag{13}$$

Replacing $d\delta/dx$ from the integral analysis result of Eq. (7), the prefactor of the first term in Eq. (12) becomes

$$\frac{U_s}{V_s}\frac{V_\infty}{U_\infty}\frac{\delta}{\delta_1} = O(1). \tag{14}$$

A simple option for this relation to be valid is by setting

$$U_s = U_\infty\frac{\delta_1}{\delta} \quad \text{and} \quad V_s = V_\infty. \tag{15}$$

Thus, the Zagarola-Smits scaling is a direct consequence of balancing the leading terms in the mean continuity equation. The analysis also reveals that the companion scale for the mean wall-normal velocity is $V_s = V_\infty$, consistent with the finding of Wei and Klewicki [19]. Next we present the evidence that the second term in Eq. (12) is indeed much smaller than the other two terms in the outer layer of ZPG TBLs.

1. *Evidence that* $-[\frac{\delta}{V_s}\frac{dU_s}{dx}]U^*$ *is small in Eq. (12)*

Setting $U_s = U_{ZS} = U_\infty\delta_1/\delta$ and $V_s = V_\infty$, the prefactor of the second term in Eq. (12) becomes

$$\frac{\delta}{V_s}\frac{dU_s}{dx} = \delta\frac{U_\infty}{V_\infty}\frac{d(\delta_1/\delta)}{dx} = \frac{\delta^2}{\delta_1}\frac{\frac{d(\delta_1/\delta)}{dx}}{\frac{d\delta}{dx}} = \frac{\frac{1}{\delta_1}\frac{d\delta_1}{dx}}{\frac{1}{\delta}\frac{d\delta}{dx}} - 1 = \alpha - 1. \tag{16}$$

For convenience α is used to denote the ratio of $(\frac{1}{\delta_1}\frac{d\delta_1}{dx})/(\frac{1}{\delta}\frac{d\delta}{dx})$. Note that V_∞/U_∞ is replaced by the integral analysis result of Eq. (7). If experimental or numerical data of $\delta(x)$ and $\delta_1(x)$ are available, then the order of magnitude of this prefactor can be discerned.

A comprehensive experimental study of ZPG TBLs has been conducted by Osterlund [21]. Covering a wide range of Reynolds numbers, between $\mathrm{Re}_\theta = 2500$ and 27 000, Osterlund measured the evolution of $\delta(x)$ and $\delta_1(x)$ as shown in Fig. 1(a). The prefactor $\alpha - 1 = \frac{\frac{1}{\delta_1}\frac{d\delta_1}{dx}}{\frac{1}{\delta}\frac{d\delta}{dx}} - 1$ is computed using Osterlund's experimental data and plotted in Fig. 1(b), which shows that this prefactor has a small value of about (-0.1) over the Reynolds number range of the experiments. Furthermore, the magnitude of this prefactor becomes even smaller as Reynolds number increases. In the outer layer, U^* is bounded and is of $O(1)$ (see Fig. 2); thus, the second term in the continuity equation, $-[\frac{\delta}{V_s}\frac{dU_s}{dx}]U^*$, is indeed small in the outer layer of the flow.

The scatter in Fig. 1(b), especially at the first and last x stations, is likely caused by the numerical differentiation of $\delta(x)$ data to obtain $d\delta/dx$. Note that this prefactor should be negative (as shown in Fig. 2). The positive values in Fig. 1(b) are not physical and are caused by the numerical differentiation error.

