Ultimate-state transition of turbulent Rayleigh-Bénard convection

Guenter Ahlers,[1,2,*] Eberhard Bodenschatz,[2,3,4,*] and Xiaozhou He[5,*]
[1] *Department of Physics, University of California, Santa Barbara, California 93106, USA*
[2] *Max Planck Institute for Dynamics and Self-Organization (MPIDS), 37077 Göttingen, Germany*
[3] *Institute for Nonlinear Dynamics, University of Göttingen, 37077 Göttingen, Germany*
[4] *Laboratory of Atomic and Solid-State Physics and Sibley School of Mechanical and Aerospace Engineering, Cornell University, Ithaca, New York 14853, USA*
[5] *Institute for Turbulence-Noise-Vibration Interaction and Control, Shenzhen Graduate School, Harbin Institute of Technology, Shenzhen, China*
(Received 6 January 2017; published 3 May 2017)

Recently Schumacher *et al.* [Phys. Rev. Fluids **1**, 084402 (2016)] used direct numerical simulation to calculate the shear stress exerted on the top and bottom viscous boundary layers (BLs) of Rayleigh-Bénard convection with a Prandtl number $\mathrm{Pr} = 0.021$ and aspect ration $\Gamma = 1$ for Rayleigh numbers Ra up to 4×10^8. By extrapolating their results to larger Ra, they concluded that the sample would undergo a transition to turbulent BLs and enter the "ultimate state" at $\mathrm{Ra}^* \simeq 10^{11}$ for $\mathrm{Pr} = 0.021$. Here we show that their result is consistent with the experimentally determined $\mathrm{Ra}^* = 2 \times 10^{13}$ for $\mathrm{Pr} = 0.82$ by He *et al.* [Phys. Rev. Lett. **108**, 024502 (2012); New J. Phys. **17**, 063028 (2015)] and the Pr dependence of Ra^* predicted by Grossmann and Lohse [Phys. Rev. E **66**, 016305 (2002)]. Thus the numerical results of Schumacher *et al.* support the interpretation of the experimentally observed transition at $\mathrm{Ra}^* = 2 \times 10^{13}$ for $\mathrm{Pr} = 0.82$ as the ultimate-state transition.

DOI: 10.1103/PhysRevFluids.2.054603

I. INTRODUCTION

Recently Schumacher *et al.* [1] (SBPS) carried out direct numerical simulations (DNS) of turbulent Rayleigh-Bénard convection (RBC; convection in a fluid confined between horizontal parallel plates and heated from below) for a cylindrical sample with aspect ratio $\Gamma \equiv D/L$ (D and L are the diameter and height respectively) equal to 1 (for recent reviews of this system see, e.g., [2–4]). The properties of RBC depend significantly on the Prandtl number $\mathrm{Pr} \equiv \nu/\kappa$ (ν and κ are the kinematic viscosity and thermal diffusivity of the fluid respectively), and SBPS studied the case $\mathrm{Pr} = 0.021$.

The temperature difference ΔT applied to a RBC sample is represented in dimensionless form by the Rayleigh number $\mathrm{Ra} \equiv \alpha g L^3 \Delta T/(\kappa\nu)$, where α is the isobaric thermal expansion coefficient and g is the gravitational acceleration. When Ra is large enough, the sample has a nearly isothermal turbulent interior (the "bulk") extending over most of the sample height. In addition to vigorous thermal and velocity fluctuations, the bulk supports a large-scale circulation (LSC) which, for $\Gamma \simeq 1$, consists of a single convection roll. When Ra is not too large ($\mathrm{Ra} < \mathrm{Ra}^*$), thin laminar (albeit fluctuating) viscous and thermal boundary layers (BLs) are found below the top and above the bottom plate, and most of ΔT is sustained by the thermal BLs. This state is known as classical RBC.

It was suggested by Kraichnan [5] (see also [6,7]) that the fluctuations in the turbulent bulk, when vigorous enough, will apply sufficient shear to the (initially laminar) BLs to drive them turbulent as well. A major contribution to the shear will come from the LSC; but fluctuations on somewhat smaller scales will also contribute. A transition is thus expected at $\mathrm{Ra} = \mathrm{Ra}^*$ to a new state which is known as the "ultimate" state [8] since it is expected to exist asymptotically as Ra diverges.

