Self-preservation relation to the Kolmogorov similarity hypotheses

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The relation between self-preservation (SP) and the Kolmogorov similarity hypotheses (Kolmogorov, The local structure of turbulence in incompressible viscous fluid for very large Reynolds numbers, Dokl. Akad. Nauk SSSR **30**, 301 (1941) [Proc. R. Soc. London A **434**, 9 (1991)]) is investigated through the transport equations for the second- and third-order moments of the longitudinal velocity increments [$\delta u(r,t) = u(x,t) - u(x+r,t)$, where x, t, and r are the spatial point and the time and longitudinal separation between two points, respectively]. It is shown that the fluid viscosity ν and the mean turbulent kinetic energy dissipation rate $\overline{\epsilon}$ (the overbar represents an ensemble average) emerge naturally from the equations of motion as controlling parameters for the velocity increment moments when SP is assumed. Consequently, the Kolmogorov length scale η [$\equiv(\nu^3/\overline{\epsilon})^{1/4}$] and velocity scale v_K [$\equiv(\nu\overline{\epsilon})^{1/4}$] also emerge as natural scaling parameters in conformity with SP, indicating that Kolmogorov's first hypothesis is subsumed under the more general hypothesis of SP. Further, the requirement for a very large Reynolds number is also relaxed, at least for the first similarity hypothesis. This requirement however is still necessary to derive the two-thirds law (or the four-fifths law) from the analysis. These analytical results are supported by experimental data in wake, jet, and grid turbulence. An expression for the fourth-order moment of the longitudinal velocity increments $\overline{(\delta u)^4}$ is derived from the analysis carried out in the inertial range. The expression, which involves the product of $(\delta u)^2$ and $\partial\delta p/\partial x$, does not require the use the volume-averaged dissipation $\overline{\epsilon}_r$, introduced by Oboukhov [Oboukhov, Some specific features of atmospheric turbulence, J. Fluid Mech. **13**, 77 (1962)] on a phenomenological basis and used by Kolmogorov to derive his refined similarity hypotheses [Kolmogorov, A refinement of previous hypotheses concerning the local structure of turbulence in a viscous incompressible fluid at high Reynolds number, J. Fluid Mech. **13**, 82 (1962)], suggesting that $\overline{\epsilon}_r$ is not, like $\overline{\epsilon}$, a quantity issuing from the Navier-Stokes equations.

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I. INTRODUCTION

The Kolmogorov theory for very-large-Reynolds-number homogeneous and isotropic turbulence [1] has not only stood as a milestone in the theory of turbulence, but has served and continues to serve as a benchmark for any new theoretical development. Kolmogorov theory encapsulated two similarity hypotheses (SHs), which were enunciated for f, the probability distribution function (PDF) of the velocity increments $\delta u_i(r,t) = u_i(x,t) - u_i(x',t)$; x and x' are two independent points and $r = x - x'$. We briefly reproduce these hypotheses here for convenience.

First similarity hypothesis (SH1). For locally isotropic turbulence, the distributions of f are uniquely determined by the quantities ν and $\overline{\epsilon}$.

Second similarity hypothesis (SH2). When the separation r is much larger than η, the distributions of f are uniquely determined by the quantity $\overline{\epsilon}$ and do not depend on ν, where ν is the fluid viscosity,

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$\overline{\epsilon}$ is the dissipation rate of the mean turbulent kinetic energy (the overbar denotes ensemble averaging; note that in the case of measurements it represents time averaging), and η is a length scale defined by Kolmogorov and obtained on dimensional grounds. For convenience, we henceforth drop the variables t, x, x', and r, keeping in mind that $u_i = u_i(x,t)$ and $u_i' = u_i(x',t)$. It is instructive, within the context of this work, to follow Kolmogorov's empirical analysis prior to the enunciation of SH1. He first transformed the variables r_i (the space separation between two points) and t (the time). Namely, he defined the dimensionless variables r_i/l and t/σ, where l and σ are, respectively, length and time characteristic scales. Then he normalized δu_i, ν, and $\overline{\epsilon}$ as follows: $\delta u_i' = \delta u_i \sigma/l$, $\nu' = \nu\sigma/l^2$, and $\overline{\epsilon}' = \overline{\epsilon}\sigma^3/l^2$. This, after enunciating the first hypothesis, allowed him to obtain $l = \eta = \sqrt{\nu\sigma} = (\nu^3/\overline{\epsilon})^{1/4}$, $\sigma = \sqrt{\nu/\overline{\epsilon}}$, $\nu' = 1$, and $\overline{\epsilon}' = 1$. We immediately see that the (Kolmogorov) scaling velocity is $v_K = \eta/\sigma = (\nu\overline{\epsilon})^{1/4}$.

A clear distinction should be made between the Kolmogorov SHs and the asymptotic theoretical results further deduced, such as the two-thirds $[\overline{(\delta u)^2} \simeq (\overline{\epsilon} r)^{2/3}]$ and four-fifths $[\overline{(\delta u)^3} \simeq (4/5)\overline{\epsilon} r]$ laws. Kolmogorov formulated the scaling laws only after enunciating his similarity hypotheses in his first paper. In a subsequent paper [2], he deduced the four-fifths law from the equations of motion, without any reference to his similarity hypotheses, but with the assumption of sufficiently large Reynolds numbers. This law serves as benchmark for any turbulence theory at high Reynolds number [3].

As pointed out in Ref. [4], Kolmogorov adroitly avoided alluding to the Navier-Stokes equations in his first paper when he proposed his two similarity hypotheses. Further, [5] pointed out that these hypotheses cannot be rigorously proved, i.e., derived purely analytically from general laws of mechanics. More recently, [6] also stated that SH1 cannot be formally proven in any deductive way. This lack of rigorous theoretical support for these SHs leaves open the issue of their validation or invalidation, particularly considering that they are formulated for very high, if not infinite, Reynolds numbers. The latter requirement makes their testing practically impossible both experimentally and numerically. The current general consensus is that these SHs require corrections due to the spatial fluctuating nature of ϵ (i.e., the small-scale intermittency). While many correction models have been proposed, these corrections too are based on phenomenological arguments and it remains to be determined whether or not they comply with the Navier-Stokes equations.

Batchelor [7] was the first to discuss Kolmogorov theory and the similarity hypotheses in the context of the Navier-Stokes equations and showed that the hypotheses are fully consistent with the Navier-Stokes equations. It is important to stress that he did not formulate these hypotheses from the Navier-Stokes equations. Instead, he considered SHs and Navier-Stokes equations as two different concepts and showed that they may be compatible. He nevertheless did recognize that Kolmogorov theory requires some form of self-preservation (SP). The hypothesis of self-preserving development of a turbulent flow assumes that all physical phenomena of the motion are reflected by terms in transport equations for any turbulent quantity that admits similar forms at all stages, the differences being described by changes of velocity and length scales that are functions of time (in decaying turbulence) or of position in the flow direction [8]. Lin [9] followed Batchelor and applied SP analysis to the von Kármán–Howarth equation [10] to show that the Kolmogorov scaling was consistent with SP. Unfortunately, like [7], he assumed the first similarity hypothesis rather than derive it, which allowed him to use the Kolmogorov scales as scaling variables. George [11] also discussed the relation of SP and Kolmogorov's theory without deriving the first similarity hypothesis. He argued that Kolmogorov's theory is at best an approximation for turbulence at finite Reynolds number, while also concluding that the theory, at least for isotropic turbulence, gives way to a higher principle, that of self-preservation at all scales, under certain conditions.

