PHYSICAL REVIEW FLUIDS **2**, 124701 (2017)

Modified Taylor vortices

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(Received 11 April 2017; published 14 December 2017)

The classic problem reported by Taylor [Philos. Mag. **46**, 671 (1923)] on the viscous decay of a planar vortex grid is here extended in three ways. First we present solutions for the vortex grid modulated in the direction normal to the grid. Second, we find a solution for viscous decay of vortices on a disk and its modulation along the axial direction. Finally, we provide solutions for the viscous decay of vortices on a cylindrical surface and its modulation along the radial direction. The exponentially decaying vortices in these modulated vortex systems represent exact unsteady three-dimensional solutions of the Navier-Stokes equations.

DOI: 10.1103/PhysRevFluids.2.124701

I. INTRODUCTION

Taylor [1] gave an exact solution of the unsteady Navier-Stokes equations describing a double array of vortices that decay exponentially in time. His result for a grid of square vortices has been generalized to a rectangular grid of vortices; see Drazin and Riley [2].

In a subsequent study, Taylor and Green [3] considered the evolution of an initial three-dimensional velocity field periodic in each direction to study its decay from larger vortices to smaller ones to ascertain the eventual evolution to turbulence. The study was analytical, carried out to a second approximation, and a comparison of results with experiments was presented. This initial velocity field given by Taylor and Green [3] has now become the canonical test bed for computational studies on the evolution of the flow field; see, for example, Brachet *et al.* [4], Kim and Moin [5], and DeBonis [6].

In this study we extend the work of Taylor [1] in three ways. First, we provide solutions for vortices modulated in the direction normal to the plane of the vortex grid. This yields vortices whose circulation varies periodically in the normal direction, thus providing a three-dimensional unsteady solution of the Navier-Stokes equations.

The second extension concerns the decay of vortices on a disk whose solutions are given by Bessel functions in the radial direction with cosine periodicity around the azimuth. We determine how the vortices may be modulated in the axial direction normal to the plane of the disk, again finding that the vortex structure varies periodically in the axial direction.

The final problem considered is motivated by the work of Haslam and Mallier [7] who attempted to show how the planar inviscid solutions for the co-rotating vortices of Stuart [8] and the counter-rotating vortices of Mallier and Maslowe [9] can be mapped onto a cylindrical surface. Schusser [10], however, showed that their results were erroneous because no consistent pressure is available. Nevertheless, we find that indeed the vortex grid of Taylor [1] can be mapped onto the surface of a cylinder. We then investigate how this flow is modulated in the radial direction, which gives rise to rather complicated solutions for which we provide large radii asymptotic behaviors.

The presentation is as follows. The analysis for modulated Taylor vortices on a plane is given in Sec. II. Section III deals with Taylor vortices on a disk and their axial modulation. In Sec. IV we analyze Taylor vortices on a cylindrical surface and their radial modulation. The paper ends with a discussion and concluding remarks in Sec. V.

II. MODULATED TAYLOR VORTICES ON A PLANE

Drazin and Riley [2] have shown that the classic viscous decay solution of Taylor for a square array of counter-rotating vortices on a plane may be generalized to rectangular arrays using the stream function

$$\psi(x,y,t) = A\cos ax\, \cos by\, e^{-\beta^2 \nu t}, \tag{1}$$

where $\beta^2 = a^2 + b^2$. A sample plot is given in Fig. 1.

An attempt can be made to discover whether vortex cubes exist so that when packed together they form a cubic array of vortices. That this cannot occur is easily verified by packing cubes together and drawing the direction of flows in orthogonal planes which shows that while counter-rotating vortices appear in two orthogonal planes, a periodic straining flow occurs in the remaining plane.

Thus, with Cartesian coordinates $(x,\ y,\ z)$ and velocities $(u,\ v,\ w)$, one is restricted to modulated vortex flows of the type

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}, \qquad w = 0. \tag{2}$$

A simple exercise shows that the equation governing $\psi(x,y,z,t)$ is given by

$$\widetilde{\nabla}^2 \psi_t - \frac{\partial(\psi, \widetilde{\nabla}^2 \psi)}{\partial(x,y)} = \nu \widetilde{\nabla}^2 \left(\widetilde{\nabla}^2 \psi + \frac{\partial^2 \psi}{\partial z^2} \right), \qquad \text{(3a)} \tag{3a}$$

where

$$\widetilde{\nabla}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \qquad \text{(3b)} \tag{3b}$$

