Smooth phase transition of energy equilibration in a springy Sinai billiard

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Statistical equilibration of energies in a slow-fast system is a fundamental open problem in physics. In a recent paper, it was shown that the equilibration rate in a springy billiard can remain strictly positive in the limit of vanishing mass ratio (of the particle and billiard wall) when the frozen billiard has more than one ergodic component [Proc. Natl. Acad. Sci. USA **114**, E10514 (2017)]. In this paper, using the model of a springy Sinai billiard, it is shown that this can happen even in the case where the frozen billiard has a single ergodic component, but when the time of ergodization in the frozen system is much longer than the time of equilibration. It is also shown that as the size of the disk in the Sinai billiard is increased from zero, thereby leading to a decrease in the time required for ergodization in the frozen system, the system undergoes a smooth phase transition in the equilibration rate dependence on mass ratio.

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I. INTRODUCTION

Although the equilibrium properties of statistical systems are quite well understood, one of the fundamental open problems in statistical physics is about how dynamical systems actually reach this state of statistical equilibrium. A partial answer is provided by the ergodic hypothesis, which states that all accessible microstates of a given system are equiprobable over sufficiently long periods [1,2]. However, there are very few dynamical systems which have been actually proven to be ergodic [3–8]; and even for ergodic systems, the time required for ergodization may be so long that it may be practically irrelevant. In the case of slow-fast ergodic systems, it has also been shown that there are adiabatic invariants which can prevent equilibration of the full system over very long periods [9,10]. Hence, there is a theoretical as well as practical need to understand the equilibration properties of systems from the dynamical perspective.

In a recent paper, it has been shown that equilibration of energies can be achieved in slow-fast systems on reasonable timescales if the frozen system has more than one ergodic component [11]. This was numerically demonstrated by studying the dynamics of a pointlike particle of small finite mass, m, in a springy billiard where one of the walls is massive, $M \gg m$, and is connected to a linear spring. Three different springy billiards were studied in that paper: springy barred rectangle, springy mushroom, and springy stadium. The total energy of the springy billiard system is conserved in each case since its an autonomous system. It was found that the partial energies of the particle, E_p, and the massive billiard wall, E_b, reached a state of equipartition and equilibration asymptotically with time for all the three springy billiards when the mass ratio m/M was nonzero. However, in the limit of a vanishing mass ratio, only the springy barred rectangle and springy mushroom retained a nonzero equilibration rate, whereas the equilibration rate for the springy stadium went to zero. It was shown that this difference in behavior can be explained through a mathematical model by taking into account the fact that the springy barred rectangle and springy mushroom have more than one ergodic component in the frozen state (called VFS systems, variable partition of the fast subspace), and the springy stadium has only one ergodic component (called EFS systems, an ergodic fast subsystem for almost all values of the slow variables). However, one similarity between the springy mushroom and the springy stadium was that in both cases, the equilibration rate varied with mass as $\sqrt{m/M}$, which is the same as what was predicted earlier for the case of uniformly hyperbolic systems [9,10]. For the case of the springy barred rectangle, the equilibration rate was found to be independent of the mass ratio as also predicted by the mathematical model [11]. Though the above result is expected to hold for systems in general which can clearly be classified as being VFS or EFS, the behavior of systems which are in between can be much more interesting. One example of such a system is the springy Sinai billiard as shown in Fig. 1, which essentially consists of a circular disk within a tetragon.

The conventional Sinai billiard consists of a circular disk at the center of a rectangle and, in fact, was one of the first dynamical billiards to be shown to be hyperbolic [3,4,12]. If one of the walls of this billiard is attached to a linear spring, it is expected to show similar equilibration properties as that of the springy Stadium [11]. In the limit of a vanishing disk radius, the springy Sinai billiard is reduced to a springy rectangular billiard, which is known to be integrable [13], and hence does not have any equilibration of energies even for a nonzero mass ratio [11]. In an integrable billiard attached to a spring, the partial energies of the particle and the oscillating bar keep varying periodically about a certain average without reaching equilibration [9–11].

