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Reciprocal theorem for convective heat and mass transfer from a particle in Stokes and potential flows

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In the study of convective heat and mass transfer from a particle, key quantities of interest are usually the average rate of transfer and the mean distribution of the scalar (i.e., temperature or concentration) at the particle surface. Calculating these quantities using conventional equations requires detailed knowledge of the scalar field, which is available predominantly for problems involving uniform scalar and flux boundary conditions. Here we derive a reciprocal relation between two diffusing scalars that are advected by oppositely driven Stokes or potential flows whose streamline configurations are identical. This relation leads to alternative expressions for the aforementioned average quantities based on the solution of the scalar field for uniform surface conditions. We exemplify our results via two applications: (i) heat transfer from a sphere with nonuniform boundary conditions in Stokes flow at small Péclet numbers and (ii) extension of Brenner's theorem for the invariance of heat transfer rate to flow reversal.

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I. INTRODUCTION

Since the seminal works of Acrivos and co-workers [1–4] and Brenner [5–7], there has been significant theoretical progress in the area of transport phenomena in creeping and potential flows (see, e.g., [8]). However, the vast majority of analytical results pertaining to convective heat and mass transfer from an object have been derived under the assumption of a uniform boundary condition on the surface of the particle. This limits the applicability range of such results, as in many practical applications convective transfer takes place from particles with nonuniform surface conditions.

In this Rapid Communication, we develop a reciprocal theorem for convective heat and mass transfer from an arbitrarily shaped particle in streaming Stokes and potential flows. The theorem establishes a reciprocal relation between two scalars (i.e., two temperature or two concentration fields) whose advection velocities differ by only a negative sign. This relation results in alternative formulas for the average heat and mass transfer from the particle and its mean surface temperature and concentration, all of which are typical average quantities of interest in heat and mass transfer problems. These formulas readily extend some of the existing analytical results to accommodate nonuniformities in boundary conditions without solving for new scalar fields. In this regard, the theorem may be viewed as heat and mass transfer counterpart of the Lorentz reciprocal theorem of hydrodynamics (see, e.g., [9]), which has been used for calculating drag, torque, and propulsion speed in Stokes flow without developing detailed flow fields (see, e.g., [10–15]).

Below we begin by explaining the derivation of the reciprocal theorem and subsequent alternative equations. Then we discuss two applications, the first of which is the heat transfer from a sphere in Stokes flow in the limit of small Péclet number followed by the extension of Brenner's flow reversal theorem [6,7]. Finally, we point out several possible generalizations of our work.

II. DERIVATION OF THE RECIPROCAL THEOREM AND ALTERNATIVE FORMULAS

Consider an unbounded steady Stokes or potential flow with velocity $\boldsymbol{u}$ past a stationary impermeable particle of arbitrary geometry. Let η and φ be two diffusing scalar fields that vanish at

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infinity and are advected, respectively, by the velocity fields $\boldsymbol{u}$ and $-\boldsymbol{u}$. Then, neglecting source terms and assuming that the fluid properties are constant, their Laplace transforms $\tilde{\eta}$ and $\tilde{\varphi}$ satisfy

$$\mathfrak{s}\tilde{\eta} - \eta_0 + \boldsymbol{u} \cdot \nabla\tilde{\eta} = D\nabla^2\tilde{\eta}, \tag{1a}$$

$$\mathfrak{s}\tilde{\varphi} - \varphi_0 - \boldsymbol{u} \cdot \nabla\tilde{\varphi} = D\nabla^2\tilde{\varphi}, \tag{1b}$$

where D is a constant denoting the diffusion coefficient, $\mathfrak{s}$ is the Laplace variable with respect to time t, and η_0 and φ_0 are initial conditions at $t = 0$. Multiplying Eq. (1a) by $\tilde{\varphi}$ and Eq. (1b) by $\tilde{\eta}$ and subtracting the resulting equations yield

$$\tilde{\eta}\varphi_0 - \tilde{\varphi}\eta_0 + \nabla \cdot (\tilde{\eta}\tilde{\varphi}\boldsymbol{u}) = D(\tilde{\varphi}\nabla^2\tilde{\eta} - \tilde{\eta}\nabla^2\tilde{\varphi}). \tag{2}$$