In the present paper we address whether the Kolmogorov theory similarity hypotheses can be deduced from the Navier-Stokes equations. This approach is different from that of simply testing if they comply with the first principles. More specifically, we seek, using a rigorous theoretical development free of any phenomenology, to find out whether or not ν and $\overline{\epsilon}$ emerge naturally as scaling parameters from the equations of motion. This approach differs significantly from the common practice that assumes this scaling and assesses whether or not it holds up against experimental and numerical data.

In the present work, and following the earlier work cited above, we also use the framework of SP and apply it to the transport equations for the second- and third-order moments of the longitudinal velocity increments. The theoretical development and results are presented in Sec. II, while in Sec. III we compare theory against some experimental results.

II. THEORETICAL CONSIDERATIONS

A. Transport equations for the second and third moments of the velocity increment δu

Starting with the Navier-Stokes equations written at two separate and independent spatial points x and x', the transport equations for δu_2 [$\equiv\overline{(\delta u)^2}$] and δu_3 [$\equiv\overline{(\delta u)^3}$] for homogeneous isotropic turbulence (HIT) can be expressed as [12]

$$\frac{\partial \delta u_2}{\partial t} + \left(\frac{\partial}{\partial r} + \frac{4}{r}\right)\delta u_3 = -4\overline{\epsilon} + 6\nu\left\{\frac{\partial^2}{\partial r^2} + \frac{4}{r}\frac{\partial}{\partial r}\right\}\delta u_2 \tag{1}$$

and [13]

$$\frac{\partial \delta u_3}{\partial t} + \left\{\left(\frac{\partial}{\partial r} + \frac{2}{r}\right)\delta u_4 - \frac{6}{r}(\overline{\delta u_2 \delta v_2})\right\} = -2\nu E_{111} + 2\nu\left\{-\frac{4}{r^2}\delta u_3 + \frac{4}{r}\frac{\partial \delta u_3}{\partial r} + \frac{\partial^2 \delta u_3}{\partial r^2}\right\} - T_{111}, \tag{2}$$

where $\overline{\epsilon} = -\frac{2}{3}\frac{\partial \overline{u^2}}{\partial t}$ is the turbulent kinetic energy dissipation rate, $\delta u_4 = \overline{(\delta u)^4}$, and $\overline{\delta u_2 \delta v_2} = \overline{(\delta u)^2(\delta v)^2}$. Also, T_{111} and E_{111} are defined as

$$T_{111} = \frac{3}{\rho}\overline{(\delta u)^2\left(\frac{\partial \delta p}{\partial X}\right)}, \tag{3a}$$

$$E_{111} = 3\overline{(\delta u)\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u'}{\partial x'}\right)^2\right\}} = 3\overline{(\delta u)\left\{\left(\frac{\partial \delta u}{\partial x}\right)^2 + \left(\frac{\partial \delta u}{\partial x'}\right)^2\right\}}. \tag{3b}$$

In the above equations and expressions we used the transformation $X = (x + x')/2$ (see, e.g., [13]) and the fact that $\partial u/\partial x' = \partial u'/\partial x = 0$. The second term on the left-hand side of both (1) and (2) is the transport term. The first term on the right-hand side of (1) is the dissipation term, whose counterpart in Eq. (2) is the first term on the right-hand side of that latter equation. The last term and the second term on the right-hand side of (1) and (2), respectively, are the viscous terms. The term T_{111} is the pressure source term and has no equivalent in Eq. (1) for HIT. Equations (1) and (2) can be further developed in a more useful form. Integrating them, and with some trivial manipulations, we have

$$\delta u_3 = 6\nu\frac{\partial \delta u_2}{\partial r} - \frac{3}{r^4}\int_0^r s^4\frac{\partial \delta u_2}{\partial t}ds - \frac{4}{5}\overline{\epsilon}r, \tag{4}$$

$$\delta u_4 = \frac{6}{r^2}\int_0^r s(\overline{\delta u_2\delta v_2})ds - \frac{1}{r^2}\int_0^r s^2 T_{111}ds - \frac{2\nu}{r^2}\int_0^r s^2 E_{111} + \frac{2\nu}{r^2}\int_0^r\left\{4 + 4s\frac{\partial}{\partial s} + s^2\frac{\partial^2}{\partial s^2}\right\}\delta u_3 ds - \frac{1}{r^2}\int_0^r \frac{\partial \delta u_3}{\partial t}ds. \tag{5}$$

In the majority of studies, the term in Eq. (4) containing the integral was dropped on the basis that the Reynolds number is large enough to allow a net scale separation between large and small scales. It is only over the past decade or so that the term was reintroduced [12], derived (4) in the context of decaying grid turbulence for investigating the contributions of the large scales, represented by the integral term.

B. Self-preservation analysis

We seek SP solutions in the form

$$\delta u_2 = u_2^2(t) f(r^*), \tag{6a}$$

$$\delta u_3 = u_3^3(t) g(r^*), \tag{6b}$$

$$\delta u_4 = u_2^4(t) h(r^*), \tag{6c}$$

$$\delta v_2 = v_2^2(t) f_v(r^*), \tag{6d}$$

$$\overline{\delta u_2 \delta v_2} = u_2^2(t) v_2^2(t) f_{uv}(r^*), \tag{6e}$$

$$T_{111} = \frac{u_2^4(t)}{l(t)} \tilde{T}_{111}(r^*), \tag{6f}$$

$$E_{111} = \frac{u_2^3(t)}{l^2(t)} \tilde{E}_{111}(r^*), \tag{6g}$$

where $r^* = r/l(t)$; $l(t)$, $u_2(t)$, $u_3(t)$, and $v_2(t)$ are a length scale and velocity scaling functions to be determined, while f, g, h, f_v, f_{uv}, $\tilde{T}_{111}$, and $\tilde{E}_{111}$ are dimensionless functions of r^* only. For convenience, we hereafter drop the variable t. We introduce the skewness and flatness factors of the longitudinal velocity increment S ($\equiv \delta u_3/\delta u_2^{3/2}$) and F ($\equiv \delta u_4/\delta u_2^2$), respectively, and the mixed flatness F_{uv} [$\equiv \overline{(\delta u)^2(\delta v)^2}/(\delta u_2)^2$] as SP controlling parameters. If SP is satisfied, we can write $S(r^*) = c_S(t)\phi_S(r^*)$, $F(r^*) = c_F(t)\phi_F(r^*)$, and $F_{uv}(r^*) = c_{F_{uv}}(t)\phi_{F_{uv}}(r^*)$, where c_S, c_F, and $c_{F_{uv}}$ are dimensionless functions of time and ϕ_S, ϕ_F, and $\phi_{F_{uv}}$ are dimensionless functions of r^*. Substituting the expression (6) and the SP forms of S and F in Eqs. (4) and (5), we obtain