---

*Member of the International Collaboration for Turbulence Research.

 ©2017 American Physical Society

An estimate by Grossmann and Lohse (GL) [9,10] yielded $\mathrm{Ra}^* = \mathcal{O}(10^{14})$ for $\mathrm{Pr} = 1$. For Pr near 1 and large Ra, one expects that the thermal and the viscous dissipation are much larger in the bulk than they are in the boundary layers and that the viscous BL thickness λ_u near the plates is larger than the thickness λ_θ of the thermal BLs (see, e.g., Fig. 2 of [10]). For that case (regime IV_u, see Table 3 of [9]) GL predicted a strong Pr dependence of Ra* given by $\mathrm{Ra}^* \propto \mathrm{Pr}^{3/2}$. Because of the influence of neighboring regimes in parameter space (e.g., regime IV_l with $\lambda_u < \lambda_\theta$), one expects that the value of the exponent may differ slightly from 3/2 and to have an effective value valid only over a finite range of Ra and Pr; but for Pr near 1 and at large Ra this effective value is indeed quite close to 1.5.

Early heat-transport measurements using cryogenic helium gas [8,11,12] with $\mathrm{Pr} \simeq 1$ indicated transitions at Ra values Ra_t in the range $10^{11} \lesssim \mathrm{Ra}_t \lesssim 10^{12}$ which were well below the theoretical predictions for Ra* but which were interpreted by the authors as the ultimate-state transition. Other measurements with helium at low temperatures [13,14] were unable to confirm these results. In contradistinction, recent measurements obtained in the "Uboot of Göttingen" using compressed sulfur hexafluoride (SF_6) at ambient temperatures ($\mathrm{Pr} = 0.82$) [15–19] (the "Göttingen experiments") found a transition at $\mathrm{Ra}^* \simeq 2 \times 10^{13}$ for $\Gamma = 1.00$ [18], consistent with the GL estimate for Ra*.

One of the results of SBPS was the shear stress applied by the LSC to the central sections of the viscous BLs over the range $3 \times 10^5 \leqslant \mathrm{Ra} \leqslant 4 \times 10^8$. By extrapolation to larger Ra and comparison with channel-flow results [20] the authors conclude that the ultimate-state transition would be reached near $\mathrm{Ra}^* = 1 \times 10^{11}$ for their $\mathrm{Pr} = 0.021$. They estimate that the uncertainty of their extrapolation is such that $3 \times 10^{10} \lesssim \mathrm{Ra}^* \lesssim 5 \times 10^{11}$. The purpose of this paper is to point out that this result disagrees with the transitions reported in the range $10^{11} \lesssim \mathrm{Ra}_t \lesssim 10^{12}$ for $\mathrm{Pr} \simeq 1$ [8,11,12] and any reasonable estimate of the Pr dependence of Ra*, and that it is in remarkably good agreement with the experimental finding $\mathrm{Ra}^* = 2 \times 10^{13}$ [18] for $\mathrm{Pr} = 0.82$ and the Pr dependence of Ra* predicted by GL [10].

II. ESTIMATE OF Ra*(Pr) BASED ON Ra*(0.82) = 2 × 10^13 AND THE GROSSMANN-LOHSE MODEL

One way to estimate Ra* at any Pr is to use the GL prediction

$$\mathrm{Ra}^* \propto \mathrm{Pr}^{3/2} \tag{1}$$

for their regime IV_u. With $\mathrm{Ra}^* = 2 \times 10^{13}$ for $\mathrm{Pr} = 0.82$ this yields $\mathrm{Ra}^* = 8 \times 10^{10}$ for $\mathrm{Pr} = 0.021$, in excellent agreement with the SBPS estimate. A similar extrapolation based on the transition values in the range from 10^{11} to 10^{12} observed in the cryogenic experiments [8,11,12] with $\mathrm{Pr} \simeq 1$ clearly would fall well below the estimate of SBPS. The analysis reported in Ref. [16] of five sets of measurements [11,12] with $0.97 \lesssim \mathrm{Pr} \lesssim 1.74$ that yielded well defined transition points gave the averaged results $\mathrm{Ra}_t = 4.4 \times 10^{11}$ and $\mathrm{Pr} = 1.42$. Using this result and Eq. (1) gives $\mathrm{Ra}^* = 8 \times 10^8$, about two orders of magnitude lower than the SBPS value.

