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Stokes resistance of a cylinder near a slippery wall

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Motivated by observations of large effective slip over nanostructured surfaces, we analyze here the hydrodynamic resistance to the rigid-body motion of a cylinder near a slippery wall—perhaps the simplest anisotropic configuration which nonetheless exhibits a nontrivial translation-rotation coupling. Focusing upon the lubrication limit, the properly scaled resistance-matrix coefficients depend only upon the ratio of the slip length to the cylinder-wall clearance. A distinct feature of a lubrication analysis with slippage is that the pressure field cannot be determined in closed form; nonetheless, the dependence upon the slip length of the resistance coefficients is explicitly obtained. The limit of large slip length, where the wall behaves as a free surface, is elucidated. Considering the companion problem of a slippery cylinder, we observe a symmetry relating the two problems.

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I. INTRODUCTION

There is an ongoing interest in the design and optimization of microstructured superhydrophobic surfaces [1–5] which allow for macroscale fluid slippage. The microstructure typically consists of a periodic array of pillars, holes, or grooves, about or in which air bubbles get trapped in a Cassie state. With the liquid being in contact with a gas-solid substrate, it experiences an effective slip as a consequence of the low resistance of the gas phase to liquid motion. Regardless of the fine details of the microscale flow, the two-phase substrate may be described on the macroscale via a Navier slip condition, relating the tangential velocity to the shear stress. For a planar surface, where the latter is proportional to the tangential velocity gradient, this condition adopts the form

$$\text{velocity} = b \times \text{velocity gradient}, \tag{1}$$

where the gradient is formed in a direction perpendicular to the wall, pointing into the liquid. The details of the microstructure are lumped into the value of the "slip length" b.

Is the modification of no slip to slip significant in the context of superhydrophobic systems? The answer is not straightforward. The macroscale description is valid provided the linear dimension characterizing the flow is large compared with the microstructure period. Typically, the slip length is comparable with that period, implying a small effect in (1). Fortunately, there are certain singular configurations which naturally lead to a slip length much larger than the period. Thus, a scaling analysis of the pillar-array geometry [6] predicts a slip length which diverges at small solid fractions of the substrate; this scaling law was later corroborated by the detailed analysis of Davis and Lauga [7]. More recently, a similar slip-length divergence was predicted for dense bubble mattresses [8,9] and the velocity amplification pertaining to pressure-driven flows was elucidated [10]. Large slip lengths ($\sim 50\ \mu$m) over nanoengineered surfaces which minimize the liquid-solid contact area have indeed been experimentally observed [11,12].

Superhydrophobic effects are pronounced when considering such "slippery" configurations *in conjunction with* small-scale flows, varying over lengths commensurate with b. This combination naturally brings into mind the canonical problem of particle-wall interaction in the lubrication limit, where the pertinent length scale is the particle-wall clearance. As that limit is naturally associated with intensive hydrodynamic forces, it constitutes the natural platform for amplification of slip effects. Furthermore, as the lubrication approximation provides a convenient framework where the singular scaling of the hydrodynamic loads naturally appears, its use can explicitly reveal the role of slip.

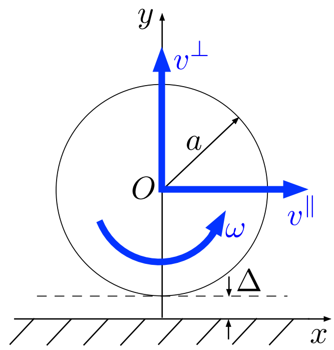

FIG. 1. Schematic of the problem.

In what follows we address the effect of slip upon the entire resistance matrix which describes the linear hydrodynamic response to all possible modes of rigid-body particle motion near the wall. For "conventional" no-slip surfaces this matrix possesses several well-known symmetry properties [13,14]; in particular, the coupling tensors which describe torque due to pure translation and force due to pure rotation are linked. Since these symmetry properties hinge upon the no-slip condition (via use of the Lorenz reciprocal theorem) it is not *a priori* evident whether they are retained in the presence of slippery surfaces. In that sense, our consideration of general (i.e., nonaxisymmetric) rigid-body motion is significantly more revealing than earlier axisymmetric analyses (e.g., the head-on approach of two spheres [15]) where such symmetry properties do not come into play. Other questions of interest have to do with the limit of large slip length and the linkage, if any, between the resistance matrix pertinent for a slippery wall and that pertinent for a slippery particle.