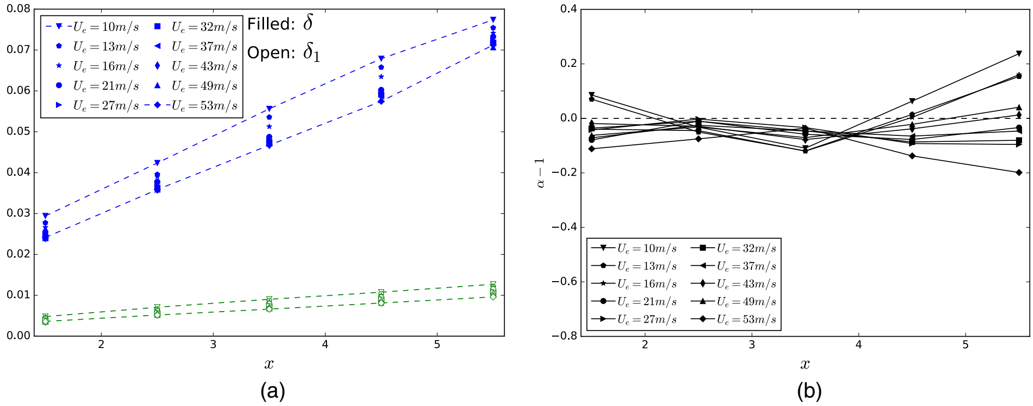

FIG. 1. (a) Boundary layer thickness, δ, and displacement thickness, δ_1, as a function of distance in the streamwise direction, x. (b) $\alpha - 1 = \frac{\frac{1}{\delta_1}\frac{d\delta_1}{dx}}{\frac{1}{\delta}\frac{d\delta}{dx}} - 1$ as a function of x. ZPG TBLs data are from experiments of Osterlund [21].

2. *Experimental and numerical data of the continuity equation*

Setting $U_s = U_{ZS} = (\delta_1/\delta)U_\infty$ and $V_s = V_\infty$, the continuity equation (12) can be written as

$$\eta\frac{\partial U^*}{\partial \eta} - [\alpha - 1]U^* + \frac{\partial V^*}{\partial \eta} = 0. \tag{17}$$

Using the direct numerical simulation (DNS) data of Ref. [22], the three terms in Eq. (17) are computed and presented in Fig. 2. Because the prefactor of the second term is small as discussed in the previous section, to better show the trend, U^* itself is plotted without the prefactor.

Also plotted in Fig. 2 is the sum of the first term and the last term $\eta\partial U^*/\partial\eta + \partial V^*/\partial\eta$, which is essentially zero in the outer layer, suggesting that the balance of the continuity equation is dominated

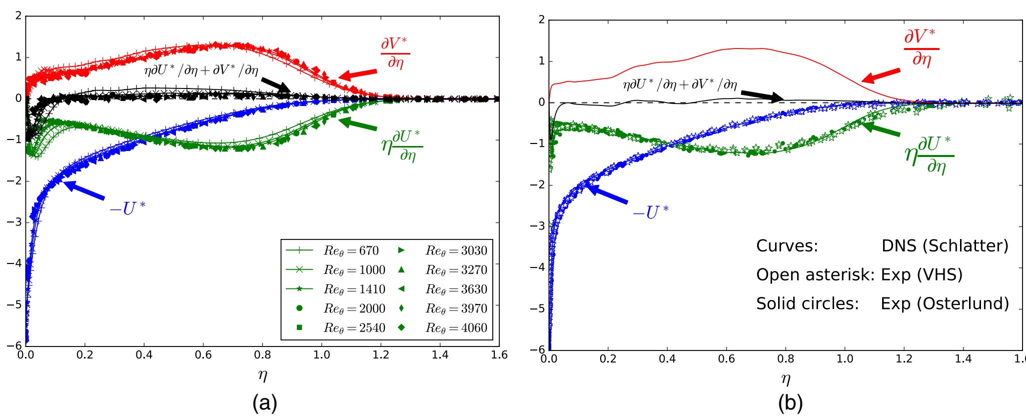

FIG. 2. Balance of the mean continuity equation, $\eta\frac{\partial U^*}{\partial\eta} - [\alpha - 1]U^* + \frac{\partial V^*}{\partial\eta} = 0$. Note the prefactor for the second term is not multiplied (if multiplied, the term would be too small to see). (a) DNS data of Ref. [22]. (b) Experimental data at higher Reynolds numbers. Only the first two terms are available (V is not measured). Solid circles data points are from experiments by Osterlund [21] ($\mathrm{Re}_\theta = 2500$ to 27 000). Open asterisk data points are from experiments by Vallikivi *et al.* [23] ($\mathrm{Re}_\theta = 8400$ to $\mathrm{Re}_\theta = 235\,000$). Curves: DNS data at $\mathrm{Re} = 4060$ [22].

by the first and the last term. Figure 2 also shows that the prefactor in the second term should be a negative number, in order to contribute to the balance of the equation in the near-wall region.