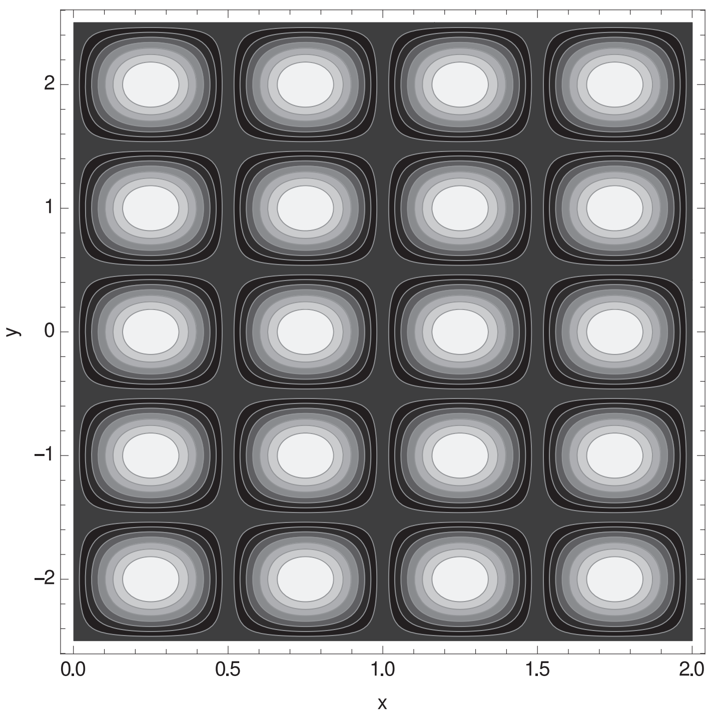


FIG. 1. Taylor vortices plotted for $a = 2\pi$ and $b = \pi$.

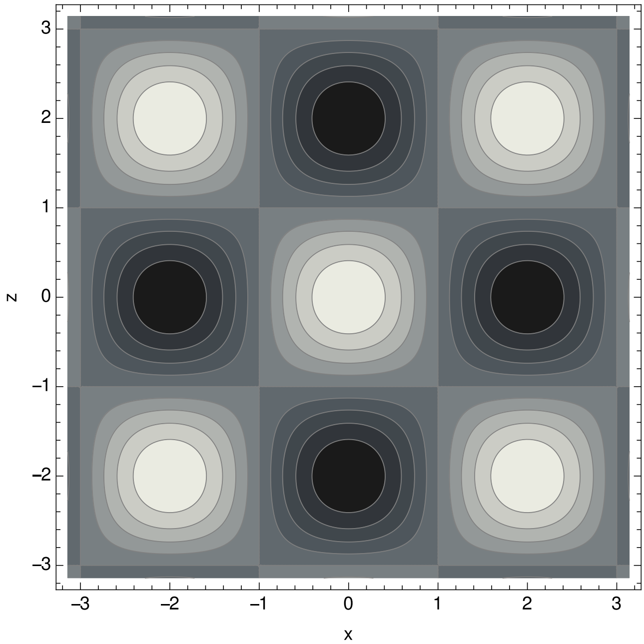


FIG. 2. Variation of the stream-function contours with x and z at $y = 0$ for the modulated vortices on a plane with $d = 2$.

The stream function is posited as

$$\psi(x, y, z, t) = A \cos ax \cos by\, F(z)\, e^{-\alpha^2 \nu t}, \tag{4}$$

where A has units (L/T). This satisfies governing Eqs. (3) provided

$$F'' + (\alpha^2 - \beta^2)\, F = 0, \qquad \beta^2 = a^2 + b^2. \tag{5}$$

The three classes of solutions that appear are

$$F(z) = B \sin \sqrt{\alpha^2 - \beta^2}\, z + C \cos \sqrt{\alpha^2 - \beta^2} z, \quad (\alpha^2 > \beta^2), \tag{6a}$$

$$F(z) = B + Cz, \quad (\alpha^2 = \beta^2), \tag{6b}$$

$$F(z) = B \cosh \sqrt{\beta^2 - \alpha^2}\, z + C \sinh \sqrt{\beta^2 - \alpha^2}\, z, \ (\alpha^2 < \beta^2). \tag{6c}$$

With $F(z)$ determined, the pressure distribution is given by

$$p = p_0 + \frac{\rho A^2}{2} (b^2 \sin^2 ax + a^2 \sin^2 by) F^2(z)\, e^{-2\alpha^2 \nu t}. \tag{7}$$

We neglect the solution given by Eq. (6c), which is unbounded in z, but note that in Eq. (6b) when $C = 0$ and $B = 1$ one recovers the generalized Taylor solution given in Eq. (1). The periodic solutions in Eq. (6a) represent the decay of Taylor vortices modulated along the coordinate normal to the plane of the vortices.

As an example, consider $B = 0$ in Eq. (6a). Then for the square vortex grid reported by Taylor [1] with $a = b = \pi/d$, the z modulation of this vortex grid will be of identical wavelength when $\sqrt{\alpha^2 - \beta^2} = \pi/d$ giving $\alpha^2 = 3\pi^2/d^2$. The resulting stream function for this case is then

$$\psi(x, y, z, t) = A \cos\left(\frac{\pi x}{d}\right) \cos\left(\frac{\pi y}{d}\right) \cos\left(\frac{\pi z}{d}\right) e^{-3\pi^2 \nu t/d^2}. \tag{8}$$

The resulting modulation is shown in Fig. 2. Note, however, that these contours do not represent streamlines that exist only in x-y planes; what is seen in this figure is the periodic z variation in intensity of the stream function.

It is interesting to point out that the initial vortex structure found here can be derived from a reduced form of the general three-dimensional velocity field given by Taylor and Green [3], but here we have provided the additional feature of its temporal decay.