$$6f'(r^*) - c_S \mathrm{Re}_l \phi(r^*) f(r^*)^{3/2} - \frac{3l^2}{\nu u_2^2} \frac{\partial u_2^2}{\partial t} \frac{1}{r^{*4}} \int_0^{r^*} s^{*4} f(s^*) ds^* + \frac{3}{\nu} l \frac{\partial l}{\partial t} \frac{1}{r^{*4}} \int_0^{r^*} s^{*5} f'(s^*) ds^* = \frac{4}{5} \overline{\epsilon} \frac{l^2}{\nu u_2^2} r^* \tag{7}$$

and

$$c_F \mathrm{Re}_l \phi_F f^2 = -\frac{\mathrm{Re}_l}{r^{*2}} \int_0^{r^*} s^{*2} \tilde{T}_{111} ds^* - \frac{6}{r^{*2}} \int_0^{r^*} s^{*2} \tilde{E}_{111} ds^* + \frac{6 c_{F_{uv}} \mathrm{Re}_l}{r^{*2}} \frac{v_2^2}{u_2^2} \int_0^{r^*} s^* \phi_{F_{uv}} f^2 ds^* + \frac{2c_S}{r^{*2}} \int_0^{r^*} \left\{ 4 + 4s^* \frac{\partial}{\partial s^*} + s^{2*} \frac{\partial^2}{\partial s^{*2}} \right\} \phi_S f^{3/2} ds^* - \frac{l^2}{\nu u_2^3} \frac{\partial c_S u_2^3}{\partial t} \frac{1}{r^{*2}} \int_0^{r^*} s^{*2} \phi_S f^{3/2} ds^* + \frac{l}{\nu} \frac{\partial l}{\partial t} \frac{c_S}{r^{*2}} \int_0^{r^*} s^{*3} \frac{\partial(\phi_S f^{3/2})}{\partial s^*} ds^*, \tag{8}$$

where s^* is a dummy variable of integration. It can be shown that when $r^* \to \infty$, the integrals in these equations approach finite values. While (7) and its versions for the centerlines of a round jet and a plane wake have been used to investigate self-preservation [14–17], here (8) is considered in the SP analysis. It is evident that, if SP is to be valid, (7) and (8) must share the same scaling parameters. Since the coefficients of the first term on the left-hand side of (7) and the second term on the right-hand side of (8) are constant then, the SP constraints are

$$\mathrm{Re}_l = C_1, \quad c_S = C_{1,a}, \quad c_F = C_{1,b}, \quad c_{F_{uv}} \frac{v_2^2}{u_2^2} = C_{1,c}, \tag{9}$$

$$\frac{l^2}{\nu u_2^2} \frac{\partial u_2^2}{\partial t} = C_2, \tag{10}$$

$$\frac{l}{\nu}\frac{\partial l}{\partial t} = C_3, \tag{11}$$

$$\frac{l^2}{\nu u_2^3}\frac{\partial c_S u_2^3}{\partial t} = C_4, \tag{12}$$

$$\frac{c_S l}{\nu}\frac{\partial l}{\partial t} = C_5, \tag{13}$$

$$\frac{\overline{\epsilon} l^2}{\nu u_2^2} = C_6, \tag{14}$$

where $\mathrm{Re}_l = u_2 l/\nu$ is a scaling Reynolds number. The above constraints, i.e., C_1, C_2, C_3, C_4, C_5, and C_6, must be independent of t, apply for all separations r^*, while the numerical values of the constants depend on the scaling variables. Note that the ratio C_5/C_3 shows also that c_S is a constant, which in turn implies that $u_3 = u_2$. Combining the first expression of (9) with (14) and using the definition of Re_l yields

$$l = (\sqrt{C_6}C_1)^{1/2}\left(\frac{\nu^3}{\overline{\epsilon}}\right)^{1/4}. \tag{15}$$

Now substituting (15) into (14) leads to

$$u_2 = \left(\frac{C_1}{\sqrt{C_6}}\right)^{1/2}(\nu\overline{\epsilon})^{1/4}. \tag{16}$$

A few comments can be formulated regarding the derivation of the length scale (15), the velocity scale (16), and Kolmogorov's first similarity hypothesis. It should be noted first that an expression similar to (15) was derived in Ref. [15] in the context a generalization of SP in HIT. Any length scale and velocity scale satisfying (15) and (16) will be scaling variables. For example, if one takes $l = \lambda$ and $u_2 = u'$ and if Re_λ is constant, then λ and u' are scaling variables in conformity with SP as observed on the centerlines of a round jet flow [18] and a cylinder wake [16] (see also Sec. III).

(i) Recognizing that $\eta = (\nu^3/\overline{\epsilon})^{1/4}$ and $v_K = (\nu\overline{\epsilon})^{1/4}$, the Kolmogorov length and velocity scales, respectively, we can write $l = C_\eta \eta$ and $u_2 = C_{v_K} v_K$. Accordingly, v_K and length η emerge as scaling variables from the analysis where no assumption other than SP has been made. This departs from the common practice where the first similarity hypothesis must be invoked in order to obtain v_K and η from dimensional arguments. We mentioned in the Introduction that prior to enunciating his first similarity hypothesis Kolmogorov introduced a length scale and a time scale, which are identified with η and η/v_K once his hypothesis is introduced. Notice that if one takes $l = \lambda$ and $u_2 = u'$, then $C_6 = 15$, and if Re_λ is constant, then λ and u' are scaling variables too

(ii) Kolmogorov's similarity hypotheses are solely based on empirical grounds based on the idea of the energy cascade phenomenology. Our analysis shows that $\overline{\epsilon}$ and ν emerge as the natural parameters for scaling the second- and third-order moments of δu from the Navier-Stokes equations when the assumption of SP is invoked. This result is interesting because it indicates convincingly that the similarity hypotheses do not lead to any violation of the equations of motion.

(iii) The emergence of $\overline{\epsilon}$ and ν as natural scaling parameters appears to be intrinsically embedded in the Navier-Stokes equations. This makes the enunciation of the first similarity hypothesis by Kolmogorov quite a remarkable achievement. Also, interestingly, the analysis is valid, in principle, at all Reynolds numbers, thus removing the requirement for a high Reynolds number for $\overline{\epsilon}$ and ν to be the scaling parameters. The only requirement is that the scaling Reynolds number Re_l remains constant during decay, regardless of its value. The same cannot be said for the second similarity hypothesis, which appears to require a very large Reynolds number before it can be satisfied. This highlights the clear distinction between the first and second similarity hypotheses and shows that the second similarity cannot be satisfied before the first one, hinting that the dissipative scales are likely to satisfy SP before those in the inertial range.