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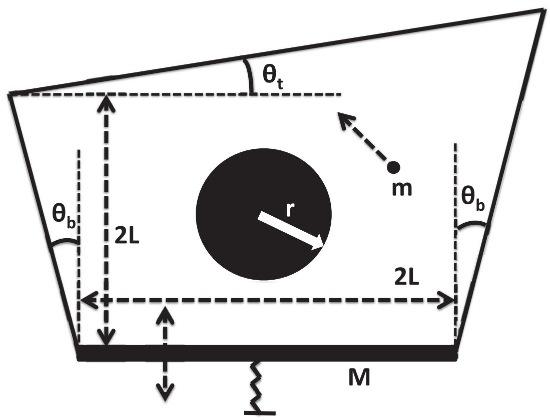

FIG. 1. Springy Sinai billiard, which consists of a particle of mass, $m \ll 1$, moving within the billiard boundaries undergoing elastic reflections at each collision with the boundaries (including the disk in between). The bottom wall of the billiard has a mass, $M = 1$, and is attached to a spring such that its natural frequency of oscillations is $\omega = 1$. When the disk radius is zero, this becomes a polygon (which is nonergodic if all its angles are rational multiples of π), and for a nonzero disk radius, this becomes a Sinai billiard, which is known to be hyperbolic. It is this transition in the billiard properties as we change the disk radius that leads to the phase transition that is demonstrated in this paper. The parameter values chosen are $L = 2$, $\theta_b = \pi/18 = \theta_t$, $0 \leqslant r \leqslant 1$, $M = 1$, and $4 \times 10^{-6} \leqslant m \leqslant 12 \times 10^{-6}$.

As shown in Fig. 1, the walls of the springy Sinai billiard have been made slanted in this work to create a tetragon. Among all possible varieties of polygons, it is known that only four are integrable: rectangle, equilateral triangle, right-angled triangle with two other angles $\pi/4$, and right-angled triangle with two other angles $\pi/3$ and $\pi/6$. If the angles of the nonintegrable polygon are rational multiples of π, then the billiard is also known to be nonergodic [13–15] and, hence, is expected to show a behavior similar to that of VFS systems [11]. Very little is known about the ergodic properties of polygons with angle(s) which are irrational multiple(s) of π, and it is one of the most important open questions in the field of dynamical billiards [16–19]. Polygonal billiards are also of importance in the study of quantum mechanics and can have very interesting solutions for the quantum energy levels with important implications in the field of quantum chaos [15].

In this paper, the springy Sinai billiard with slanted walls has been studied and found to have several very interesting properties so far not reported in any other slow-fast system. When the radius of the disk is nonzero, the system is known to be ergodic, and the timescale of ergodization decreases as the disc radius increases [4]. Hence, when the disk radius of this springy Sinai billiard is large enough, it is expected to behave like an EFS system, but it is *a priori* not clear what may happen when the disk radius is small enough. This is because for a small disk radius, the dynamical properties of the nonergodic nonintegrable tetragon may become more dominant. Hence, the system might behave like an EFS system for all nonzero values of the disk radius, or it might undergo a phase transition to a VFS-like system when the disk radius is small enough. This question is numerically studied in this paper, and the latter possibility is found to be true, i.e., the springy Sinai billiard with slanted walls indeed undergoes a smooth phase transition from an EFS system to a VFS-like system as the disk radius is decreased. Interestingly, for low values of the disk radius, the equilibration rate dependence on the mass ratio is also found to be nonmonotonic. This hints at the possibility that this dynamical system can be a very good candidate for the discovery of interesting dynamical properties not commonly found in other springy billiards.