III. VORTEX DECAY ON DISK SURFACES

Consider cylindrical coordinates (r, θ, z) with velocities $(v_r, v_\theta, 0)$. Following Goldstein [11] we introduce the stream function $\psi(r,\theta,z,t)$ defined as

$$v_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \qquad v_\theta = -\frac{\partial\psi}{\partial r}. \tag{9}$$

Here the unsteady viscous flow is governed by the equation

$$\widehat{\nabla}^2\psi_t - \frac{1}{r}\frac{\partial(\psi,\widehat{\nabla}^2\psi)}{\partial(r,\theta)} = \nu\widehat{\nabla}^2\left(\widehat{\nabla}^2\psi + \frac{\partial^2\psi}{\partial z^2}\right), \tag{10a}$$

where

$$\widehat{\nabla}^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}. \tag{10b}$$

A. Vortex decay on a disk

We first consider vortex decay on a fixed disk located at an arbitrary $r-\theta$ plane, let us say at $z = 0$. In this case, the term with second derivatives in z in Eq. (10a) is zero. Inserting the solution ansatz

$$\psi(r,\theta,t) = e^{-\alpha^2\nu t}\Psi(r,\theta) \tag{11}$$

into Eqs. (10) shows that solutions may be obtained when

$$\widehat{\nabla}^2\Psi + \alpha^2\Psi = 0. \tag{12}$$

A separable solution, regular at the origin, gives the stream function

$$\psi(r,\theta,t) = A\,J_n(\alpha r)\cos n\theta\, e^{-\alpha^2\nu t}, \tag{13}$$

where A has units (L^2/T). This is the cylindrical analog of the problem originally reported by Taylor [1]. A fundamental difference here is that the positive value of α^2 is arbitrary. A sample plot is given in Fig. 3(a).

B. Axially modulated vortex decay on disk surfaces

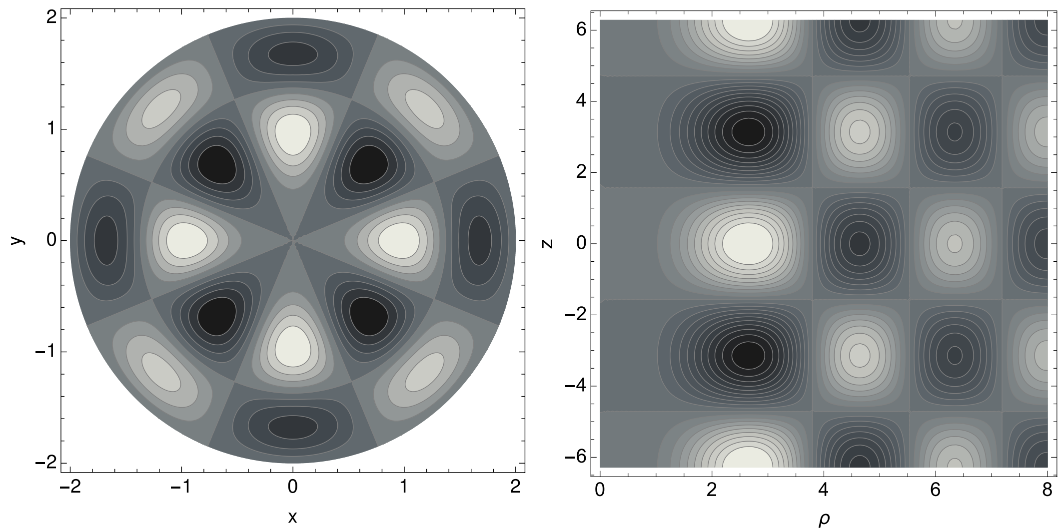
We now consider the axial modulation of these vortices. Here we assume

$$\psi(r,\theta,z,t) = -A\,F(r)\cos n\theta\, G(z)\, e^{-\alpha^2\nu t} \tag{14}$$

and insert this into the governing Eqs. (10). The nonlinear terms are zero, giving

$$G_{zz} \pm k^2 G = 0, \tag{15a}$$

$$r^2F'' + rF' + (\beta^2r^2 - n^2)F = 0, \qquad \beta^2 = \alpha^2 \pm k^2, \tag{15b}$$



(a) (b)

FIG. 3. (a) Vortices on a disk plotted for $n = 4$ and $r = 2$. (b) Variation of the stream-function contours with r and z on a disk for $\theta = \pi/2$, $k = 1$, and $n = 4$.

where the primes in Eq. (15b) denote differentiation with respect to r. The pressure obtained from the Navier-Stokes equations is then

$$p = p_0 = \frac{\rho A^2}{2} \sin^2 n\theta\, G^2(z)\left[F'^2 + \left(\beta^2 - \frac{n^2}{r^2}\right)\right] e^{-2\alpha^2 \nu t}. \tag{16}$$

For solutions bounded at large z one must have $0 < k^2 < \alpha^2$ and we choose $G(z) = \cos kz$. The solution of Bessel's Eq. (15b), regular at the origin, then provides the stream function [see Fig. 3(b)]

$$\psi(r,\theta,z,t) = -A J_n(\beta r) \cos n\theta \cos kz\, e^{-\alpha^2 \nu t}, \qquad \beta^2 = \alpha^2 - k^2. \tag{17}$$

IV. VORTEX IN CYLINDRICAL SYSTEMS

Although Haslam and Mallier [7] were not successful in mapping the planar inviscid solutions for the co-rotating vortices of Stuart [8] and the counter-rotating vortices of Mallier and Maslowe [9] onto a cylindrical surface, we find that a rectangular grid of Taylor vortices can indeed be mapped onto the surface of a cylinder. Taking cylindrical coordinates (r, θ, z) with velocities $(0, v_\theta, v_z)$, the stream function

$$v_\theta = \frac{\partial \psi}{\partial z}, \qquad v_z = -\frac{1}{r}\frac{\partial \psi}{\partial \theta} \tag{18}$$

satisfies the equation of continuity for incompressible flow. We note that the vorticity equation can be expressed as