According to Green's second identity[1]

$$\int_V (\tilde{\varphi}\nabla^2\tilde{\eta} - \tilde{\eta}\nabla^2\tilde{\varphi})dV = \int_{S_p+S_\infty} [\tilde{\varphi}(\boldsymbol{n} \cdot \nabla\tilde{\eta}) - \tilde{\eta}(\boldsymbol{n} \cdot \nabla\tilde{\varphi})]dS, \tag{3}$$

where V, S_p, and S_∞ denote the domain volume, surface of the particle, and bounding surfaces at infinity, respectively. Substituting Eq. (2) into Eq. (3) and applying the divergence theorem, we obtain

$$\int_V (\tilde{\eta}\varphi_0 - \tilde{\varphi}\eta_0)dV + \int_{S_p} \tilde{\eta}\tilde{\varphi}(\boldsymbol{n} \cdot \boldsymbol{u})dS = \int_{S_p} D[\tilde{\varphi}(\boldsymbol{n} \cdot \nabla\tilde{\eta}) - \tilde{\eta}(\boldsymbol{n} \cdot \nabla\tilde{\varphi})]dS. \tag{4}$$

Integrals over S_∞ are zero since $\tilde{\eta}\tilde{\varphi}$, $\tilde{\varphi}(\boldsymbol{n} \cdot \nabla\tilde{\eta})$, and $\tilde{\eta}(\boldsymbol{n} \cdot \nabla\tilde{\varphi})$ decay faster than the inverse distance squared in the far field [see also the derivation of Eqs. (2.10)–(2.20) in Ref. [6]].

Given $\eta_0 = \varphi_0 = 0$ everywhere in V except on S_p, the first integral on the left-hand side of Eq. (4) is zero. The second integral also vanishes since $\boldsymbol{n} \cdot \boldsymbol{u}$ is zero on S_p due to the impenetrability condition. Hence,

$$\int_{S_p} \tilde{\varphi}(\boldsymbol{n} \cdot \nabla\tilde{\eta})dS = \int_{S_p} \tilde{\eta}(\boldsymbol{n} \cdot \nabla\tilde{\varphi})dS. \tag{5}$$

Equation (5) establishes a reciprocal relation between η and φ in the Laplace domain. In the following we show that this equation leads to alternative expressions for two average quantities that are typically sought in heat and mass transfer problems. We use η to denote the solution to problems with variations on the surface, whereas we let φ represent the solution to auxiliary problems with uniform boundary conditions.

Assuming $\tilde{\varphi}$ is uniform on S_p, from Eq. (5) we have

$$\bar{\tilde{q}}_\eta = \left(\frac{1}{\bar{S}_p\tilde{\varphi}_s}\right)\int_{S_p} \tilde{\eta}_s\tilde{q}_\varphi dS, \tag{6}$$

with $\bar{\tilde{q}}_\eta\bar{S}_p = -\mathscr{D}\int_{S_p}(\boldsymbol{n} \cdot \nabla\tilde{\eta})dS$ and $\tilde{q}_\varphi = -\mathscr{D}(\boldsymbol{n} \cdot \nabla\tilde{\varphi})$. Here $\mathscr{D}$ is the proportionality constant that relates the flux to the normal gradient and $\bar{S}_p$ is the surface area of the particle. Also, the subscript s denotes a surface value and the overbar represents a surface average. The left-hand side of Eq. (6) represents the Laplace transform of the time-varying average convective flux from the particle corresponding to an arbitrarily nonuniform surface distribution $\tilde{\eta}_s$. The right-hand side of Eq. (6)

[1] See [16,17] for other applications of Green's second identity in heat transfer.

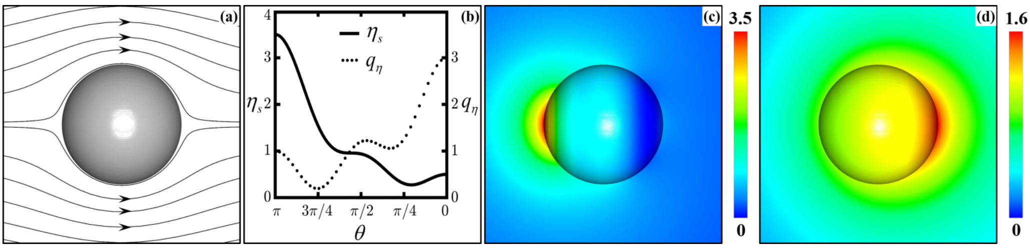

FIG. 1. Uniform Stokes flow past a sphere with varying temperature and heat flux boundary conditions. (a) Flow streamlines directed from the left to the right in the meridian plane. (b) Distributions of the prescribed surface temperature η_s and heat flux q_η corresponding to the normalized temperature fields illustrated in (c) and (d), respectively. Here η_s and q_η are azimuthally independent, i.e., they are not functions of ϕ. Also, Pe = 0.5 and the mean boundary values $\bar{\eta}_s = \bar{q}_\eta = 1$.