(iv) The derivation of the expressions (15) and (16), which uses the assumption of SP, illustrates how the first similarity hypothesis is subsumed under the hypothesis of SP. Any set of velocity and length scales that satisfy (15) and (16), respectively, complies with SP. Further, considering that η is the smallest scaling length, it is reasonable to expect that turbulence in the dissipative range would certainly evolve in SP even when SP is achieved over a limited range of scales only. This argument is consistent with the results in Ref. [19], where it is shown that almost irrespectively of the flow, or flow region, the Kolmogorov scaling of velocity spectra holds at Re_λ as low as 30, in the dissipative range.

C. Remarks for the inertial range

We remark that the above analysis does not immediately lead to the second similarity hypothesis, namely, that $\overline{\epsilon}$ is the only scaling parameter in the range of scales much larger than η. It is now well established, both numerically and experimentally, that the first and second terms on the right-hand side of (4) are practically zero in the separation range $\lambda \leqslant r \leqslant L$ (i.e., the inertial range) when the Reynolds number is large. Less information is given for the terms in Eq. (5). However, Ref. [20] showed that the time derivative and the viscous terms of (5) are negligible in the inertial range. Thus, in the inertial range (4) and (5) reduce to the expressions

$$\delta u_3 = -\frac{4}{5}\overline{\epsilon} r, \tag{17}$$

$$\delta u_4 - \frac{6}{r^2}\int_0^r s(\overline{\delta u_2 \delta v_2})ds = \frac{6}{r^2}\int_0^r \left\{ -s^2 \frac{1}{2\rho}\overline{(\delta u)^2\left(\frac{\partial \delta p}{\partial X}\right)} - s^2[\overline{(\delta u)\widetilde{\epsilon}}]\right\} ds, \tag{18}$$

where we defined $\widetilde{\epsilon} = \nu\{(\frac{\partial u}{\partial x})^2 + (\frac{\partial u'}{\partial x'})^2\}$, for convenience. We immediately recognize the four-fifths law in the expression (17). Note also that in the inertial range Eq. (7) reduces to

$$f^{3/2} = \left\{\frac{4}{5}\frac{1}{c_S \phi(r^*)}\left(\frac{\overline{\epsilon} l}{u_2^3}\right)\right\} r^* \tag{19}$$

or

$$u_2^3 f^{3/2} = (\delta u_2)^{3/2} = -\frac{4}{5S}\overline{\epsilon} r, \tag{20}$$

which is the Kolmogorov two-thirds law. The present derivation of this law requires only the existence of an inertial range. This departs from the derivation of Kolmogorov [1], who used empirical scaling arguments after invoking his second similarity hypothesis, to reach this result. Interestingly though, in his second paper [2] he derived the four-fifths law from (4) without relying on the second similarity hypothesis, and by combining (17) with $S\ (=\delta u_3/\delta u_2^{3/2})$, obtained the two-thirds law (20), demonstrating that his similarity hypotheses are indeed not required to obtain both (17) and (20).

Expression (18) warrants some comments. If we define an instantaneous pseudodissipation rate as $\epsilon_{\mathrm{pd}} = \nu(\frac{\partial u}{\partial x})^2$, we can write (18) as

$$\delta u_4 - \frac{6}{r^2}\int_0^r s(\overline{\delta u_2 \delta v_2})ds = \frac{6}{r^2}\int_0^r \left\{ -s^2 \frac{1}{2\rho}\overline{(\delta u)^2\left(\frac{\partial \delta p}{\partial X}\right)} - s^2\overline{(\delta u)(\epsilon_{\mathrm{pd}} + \epsilon'_{\mathrm{pd}})}\right\} ds. \tag{21}$$

Although they did not show the distribution of E_{111}, Hill and Boratav [20] stated that the term is negligible in the inertial range, which, if confirmed (see Sec. V), leads to

$$\delta u_4 - \frac{6}{r^2}\int_0^r s(\overline{\delta u_2 \delta v_2})ds \simeq -\frac{3}{r^2}\int_0^r s^2\left\{\frac{1}{\rho}\overline{(\delta u)^2\left(\frac{\partial \delta p}{\partial X}\right)}\right\} ds. \tag{22}$$

Further, Yakhot [21] argues that, for very a large Reynolds number, the viscous and dissipation terms in the transport equations of odd-order moment δu_{2n+1} must be zero for symmetry reasons. This theoretical prediction by [21] is well confirmed by the high-resolution direct numerical simulation (DNS) data of forced HIT [22]. However, the dissipation term is not zero in the transport equations of even-order moment δu_{2n}. This is well verified for the transport equation for δu_{2n}, but also, as shown recently in Ref. [23], for δu_4 Hill and Boratav [20] showed that on the left-hand side of (2) (see also [22]), the third term behaves like the second term or equivalently that $\frac{6}{r^2}\int_0^r s(\overline{\delta u_2 \delta v_2})ds \sim \overline{\delta u_4}$, which leads to

$$\delta u_4 \simeq -\frac{3C_{\delta u_4}}{r^2}\int_0^r s^2\left\{\overline{\frac{1}{2\rho}(\delta u)^2\left(\frac{\partial \delta p}{\partial X}\right)}\right\}ds \tag{23}$$

and provides an expression for δu_4 in the inertial range, where $C_{\delta u_4}$ is a constant independent of r.

This is in agreement with the DNS of [22], showing that

$$1-\frac{\frac{6}{r}(\overline{\delta u_2 \delta v_2})}{\left(\frac{\partial}{\partial r}+\frac{2}{r}\right)\delta u_4} \simeq \frac{-T_{111}}{\left(\frac{\partial}{\partial r}+\frac{2}{r}\right)\delta u_4} \simeq \text{const.} \tag{24}$$

Expression (23) shows that the scaling of δu_4 reduces to the scaling of the right-hand side of (23). Gotoh and Nakano [22] developed an analytical expression for the integrant of the right-hand side of (23) based on conditional averaging and the Bernoulli equation. Yakhot [24] proposed a modified version of the model in Ref. [22]. However, both models rely on empirical arguments based on internal intermittency, which we briefly discuss below.

We notice that (23) does not involve $\overline{\epsilon}$. Even in Eq. (21) $\overline{\epsilon}$ is missing; it is the product of δu and the sum of ϵ_{pd} and ϵ'_{pd} taken at two locations separated by a distance r that is involved. Now, if we follow [13] we can write the sum of the pressure and dissipation terms in the transport equation for δu_n as

$$T_n + E_n = n\overline{\frac{1}{\rho}(\delta u)^{n-1}\left(\frac{\partial \delta p}{\partial X}\right)} + 2\overline{(\delta u)^{n-2}(\epsilon_{pd}+\epsilon'_{pd})}. \tag{25}$$

Setting $n=2$, the first term on the right-hand side of (25) is zero while the second term becomes $4\overline{\epsilon}$. This draws our attention to Landau's critical remark concerning the effect of fluctuations of ϵ on the small-scale properties of turbulence (see [25]; see also [5] for a detailed account). Kolmogorov proposed a revised version of his initial similarity hypotheses taking Landau's remark into consideration [26]. He based his refined similarity hypotheses on a method proposed by Oboukhov [27], who, on a phenomenological basis, introduced $\overline{\epsilon}_r$, a dissipation rate averaged over a sphere of radius r,

$$\overline{\epsilon}_r(x,t) = \frac{6}{v_r}\int_0^{v_r}\epsilon(x,t)dv_r \tag{26}$$