To determine the dependence on the Reynolds numbers, the first term and second term from two high Reynolds number experiments, Osterlund [21] and Vallikivi *et al.* [23], are plotted in Fig. 2(b) along with data from a moderate Reynolds number numerical simulation at $\mathrm{Re}_\theta = 4060$ [22]. Figure 2(b) shows that the dependence on Reynolds number is weak.

In short, experimental and numerical data shown in Fig. 2 support the assumption made in the previous section: $[\frac{U_s}{V_s}\frac{d\delta}{dx}]\eta\frac{\partial U^*}{\partial \eta} \approx \eta\frac{\partial U^*}{\partial \eta}$ is a leading order of magnitude term in the continuity equation that balances the other leading order of magnitude term $\frac{\partial V^*}{\partial \eta}$ in the outer region. This balance leads to the ZS scaling for the mean streamwise velocity deficit and the companion scale $V_s = V_\infty$ for the mean wall-normal velocity.

B. Similarity and order-of-magnitude analysis of the mean momentum equation

Replacing $\partial U/\partial x$ with $\partial V/\partial y$ and substituting the derivatives in Eq. (10) into Eq. (4), the mean momentum equation becomes

$$0 = (U_\infty - U_s U^*)\frac{V_s}{\delta}\frac{\partial V^*}{\partial \eta} - (V_s V^*)\left(-\frac{U_s}{\delta}\right)\frac{\partial U^*}{\partial \eta} + \nu\left(\frac{-U_s}{\delta^2}\right)\frac{\partial^2 U^*}{\partial \eta^2} + \left(\frac{R_{12,s}}{\delta}\right)\frac{\partial R_{12}^*}{\partial \eta}. \tag{18}$$

Equation (18) is then multiplied by $\delta/(U_s V_s)$ to turn it into a nondimensional form:

$$0 = \left[\frac{U_\infty}{U_s}\right]\frac{\partial V^*}{\partial \eta} - U^*\frac{\partial V^*}{\partial \eta} + V^*\frac{\partial U^*}{\partial \eta} - \left[\frac{\nu}{\delta V_s}\right]\frac{\partial^2 U^*}{\partial \eta^2} + \left[\frac{R_{12,s}}{U_s V_s}\right]\frac{\partial R_{12}^*}{\partial \eta}. \tag{19}$$

It is well known that the viscous force term, the fourth term on the right, is negligible in the outer layer. Here we show that the leading order of magnitude terms are the first and the last terms. Since these two terms are proposed as leading order, their prefactors need to have the same nominal order of magnitude:

$$O\left(\frac{U_\infty}{U_s}\right) = O\left(\frac{R_{12,s}}{U_s V_s}\right), \tag{20}$$

and using the scaling relations from Eq. (15), we find

$$O(R_{12,s}) = O(U_\infty V_s) = O(U_\infty V_\infty). \tag{21}$$

1. *Balance of the mean momentum equation*

Setting $U_s = U_{\mathrm{ZS}}$, $V_s = V_\infty$ and dividing by δ/δ_1, the mean momentum balance equation (19) becomes

$$0 = \frac{\partial V^*}{\partial \eta} + \left[\frac{\delta_1}{\delta}\right]\left(-U^*\frac{\partial V^*}{\partial \eta} + V^*\frac{\partial U^*}{\partial \eta}\right) - \left[\frac{\delta_2}{\delta}\frac{U_\infty^+}{\delta^+}\right]\frac{\partial^2 U^*}{\partial \eta^2} + \left[\frac{R_{12,s}}{U_\infty V_\infty}\right]\frac{\partial R_{12}^*}{\partial \eta}, \tag{22}$$

where $U_\infty^+ \equiv U_\infty/u_\tau$ and $\delta^+ \equiv \delta u_\tau/\nu$.