II. BILLIARD MODEL

The springy Sinai billiard shown in Fig. 1 consists of a circular disk of radius, r, contained within a tetragon. In this paper, the values taken are $L = 2$, $\theta_b = \pi/18 = \theta_t$, $0 \leqslant r \leqslant 1$, $M = 1$, and $4 \times 10^{-7} \lesssim m \lesssim 9 \times 10^{-5}$. Simulations were performed for other values of θ_t, θ_b, and qualitatively similar results were found as those reported in this paper. The spring attached to the massive billiard boundary has a spring constant such that its angular frequency of oscillations is $\omega = 2\pi$. Numerical simulations are performed for an ensemble of 10 000 particles using the same algorithm described in Ref. [11]. The particle moves inside the billiard in straight lines and undergoes elastic collisions at the boundaries. The particle velocity after collision with a static wall is simply given by the law of elastic collisions, where the angle of incidence is equal to the angle of reflection. In this case, the particle velocity undergoes only a change in direction, and its speed remains the same. When the particle undergoes elastic collisions with the oscillating bar, the time and position of collision are calculated using a combination of bisection and Newton method. The equilibration rate is estimated using a linear least-squares fit of $\log |E_b - 0.5|$ over a time interval in which the bar energy, E_b, changes by a factor of e. In each simulation, there is only a single particle in the springy billiard, and then an average is taken over 10 000 different randomly chosen initial conditions. The total energy of the system stays constant at $E = 1$, which is the sum of the particle kinetic energy, E_p, and the energy of the oscillating bar, E_b. The energy E_b is a sum of the kinetic energy of the bar and the potential energy of the attached spring. There is an exchange of energy between E_p and E_b each time there is a collision between the oscillating bar and the particle. Between such collisions, the values of E_p and E_b remain unchanged.

III. RESULTS

Figure 2 shows the plot of the bar energy, E_b, with time for a few values of r/L and m/M each. As can be clearly seen, the bar energy reaches its equilibrium value of 0.5 in all cases, which is the expected behavior for billiards which are nonintegrable [11]. The equilibration proceeds approximately as an exponential function, and so E_b can be written as

$$E_b(t) \approx 0.5 + [E_b(0) - 0.5]e^{-\gamma t}, \tag{1}$$

where γ is the equilibration rate and depends on $E_b(0)$, m/M as well as the billiard parameters. In this paper, we have kept all other billiard parameters fixed, except for the disk radius, r.

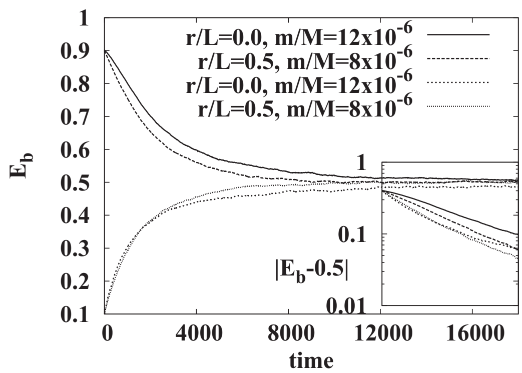

FIG. 2. Variation of energy of the oscillating bar, E_b, with time for a few values of r/L and m/M. As can be seen, the bar energy tends towards the equilibration at $E_b = 0.5$ irrespective of its starting value. The inset shows a plot of $|E_b - 0.5|$ on the logarithmic scale, which turns out to be close to straight lines, thereby implying that the bar energy converges to its equilibrium value exponentially in time.

Figure 3 shows a plot of the equilibration rate, γ, versus the mass ratio, m/M, for a few values of r/L when $E_b(0) = 0.9$. As can be seen, for larger values of r/L, the equilibration rate increases with an increase in m/M as a power law and tends to zero in the limit of a vanishing mass ratio as is expected of EFS systems. However, for lower values of r/L, γ becomes nonmonotonic and has a nonzero value in the limit of a vanishing mass ratio, which is typical of VFS systems. For some values of the disk radius around $r/L \sim 0.12$, we also see that the equilibration rate is independent of the mass ratio, which is similar to the behavior found for the springy barred rectangle [11]. A qualitatively similar behavior is observed when $E_b(0) = 0.1$ as shown in Fig. 4. Hence, it is reasonable to conclude that this is typical behavior of the springy Sinai billiard. This result is significant since it is usually believed that ergodic systems should not have equilibration of energies