(the integration is over a sphere v_r of radius r [26]), which can be used to form a Reynolds number Re_r $[\equiv(r^{4/3}\overline{\epsilon}_r^{1/3})/\nu = (r/\eta_r)^{4/3}]$. The second refined similarity states that if Re_r is very large then f_r becomes independent of Re_r, i.e., it is universal, where f_r is the PDF for $\delta u/(r\overline{\epsilon}_r)^{1/3}$, which by virtue of the first refined similarity hypothesis depends only on Re_r when $r \ll L$, where L is a characteristic length of the large-scale motion. These refined similarity hypotheses lead to

$$\delta u_2 \sim (\overline{\epsilon}_r r)^{2/3}\beta_2(\text{Re}_r) \tag{27}$$

and

$$\delta u_3 \sim (\overline{\epsilon}_r r)\beta_3(\text{Re}_r), \tag{28}$$

where $\beta_2(\text{Re}_r)$ and $\beta_3(\text{Re}_r)$ are supposed to be universal functions of the single parameter Re_r. One critical aspect of the formulation of the refined similarity hypotheses, other than the fact that it is based on phenomenological arguments, is $\overline{\epsilon}_r$ and in particular the form of its PDF, which since the work of Kolmogorov [26] has been the object of intensive research. Nevertheless, if one accepts that the fluctuations of ϵ are an intrinsic property of the small-scale turbulence, then one must also accept that the effects of these fluctuations should be reflected in the small-scale statistical results based on a two-point analysis of the Navier-Stokes equations. Interestingly, Landau and Lifshitz [25] noted that "[the] question whether fluctuations of ϵ should be reflected in the form of the correlation functions in the inertial range can scarcely be resolved with certainty until we have a consistent theory of turbulence." Expression (25), which, incidentally, involves correlations between δu and ϵ_{pd} at two separations, does seem to be in line with this comment, but only for the transport equations of the even-order moments of δu. For the odd-order moment transport equations, the contribution from the viscous terms is negligible in the inertial range [13,21,22]. The important point to remark here is that $\overline{\epsilon}_r$ does not appear in any equations, implying that it is not a natural scaling parameter emerging from the Navier-Stokes equations. This corroborates the results in Ref. [23], where it was concluded that "the relation of the specific assumption $\overline{\epsilon}_r^{n/3}$ for the nth-order structure functions in the inertial range to the dissipation parameters derived from the Navier-Stokes equations is missing."

Quite interestingly, the expressions (17) and (18) challenge the need for introducing phenomenological arguments to account for the spatial fluctuations of ϵ on the scaling of δu_n, at least when $n = 3$ and 4, and show in fact that the similarity hypotheses do not conflict with the internal intermittency phenomenon. In hindsight, the warning against generalizing the scaling laws of the second- and third-order moments of δu to higher moments [7] is a call for caution when we try to generalize Kolmogorov theory to $\overline{(\delta u)^n}$ with $n > 3$. Batchelor [7] stated that this generalization is very doubtful, because the dynamical relation between the second- and third-order moments of δu [i.e., Eq. (1)] is the only one in which the pressure does not appear (he expressed the statement in terms of double- and triple-velocity correlation). Expressions (18) and (25) show that such a generalization is *a priori* not possible. Gotoh and Nakano [22] argue that the pressure gradient acts in a way to resist to stretching and squeezing of fluid elements, leading to a weakening of the effect of intermittency. Kraichnan [28] also envisaged this possibility.

A word of caution is warranted regarding the above analysis of the inertial range. The analysis assumes that the Reynolds number is large enough for the inertial range to exist or at least that the contributions from the viscous term and the large-scale term (here represented by the time derivative terms) are negligible in Eqs. (4) and (5). Unless this is verified, both (17) and (18) cannot be correct.

III. EXPERIMENTAL RESULTS

A. Scaling with $\overline{\epsilon}$ and ν

The analysis in Sec. II shows that $\overline{\epsilon}$ and ν are the relevant natural parameters under SP and that any scaling set (l,u_2) must be of the form (15) and (16) (or $l = C_\eta \eta$ and $u_2 = C_{v_K} v_K$). To test the analytical results, one can, for example, plot the ratios λ/η and u'/v_K. First, we note that if we take $l = \lambda$ and $u_2 = u'$ in Eqs. (15) and (16) and combine with (9), we obtain

$$\frac{\lambda}{\eta} = (\sqrt{15}\mathrm{Re}_\lambda)^{1/2}, \tag{29}$$

$$\frac{u'}{v_K} = (\mathrm{Re}_\lambda/\sqrt{15})^{1/2} \tag{30}$$

since under local isotropy condition $C_6 = 15$. Thus, under SP these ratios must be constant as the turbulence decays, since Re_λ is constant under SP. Figure 1 reports such plots where the measurements were carried out on the centerlines of the turbulent round jet [18] and a cylinder wake [16]. While the jet data cover a shorter distance than the wake data, they nevertheless indicate that both λ/η and u'/v_K approach a plateau when $x/D \geqslant 30$ (D is the nozzle diameter). The wake data show clear evidence of a plateau for both λ/η and u'/v_K when $x/D \geqslant 200$ (D is the cylinder diameter). Note that, according to (29) and (30), the ratio $(\lambda/\eta)_w$ for the wake should be equal to $(\mathrm{Re}_{\lambda,j}/\mathrm{Re}_{\lambda,w})^{1/2}(\lambda/\eta)_j$, where the subscripts w and j refer to wake and jet (a similar relation holds for u'/v_K). In the present case $(\mathrm{Re}_{\lambda,j}/\mathrm{Re}_{\lambda,w})^{1/2} = 3$. The presence of a plateau in these distributions indicates that SP is well satisfied over a wide range of separations, but it also vindicates the first similarity hypothesis. This is quite remarkable in particular for the wake flow where Re_λ is about 45, which is relatively low, and certainly not in conformity with the requirement of high Re for the SH1

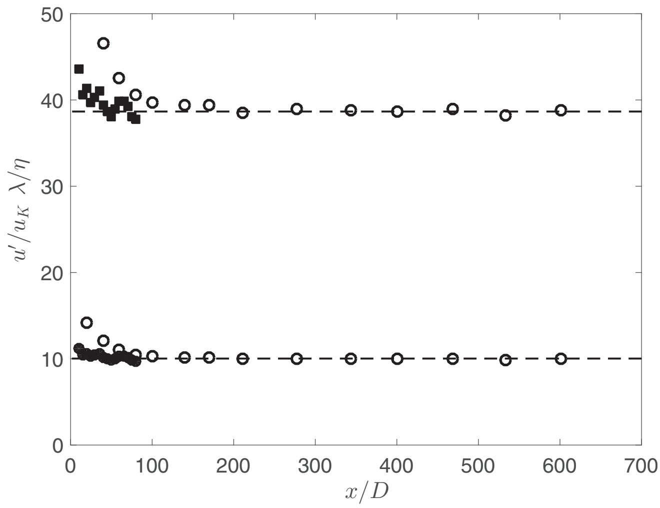

FIG. 1. Variations of λ/η (squares) and u'/v_K (circles) on the centerline of an axisymmetric turbulent jet (closed symbols, $\mathrm{Re}_\lambda \simeq 400$) and the centerline of a cylinder wake (open symbols, $\mathrm{Re}_\lambda \simeq 45$). Note that, following (29) and (30), we multiplied both λ/η and u'/v_K for the wake by the factor $(\mathrm{Re}_{\lambda,j}/\mathrm{Re}_{\lambda,w})^{1/2}$. The dashed straight lines are only used as a visual guide.

to hold. This indicates that SH1 requires a form of SP, as noted by [7], but requires neither Re_λ to be large nor the turbulence to be globally homogeneous and isotropic. This is consistent with [19].

