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Onset of cellular motion in Taylor-Couette flow

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Taylor-Couette flow between concentric rotating cylinders provides a prime example of a pitchfork bifurcation in hydrodynamic stability. However, more than 40 years ago, Benjamin pointed out difficulties in establishing quantitative links between models where periodicity is assumed along the length of the cylinders and calculations using the Navier-Stokes equations. A new model was recently advanced where it is claimed that the bifurcation which is relevant to the onset of cellular motion is the one between neighboring states. Here, we provide experimental data in support of this idea which explain why the onset of cellular motion is observed to be sharp whereas the periodic model bifurcation is highly imperfect.

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Introduction. Taylor-Couette flow concerns the motion of fluid in the gap between a pair of concentric cylinders when one or both of the cylinders is rotated. Taylor [1] studied the problem as a laboratory model to study the Rayleigh criterion for the stability of rotary fluid motion. He observed several types of instability including steady secondary cellular motion and spiralling flows. Taylor's seminal experimental and theoretical work is a cornerstone of hydrodynamic stability theory since it provides a detailed quantitative comparison between observation and theory on the important concept of the exchange of stability. Although there are many experimental, theoretical, and numerical investigations into the problem [2,3], it is only relatively recently that the onset of cellular motion has been reappraised [4]. The focus of our work is on the onset of cells in Taylor-Couette flow, which is widely accepted as an example of a pitchfork bifurcation [5]. It is disconnected by imperfections in an experiment, but for large aspect ratios the disconnection is small and the onset is sharp. However, the saddle node of the disconnected branch is a factor of 3–10 above the onset. This is inconsistent with the observed behavior being a simple unfolding of the model bifurcation with idealized boundary conditions. A solution to this long-standing conundrum was recently advanced [4] which involves the unfolding of the bifurcation between states with N and $N \pm 2$ cells. Here, we provide experimental and numerical evidence in support of this interpretation and we provide numerical evidence to show that this will be the case for modest aspect ratios.

We focus on the case where the inner cylinder alone rotates and the outer is held fixed, and the first instability is to a secondary cellular flow. The dynamical parameter which controls the flow is the Reynolds number Re, which is defined here as $\omega_i r_i d/\nu$, where ω_i is the rotation rate of the inner cylinder of radius r_i, $d = r_i - r_o$ is the gap between the two cylinders, and ν is the kinematic viscosity of the fluid. At small Re the fluid is mainly dragged round by the inner cylinder and the flow field is primarily featureless. It becomes structured at the onset of steady secondary cellular motion above a certain small range of Re. The flow has the form of tori stacked upon each other along the length of the cylinder, as in Fig. 1. The flow rotates alternately in clockwise and anticlockwise directions in each torus and the flow is steady and axisymmetric. The geometrical parameters which control the flow are the aspect ratio $\Gamma = (r_o - r_i)/\ell$, where ℓ is the length of the cylindrical gap,

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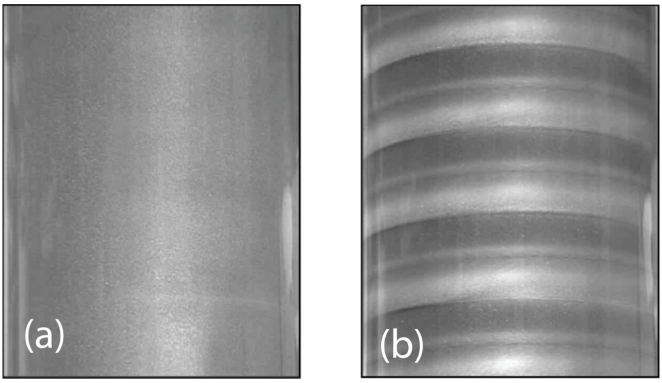

FIG. 1. Front view of visualization of Taylor-Couette flow (a) at $\mathrm{Re} = 65.0$ and (b) $\mathrm{Re} = 80.0$. The cylinders had a radius ratio $\eta = 0.5$ and an aspect ratio $\Gamma = 24.0$. The critical value of Re is ~68.15.