The ratio between the displacement thickness and the boundary layer thickness, δ_1/δ, appears as the prefactor for the second and third terms. Therefore, the magnitude of δ_1/δ determines the contribution of these two terms to the force balance. Using the DNS data [22] and experimental data [3,21,23], this ratio δ_1/δ is presented in Fig. 3, which shows that δ_1/δ decreases with Reynolds numbers, but very slowly (close to a logarithmic-like fashion).

The magnitude of each term in Eq. (22) can be estimated as the following:

(1) $\frac{\partial V^*}{\partial \eta}$: $V^* = V/V_\infty$ is a regular function that varies between 0 and 1 over the range of $\eta = 0$ to 1, so $\frac{\partial V^*}{\partial \eta}$ will be $O(1)$.

(2) $[\frac{\delta_1}{\delta}](-U^*\frac{\partial V^*}{\partial \eta} + V^*\frac{\partial U^*}{\partial \eta})$: The terms inside the parentheses are both $O(1)$ in the outer layer, but the magnitude of the prefactor is small ($\sim$0.1–0.2) and decreases with Reynolds number. In the case

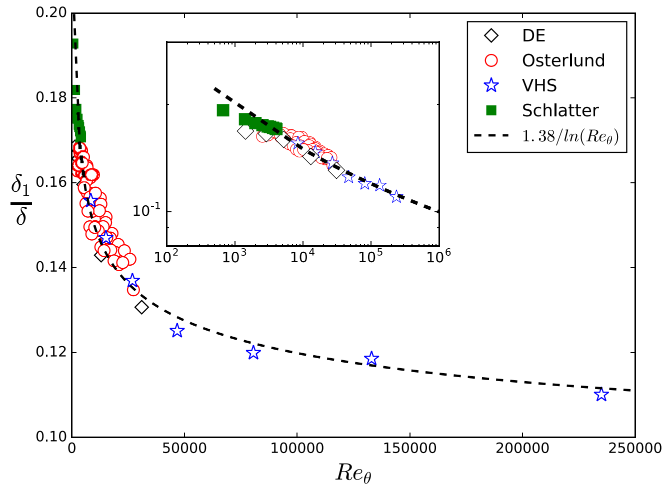

FIG. 3. Ratio of the displacement thickness (δ_1) to the boundary layer thickness (δ), δ_1/δ. Data points are from three experiments by DeGraaff and Eaton (DE) [3], Osterlund [21], Vallikivi *et al.* (VHS) [23], and the DNS of Schlatter [22].

of equilibrium TBLs, it can be shown that δ_1/δ tends to zero as the Reynolds number approaches infinity. The same behavior can be expected for nonequilibrium TBLs and for quasi-equilibrium TBLs such as the ZPG TBLs. However, Fig. 3 shows that δ_1/δ has a non-negligible value of about 0.11 at $\mathrm{Re}_\theta = 250{,}000$. Therefore, at finite Reynolds numbers, these advection terms are small but non-negligible, as can be seen directly in Fig. 4.

(3) $-[\frac{\delta_2}{\delta}\frac{U_\infty^+}{\delta^+}]\frac{\partial^2 U^*}{\partial\eta^2}$: The derivative itself is bounded in the outer layer, but the prefactor is much less than one ($\delta_2 < \delta$, and $U_\infty^+ << \delta^+$). Thus, this viscous term is much smaller in the outer layer.