It may be argued that the above analysis is not strictly valid because the pure exponent 3/2 of regime IV_u of GL is not valid over the involved Pr and Ra range, and that consideration of the effective values of relevant exponents may give a different result. To explore this possibility, we consider the shear Reynolds number Re_s of the viscous boundary layers directly within the GL model without resorting to any power-law dependence. The shear Reynolds number is related to the bulk Reynolds number Re of the LSC flow field adjacent to the BLs by [10]

$$\mathrm{Re}_s = a\mathrm{Re}^{1/2}\,. \tag{2}$$

Here a is a constant with a value that is somewhat uncertain and variously was estimated to range from 0.25 [10] to nearly 1 [21]. In analogy to wall-bounded shear flow [22], the viscous BLs are expected to undergo a transition from a laminar to a turbulent state when Re_s reaches a critical value Re_s^*. Initially Re_s^* was estimated to be 420 [10,23], but this value also is not known very well [10,22].

In order to estimate the value Re^* of Re and the corresponding Rayleigh number Ra^* at the ultimate-state transition for any Pr, we first need the Ra- and Pr-independent ratio Re_s^*/a. In view of the large uncertainties of both a and Re_s^*, we used the GL model in the form of Eqs. (2.1) and (2.2) of Ref. [21] with the coefficients given in that reference and the experimental result $\mathrm{Ra}^* = 2 \times 10^{13}$ for $\mathrm{Pr} = 0.82$ [18]. This yielded $\mathrm{Re}^* = 3.2 \times 10^5$ and [with Eq. (2)] $\mathrm{Re}_s^*/a = 562$. We note that, with a somewhat less than 1, this corresponds to a reasonable value for Re_s^*.

We can now use the GL model for any Pr to determine the value of Ra that yields a value of Re such that Eq. (2) yields $\mathrm{Re}_s^*/a = 562$. For $\mathrm{Pr} = 0.021$ we find $\mathrm{Ra}^* = 9.7 \times 10^{10}$. This result is nearly the same as the one based on the pure power law, Eq. (1), for regime IV_u. It is in the middle of the range $3 \times 10^{10} \lesssim \mathrm{Ra}^* \lesssim 5 \times 10^{11}$ estimated by SBPS from their extrapolation.

A similar analysis based on the values $\mathrm{Ra}_t = 4.4 \times 10^{11}$ and $\mathrm{Pr} = 1.42$ [16] corresponding to the cryogenic experiments [8,11,12] gives $\mathrm{Re}^* = 4.0 \times 10^4$ and $\mathrm{Re}_s^*/a = 201$. We note that, even with a as large as 1, this corresponds to a rather small value for Re_s^*. For $\mathrm{Pr} = 0.021$ this analysis then yields $\mathrm{Ra}^* = 1.1 \times 10^9$, only slightly larger than the value 8×10^8 based on Eq. (1). As mentioned, either result is well below the lower limit allowed by the SBPS estimate.

III. SUMMARY AND CONCLUSION

In this paper we used the Grossmann-Lohse model [9,10,21] and the experimental value $\mathrm{Ra}^* = 2 \times 10^{13}$ from the Göttingen experiment [18] for the ultimate-state transition at $\mathrm{Pr} = 0.82$ and $\Gamma = 1.00$ to estimate the value $\mathrm{Ra}^* = 9.7 \times 10^{10}$ for $\mathrm{Pr} = 0.021$. This result agrees well with the numerical estimate $3 \times 10^{10} \lesssim \mathrm{Ra}^* \lesssim 5 \times 10^{11}$ for $\mathrm{Pr} = 0.021$ of SBPS [1]. This agreement lends support to the claim that the transition observed in the Göttingen experiment is indeed the transition to the ultimate state of RBC. On the other hand, a similar extrapolation of the cryogenic results [8,11,12] yields a transition at $\mathrm{Ra} \simeq 1.1 \times 10^9$ for $\mathrm{Pr} = 0.021$, which is inconsistent with the numerical work of SBPS and the GL model.

---

[1] J. Schumacher, V. Bandaru, A. Pandey, and J. D. Scheel, Transitional boundary layers in low-Prandtl-number convection, Phys. Rev. Fluids **1**, 084402 (2016).