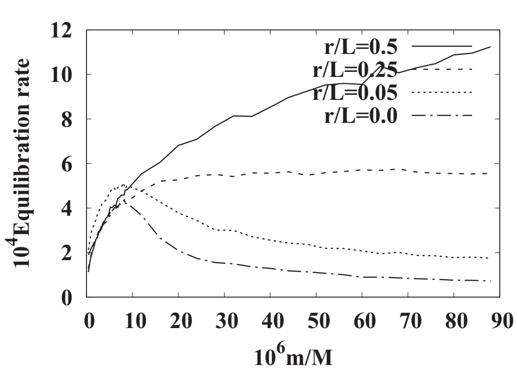

FIG. 3. Equilibration rate, γ, dependence on the mass ratio, m/M, for a few values of the disk radius, r/L, for $E_b(0) = 0.9$. For low values of r/L, γ is nonmonotonic in m/M and has a nonzero value in the limit of a vanishing mass ratio. However, as the value of r/L crosses a certain critical value, γ shows it is monotonic in m/M and tends towards a zero limiting value for a large enough radius. For intermediate values of $r/L \sim 0.25$, the value of γ becomes independent of the mass ratio beyond a certain threshold, which is similar to the behavior observed in the springy barred rectangle [11].

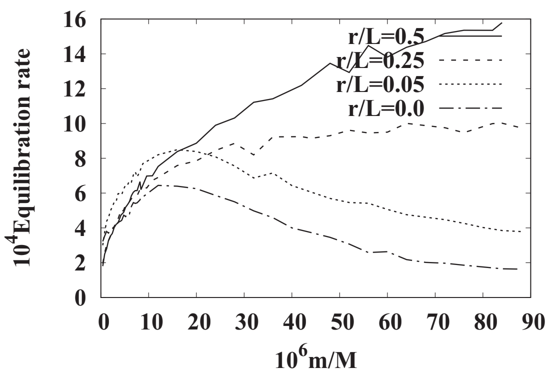

FIG. 4. Equilibration rate, γ, dependence on the mass ratio, m/M, for a few values of the disk radius, r/L, for $E_b(0) = 0.1$. The behavior is found to be qualitatively similar to that for $E_b(0) = 0.9$ as shown in Fig. 3. For low values of r/L, γ is nonmonotonic in m/M and has a nonzero value in the limit of the vanishing mass ratio. However, as the value of r/L crosses a certain critical value, γ shows it is monotonic in m/M and tends towards a zero limiting value for a large enough radius.

in the limit of a vanishing mass ratio [9,10]. But this result shows that if the ergodicity is weak, then even ergodic systems can have equilibration of energies in this limit.

The simulation codes written in FORTRAN 77 using MPI are available at Ref. [20].

IV. DISCUSSION AND CONCLUSION

These results are particularly relevant to practical systems, since most of them are neither strictly VFS or EFS and actually fall somewhere in between, in the same sense as most real systems have a mixed phase space. Hence, most real systems are expected to show this kind of a smooth phase transition as the relevant parameters are varied. The criteria for observing a similar behavior are that for some set of parameters, the system should become strongly ergodic, and for some other set of parameters, the system should become strongly nonergodic, while remaining nonintegrable for all parameter values. The simultaneous requirement of nonergodicity and nonintegrability for some parameter values is important since in the springy Sinai billiard shown in Fig. 1, if the bounding polygon is a rectangle (nonergodic, but integrable) instead of an arbitrary tetragon, then the phase transition will not be observed. This is because there is no equilibration of energies in an integrable system with a linear spring and the bar energy keeps oscillating about a certain mean value all the time. However, there might be interesting equilibration effects even in integrable billiards when the spring becomes nonlinear [21].