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \boldsymbol{\nabla} \times (\boldsymbol{\omega} \times \mathbf{u}) = \nu \nabla^2 \boldsymbol{\omega}, \tag{19}$$

where ν is the kinematic viscosity. We will use the following operators:

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \nabla_T^2, \qquad \text{with} \quad \nabla_T^2 = \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}. \tag{20}$$

Let us consider the r component of this equation,

$$\frac{\partial \omega_r}{\partial t} + \mathbf{e}_r \cdot \nabla \times (\boldsymbol{\omega} \times \mathbf{u}) = \nu\left(\nabla^2 \omega_r - \frac{\omega_r}{r^2} - \frac{2}{r^2}\frac{\partial \omega_\theta}{\partial \theta}\right), \tag{21}$$

where ω_r is the radial component of the vorticity given by

$$\omega_r = -\nabla_T^2 \psi, \tag{22}$$

and ω_θ is the azimuthal component of the vorticity, which can be written as

$$\omega_\theta = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial \theta}\right). \tag{23}$$

Then Eq. (21) can be expressed as

$$\nabla_T^2 \frac{\partial \psi}{\partial t} - \mathbf{e}_r \cdot \nabla \times (\boldsymbol{\omega} \times \mathbf{u}) = \nu\left[\nabla^2 \nabla_T^2 \psi - \frac{1}{r^2}\nabla_T^2 \psi + \frac{2}{r^2}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial^2 \psi}{\partial \theta^2}\right)\right]. \tag{24}$$

A little bit of algebra shows that the equation for $\psi(r,\theta,z)$ is given by

$$\nabla_T^2 \frac{\partial \psi}{\partial t} - \frac{1}{r}\frac{\partial\left(\psi, \nabla_T^2 \psi\right)}{\partial(\theta,z)} = \nu\left[\left(\nabla^2 - \frac{1}{r^2}\right)\nabla_T^2 \psi + \frac{2}{r^2}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial^2 \psi}{\partial \theta^2}\right)\right]. \tag{25}$$

A. Vortex on a cylindrical surface

Consider first the problem of vortex flow on the fixed cylindrical surface $r = R$, in which case $\psi = \psi(R,\theta,z)$. Then, the stream function is governed by the equation

$$\nabla_T^2 \psi_t - \frac{1}{R}\frac{\partial\left(\psi, \nabla_T^2 \psi\right)}{\partial(\theta,z)} = \nu \nabla_T^2\left(\nabla_T^2 \psi - \frac{\psi}{R^2}\right), \tag{26}$$

where the t subscript denotes differentiation with respect to time.

A decaying vortex solution may be posited in the form

$$\psi(R,\theta,z,t) = A\, R\, \sin n\theta\, F(z)\, e^{-\alpha^2 \nu t}, \tag{27}$$

in which A, with units (L/T), measures the strength of the flow. Inserting ansatz Eq. (27) into Eq. (26) reveals that the nonlinear terms are zero when

$$F\, F''' - F'\, F'' = 0, \tag{28}$$

which has periodic solutions

$$F(z) = (\sin kz,\ \cos kz). \tag{29}$$

The remaining linear unsteady and viscous terms are in balance when α satisfies the equation

$$F^{iv} + \left(\alpha^2 - \frac{(2n^2+1)}{R^2}\right)F'' + \frac{n^2}{R^2}\left(\frac{n^2}{R^2} - 1 - \alpha^2\right)F = 0. \tag{30}$$

In the sequel we take $F(z) = \cos kz$. Inserting this solution into Eq. (30) yields

$$\overline{k}^4 + (2n^2 + 1 - \overline{\alpha}^2)\,\overline{k}^2 + n^2(n^2 - 1 - \overline{\alpha}^2) = 0, \tag{31}$$

where the dimensionless forms of axial wavelength $\overline{k} = Rk$ and time exponent factor $\overline{\alpha} = R\alpha$ have been introduced. The stream function is plotted in Fig. 4.

The radial pressure gradient required to maintain this flow is

$$\left.\frac{\partial p}{\partial r}\right|_{r=R} = \rho A^2 R k^2 \sin^2 n\theta \sin^2 kz\, e^{-2\alpha^2 \nu t} + \frac{2A\rho\nu n k}{R}\cos n\theta \sin kz\, e^{-\alpha^2 \nu t}, \tag{32}$$

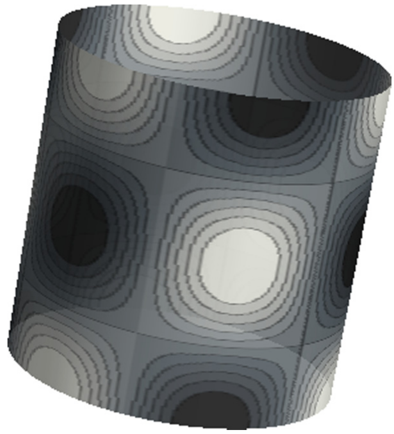


FIG. 4. Contours for the stream function $\psi = \sin(3\theta)\cos(\pi z)$ on a cylinder showing the presence of vortices.

and the pressure acting over the cylindrical surface is

$$p = p_0 + \frac{\rho A^2}{2}(n^2 \sin^2 kz - \bar{k}^2 \sin^2 n\theta)\, e^{-2\alpha^2 \nu t} - \frac{A\rho\nu n}{\bar{k}R}(\bar{\alpha}^2 - \bar{k}^2 - n^2)\cos n\theta \sin kz\, e^{-\alpha^2 \nu t}. \quad (33)$$