II. PROBLEM FORMULATION

With the goal of addressing the above questions in the simplest anisotropic geometry involving translation-rotation coupling, we consider the two-dimensional configuration of an infinite cylinder (radius a) which is undergoing rigid-body motion in a viscous liquid (viscosity μ) near an adjacent wall; in the absence of slip this problem was analyzed by Jeffrey and Onishi [16]. In the quasisteady Stokes-flow description, the velocity field depends only upon the instantaneous cylinder-wall separation distance Δ. We employ Cartesian coordinates, with the x axis lying along the wall and the y axis passing through the instantaneous center O of the cylinder cross section. The cylinder rotates about O with an angular velocity $\hat{\mathbf{e}}_z\omega$; in addition, its center O translates with the rectilinear velocity $\hat{\mathbf{e}}_x v^{\parallel} + \hat{\mathbf{e}}_y v^{\perp}$. The problem is described in Fig. 1.

We assume that the Navier condition (1) applies at the wall, while the cylinder boundary is a no-slip surface. With condition (1) being linear and homogeneous, the Stokes-flow properties which follow from linearity and tensorial symmetry [17] are still retained. Thus, translation and rotation parallel to the wall may result in torque and drag, but not in a lift force. The hydrodynamic forces (per unit length) in the x and y directions and torque (per unit length) about O in the z direction must accordingly adopt the respective forms

$$-\mu(f^{\parallel}v^{\parallel} + ad\omega), \tag{2a}$$

$$-\mu f^{\perp}v^{\perp}, \tag{2b}$$

$$-\mu(acv^{\parallel} + a^2 t\omega). \tag{2c}$$

In writing these we have factored out the dimensional dependence upon viscosity and particle size. The dimensionless resistance coefficients $(f^{\parallel}, d, f^{\perp}, c, t)$ are accordingly functions of the geometry, namely, the ratio $\epsilon = \Delta/a$; in addition, they depend upon the ratio $B = b/\Delta$ which enters through condition (1). Note that in the conventional no-slip framework the symmetry properties of the coupling tensors necessitate $c = d$ [13]. Here we do not assume *a priori* that these two coupling coefficients are equal.

III. LUBRICATION LIMIT

We consider near-contact configurations where Δ and b are comparable, both small relative to a. In terms of governing dimensionless parameters, this scenario is represented by the asymptotic limit $\epsilon \ll 1$ with $B = O(1)$. Making use of the standard stretched coordinates, $X = x/(a\Delta)^{1/2}$ and $Y = y/\Delta$ [16], the cylinder shape in the narrow-gap region is given by $Y = H(X) + O(\epsilon)$ in which $H(X) = 1 + X^2/2$. The corresponding unit vector which is normal to the cylinder (pointing into the liquid) is $(\epsilon^{1/2}X\hat{\mathbf{e}}_x - \hat{\mathbf{e}}_y)[1 + O(\epsilon)]$.

In the near-contact limit the resistance to perpendicular motion is $O(\epsilon^{-3/2})$, and is conveniently written

$$f^{\perp} = \epsilon^{-3/2}F^{\perp}. \tag{3}$$

The remaining coefficients are $O(\epsilon^{-1/2})$,

$$f^{\parallel} = \epsilon^{-1/2}F^{\parallel}, \tag{4a}$$

$$d = \epsilon^{-1/2}D, \tag{4b}$$

$$c = \epsilon^{-1/2}C, \tag{4c}$$

$$t = \epsilon^{-1/2}T. \tag{4d}$$

Our goal is the calculation of the five rescaled coefficients $(F^{\parallel}, D, F^{\perp}, C, T)$. Toward this end, we exploit the underlying linearity and consider separately the three independent problems of rotation, parallel translation, and perpendicular translation. We employ a unified dimensionless notation, with $\mathcal{U}$ representing the appropriate scale for velocities in the x direction. Thus, in the rotation and parallel-translation problems it is respectively chosen as $a\omega$ and $v^{\parallel}$, while in the perpendicular-translation problem it is given by $\epsilon^{-1/2}v^{\perp}$.

Writing the pressure and (x, y) velocity components as

$$\epsilon^{-3/2}\frac{\mu\mathcal{U}}{a}P, \tag{5a}$$

$$\mathcal{U}U, \tag{5b}$$

$$\epsilon^{1/2}\mathcal{U}V, \tag{5c}$$

we hereafter consider only leading-order terms in ϵ; the $O(1)$ dimensionless variables (U, V, P) are accordingly functions of X and Y (as well as B), but not of ϵ. In our unified notation, the differential equations governing these leading-order variables are the same in all three problems. Thus, the Stokes equation in the y direction necessitates that P is independent of Y and is accordingly a function of X alone, say $P(X)$. The Stokes equation in the x direction then reads

$$\frac{\partial^2 U}{\partial Y^2} = P'(X), \tag{6}$$

where the prime denotes differentiation. This equation needs to be solved in conjunction with the continuity equation

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0. \tag{7}$$

The difference between the three problems enters only through the boundary conditions.