Although springy billiards can be differentiated based on their equilibration properties in the limit of a vanishing mass ratio, one may ask whether this limit is actually achievable in a physical system and whether this limit can be directly simulated. The limit of a vanishing mass ratio can be treated in two ways. One is to keep the initial ratio of E_b/E and E_p/E fixed as the mass ratio goes to zero (the study of statistical equilibration), and another is to keep E_p fixed at a finite value and let E_b go to infinity (Fermi acceleration [14,22–24]). The limit of a vanishing mass ratio can be and has been directly simulated in the case of Fermi acceleration, but cannot be done in the case of equilibration studies. In this context, it is important to note that the results obtained in this paper can also have very interesting implications in the study of Fermi acceleration in dynamical billiards. Fermi acceleration is the study of particle dynamics within billiards in a similar manner as studied in this work, with the only difference that there the billiard wall is infinitely massive, and, hence, the total energy of the system is not a conserved quantity. So, instead of equilibration, what is observed is an unbounded increase of energy of the particle ensemble with time if the underlying billiard is nonintegrable. In most such systems studied so far, this energy growth has been found to be either exponential or quadratic in time. The prevailing understanding is that the energy growth rate is quadratic-in-time if the underlying frozen billiard is ergodic [22] and exponential-in-time if it is nonergodic [14,23]. One of the open questions in this area is whether polygons, which are known to be pseudo-integrable [15], in general have an exponential growth of energy or not. Some indirect evidence has been found which indicates an exponential-in-time growth of energy in polygons [14], but it is not yet well established. And as shown in Ref. [11], there is a strong connection between equilibration rates in springy billiards in the limit of a vanishing mass ratio and exponential acceleration when the same system is studied in the context of Fermi acceleration. The results reported in this paper provide more evidence to support the case of exponential energy growth in polygons in general.

This work primarily has three limitations, which can serve as fruitful directions for future work. First, there is a lack of a suitable theoretical explanation for the phase transition, but it is important to note that the theoretical model presented in Ref. [11] is also applicable to the springy Sinai system presented in this manuscript. As described in Ref. [11], in order to make theoretical predictions about the equilibration rate in springy billiards, it is necessary to know the ergodic partitions or components of the billiard under consideration and the rate at which the particle jumps between these ergodic components as the billiard wall oscillates. Although this information was available for the springy systems considered in Ref. [11], it is not available for the springy Sinai billiard, mainly because, in this case, the ergodic components are not that well separated as compared to the springy barred rectangle or the springy mushroom. Also, as explained in Ref. [11], the theoretical prediction for equilibration rate of springy systems is generally possible only in the limit of a vanishing mass ratio, and not at a finite mass ratio. This is because the necessary information mentioned above is generally not available at a finite mass ratio (the springy barred rectangle in Ref. [11] is an important exception). Due to these reasons, a theoretical explanation for the equilibration behavior of the springy Sinai system is extremely hard and is unlikely to be available in the near future. Now, of course, due to lack of a theoretical explanation, one may question the validity of the smooth phase transition in this system. Here it is important to note that the phase transition in a springy Sinai billiard is actually expected based on the already established results and is in accordance of our current understanding of such systems. As mentioned in this paper, when the disk radius is zero, the system becomes a rational polygon, which is known to be nonergodic, whereas the system is ergodic for a nonzero disk radius. And these two kinds of systems have already been shown to have very different equilibration properties [11]. The important contribution of this work is to present a dynamical system which displays both kinds of equilibration properties based on system parameters.

Second, it is not clear why the equilibration rate, γ, is nonmonotonic with respect to the mass ratio for low values of r/L. Perhaps there is some kind of resonance phenomenon taking place for certain values of m/M for lower values of the disk radius, which leads to a maximization of the equilibration rate.