and $\eta = r_i/r_o$. The images in Fig. 1 suggest that it is appropriate to assume periodicity along the length of the cylinder in models. Under these conditions the onset of cellular motion is a pitchfork bifurcation [6]. Numerical techniques can be used to calculate the steady solution branches and the presence of a simple pitchfork under appropriate boundary conditions is confirmed [7]. Specifically, at a critical value of Re_c, the trivial solution of rotary Couette flow becomes unstable at a pitchfork bifurcation and exchanges stability with a pair of cellular states of a well-defined wavelength along the cylindrical gap. Each state in the pair can be translated into the other by a shift of half a wavelength.

Imposition of fixed end conditions disconnects the model bifurcation. It was pointed out by Benjamin [8,9] that links between the periodic boundary conditions of models and the finite domain of experiments are subtle. However, quantitative agreement between the results of numerical calculations of the Navier-Stokes equations and experiments can be obtained [7,10].

The effects of ends can be clarified using boundary conditions with the homotopy parameter τ, $0 \leqslant \tau \leqslant 1$ [11]. On the end plates $z = 0$ or ℓ the velocity is decomposed into normal and tangential components u_N and u_T, with the requirement

$$u_N = 0,(1-\tau)\frac{\partial u_T}{\partial N} + \tau u_T = 0. \tag{1}$$

If $\tau = 0$, then (1) yields idealized boundary conditions that imply the end plates are planes of symmetry; consequently, a periodic solution in an infinite cylinder can be constructed by repeated reflection. If $\tau = 1$, then (1) yields physical no-slip boundary conditions. The outline of this Rapid Communication is as follows. In the next section we discuss the numerical and experimental methods. Results are then presented for the onset of 16 steady cells and conclusions are drawn.

The bifurcation diagram in Fig. 2 was calculated for $\eta = 0.9$, $\gamma = 16$ over the range of Re, 0–1600. The measure used to distinguish between the various states is the midgap, midplane radial velocity component as this provides a simple, physically relevant quantity. The pitchfork bifurcation in Fig. 2(a) was calculated for a domain length of 16 with $\tau = 0$. An exchange of stability between Couette flow and a pair of cellular states can be seen. When τ is set to 1, no-slip boundary conditions pertain at the ends of the domain. The result as shown in Fig. 2(b) is that the bifurcation is disconnected. One branch indicates a sharp continuous onset of 16 cells and was termed primary by Benjamin [8]. This is the lower branch labeled normal 16 cells in Fig. 2(b). There is also a disconnected branch which corresponds to one of Benjamin's "anomalous modes" [8] which is terminated by a saddle-node bifurcation at its lower end. This is the upper branch labeled anomalous 16 cells in Fig. 2(b). N.B. In the projection shown here the stable and unstable parts of the anomalous branch are superposed.