IV. ROTATION

We start with the problem of particle rotation, where the boundary conditions governing U are

$$U = B\frac{\partial U}{\partial Y} \quad \text{at} \quad Y = 0, \tag{8a}$$

$$U = 1 \quad \text{at} \quad Y = H, \tag{8b}$$

while those governing V are

$$V = 0 \quad \text{at} \quad Y = 0, \tag{9a}$$

$$V = X \quad \text{at} \quad Y = H. \tag{9b}$$

The solution of (6) and (8) is

$$U = \frac{P'}{2}\left(Y^2 - \frac{Y+B}{H+B}H^2\right) + \frac{Y+B}{H+B}, \tag{10}$$

where $P(X)$ remains to be determined. There are two methods to determine $P(X)$. In the direct way (10) is substituted into (7); conditions (9) then provide a second-order equation governing P, which is to be solved subject to the decay conditions,

$$P \to 0 \quad \text{as} \quad X \to \pm\infty, \tag{11}$$

which in turn follow from the need to match the $O(\mu\mathcal{U}/a)$ pressure outside the gap. A more convenient method makes use of the constancy of the volumetric flux $\int_0^H U\,dY$ through the gap. Setting this flux equal to a constant, say Q, we immediately obtain P' as

$$P' = \frac{6(H+2B)}{H^2(H+4B)} - \frac{12(H+B)}{H^3(H+4B)}Q. \tag{12}$$

At this point the standard procedure involves integration to obtain $P(X)$, with Q and the additional integration constant being determined from (11). In the present problem, where B is arbitrary, such an approach is unwieldy. In what follows, we employ an alternative procedure for obtaining the resistance coefficients in closed form without evaluating $P(X)$ [18]. To that end, the flux Q appearing in (12) must be determined. This is accomplished by writing (11) as a condition on P',

$$\int_{-\infty}^{\infty} P'(X)dX = 0, \tag{13}$$

which when applied to (12) readily gives

$$Q = \frac{4B}{3}\frac{(2B+1)\sqrt{1+4B}-1}{(2B^2+2B-1)\sqrt{1+4B}+1}. \tag{14}$$

Back substitution into (12) thus fully determines P'.

Consider the hydrodynamic torque (per unit length) about O in the clockwise direction. Scaled by $\epsilon^{-1/2}\mu a\mathcal{U}$, it is given by the integral

$$\int_{-\infty}^{\infty} \left.\frac{\partial U}{\partial Y}\right|_{Y=H} dX. \tag{15}$$

A comparison of (2c) and (4d) with the present dimensionless notation reveals that this integral coincides with T. With (10) depending only upon P', substitution of (12) and (14) gives

$$T = 2\sqrt{2}\pi(2+B)\frac{4B^2-3B-1+(B+1)\sqrt{1+4B}}{8B^3+10B^2-2B-1+\sqrt{1+4B}} \tag{16}$$

with the respective limits

$$\lim_{B\to 0} T = 2\pi\sqrt{2}, \tag{17a}$$

$$\lim_{B\to\infty} T = \pi\sqrt{2}. \tag{17b}$$

The value attained as $B \to 0$ recovers the classical result of a no-slip wall [16]. Note the finite torque in limit $B \to \infty$, at which the wall behaves as a free surface [see 8(a)].

The hydrodynamic force (per unit length) on the cylinder in the negative x direction, scaled by $\epsilon^{-1/2}\mu\mathcal{U}$, is affected by the small projection of the large pressure [cf. (15)]:

$$\int_{-\infty}^{\infty}\left(\left.\frac{\partial U}{\partial Y}\right|_{Y=H} + XP\right)dX. \tag{18}$$

A comparison with (2a) and (4b) reveals that this scaled force coincides with D. It may appear that knowledge of $P(X)$ is required for the evaluation of (18). Noting however that P is an odd function of X and making use of (11), we write the integral of XP in (18) as

$$-2\int_0^{\infty} dX\, X \int_X^{\infty} d\xi\; P'(\xi). \tag{19}$$

Interchanging the integration order then gives $-\int_0^{\infty}\xi^2 P'(\xi)d\xi$—a definite integral that involves only P'. Substitution of (10) and (12) in conjunction with (14) gives

$$D = -2\sqrt{2}\pi\,\frac{1+6B+8B^2-(1+4B+2B^2)\sqrt{1+4B}}{1+2B-10B^2-8B^3-\sqrt{1+4B}}. \tag{20}$$

As $B \to 0$ the coupling coefficient vanishes, in agreement with the classical result [16]. Note the additional vanishing of D as $B \to \infty$.