For integer values of n and selected values of $\bar{k}$ the decay exponent factor in Eq. (31) is found as

$$\bar{\alpha}^2 = (\bar{k}^2 + n^2 - 1) + \frac{2\bar{k}^2}{(\bar{k}^2 + n^2)}. \quad (34)$$

Now consider two cases. First suppose $\bar{k} = n$. Then one finds

$$\bar{\alpha}^2 = 2n^2 + 1, \quad (35)$$

showing that $3 \leqslant \bar{\alpha}^2 < \infty$ as $1 \leqslant n < \infty$. In the second case we assume that $\bar{\alpha}^2 = 2$ similar to what happens for the classic Taylor problem. In this case

$$\bar{k}^2 = \frac{1 - 2n^2 \pm \sqrt{8n^2 + 1}}{2}. \quad (36)$$

The minimum value $n = 1$ has the interesting feature that when the axial variation is exponential rather than periodic one finds $\bar{k}_1^2 = -2$.

B. Radially modulated vortices on cylindrical surfaces

Now consider the more general case where vortices on cylindrical surfaces are modulated in the radial direction. We still assume that there is no radial velocity, yet $\psi = \psi(r,\theta,z)$. In this case, we use a stream function of the form

$$\psi(r,\theta,z,t) = AG(r)\sin n\theta \cos kz e^{-\beta k^2 \nu t}. \quad (37)$$

Inserting this ansatz into Eq. (25), the convective term is identically zero and the equation for $G(r)$ becomes linear with variable coefficients, *viz.*

$$[n^2 - k^2r^2 - n^4 + n^2k^2r^2(\beta - 2) + k^4r^4(\beta - 1)]G(r) + r(k^2r^2 - n^2)G'(r) + r^2(n^2 + k^2r^2)G''(r) = 0 \quad (38)$$

where primes denote differentiation with respect to r. The boundary is placed at the radius $r = R$ as before. The problem has four parameters, namely k, n, β, and R. However, the number of parameters can be reduced if we introduce the dimensionless parameter

$$\kappa = kR/n, \tag{39}$$

which is a quotient between the characteristic azimuthal arc of the perturbation and its axial wavelength. We also nondimensionalize the radial variable as

$$\rho = \kappa r/R, \tag{40}$$

so that $G(r) \to H(\rho)$. Note that the dimensional quantity measuring r in our case is n/k. Therefore, for this approach, we exclude the case $n = 0$ that should be treated separately. The resulting equation can be written as

$$L[H] = [1-\rho^2 + n^2(1+\rho^2)(\beta\rho^2-\rho^2-1)]H(\rho) + \rho(\rho^2 - 1)H'(\rho) + \rho^2(1 + \rho^2)H''(\rho) = 0, \tag{41}$$

where a prime denotes differentiation with respect to ρ. Then we have a family of solutions determined by three dimensionless parameters: n, β, and the position of the radial boundary at $\rho = \kappa$. The advantage of this approach is that not only are the parameters reduced, but one of them is shifted to the boundary condition, while β and n are the sole remaining constants in the differential equation. In the following, we impose the condition $H(\kappa) = 0$, i.e., $\mathbf{u} = 0$ at the radius of the cylinder, since it is assumed to be at rest. We also consider $H'(\kappa) = 1$, without loss of generality. If this condition is changed to $H'(\kappa) = D$, an arbitrary constant value, it suffices to multiply the solutions obtained with $H'(\kappa) = 1$ by D to get the new solutions. This result shows that a change in the condition mentioned above could alter the maximum of the solution but neither the zeros nor the general behavior.

It is possible to obtain the solutions of the problem in a complete analytical way in terms of Heun's confluent functions. To prove this assertion, we set $H(\rho) = \bar{Q}(\rho)\rho^s$ and then use the transformation of dependent variable $\zeta = -\rho^2$. The resulting equation for $Q(\zeta)$ is

$$\frac{d^2Q}{d\zeta^2} + \left(\frac{1}{\zeta - 1} + \frac{s}{\zeta}\right)\frac{dQ}{d\zeta} - \left(\frac{\delta}{\zeta} + \frac{1 + n^2 - s^2}{4(\zeta - 1)\zeta} + \frac{(1 - s)^2 - n^2}{4(\zeta - 1)\zeta^2}\right)Q = 0, \tag{42}$$

once one defines $\delta = n^2(\beta - 1)/4$. The value of s is arbitrary. However, if one selects $s = 1 \pm n$, Eq. (42) can be transformed into the confluent Heun equation,

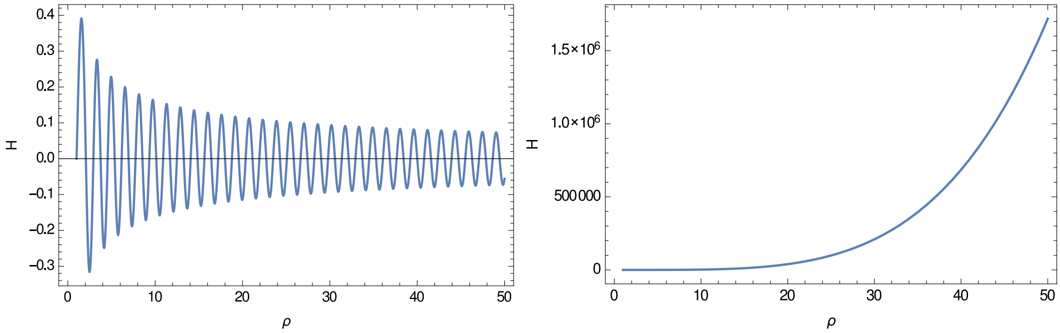
$$\frac{d^2Q}{d\zeta^2} + \left(\frac{1}{\zeta - 1} + \frac{1 \pm n}{\zeta}\right)\frac{dQ}{d\zeta} - \left(\frac{\delta}{\zeta} \mp \frac{n}{2(\zeta - 1)\zeta}\right)Q = 0. \tag{43}$$