And, third, the functional dependence of the equilibration rate on m/M is unclear for lower values of the disk radius. This information is required so as to be able to predict the value of the equilibration rate in the limit of a vanishing mass ratio. We can graphically see that this limiting value is most likely nonzero, but a proper empirical estimation is needed in order to be sure.

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[1] J. R. Dorfman, *An Introduction to Chaos in Nonequilibrium Statistical Mechanics* (Cambridge University Press, Cambridge, 2001).

[2] C. C. Moore, Ergodic theorem, ergodic theory, and statistical mechanics, Proc. Natl. Acad. Sci. USA **112**, 1907 (2015).

[3] Y. G. Sinai, Dynamical systems with elastic reflections, Russ. Math. Surv. **25**, 137 (1970).

[4] Y. G. Sinai and N. Chernov, Ergodic properties of certain systems of two-dimensional discs and three-dimensional balls, Russ. Math. Surv. **42**, 181 (1987).

[5] N. Simanyi and D. Szasz, Hard ball systems are completely hyperbolic, Ann. Math. **149**, 35 (1999).

[6] A. Kaplan, N. Friedman, M. Andersen, and N. Davidson, Observation of Islands of Stability in Soft Wall Atom-Optics Billiards, Phys. Rev. Lett. **87**, 274101 (2001).

[7] A. Rapoport, V. Rom-Kedar, and D. Turaev, Stability in high dimensional steep repelling potentials, Commun. Math. Phys. **279**, 497 (2008).

[8] V. Rom-Kedar and D. Turaev, Billiards: A singular perturbation limit of smooth Hamiltonian flows, Chaos **22**, 026102 (2012).

[9] A. I. Neishtadt and Y. G. Sinai, Adiabatic piston as a dynamical system, J. Stat. Phys. **116**, 815 (2004).

[10] P. Wright, The periodic oscillation of an adiabatic piston in two or three dimensions, Commun. Math. Phys. **275**, 553 (2007).

[11] K. Shah, D. Turaev, V. Gelfreich, and V. Rom-Kedar, Equilibration of energy in slow-fast systems, Proc. Natl. Acad. Sci. USA **114**, E10514 (2017).
[12] L. A. Bunimovich, Mechanisms of chaos in billiards: Dispersing, defocusing and nothing else, Nonlinearity **31**, R78 (2018).
[13] E. Gutkin, Billiards in polygons, Physica D **19**, 311 (1986).
[14] K. Shah, Energy growth rate in smoothly oscillating billiards, Phys. Rev. E **83**, 046215 (2011).
[15] R. J. Richens and M. V. Berry, Pseudo-integrable systems in classical and quantum mechanics, Physica D **2**, 495 (1981).
[16] E. Gutkin and N. Haydn, Topological entropy of generalized polygon exchanges, Bull. Am. Math. Soc. **32**, 50 (1995).
[17] J. Bobok and S. Troubetzkoy, Topologically weakly mixing polygonal billiards, arXiv:1702.08260 (2017).
[18] E. Gutkin, Billiard dynamics: A survey on emphasis on open problems, Regular Chaotic Dyn. **8**, 1 (2003).
[19] K. Dingle, J. S. W. Lamb, and J. Lazaro-Cami, Knudsen's law and random billiards in irrational triangles, Nonlinearity **26**, 369 (2013).
[20] https://tinyurl.com/y5ryzo2k and https://tinyurl.com/y4js9fh5.
[21] P. Schmelcher, Driven power-law oscillator, Phys. Rev. E **98**, 022222 (2018)
[22] V. Gelfreich and D. Turaev, Fermi acceleration in non-autonomous billiards, J. Phys. A **41**, 212003 (2008).
[23] K. Shah, D. Turaev, and V. Rom-Kedar, Exponential energy growth in a Fermi accelerator, Phys. Rev. E **81**, 056205 (2010).
[24] V. Gelfreich, V. Rom-Kedar, K. Shah, and D. Turaev, Robust Exponential Acceleration in Time-Dependent Billiards, Phys. Rev. Lett. **106**, 074101 (2011).