The ratios λ/η and u'/v_K measured in grid turbulence are reported in Fig. 2 for different grid Reynolds numbers ($\mathrm{Re}_M = 37\,500$, $13\,000$, 5800, 4180, and 1200; see [29] for details of the measurements). Both ratios present a systematic decrease with increasing x/M. For $\mathrm{Re}_M = 1200$,

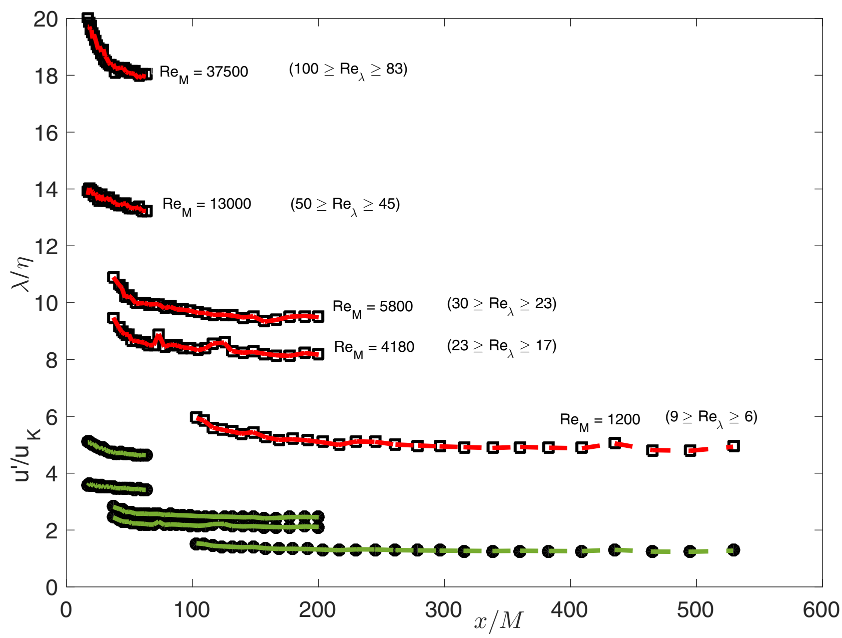

FIG. 2. Variations of λ/η (open squares) and u'/v_K (closed circles) in decaying grid turbulence for several values of Re_M. The red dashed line shows Eq. (29) and the green dashed line Eq. (30).

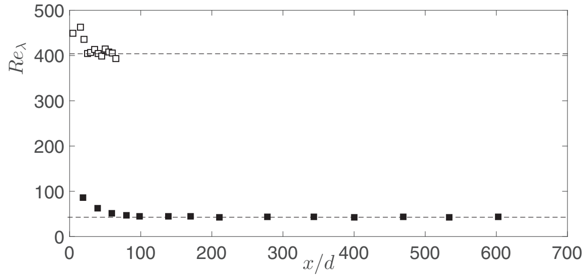

FIG. 3. Variations of Re_λ on the centerline of an axisymmetric turbulent jet (open squares) and the centerline of a cylinder wake (closed squares). The dashed straight lines are only used as a visual guide.

the ratios seem to approach a plateau when $x/M \geqslant 300$. At this Re_M, the values of Re_λ are quite low and it is likely that turbulence approaches its final stage of decay, which is excepted to be in SP [30] (see also [31]). Nevertheless, albeit slow due to the weak Re_λ variations, the variations of λ/η and u'/u_K with increasing x/M become visible when the plotting scales are enlarged. The variations of these ratios are consistent with a lack of SP, which is reflected by the nonconstancy of Re_λ with increasing x/M. The ratios were also calculated using the right-hand sides of (29) and (30) for a self-consistency test. Since λ and u' can be scaling variables complying with SP on the wake and jet centerlines, then Re_λ must be constant, which is well illustrated in Fig. 3. In grid turbulence Re_λ decreases with increasing x/M [29] as can be seen in Fig. 4, reflecting a lack of SP. As for the variations of λ/η and u'/u_K at low Reynolds numbers, the Re_λ variation becomes evident when an enlarged scale is used. Note that for the sake of simplicity of analysis, we have developed the theoretical arguments in the context of HIT. Nonetheless, Figs. 1 and 2 strongly support the applicability of the present theory for flows that are not necessarily isotropic, such as jet and wake centerlines (see also [16,29]). There are two reasons.

(i) The present analysis can be developed for the transport equation of the second-order moments, in flows slightly inhomogeneous but anisotropic, as already underlined in, e.g., [19] (the transport

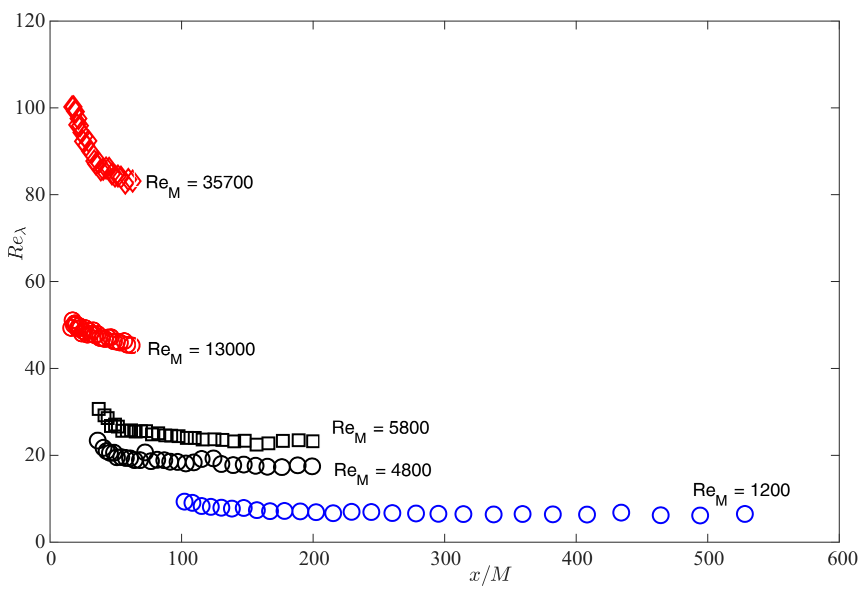

FIG. 4. Variations of Re_λ in decaying grid turbulence for several values of Re_M.