(4) $[\frac{R_{12,s}}{U_\infty V_\infty}]\frac{\partial R_{12}^*}{\partial\eta}$: This term has to be a leading order of magnitude term to balance the $\partial V^*/\partial\eta$ term in Eq. (22). A proper scaling for R_{12} is then $R_{12,s} = U_\infty V_\infty$ for the prefactor of this term to be

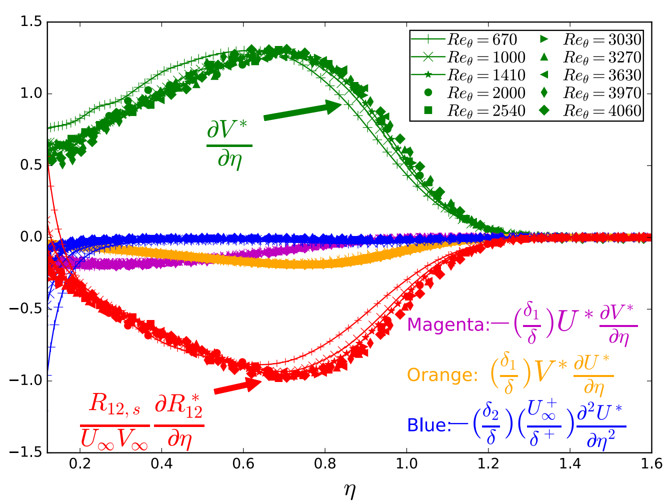

FIG. 4. Force terms in the streamwise mean momentum equation (22). The data are from DNS of Schlatter *et al.* [22].

$O(1)$. Furthermore, R_{12}^* is a regular function that varies between $O(1)$ and 0 over the range of $\eta = 0$ to 1. Therefore the whole term is $O(1)$ balancing the $\partial V^*/\partial\eta$ term in Eq. (22).

From the integral analysis result of Eq. (8), it is known that $U_\infty V_\infty = Hu_\tau^2$. Thus, a proper scaling for the Reynolds shear stress is

$$R_{12,s} = U_\infty V_\infty = Hu_\tau^2. \tag{23}$$

The traditional scaling for R_{12} in ZPG TBLs is u_τ^2. It is known that the shape factor H in ZPG TBLs decreases from about 1.4 at low Reynolds number to 1 at infinite Reynolds number. Therefore, the traditional scaling of u_τ for R_{12} is consistent with the present analysis.

2. Experimental and numerical data of the mean momentum equation

Force terms in Eq. (22) are computed from the DNS data of Schlatter *et al.* [22] and presented in Fig. 4. In the outer region, Fig. 4 shows that the force balance is dominated by two terms: a driving force given by $\partial V^*/\partial\eta$ and a drag force given by $\partial R_{12}^*/\partial\eta$. Figure 4 shows that, over the Reynolds number range of the DNS study, $\partial V^*/\partial\eta$ peaks around $\eta = 0.7$, at a magnitude around 1.3, which is of $O(1)$. Figure 4 shows that the peak value of $\partial R_{12}^*/\partial\eta$ is about -1.

All the data in Fig. 4 are low to moderate Reynolds number DNS data. In the outer layer, it was found that $\eta\partial U^*/\partial\eta$ is balanced with $\partial V^*/\partial\eta$ [see Fig. 2(b)]. Additionally, the high Reynolds number data of Osterlund [21] and VHS [23] showed the profiles for $\eta\partial U^*/\partial\eta$ were collapsed and thus self-similar. Therefore, we can argue that the data for $\partial V^*/\partial\eta$ should be collapsed, even though higher Reynolds number data has not been obtained for this measurement.

IV. SUMMARY

In this Rapid Communication, the Zagarola-Smits scaling for the mean streamwise velocity deficit in the outer layer is derived directly from a combination of integral, similarity, and order-of-magnitude analysis of the continuity equation for ZPG TBLs. The analysis is supported by the numerical and experimental data for the continuity equation. The analysis also reveals that a proper scaling for the mean wall normal velocity V is V_∞. Extending the analysis to the mean momentum balance equation for ZPG TBLs reveals that the Reynolds shear stress scales as $R_{12,s} = U_\infty V_\infty = Hu_\tau^2$. Strictly speaking, the present analysis of the ZPG TBLs is only valid when the Reynolds number tends to infinity since it assumes self-similarity of the velocity moments. However, data indicate that the whole analysis works at moderate Reynolds numbers as well.

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