[2] G. Ahlers, S. Grossmann, and D. Lohse, Heat transfer and large scale dynamics in turbulent Rayleigh-Bénard convection, Rev. Mod. Phys. **81**, 503 (2009).

[3] D. Lohse and K.-Q. Xia, Small-scale properties of turbulent Rayleigh-Bénard convection, Annu. Rev. Fluid Mech. **42**, 335 (2010).

[4] F. Chillà and J. Schumacher, New perspectives in turbulent Rayleigh-Bénard convection, Eur. Phys. J. E **35**, 58 (2012).

[5] R. Kraichnan, Small-scale structure of a scalar field convected by turbulence, Phys. Fluids **11**, 945 (1968).

[6] E. A. Spiegel, Convection in stars, Annu. Rev. Astron. Astrophys. **9**, 323 (1971).

[7] S. Grossmann and D. Lohse, Multiple scaling in the ultimate regime of thermal convection, Phys. Fluids **23**, 045108 (2011).

[8] X. Chavanne, F. Chillà, B. Castaing, B. Hébral, B. Chabaud, and J. Chaussy, Observation of the Ultimate Regime in Rayleigh-Bénard Convection, Phys. Rev. Lett. **79**, 3648 (1997).

[9] S. Grossmann and D. Lohse, Scaling in thermal convection: A unifying view, J. Fluid Mech. **407**, 27 (2000).

[10] S. Grossmann and D. Lohse, Prandtl and Rayleigh number dependence of the Reynolds number in turbulent thermal convection, Phys. Rev. E **66**, 016305 (2002).

[11] X. Chavanne, F. Chilla, B. Chabaud, B. Castaing, and B. Hebral, Turbulent Rayleigh-Bénard convection in gaseous and liquid He, Phys. Fluids **13**, 1300 (2001).

[12] P.-E. Roche, F. Gauthier, R. Kaiser, and J. Salort, On the triggering of the Ultimate Regime of convection, New J. Phys. **12**, 085014 (2010).
[13] X. Z. Wu, Ph.D. thesis, University of Chicago, 1991.
[14] J. J. Niemela, L. Skrbek, K. R. Sreenivasan, and R. Donnelly, Turbulent convection at very high Rayleigh numbers, Nature (London) **404**, 837 (2000).
[15] X. He, D. Funfschilling, H. Nobach, E. Bodenschatz, and G. Ahlers, Transition to the Ultimate State of Turbulent Rayleigh-Beńard Convection, Phys. Rev. Lett. **108**, 024502 (2012).
[16] G. Ahlers, X. He, D. Funfschilling, and E. Bodenschatz, Heat transport by turbulent Rayleigh-Bénard convection for $Pr \simeq 0.8$ and $3 \times 10^{12} \lesssim Ra \lesssim 10^{15}$: Aspect ratio $\Gamma = 0.50$, New J. Phys. **14**, 103012 (2012).
[17] X. He, D. Funfschilling, E. Bodenschatz, and G. Ahlers, Heat transport by turbulent Rayleigh-Bénard convection for $Pr \simeq 0.8$ and $4 \times 10^{11} \lesssim Ra \lesssim 2 \times 10^{14}$: Ultimate-state transition for aspect ratio $\Gamma = 1.00$, New J. Phys. **14**, 063030 (2012).
[18] X. He, D. van Gils, E. Bodenschatz, and G. Ahlers, Reynolds numbers and the elliptic approximation near the ultimate state of turbulent Rayleigh-Bénard convection, New J. Phys. **17**, 063028 (2015).
[19] X. He, E. Bodenschatz, and G. Ahlers, Azimuthal diffusion of the large-scale-circulation plane, and absence of significant non-Boussinesq effects, in turbulent convection near the ultimate-state transition, J. Fluid Mech. **791**, R3 (2016).
[20] J. Kim, P. Moin, and R. Moser, Turbulence statistics in fully developed channel flow at low Reynolds number, J. Fluid Mech. **177**, 133 (1987).
[21] J. Stevens, E. van der Poel, S. Grossmann, and D. Lohse, The unifying theory of scaling in thermal convection: The updated prefactors, J. Fluid Mech. **730**, 295 (2013).
[22] A. J. Smits, B. McKeon, and I. Marusic, High-Reynolds Number Wall Turbulence, Annu. Rev. Fluid Mech. **43**, 353 (2011).
[23] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Pergamon, Oxford, 1963).