is a surface integral that only involves $\tilde{\eta}_s$ and the local convective flux $\tilde{q}_\varphi$. The latter denotes the transfer due to the reversed flow $-\boldsymbol{u}$ and subject to a uniform surface distribution $\tilde{\varphi}_s$.

Similarly, assuming $\tilde{q}_\varphi$ is uniform on S_p, i.e., $\boldsymbol{n} \cdot \nabla\tilde{\varphi}$ is uniform, we obtain

$$\bar{\tilde{\eta}}_s = \left(\frac{1}{\bar{S}_p \tilde{q}_\varphi}\right) \int_{S_p} \tilde{q}_\eta \tilde{\varphi}_s dS, \tag{7}$$

where $\bar{\tilde{\eta}}_s \bar{S}_p = \int_{S_p} \tilde{\eta}\, dS$, with $\bar{\tilde{\eta}}_s$ being the instantaneous mean distribution of $\tilde{\eta}$ on S_p associated with a nonuniform flux $\tilde{q}_\eta$. The right-hand side of Eq. (7) involves the integral over S_p of the varying flux boundary condition $\tilde{q}_\eta$ times the surface distribution $\tilde{\varphi}_s$, which corresponds to the flow in the opposite direction (i.e., $-\boldsymbol{u}$) and uniform surface flux $\tilde{q}_\varphi$.

Simply put, Eqs. (6) and (7) relate the average quantities of interest $\bar{\tilde{q}}_\eta$ and $\bar{\tilde{\eta}}_s$ to the boundary information $\tilde{\eta}_s$ and $\tilde{q}_\eta$, respectively, and the solution of a simpler problem φ, which is often already known. Remember that conventionally $\bar{\tilde{q}}_\eta$ and $\bar{\tilde{\eta}}_s$ are directly calculated from $\int_{S_p}(\boldsymbol{n} \cdot \nabla\tilde{\eta})dS$ and $\int_{S_p} \tilde{\eta}\, dS$, respectively. However, evaluating these integrals requires detailed knowledge of η, which is analytically much more challenging to obtain than φ. Next we discuss illustrative applications of Eqs. (6) and (7).

III. HEAT TRANSFER FROM A SPHERE IN STOKES FLOW AT LOW PÉCLET NUMBERS

Consider a uniform Stokes flow past a sphere of radius a whose far field velocity is $\boldsymbol{u}_\infty = u_\infty \boldsymbol{e}_z$, where $u_\infty = |\boldsymbol{u}_\infty|$ is a constant and $\boldsymbol{e}_z$ is the unit vector in the z direction (see Fig. 1). Here the positive z direction corresponds to $\theta = 0$ in the spherical coordinates (r, θ, and ϕ). First, we examine the heat transfer due to a time-independent surface temperature, which can be written in a general form as

$$\eta_s = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_l^m e^{im\phi} P_l^m(\mu), \tag{8}$$

where $\{A_l^m\}$ and $\{P_l^m\}$ are constant coefficients and associated Legendre polynomials of degree l and order m, respectively, and $\mu = \cos\theta$. Let φ be the solution of the temperature field that satisfies $\varphi_s = \text{const}$ and is advected by $-\boldsymbol{u}$. Following Acrivos and Taylor [1],

$$(\boldsymbol{n} \cdot \nabla\varphi)|_{r=a} = \left(\frac{\varphi_s}{a}\right)\left[\sum_{k=0}^{2} f_k(\text{Pe}) P_k^0(\mu) + O(\text{Pe}^3)\right], \tag{9}$$

where $\mathrm{Pe} = u_\infty a/D$ is the Péclet number (here D is the thermal diffusivity of the fluid),