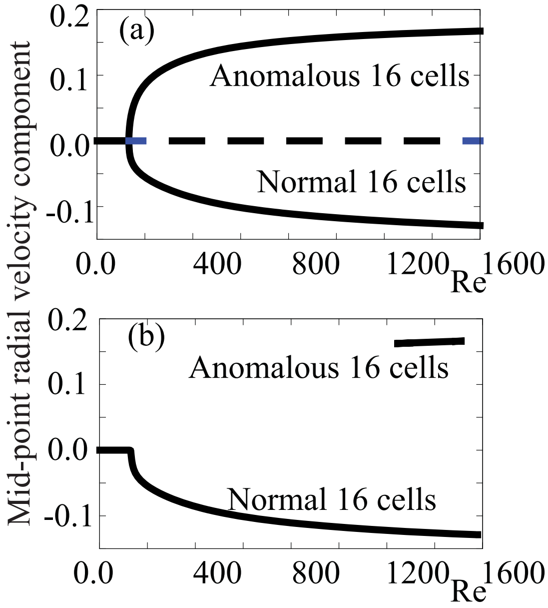

FIG. 2. Numerical calculations of the normal and anomalous 16 cell branches at $\Gamma = 16$, $\eta = 0.9$. (a) A pitchfork with $\tau = 0$. (b) A disconnected pitchfork with no-slip end conditions. The limit point for the anomalous branch is at $\mathrm{Re} \approx 1230.67$ and the onset of the normal flow is at $\mathrm{Re} \approx 131.06$. N.B. Both branches can be extended to higher values of Re.

An anomalous mode has the same number of cells as the corresponding primary state, but it is translated along the flow domain by half a wavelength and all cells rotate in the opposite sense to the primary. For this value of η, the anomalous modes exist at relatively high values of Re and will not be observable as a steady flow [12] since the motion will be time dependent at this high value of Re. However, anomalous modes are observed for smaller values of radius ratio η, and saddle-node points are in excellent quantitative agreement with numerical results [7,13].

The disconnection of the pitchfork in Fig. 2 is large and the saddle-node or limit point is a factor ~10 above the bifurcation point of the periodic model. Disconnections by factors of ~10 imply that there ought to be a very smooth development of the primary branch with an increase of Re if the onset of cells is to be considered as a simple imperfection added to the bifurcation of the periodic model [14]. However, the surprising feature observed in Taylor-Couette flow experiments is that the onset of cells is generally sharp for $\Gamma \gtrsim 12$ [4] and is in excellent accord with theory. Hence the ends do not simply disconnect the pitchfork of the periodic model. This is an oversimplified interpretation of the facts since the sharp onset of cells and large disconnection of the anomalous branch are inconsistent. A proposal for the resolution of this puzzle has been advanced [4] who argue that N and $N \pm 2$ neighboring primary cellular states emerge from a mixed mode state through an approximate pitchfork bifurcation at a specific value of Γ. To date there has been no direct experimental evidence to support this proposal and here we provide data to test this idea.

Experiment. The experimental apparatus is shown schematically in Fig. 3 and comprised two concentric cylinders (radii r_i,r_o) where the inner rotated at speed ω_i. The gap between the cylinders ($d = r_o - r_i$) was filled with silicone oil which had a measured kinematic viscosity $\nu = 10.6$ cS at the thermostatically controlled working temperature of 24.00 ± 0.01°C. The inner cylinder was machined from stainless steel and had a radius of $r_i = (12.50 \pm 0.01)$ mm, while the outer cylinder was optically polished glass with a radius of $r_o = (25.00 \pm 0.01)$ mm. The motor which drove the inner cylinder round had a phase-locked loop control with an accuracy of 10^4 (10^7) in the short