V. PARALLEL TRANSLATION

In the problem of parallel translation the conditions governing U are again given by (8). This velocity component is accordingly given by the form (10). The difference with the rotation problem enters through the flux-constancy condition, which here applies in the particle-fixed reference frame where the pertinent velocity is $U-1$. Imposing that $\int_0^H (U-1)dY$ is a constant, say $\tilde{Q}$, we find

$$P' = -\frac{6}{H(H+4B)} - \frac{12(H+B)}{H^3(H+4B)}\tilde{Q}, \tag{21}$$

which differs in form from (12). [If one were to calculate P' using the direct approach, the difference would have entered through the homogeneous conditions satisfied by V, replacing (9).] Applying (13) we here find

$$\tilde{Q} = -\frac{8B}{3}\,\frac{\sqrt{1+4B}-1}{(2B^2+2B-1)\sqrt{1+4B}+1}. \tag{22}$$

Back substitution into (21) then determines P'.

A comparison of (2) and (4) with the present problem gives

$$C = \int_{-\infty}^{\infty}\left.\frac{\partial U}{\partial Y}\right|_{Y=H} dX, \tag{23a}$$

$$F^{\parallel} = \int_{-\infty}^{\infty}\left(\left.\frac{\partial U}{\partial Y}\right|_{Y=H} + XP\right)dX. \tag{23b}$$

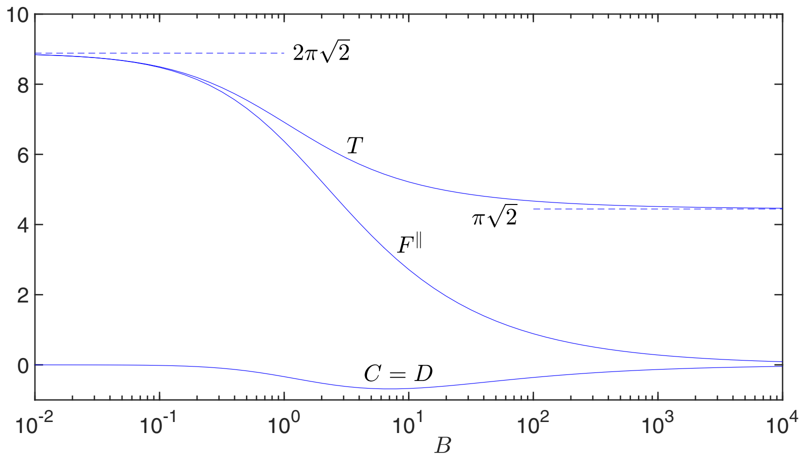

FIG. 2. Variation with B of dimensionless resistance coefficients for rotation perpendicular to and translation parallel to the wall. The limits (17) are indicated.

These coefficients are evaluated using a procedure similar to that employed in the above calculation of T and D. We find that $C = D$ and

$$F^{\parallel} = 4\sqrt{2}\pi \frac{(1 + 2B + 2B^2)\sqrt{1 + 4B} - 1 - 4B}{\sqrt{1 + 4B} - 1 - 2B + 10B^2 + 8B^3}. \tag{24}$$

For $B \to 0$ $F^{\parallel} \to 2\sqrt{2}\pi$, in agreement with Ref. [16]. On the other hand, $F^{\parallel}$ vanishes in the free-surface limit $B \to \infty$. To understand why, we note that in that limit conditions (8) degenerate to

$$\frac{\partial U}{\partial Y} = 0 \quad \text{at} \quad Y = 0, \tag{25a}$$

$$U = 1 \quad \text{at} \quad Y = H. \tag{25b}$$

The flow problem then possesses a trivial solution where the liquid rigidly moves with cylinder, $U \equiv 1$. As this rigid-body motion generates no pressure gradient, $P \equiv 0$. With this simple flow it is readily seen from (23) that $F^{\parallel}$ vanishes. This vanishing merely implies a smaller pressure scaling, a situation familiar from other lubrication problems involving free surfaces [19]. Since conditions (25) also apply at the free-surface limit of the rotation problem, it may appear as though the above rigid-body fluid motion applies there as well. The accompanying $V \equiv 0$ is, however, incompatible with conditions (9). This explains in retrospect the nonzero limit (17b).