$$f_0 = -1 - \frac{1}{2}\left(\mathrm{Pe} + \frac{1}{2}\mathrm{Pe}^3 \ln \mathrm{Pe}\right) - \frac{1}{2}\mathrm{Pe}^2 \ln \mathrm{Pe} - \mathrm{Pe}^2\left(\frac{\gamma}{2} + \frac{121}{960}\right), \tag{10a}$$

$$f_1 = -\frac{3}{8}\left(\mathrm{Pe} + \frac{1}{2}\mathrm{Pe}^3 \ln \mathrm{Pe}\right) + \frac{9}{16}\mathrm{Pe}^2, \tag{10b}$$

$$f_2 = \frac{33}{448}\mathrm{Pe}^2, \tag{10c}$$

and γ is the Euler constant. Equation (9) is derived under the assumption of small Pe and vanishing viscous dissipation. Since the problem is steady state, we can drop the tilde from Eqs. (6) and (7). Hence, substituting Eqs. (8) and (9) into Eq. (6), we find

$$\frac{\bar{q}_\eta a}{\mathscr{D} A_0^0} = -\left(f_0 + \frac{A_1^0}{3A_0^0} f_1 + \frac{A_2^0}{5A_0^0} f_2\right) + O(\mathrm{Pe}^3), \tag{11}$$

where $\mathscr{D}$ is the thermal conductivity of the fluid. Remarkably, we see that, at small Pe, only the first three coefficients corresponding to $m = 0$ in Eq. (8) (including A_0^0, which is equal to the average surface temperature $\bar{\eta}_s$) contribute to q_η. These coefficients are related to, respectively, the heat monopole, dipole, and quadrupole.

Following a similar procedure, we can find $\bar{\eta}_s$ resulting from a time-independent nonuniform flux distribution

$$q_\eta = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} B_l^m e^{im\phi} P_l^m(\mu), \tag{12}$$

with $\{B_l^m\}$ being constant coefficients. Only this time, we use the low-Péclect-number solution of Bell *et al.* [18] for a constant heat flux q_φ. Accordingly,

$$\varphi_s = \left(\frac{q_\varphi a}{\mathscr{D}}\right)\left[\sum_{k=0}^{2} g_k(\mathrm{Pe}) P_k^0(\mu) + O(\mathrm{Pe}^3 \ln \mathrm{Pe})\right], \tag{13}$$

where

$$g_0 = 1 - \frac{1}{2}\mathrm{Pe} - \frac{1}{2}\mathrm{Pe}^2 \ln \mathrm{Pe} + \left(\frac{193}{1920} - \frac{\gamma}{2}\right)\mathrm{Pe}^2, \tag{14a}$$

$$g_1 = -\left(\frac{3}{16}\mathrm{Pe} - \frac{3}{8}\mathrm{Pe}^2\right), \tag{14b}$$

$$g_2 = \frac{29}{896}\mathrm{Pe}^2. \tag{14c}$$

Replacing q_η and φ_s in Eq. (7) yields

$$\frac{\bar{\eta}_s \mathscr{D}}{a B_0^0} = g_0 + \frac{B_1^0}{3B_0^0} g_1 + \frac{B_2^0}{5B_0^0} g_2 + O(\mathrm{Pe}^3 \ln \mathrm{Pe}). \tag{15}$$

As expected, the same three coefficients (i.e., the average heat flux B_0^0, B_1^0, and B_2^0) appear in Eq. (15). Thus, in this problem, the mean surface temperature is independent of the detailed flux distribution in the ϕ direction, as $\{B_l^m\}$ for $m \geqslant 1$ do not contribute to $\bar{\eta}_s$ [see also Eq. (12)]. By the same token, the total rate of heat transfer does not depend on the detailed azimuthal distribution