Solutions of this type of equation provide a generalization of those of the hypergeometric equation and have been studied in several monographs; see Ronveaux [12] and Slavyanov and Lay [13]. In particular, since n is an integer, its solution is the confluent Heun function $H_c(0,0,\pm n,\delta,0,1 - \zeta)$. Therefore, the general solution of this equation can be expressed as

$$H(\rho) = C_1 H_c(0,0,n,\delta,1 + \rho^2)\rho^{1+n} + C_2 H_c(0,0,-n,\delta,1 + \rho^2)\rho^{1-n}. \tag{44}$$

This general form can be used to solve the full problem but it is difficult to handle; we will show that very good approximations are possible in terms of more familiar functions. Equation (41) can also be solved numerically; doing so for $n = 4$ and $\kappa = 1$ shows an important change in behavior as β is varied; see Fig. 5.

The results show a structure of vortices decreasing in intensity as one moves away from the cylinder with alternating directions of rotation across the nodes in ρ. The solutions have a series of cells as shown in Fig. 6. For example, the results for $n = 4$, $\beta = 2$, and $\kappa = 1$ are shown in Fig. 5(a). Figure 7 shows how the cells are altered as β decreases. They become larger as one approaches



(a) (b)

FIG. 5. Solutions $H(\rho)$ of Eq. (41) for $n = 4, \kappa = 1$: (a) $\beta = 2$, (b) $\beta = 1$.

$\beta = 1$ and then, for smaller β's, the flows occupy wedges devoid of periodic radial structure. This is consistent with the solution shown in Fig. 5(b).

V. ASYMPTOTIC SOLUTION FOR LARGE ρ

The full Eq. (41) described above can be written as

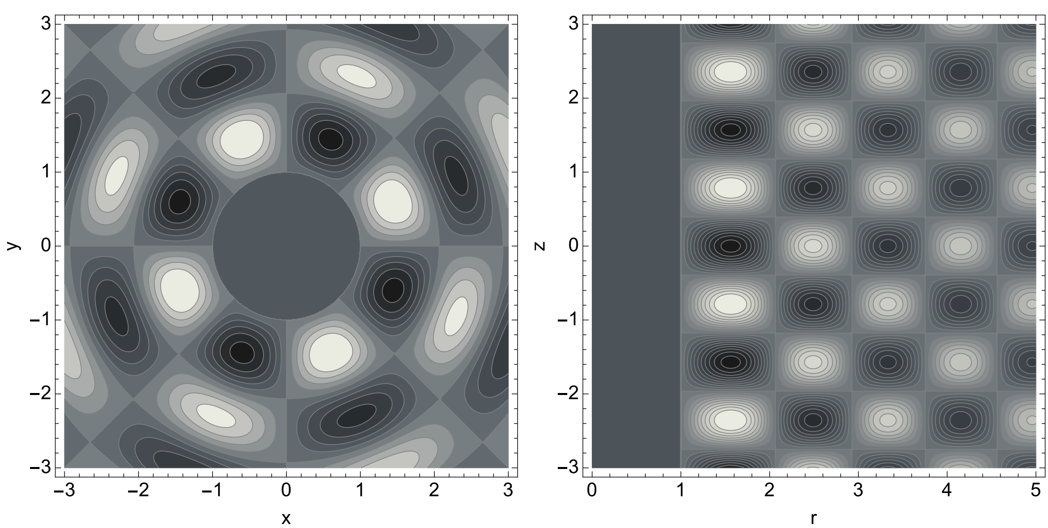
$$L[H] = (1 + \rho^2)\tilde{L}[H] + 2(H - \rho H') = 0, \tag{45}$$

where

$$\tilde{L}[H] = (n^2(\beta - 1)\rho^2 - n^2 - 1)H + \rho(H' + \rho H''[\rho]). \tag{46}$$