The variation with B of the resistance coefficients describing rotation perpendicular to and translation parallel to the wall is portrayed in Fig. 2.

VI. PERPENDICULAR TRANSLATION

We now consider the problem of translation perpendicular to the wall with velocity $v^{\perp}$. In the corresponding dimensionless formulation, the boundary conditions governing U, the x component of the fluid velocity, are homogeneous:

$$U = B\frac{\partial U}{\partial Y} \quad \text{at} \quad Y = 0, \tag{26a}$$

$$U = 0 \quad \text{at} \quad Y = H, \tag{26b}$$

while those governing V, the y component of the fluid velocity, are

$$V = 0 \quad \text{at} \quad Y = 0, \tag{27a}$$

$$V = 1 \quad \text{at} \quad Y = H. \tag{27b}$$

The solution of (6) and (26) is

$$U = \frac{P'}{2}\left(Y^2 - \frac{Y+B}{H+B}H^2\right). \tag{28}$$

As before, we determine P' using a statement of integral mass conservation. Here, in view of the obvious symmetry about $X = 0$, this statement adopts the form

$$\int_0^H U\, dY = -X, \tag{29}$$

where the right-hand side accounts for the volumetric flux in the y direction associated with 27(b). This immediately gives

$$P' = \frac{12X(H+B)}{H^3(H+4B)}. \tag{30}$$

Note that, unlike the preceding problems, global conservation provides an equation for P' with no integration constants. This is consistent with the underlying symmetry about $X = 0$, which renders one of the large-$|X|$ conditions (11) redundant. Here integration of P' is possible in closed form, whereby use of (11) yields

$$P = \frac{3}{16B^2}\left[\ln\frac{2+8B+X^2}{2+X^2} - 8B\frac{6+4B+3X^2}{(2+X^2)^2}\right]. \tag{31}$$

Recalling that the velocity scale in the perpendicular problem has been chosen as $\epsilon^{-1/2}v^{\perp}$, comparison with (2b) and (3) readily gives the pertinent resistance coefficient as

$$F^{\perp} = -\int_{-\infty}^{\infty} P\, dX. \tag{32}$$

Substitution of (31) then furnishes the expression

$$F^{\perp} = \frac{3\pi(3+6B+2B^2-3\sqrt{1+4B})}{4\sqrt{2}B^2}. \tag{33}$$

Here we find

$$\lim_{B\to 0} F^{\perp} = 3\pi\sqrt{2}, \tag{34a}$$

$$\lim_{B\to\infty} F^{\perp} = \frac{3\pi\sqrt{2}}{4}. \tag{34b}$$

The first limit is in agreement with Ref. [16]. The finite limit at large B is expected, since the global conservation (29) necessitates a nonzero pressure gradient for any value of B. The variation with B of the resistance to perpendicular translation is portrayed in Fig. 3. The use of a separate figure is intended to prevent the misleading impression of comparable resistance to parallel and perpendicular translation—recall the distinct $\epsilon^{-3/2}$ scaling in (3).

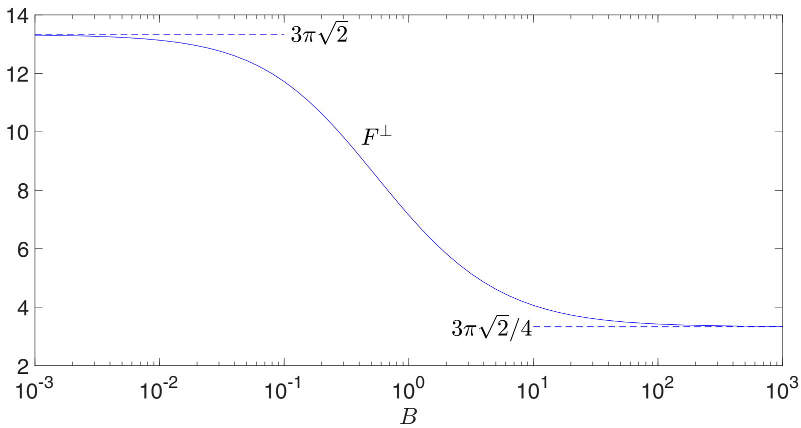

FIG. 3. Schematic of the problem. The limits (34) indicated.