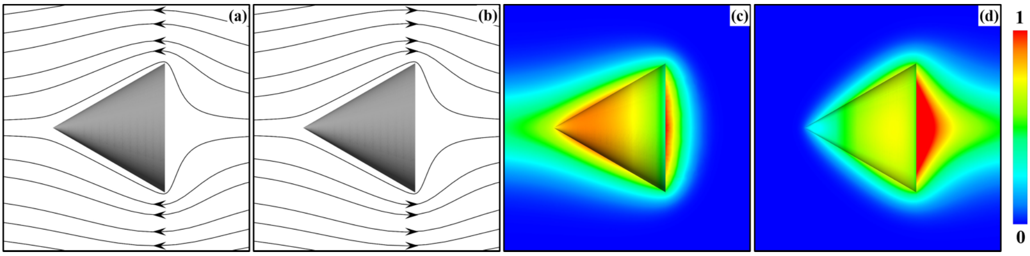

FIG. 2. Cross-sectional streamlines for a uniform Stokes flow past a cone from (a) the right to the left and (b) the left to the right. (c) and (d) The steady-state distribution of temperature in the meridian plane corresponding to the flow field in (a) and (b), respectively. In both cases, a constant heat flux is imposed at the surface of the cone and Pe = 100.

of the prescribed surface temperature [see Eq. (11)]. A key to better understand this is that, here, $u_\phi = 0$ and therefore the only transport mechanism in the ϕ direction is diffusion (see also [16]).

A natural question to ask at this point is how well Eqs. (11) and (15) work for not very small Pe. To answer this question, we carry out a series of simulations for a wide range of Péclet numbers (see, e.g., Fig. 1). We find that the analytical results for nonuniform boundary conditions differ by only a few percent or less from that of simulations up to $\mathrm{Pe} \sim O(1)$. This indicates that, despite their simplicity, Eqs. (11) and (15) are accurate over a reasonably broad range of Pe.

IV. INVARIANCE TO FLOW REVERSAL

A closer inspection of Eq. (6) reveals that if $\tilde{\eta}_s = \tilde{\varphi}_s$ is uniform then $\bar{\tilde{q}}_\eta = \bar{\tilde{q}}_\varphi$. Since η and φ are transported by, respectively, $\boldsymbol{u}$ and $-\boldsymbol{u}$, this means that the instantaneous rate of convective transfer from an arbitrarily shaped particle with a prescribed uniform surface distribution in streaming Stokes and potential flows does not change if the flow direction is reversed. The invariance of the steady-state heat and mass transfer to the flow reversal was first uncovered by Brenner [6,7] and is known as Brenner's flow reversal theorem. Interestingly, the theorem applies to the entire range of Pe, which is somewhat counterintuitive particularly for $\mathrm{Pe} \gg 1$ (where a temperature or concentration boundary layer forms around the particle). It was shown later that the steady-state condition can be relaxed [19].

Equation (7) handily extends Brenner's theorem to include the invariance of the instantaneous average surface temperature (or concentration) to the flow reversal for a prescribed uniform heat (or mass) flux at the particle surface (i.e., if $\tilde{q}_\eta = \tilde{q}_\varphi$ is uniform then $\bar{\tilde{\eta}}_s = \bar{\tilde{\varphi}}_s$). To test the prediction of Eq. (7), we numerically solve for the temperature field around a cone with a uniform surface flux in Stokes flow for $\mathrm{Pe} = 10^{-1}$–10^{4} (see, e.g., Fig. 2). In all cases, the relative difference between the average surface temperatures in the original and reversed flows is less than 0.5%.

The preceding statements for the invariance of the average quantities $\bar{\tilde{q}}_\eta$ and $\bar{\tilde{\eta}}_s$ to the flow reversal are valid only when the boundary conditions imposed on the surface of the particle are uniform. However, for axisymmetric flows past a body of revolution, we can show that if the nonuniformities are restricted to the azimuthal direction, then the insensitivity of $\bar{\tilde{q}}_\eta$ and $\bar{\tilde{\eta}}_s$ to the change in the flow direction still holds. Below we prove this for $\bar{\tilde{q}}_\eta$ and leave the analogous derivation for $\bar{\tilde{\eta}}_s$ to the reader.