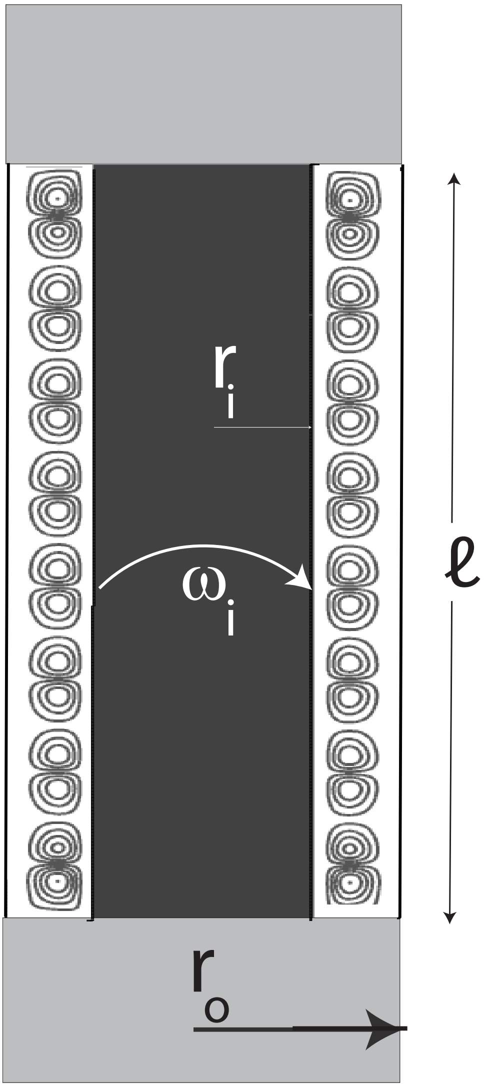

FIG. 3. A schematic diagram of the apparatus. Streamlines from a calculation of 16 cells at $\mathrm{Re} = 82.0$, $\Gamma = 16$ have been inserted in a cross-sectional plane through the apparatus.

(long) term average. The geometric parameters are the aspect ratio $\Gamma = \ell/d$ and the radius ratio $\eta = r_1/r_2 = 0.5$. We used laser Doppler velocimetry (LDV) to measure the radial velocity (u) locally at any position (r,z) within a chosen cross section of the flow domain. The measurements were mainly made on the midplane at midgap. Velocity profiles were measured by moving the LDV at a constant low speed along the length of the flow domain ℓ.

Numerical methods. Steady axisymmetric Navier-Stokes equations with either no-slip or $\tau = 0$ boundary conditions were solved numerically using the software suite ENTWIFE [15]. The finite element meshes used generally had 160 × 10 elements in the z and r directions, respectively, and suitable corner refinements were used [16]. Grid resolution was tested using grids of 240 × 20 elements and calculations of bifurcation points were found to be better than 0.1%. A review of the various techniques employed is provided by Ref. [16]. The features used were arclength continuation to follow solution branches and extended systems methods to calculate of the paths of bifurcation

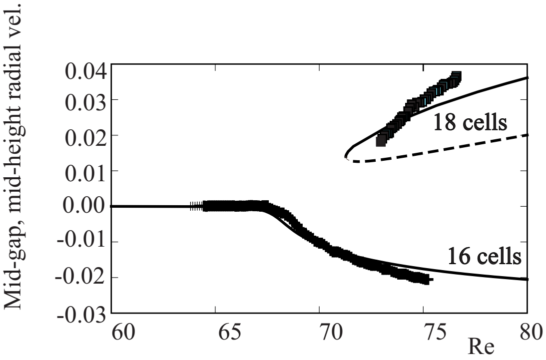

FIG. 4. Numerical (solid/dashed lines) and experimental (crosses) results for the disconnected bifurcation between 16 and 18 cell flows. The midgap radial flow component is shown plotted vs Re. The value of $\Gamma = 16.71$. N.B. The anomalous modes exist for Re $\gtrapprox 190$ and are therefore off the scale used here.

points as a function of Re and Γ. In particular, we calculated the paths of limit points, transcritical bifurcations, and hysteresis points and pitchfork bifurcations under $\tau = 0$ boundary conditions.

Results. The aspect ratio range investigated was $16 \leqslant \Gamma \leqslant 18$. For $\Gamma \leqslant 17.68$, 16 cells developed sharply close to the critical value of $\mathrm{Re}_c = 68.14$ of the periodic model for $\eta = 0.5$. An example of such an evolution is given in Fig. 4, which was obtained at $\Gamma = 16.71$. It can be seen that there is strong quantitative accord between the experimental and numerical results. The second branch which can be seen in Fig. 4 is the branch corresponding to an 18 cell flow. It is terminated by a saddle-node bifurcation at Re ≈ 71.28 (numerical), although the experimental value is slightly above this at Re ≈ 74.24. The small difference between numerical and experimental results on the secondary branch reflects the influence of imperfections in the experiment.