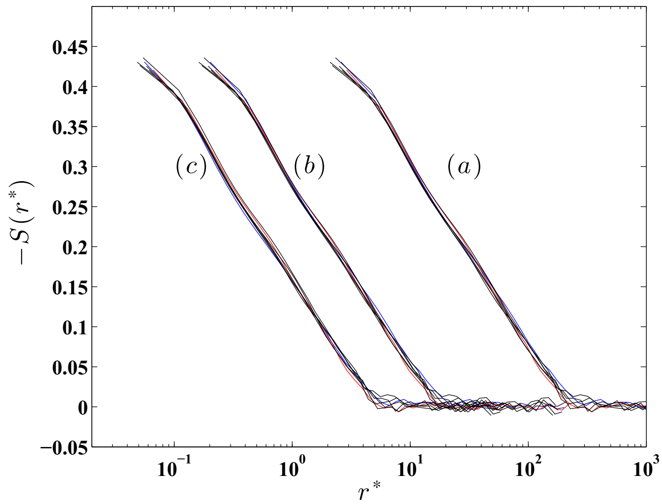

FIG. 5. Skewness of the velocity increments δu on a centerline of a turbulent cylinder far wake over the range $x/D \simeq 200$–600 (there are seven locations equidistributed along that range). The separation r is normalized by η [curve (a)], λ [curve (b)], and L_0 [curve (c)]. Data were extracted from [32].

equation for the third-order moments in anisotropic turbulence is more complex and not considered here). The scaling parameters that emerge from the analysis are then ν and $\overline{\epsilon} + \overline{\epsilon}^+$, the mean dissipation rates at the two space points (the plus superscript represents the position at $x + r$). Along each spatial direction, if turbulence is locally homogeneous, then $\overline{\epsilon} \sim \overline{\epsilon}^+$. For strongly inhomogeneous flows, it is expected that when very small scales are approached, the $\overline{\epsilon}$ and $\overline{\epsilon}^+$ will become equal. Once again, at least for the second-order statistics, the similarity parameters are those deduced by Kolmogorov on phenomenological basis.

(ii) The theory validation we provide here is based on hot-wire measurements, which correspond to selecting one spatial direction, that of the main stream. Taylor's hypothesis was employed to calculate spatial statistics, which are thereof artificially homogeneous. Therefore, our results only concern a single, homogeneous, flow direction x. These are optimal conditions for the theory to be tested.

The first and second conditions of (9) indicate that for a given Re_λ, $S(r^*)$ should collapse onto a single curve as turbulence decays. These SP conditions also apply for a non-HIT such as in a cylinder wake [16] and a jet flow [17]. To test whether the distribution $S(r^*)$ remains unchanged during the decay when SP is achieved we report in Figs. 5 and 6 the distributions of $S(r^*)$ on a centerlines of a cylinder wake (extracted from [16]) and a turbulent round jet (from [17]). In Fig. 7 we report $S(r^*)$ measured in grid turbulence (from [29]). For the cylinder wake data we show the same $S(r)$ distributions where the separation r is normalized, respectively, by η, λ, and L_0 (the half-width wake, which [16] showed to be a scaling length), while for the jet and the grid turbulence we only used η. As it could have been anticipated from the results of Fig. 1, there is a relatively good collapse of all the distributions for the wake and jet flows. For the wake the collapse is the same regardless of the length scale used to normalize r (although not shown here, this is also observed for the jet [see, for example, plots of $\overline{(\delta u)^2}$] and $\overline{(\delta u)^3}$ in Ref. [17]). Notice that the collapse is observed up to the separation r^* where $S(r^*)$ becomes zero, confirming further that SP is satisfied over a very wide range of scales, covering the dissipative, scaling, and large scales. For the grid turbulence (Re_λ varies from about 48.5 to about 45 for $\mathrm{Re}_M = 13\,000$ and from about 19 to about 17 for $\mathrm{Re}_M = 4174$), the $S(r^*)$ distributions do not present a collapse, or at least not as good as for the wake and jet, in agreement with the fact that grid turbulence does not decay in conformity with SP [29]. Note the low values of $S(r^*)$ for the grid turbulence at $\mathrm{Re}_M = 4127$ that express the very weak energy transfer. Djenidi *et al.* [33] showed that the contribution of the energy transfer in the scale-by-scale energy

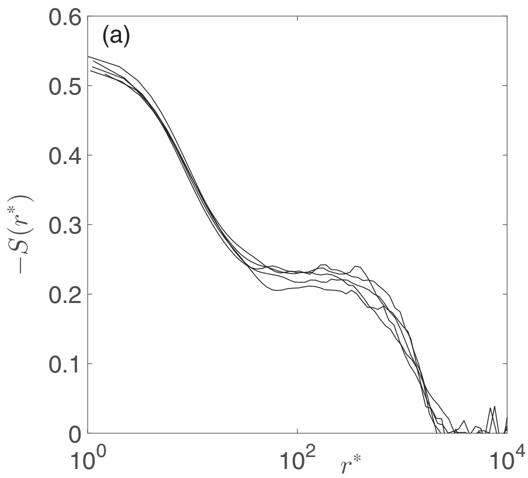

FIG. 6. Skewness of the velocity increments δu on a centerline of a turbulent round jet over the range $x/D \simeq 30$–80 for (a) $\mathrm{Re}_D = 50\,000$ and (b) $\mathrm{Re}_D = 105\,000$. The separation r is normalized by η.

budget becomes smaller than the contributions from the viscous and (large-scale) nonhomogeneous terms at all scales and that the Kolmogorov normalized spectra deviate from those at higher Re_λ, which is indicative of a breakdown of Kolmogorov's first similarity hypothesis.

Worthy of interest is the apparent plateau with a magnitude of about 0.23 in the $S(r^*)$ distributions of the jet flow suggesting a nascent scaling range. Such a plateau is absent from both the wake and grid turbulence because of the low value of Re_λ. Interestingly, Antonia and Burattini [34] showed that for an inertial range to exist Re_λ should exceed about 10^3 when forcing is applied and 10^6 when the turbulence is decaying. Antonia *et al.* [35] compiled a series of $S(r/\eta)$ distributions in several flows that include decaying grid turbulence, fully developed channel flows, and plane and round jets where Re_λ ranges from 33 to about 1000 (see their Fig. 9). None of the curves have a clear inertial range plateau. They only approach it as Re_λ increases. Even $S(r)$ obtained with the eddy-damped quasinormal simulation of decaying HIT at $\mathrm{Re}_\lambda = 25\,000$ do not exhibit an actual plateau [36].