Consider cylindrical coordinates r, ϕ, and z, where ϕ denotes the azimuthal angle and the z direction coincides with the symmetry axis of the body. Let $\tilde{\alpha}$ and $\tilde{\beta}$ be the Laplace transforms of two scalar fields that satisfy the boundary condition of uniform $\tilde{\alpha}_s = \tilde{\beta}_s$ and advected, respectively, by $\boldsymbol{u}$ and $-\boldsymbol{u}$. Also, suppose that $\tilde{\eta}$ and $\tilde{\varphi}$ are the Laplace transforms of two scalar fields that satisfy the boundary condition $\tilde{\eta}_s = \tilde{\phi}_s = \Phi(\phi,\mathfrak{s})$ and are transported, respectively, by $\boldsymbol{u}$ and $-\boldsymbol{u}$. Here Φ

is an arbitrary function of ϕ and $\mathfrak{s}$. Then, according to Eq. (6),

$$\tilde{\tilde{q}}_\eta = \left(\frac{1}{\bar{S}_p\tilde{\beta}_s}\right)\int_{\boldsymbol{R}_0}^{\boldsymbol{R}_1}\int_0^{2\pi}\Phi(\phi,\mathfrak{s})\tilde{q}_\beta r\, d\phi|d\boldsymbol{R}|, \tag{16a}$$

$$\tilde{\tilde{q}}_\varphi = \left(\frac{1}{\bar{S}_p\tilde{\alpha}_s}\right)\int_{\boldsymbol{R}_0}^{\boldsymbol{R}_1}\int_0^{2\pi}\Phi(\phi,\mathfrak{s})\tilde{q}_\alpha r\, d\phi|d\boldsymbol{R}|, \tag{16b}$$

$$\int_{\boldsymbol{R}_0}^{\boldsymbol{R}_1}\int_0^{2\pi}\tilde{q}_\alpha r\, d\phi|d\boldsymbol{R}| = \int_{\boldsymbol{R}_0}^{\boldsymbol{R}_1}\int_0^{2\pi}\tilde{q}_\beta r\, d\phi|d\boldsymbol{R}|, \tag{16c}$$

where the surface of the body is generated by rotation of the meridian curve $\widehat{\boldsymbol{R}_0\boldsymbol{R}_1}$, $|d\boldsymbol{R}|$ is an element of arc length in the meridian plane, and $\boldsymbol{R} = r\boldsymbol{e}_r + z\boldsymbol{e}_z$, with $\boldsymbol{e}_r$ and $\boldsymbol{e}_z$ being the unit vectors in the r and z directions, respectively. Given the fact that $\tilde{q}_\alpha$ and $\tilde{q}_\beta$ are independent of ϕ and $\tilde{\alpha}_s = \tilde{\beta}_s$, Eqs. (16) result in $\tilde{\tilde{q}}_\eta = \tilde{\tilde{q}}_\varphi$.

V. SUMMARY

We introduced a reciprocal theorem for convective heat and mass transfer from an arbitrarily shaped particle in streaming Stokes and potential flows. The theorem gave rise to alternative expressions for two average quantities of interest, namely, the average heat (or mass) transfer from the particle $\tilde{\tilde{q}}_\eta$ and the mean temperature (or concentration) distribution at the particle surface $\tilde{\tilde{\eta}}_s$. Notably, the surrogate equations allow us to accommodate nonuniformities in boundary conditions without solving for new temperature (or concentration) fields. We presented two exemplary applications of the formulations. In particular, for Stokes flow past a sphere, we analytically calculated $\tilde{\tilde{q}}_\eta$ and $\tilde{\tilde{\eta}}_s$ in the limit of small Péclet number corresponding to, respectively, arbitrary distributions of temperature and heat flux at the surface of the particle. We also extended Brenner's flow reversal theorem to include the invariance of the average surface temperature (or concentration) when a uniform heat flux is imposed. Further, we showed that, while Brenner's original theorem applies to only constant boundary conditions, the invariance of $\tilde{\tilde{q}}_\eta$ and $\tilde{\tilde{\eta}}_s$ to the flow reversal is equally valid for axisymmetric flows past a body of revolution with an azimuthally nonuniform boundary condition. Overall, our findings provide additional insight into systems in which convective transport processes occur in Stokes and potential flows.

Finally, we envisage several generalizations of our derivations. For instance, $\boldsymbol{u}$ could represent a linear inertial flow (see, e.g., [20,21]) or a streaming flow through a porous medium governed by linear Brinkman or Darcy equations. One might also let $\boldsymbol{u}_\infty$ correspond to a combination of general straining and pure rotational flows (see, e.g., [4,22,23]). Other alternatives would be bounded and periodic flows away from the particle. Mixed boundary conditions and slip velocity at the surface of the particle might be considered as well (see, e.g., [6,24]). Finally, the derivations could be extended to the heat and mass transfer from multiple particles in the presence of a linear source term in the transport equation (1).

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