When $\Gamma = 17.68$, there is an approximate pitchfork where there is an exchange of stability between a weak mixed mode state and 18 and 16 steady cells. The experimental and numerical results for this case shown in Fig. 5 are in excellent quantitative accord. When $\Gamma > 17.68$, 18 cells form and similarly 16 cells form when $\Gamma < 17.68$. In principle, the switch in priority will involve a transcritical bifurcation and cusp [9,17] and these features can be resolved numerically but not experimentally. They occur over a $<0.1\%$ range of the parameters. Instead, the idea of an approximate symmetry [4] with an approximate pitchfork bifurcation between neighboring states is apposite.

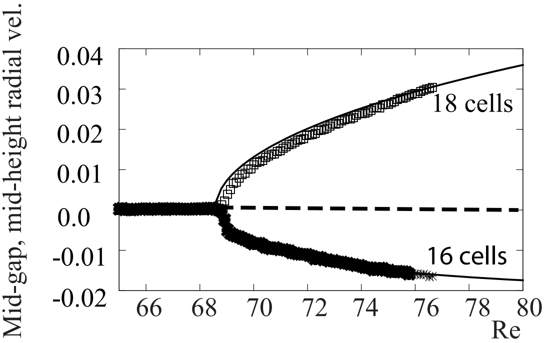

FIG. 5. Numerical (solid lines) and experimental (crosses) results for the approximate pitchfork bifurcation between 16 and 18 cell flows. The midgap radial flow component is shown plotted vs Re. The value of $\Gamma = 17.68$. N.B. The anomalous modes exist for Re $\gtrapprox 190$ and are therefore off the scale used here.

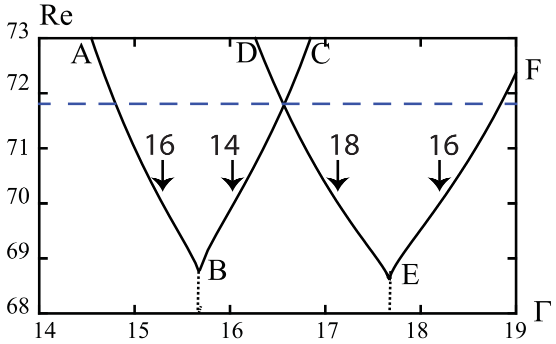

FIG. 6. Calculated paths of limit points in (Re, Γ) parameter space. AB and EF are paths of limit points for 16 cell flows. BC and DE are limit point paths for 14 and 18 cells, respectively. The 16 cell flow is primary in the aspect ratio range $\Gamma \approx 15.81$ (B) to $\Gamma \approx 17.68$ (E). N.B. The paths BC and DE do not intersect. The apparent intersection is an artifact of projection.

The robustness of these delicate features is clear and a numerical investigation of the solution structure over a wider parameter range supports this. Specifically, the paths of limit points for the N and $N \pm 2$ states were calculated and the results are presented in Fig. 6. A striking feature is that they are qualitatively similar to the unfolding of a pair of pitchfork bifurcations between 14 : 16 and 16 : 18 cell states where the cusps at B and E are aligned with the Re axis. The y scale covers a very small range of Re ($\sim$2%) and the horizontal dashed line passes through the points at which there is a change in the secondary state from $N - 2$ to $N + 2$ as Γ increases, i.e., it indicates the largest disconnection in Re of the bifurcation. Hence, the disconnection of the approximate pitchfork between neighboring states is always small so the onset of cells is sharp, even at modest aspect ratios.

Conclusion. Striking experimental and numerical evidence has been provided in support of the hypothesis that the onset of steady Taylor vortices arises at a bifurcation where the approximate symmetry between neighboring states is broken. Detailed quantitative measurements of the bifurcation at modest aspect ratios are in excellent accord with numerical results. An implication of these results is the interaction between neighboring states needs to be considered in models of the onset of time dependence including low-dimensional chaos [18]. The inclusion of the symmetries which are appropriate for a finite domain is essential to enable a connection between low-order models and the Navier-Stokes equations.

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