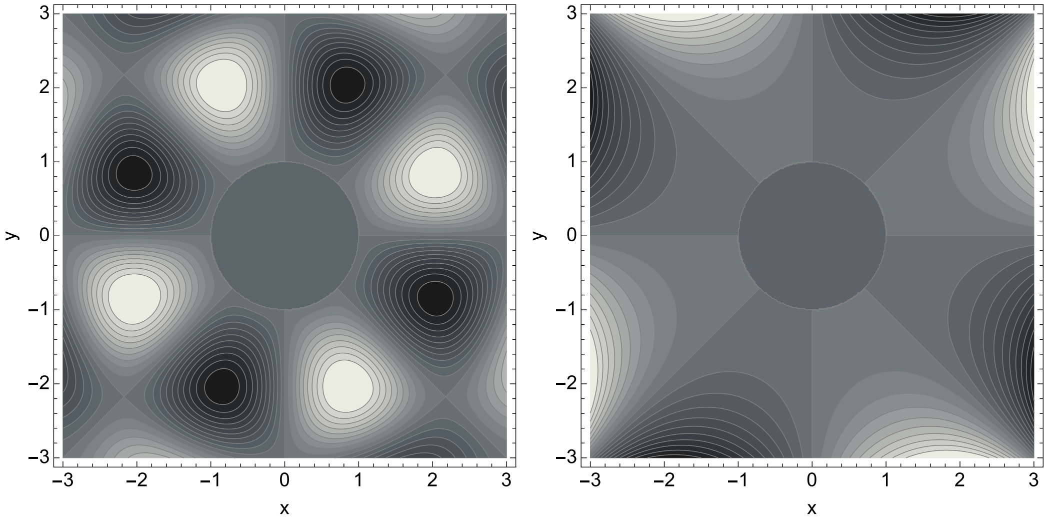
It is possible to simplify Eq. (45) for $\rho \gg 1$, retaining terms to order ρ^2. The asymptotic solution $\tilde{H}$ will satisfy $\tilde{L}[\tilde{H}] = 0$.

Note that we have retained terms in $n^2 + 1$, which could be assumed to be smaller than those of $n^2(\beta - 1)\rho^2$ in general. The reason behind this apparent inconsistency is that we want to consider



(a) (b)

FIG. 6. Contours of the stream function for $\beta = 2$ and $n = 4$ in the plane (a) $z = 0$, (b) $\theta = \pi/3$.



(a) (b)

FIG. 7. Contours of the stream function in the plane $z = 0$ for $n = 4$ and (a) $\beta = 1.4$, (b) $\beta = 0.95$.

the case when $\beta \longrightarrow 1$ for $\rho \longrightarrow \infty$ and this term becomes dominant in this limit. For other cases, their inclusion will be irrelevant.

The resulting equation for $\rho \gg 1$ can be simplified with the following change of variables assuming $\beta > 1$:

$$\chi = n\sqrt{\beta - 1}\rho \quad \text{and} \quad \sigma^2 = 1 + n^2, \tag{47}$$

which leads to

$$(-\sigma^2 + \chi^2)\tilde{H}(\chi) + \chi\left(\frac{d\tilde{H}(\chi)}{d\chi} + \chi\frac{d^2\tilde{H}(\chi)}{d\chi^2}\right) = 0, \tag{48}$$

which is a Bessel equation. Then, the asymptotic solution for $\beta > 1$ can be expressed as the sum of the usual Bessel and Neumann functions, namely

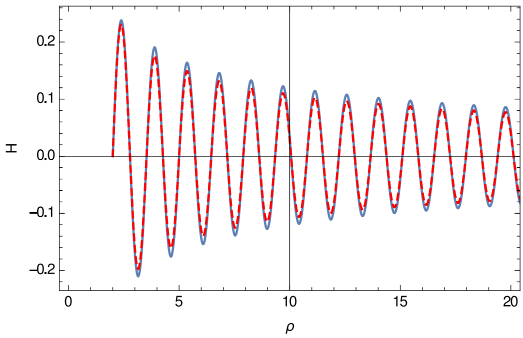
$$\tilde{H}(\chi) = AJ_\sigma(\chi) + BN_\sigma(\chi). \tag{49}$$

This leads to the oscillatory behavior observed in the full solution and it should be noted that the roots of the full and asymptotic solutions coincide. However, when $\beta < 1$, the variable is imaginary and the equation becomes a modified Bessel function whose solutions are of the Kelvin type. This fact proves analytically that the limit between oscillatory and exponential behavior occurs at the critical value $\beta = 1$ independent of n.

There are some other interesting consequences. Let us compare the asymptotic and full solutions for different values of κ in Fig. 8. Although the zeros are the same, the envelope is smaller for the asymptotic solution. This means that the sizes of the cells are practically the same but the intensity of the flow is underestimated by the asymptotic solution at small ρ. Furthermore, the difference between the full and approximate solution increases as κ decreases.

To improve the asymptotic solution, we will use the fact that the zeros of both $\tilde{H}$ and H coincide. Then, the solution of the full equation can be written in terms of the asymptotic one as $H(\rho) = f(\rho)\tilde{H}(\rho)$. Writing the identity

$$L[H] - f\tilde{L}[\tilde{H}] = L[f\tilde{H}] - f\tilde{L}[\tilde{H}] = 0 \tag{50}$$



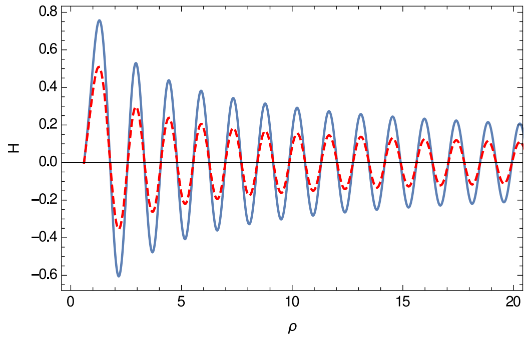


FIG. 8. Dashed red lines correspond to the asymptotic solutions and the continuous blue ones are the full solutions.