VII. SLIPPERY CYLINDER

Consider now the "complementary" situation where the Navier condition (1) applies at the cylinder boundary while the wall exhibits a conventional no slip. In the rotation and parallel-translation problems conditions (8) are replaced by

$$U = 0 \quad \text{at} \quad Y = 0, \tag{35a}$$

$$U = 1 - B\frac{\partial U}{\partial Y} \quad \text{at} \quad Y = H. \tag{35b}$$

While intuition may suggest that the exchange of the slippery surface is immaterial in the lubrication limit, a closer look reveals a fundamental difference. Consider indeed the parallel-translation problem in the free-surface limit $B \to \infty$. For the case of slippery wall, that limit has resulted in a local plug flow, moving with the cylinder, leading in turn to a vanishing drag force at leading order. When considering that limit in the case of a slippery cylinder, it is convenient to employ a reference frame which translates with the cylinder. In that frame the x component of the velocity must attain a -1 value at the wall and satisfy a shear-free condition at the cylinder. A naive guess of a plug-flow solution with velocity -1 in the x direction is, however, incompatible with impermeability at the cylinder. We accordingly anticipate here a nonzero drag in the free-surface limit.

The resistance coefficients corresponding to conditions (35) have been calculated using the procedure described in the context of a slippery wall. We again find that $C = D$. More importantly, when we evaluate (16), (20), and (24) we find the following surprising linkage with their slippery-wall counterparts:

$$\begin{pmatrix} F^{\parallel} \\ D \\ T \end{pmatrix}_{\text{slippery cylinder}} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} F^{\parallel} \\ D \\ T \end{pmatrix}_{\text{slippery wall}}. \tag{36}$$

The value attained by $F^{\parallel}$ in the free-surface limit, $\lim_{B\to\infty} F^{\parallel} = \pi\sqrt{2}$, thus follows from (17b). The prediction of nonzero drag is consistent with the preceding discussion.

For translation perpendicular to the wall, conditions (26) are replaced by $U = 0$ at $Y = 0$ and $U = -B\,\partial U/\partial Y$ at $Y = H$. Here we find that $F^{\perp}$ is the same as in the case of a slippery wall.

VIII. CONCLUDING REMARKS

Using the lubrication approximation, we have calculated the resistance matrix pertinent to two-dimensional motion of a cylinder near a slippery wall. In the appropriate dimensionless notation the torque due to parallel translation is the same as the force due to rotation. For no-slip surfaces, such an equality is a consequence of a well-known symmetry, proved by Brenner [20]; it is, however, not *a priori* evident that it should hold in the presence of slip. An extension of Brenner's "mechanical" proof [20] to slippery surfaces appears quite challenging, especially for arbitrary geometries where the right-hand side of (1) must be generalized to incorporate the appropriate projection of the rate-of-strain tensor. It may be, however, that an "irreversible-thermodynamics"-type argument, based upon the Onsager reciprocal relations, may prove useful. These arguments, based upon the linear structure of the problem [21,22], typically make no assumption regarding the form of the boundary conditions (beyond being linear and homogeneous). They are accordingly valid when the Navier condition (1), or a generalization thereof, applies.

Another symmetry we have observed has to do with the shift of the slippery surface from the wall to the cylinder boundary. We have not been able to show that relation (36) follows from the very problem formulation. In the case of no-slip surfaces, the equality of the dimensionless rotation torque and translation force has appeared as a mere coincidence [16]; we now realize that this equality simply represents the $B \to 0$ limit of symmetry (36).

An obvious extension of the present contribution is an analysis of particle motion near walls which are endowed with anisotropic slip properties. For such walls, the slip length b appearing in (1) becomes a rank-2 tensor, allowing for a significant difference of the slip-length value in the two principal directions of that tensor. Indeed, the recent analysis of Schnitzer [8,9] reveals that the slip length for longitudinal flow over a dense bubble mattress is much larger than the microstructure pitch, a singularity associated with the need of the small solid portions of the surface to support the imposed shear. Since no comparable singularity appears in the case of a transverse flow, such mattresses may be modeled using Navier slip in one direction and no slip in the orthogonal direction. With such slip anisotropy, even the simplest problem of a sphere moving perpendicular to the wall is rather unconventional: On one hand, tensorial symmetry readily implies that the hydrodynamic force experienced by the sphere is also perpendicular to the wall (pure drag). On the other hand, the anisotropy excludes axial symmetry, implying that the partial differential equation governing the lubrication pressure cannot be reduced to an ordinary one.

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