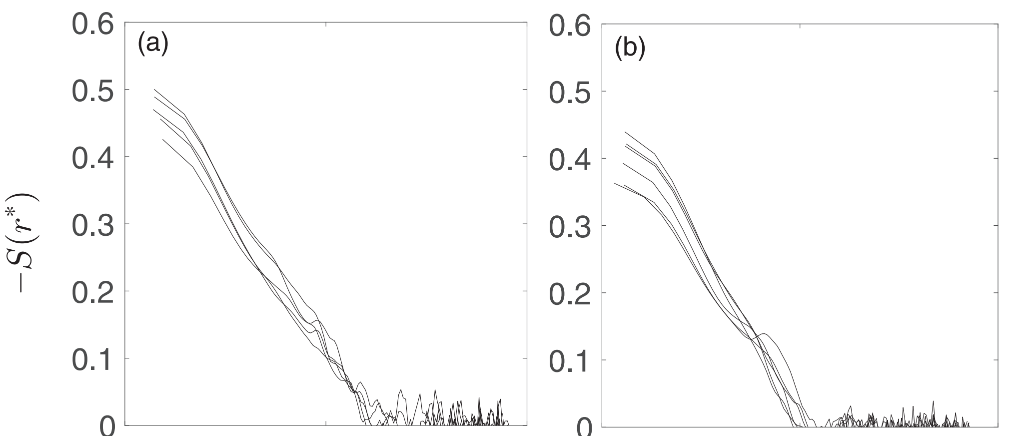

FIG. 7. Skewness of the velocity increments δu in grid turbulence over a range (a) $x/M \simeq 22$–55 for $\mathrm{Re}_M = 13\,000$ and (b) $x/M \simeq 55$–175 for $\mathrm{Re}_M = 4174$. The separation r is normalized by η.

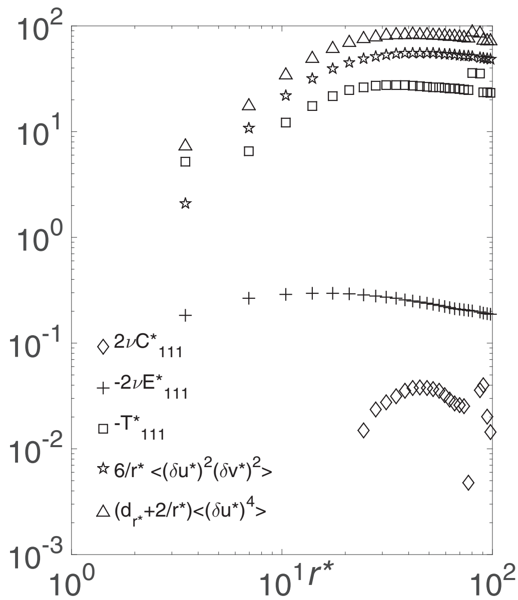

FIG. 8. Budget terms in the transport equation for δu_3 in grid turbulence at $\mathrm{Re}_\lambda = 100$. The asterisk superscript represents Kolmogorov normalization.

These observations vindicate Antonia and Burattini [34] and show that testing the four-fifths law (or two-thirds law) is particularly difficult, if not impossible.

B. Budget of the transport equation (5)

To assess the validity of expression (23), we report in Fig. 8 terms in Eq. (2), for decaying grid turbulence. The first term, reflecting the decay, is negligible and therefore not represented. All the other terms are evaluated and represented in the figure, except the term T_{111}, which is obtained by difference. For convenience, we call $2\nu C_{111}$ the second term of the right-hand side of (2) (i.e., the viscous term). The figure shows that both terms $2\nu C_{111}$ and $-2\nu E_{111}$ are negligible. The first term (called here the transport term) in the curly brackets on the left-hand side of (2) is larger than the second term (called the mixed term). The pressure term $-T_{111}$ is smaller than these two terms, in agreement with the experimental and DNS data of [20], but also with the DNS results of [22] at a higher Reynolds number ($\mathrm{Re}_\lambda = 460$). The ratio of the transport and mixed terms is 1.5 for $\mathrm{Re}_\lambda = 100$; Gotoh and Nakano [22] obtained a ratio of about 1.2.

The remarkable conclusion one can draw from these results is that expression (23) is relatively well verified even when Re_λ is not large, confirming that this expression is valid when $2\nu C_{111}$ and $2\nu E_{111}$ are negligible. Note that in the dissipative range when $r \to 0$ Eq. (23) reduces to $2\nu C_{111} = 2\nu E_{111}$ because these two terms are proportional to r, while the others behave like r^3. However, this is not obvious from the figure, most likely because statistics of very small scales are not reliable. This is due to the spatial resolution limitation, which impacts on the calculation of the derivative of the third-order structure function.

IV. CONCLUDING DISCUSSION

The following result has been obtained by applying a self-preservation analysis to the two-point transport equations of the second- and third-order moments of the longitudinal velocity increments in decaying HIT: The variables ν and $\overline{\epsilon}$ are found to be the controlling parameters for the behavior of the second-order and third-order moments of the velocity increments without invoking the first similarity hypothesis, underlying a relation between SP and the Kolmogorov theory similarity hypotheses. We have shown that ν and $\overline{\epsilon}$ are recovered from the Navier-Stokes equations, with the Reynolds number not required to be very large, but constant in the case of decaying turbulence. As a consequence of the above result, the Kolmogorov length and velocity scales (formed with ν and $\overline{\epsilon}$) emerge as the natural scaling variables compliant with SP. One can then expect that when SP holds at all scales of motion, any set of scaling parameters should be proportional to the Kolmogorov variables. When SP is incomplete, i.e., SP is tenable over a range of scales $\eta \leqslant l \leqslant l_0$ (l_0 is, for example, smaller than the integral length scale L), the analysis indicates that the Kolmogorov scales would be appropriate scaling variables within that range of scales. Measurements on the centerlines of a round jet and a plane wake and in a grid turbulence corroborate these results. These analytical results are also reported in Refs. [31] and [19].

Regarding SH2, one may tentatively argue that it can be considered as a limiting case of the first one when $\nu \to 0$. Of course, this would imply that $\overline{\epsilon}$ remains finite in the limit of infinite Reynolds number, an issue that remains to be verified, although commonly believed to be true. Note that the singularity of the Navier-Stokes equations for $\nu = 0$ is still an open issue.

An interesting observation that stems from the present analysis is that the quantity $\overline{\epsilon}_r$, introduced by Oboukhov [27] and used by Kolmogorov [26] in his refined similarity hypotheses to account for the internal intermittency, does not appear to emerge from the Navier-Stokes equations. Further, it is found that the local value of ϵ, which according to Landau's remark may affect the scaling, is well accounted for in the equations [see, for example, the term E_{111} in Eq. (5); see also [13] for equations of higher-order moments of velocity increments where similar terms appear].

It is important to stress the difference between this present work and the earlier ones (see, e.g., [7,9]). Conversely to these studies, the Kolmogorov scales η and v_K are never introduced or invoked as scaling parameters in our analysis; they emerge as outputs rather than inputs. This result is far from being trivial. Indeed, so far, the scaling based on η and v_K is invariably introduced solely on the basis of SH1. The present study provides mathematical grounds for the Kolmogorov similarity hypotheses, which justify the use of the Kolmogorov scaling, albeit under SP conditions.

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