implies that

$$2f(\tilde{H} - \rho\tilde{H}') + \rho\{2\rho(1+\rho^2)f'\tilde{H}' + \tilde{H}[(\rho^2-1)f' + \rho(1+\rho^2)f'']\} = 0, \tag{51}$$

but the equation can be approximated close to the zeros of $\tilde{H}$ as

$$2\rho[-f + \rho(1+\rho^2)f']\tilde{H}' \approx 0. \tag{52}$$

A first estimation of $f(\rho)$ can be obtained by solving this equation and the result is that

$$f(\rho) = C\frac{\rho}{\sqrt{1+\rho^2}}, \tag{53}$$

where C is a constant to be adjusted. As a consequence, a better approximation of the full solution is

$$H(\rho) \approx \frac{\rho}{\sqrt{1+\rho^2}}[A\, J_\sigma(n\sqrt{\beta-1}\rho) + B\, N_\sigma(n\sqrt{\beta-1}\rho)], \tag{54}$$

with $\sigma^2 = 1 + n^2$ and the constants A and B determined by the boundary conditions. After some simplification, one finds

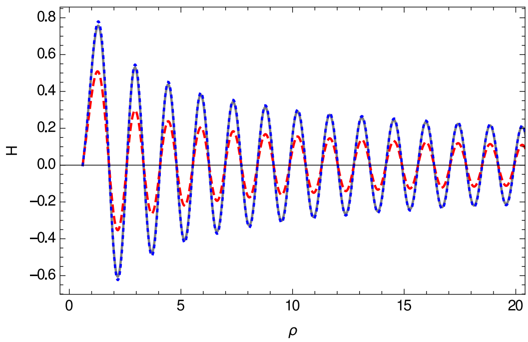
$$H(\rho) \approx \frac{\pi\rho}{2}\sqrt{\frac{1+\kappa^2}{1+\rho^2}}[J_\sigma(\eta\kappa)N_\sigma(\eta\rho) - J_\sigma(\eta\rho)N_\sigma(\eta\kappa)], \tag{55}$$

with $\eta = n\sqrt{\beta-1}$. This result is compared with the full solution Eq. (41) for different values of κ in Fig. 9.

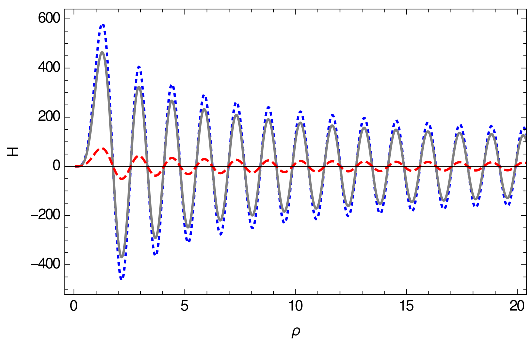
The approximate and full solutions are barely distinguishable at $\kappa = 0.6$, but differences begin to arise as κ decreases.

VI. SUMMARY AND CONCLUSION

We have extended the classic problem of a double array of exponentially decaying vortices given by Taylor [1] in three ways. First, a solution is found that modulates the vortex grid in the direction normal to the plane of vortices. In the second variation, a solution is provided for a grid of vortices on a disk described by Bessel functions in the radial direction with $\cos(n\theta)$ dependence. This vortex array is also modulated periodically in the axial direction. Although the problems on planes and disks are rather elementary, we have obtained for both cases three-dimensional unsteady solutions of the incompressible Navier-Stokes equations.



(a)



(b)

FIG. 9. The dashed red lines are $\tilde{H}$, the dotted blue lines are the full solutions H, and the continuous gray curves correspond to the approximate solutions given by Eq. (55).

The third problem, motivated by the work of Haslam and Mallier [7], guided us in mapping the double array of Taylor vortices onto a cylindrical surface. This is apparently the first time that vortices have been mapped onto the surface of a cylinder. The challenging aspect of this problem is to deduce the features of the radial variation of the vortices. By proper analysis, the problem can be reduced to three dimensionless parameters: n the index for the azimuthal variation of solutions, β a measure of the exponential decay, and ρ a dimensionless radial coordinate related to the ratio of the axial wave number k to n. We find the interesting feature that decaying oscillatory solutions for $\beta > 1$ degenerate to nonperiodic solutions for $\beta < 1$. We described in detail the possible solutions and have provided analytical results that are good approximations of the numerical solutions.

We note that the circulation around a vortex cell described by Eq. (1) is given by

$$\Gamma = 4A\left(\frac{a^2 + b^2}{ab}\right). \tag{56}$$

We have also calculated the circulation around the innermost and the next outer vortex cells for vortices on a disk to see if possibly there might be a universal circulation for this array. However, we find that the circulation around neighboring vortex cells decreases as the cell positions increase with radius, so no universality is found.

Finally, our results suggest that one might consider a similar case on a sphere leading to some type of vortex grid on its surface. Indeed, Crowdy [14] has done a stereographic projection of Stuart vortices on a sphere. For a physically relevant relation between two parameters, he was able to obtain an explicit solution for the stream function that produces a row of vortices along latitudinal circles for an arbitrary number of vortices $N \geqslant 1$ and nice plots of these vortices are presented for $N = 2,4,10$ at $\pi/3$ north latitude and for $N = 16$ on the equator. We believe that the extension of our approach to a vortex grid on a sphere may provide new solutions and it is a problem worth studying in the future.

ACKNOWLEDGMENT

A.G.G. acknowledges support from Consejo Nacional de Investigaciones Cientficas y Técnicas and Agencia Nacional de Promoción Científica y Tecnológica de la República Argentina.

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