

Recurrence time correlations in random walks with preferential relocation to visited places

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Random walks with memory usually involve rules where a preference for either revisiting or avoiding those sites visited in the past are introduced somehow. Such effects have a direct consequence on the transport properties as well as on the statistics of first-passage and subsequent recurrence times through a site. A preference for revisiting sites is thus expected to result in a positive correlation between consecutive recurrence times. Here we derive a continuous-time generalization of the random walk model with preferential relocation to visited sites proposed in *Phys. Rev. Lett.* **112**, 240601 (2014) to explore this effect, together with the main transport properties induced by the long-range memory. Despite the long-range memory effects governing the process, our analytical treatment allows us to (i) observe the existence of an asymptotic logarithmic (ultraslow) growth for the mean square displacement, in accordance to the results found for the original discrete-time model, and (ii) confirm the existence of positive correlations between first-passage and subsequent recurrence times. This analysis is completed with a comprehensive numerical study which reveals, among other results, that these correlations between first-passage and recurrence times also exhibit clear signatures of this ultraslow relaxation dynamics.

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I. INTRODUCTION

Purposeful memory is one of the essential ingredients that serves us to distinguish the behavior of higher organisms from that of simpler (living or nonliving) entities; this is an idea which has been recognized for a very long time and has represented a matter of debate in science, psychology, and epistemology for decades. Nowadays, quantitative approaches to animal and human behavior have progressively become a field of active research in biological physics. The study of neuronal patterns in the brain with the help of imaging techniques and its modelization through mathematical tools from network theory and/or population dynamics represent probably the most renowned example [1,2], but the monitoring and subsequent analysis of animal movements or trajectories in connection to statistical physics and dynamical systems [3–5] has also gained its attention as an effective field of study.

In particular, for those behavioral processes related to dispersal and/or navigation of living beings, it is clear that random walks represent a convenient description in order to condense and/or account for the properties of real trajectories and individual or collective space use [6–8]. Simpler approaches within this area just neglect any effects from spatial memory, so the position of the individual walker at time t , $X(t)$, is then a Markovian random variable. However, a rich bibliography on random walks with memory rules of a different sort has been developed throughout the years and connected to biological movement, too. Some examples of this include models in which walkers tend to exhibit locally some kind of directional memory or persistence (these could be loosely termed as *annealed* memory models); this is the case of persistent random walks [6,9,10], continuous-time random walks (CTRW) [11], Lévy walks [12–15], etc. On the other side, *quenched* memory models represent a more

complex situation in which local information about the sites visited is stored by the walker somehow, and so future rules of advance will explicitly depend on it, which in general introduces strong departures from a Markovian situation. Some well-known frameworks falling within this class are the different versions and generalizations of the self-avoiding random walk (though originally this model was proposed to describe polymer growth, not biological movement) [16], elephant random walks (which were probably the first class of solvable models proposed with long-range memory effects) [17], or, more recently, models with preferential relocation to already visited sites [18,19] or preferential persistence for familiar paths [20,21].

Though a large amount of realism can be gained by introducing memory in the description of animal and human movement, it is clear that mathematical treatment becomes then cumbersome due to its non-Markovian character. First-passage and coverage properties, for instance, of non-Markovian processes represent a formidable problem for which it is very difficult to extract analytical results unless additional assumptions are considered [22]. For the case of *annealed* memory some exceptions can be found, like those works where nonstationary random walk patterns are considered [23], but for *quenched* memory it is very difficult to find references in the literature where this has been even addressed; we can cite the recent work by Kearney and Martin [24] on the first-passage properties of Pólya urns and their connection to random walks as one of the few exceptions.

Taking all this context into account, our aim is to propose the study of first-passage and recurrence times statistics for *quenched* memory random walks roughly describing the movement of biological organisms with different memory (cognitive) abilities. While a whole analytical treatment of such situations is unattainable, still some general properties

of interest can be derived. If Θ_n represents a random variable describing the time at which the individual hits a site for n th time, then we will focus here on the random variables

$$T_n = \Theta_n - \Theta_{n-1} \quad (n \geq 1)$$

representing the random times between consecutive hittings, with $\Theta_0 = 0$. So, a consequence of introducing memory effects in the trajectories could be the emergence of correlations between T_1, T_2, \dots . Note that all these variables would be independent for the simplest case of Markovian random walks, since memory is lost after hittings. So, characterizing the correlations between these recurrence times could provide a way to classify memory random walks.

As a first step towards this aim, we here study the correlation properties between first-passage and successive recurrence times in a generalized (continuous-time) version of the random walk with preferential relocation to visited sites [18,25]. This preferential relocation process, as reported in the original work, tries to capture some basic properties of foraging in higher animals driven by a tendency to revisit sites where resources (e.g., food,...) have been successfully detected previously (so assuming that these resources are never depleted and/or can be replenished in a relatively short time). Hence this process is obviously expected to yield a positive correlation between the variables T_n , though this idea has never been explored previously as far as we know.

The present article is structured as follows. In Sec. II we derive the CTRW master equation for random walks with preferential relocation to visited sites, and justify the interest of such generalization if compared to the original (time-discrete) version. In Sec. III, we study the dispersal properties through the mean square displacement (MSD) of the corresponding random walkers to check that our results are in agreement with those from the original model. In Sec. IV we formally present the hitting and recurrence problem for this case and provide analytical justification to support the existence of positive correlations between first-passages and subsequent recurrence times for a particularly simplified case. Also, we show results from Monte Carlo simulations in order to understand the main properties of the process. Finally, the conclusions from our study are presented in Sec. V.

II. CONTINUOUS-TIME MODEL

A. CTRW framework

We start by revising briefly the classical CTRW formalism to facilitate understanding of the model presented below. The CTRW is based on the idea that the walker performs jumps of random (i.i.d.) lengths separated by random (i.i.d.) waiting times. A possible mathematical derivation is based on the combination of two balance equations. The first one states that the probability of reaching a position x at the m th step, denoted by $j_m(x)$, satisfies the mesoscopic balance equation

$$j_m(x) = \int_{-\infty}^{\infty} j_{m-1}(x-z)\Phi(z)dz + \delta_{m,0}\delta(t)j_0, \quad (1)$$

where $\Phi(x)$ is the jump kernel, which determines the probability distribution function of the jump lengths, and the last term stands for the initial condition [where $\delta_{m,0}$ is a Kronecker delta function, $\delta(t)$ is a Dirac delta function, and $j_0 = \delta(x-x_0)$

with x_0 the position of the site occupied at $t=0$]. One can also include time explicitly within this expression to write

$$j_m(x,t) = \int_0^t \int_{-\infty}^{\infty} j_{m-1}(x-z,t-\tau)\varphi(\tau)\Phi(z)dzd\tau + \delta_{m,0}\delta(t)j_0, \quad (2)$$

where $j_m(x,t)$ corresponds to the probability that the m th step is done to position x at time t [so $j_m(x) = \int_0^{\infty} j_m(x,t)dt$] and $\varphi(t)$ is the probability distribution function of waiting times between consecutive jumps.

A second equation is introduced through the probability of being at position x at time t after m steps have been made, $p_m(x,t)$; this expression reads

$$p_m(x,t) = \int_0^t j_m(x,t-\tau)\phi(\tau)d\tau, \quad (3)$$

where $\phi(\tau)$ is the probability that the walker has not jumped in a time τ since it arrived to x , so this satisfies $\phi(t) = \int_t^{\infty} \varphi(\tau)d\tau$. By combining (2) and (3) one obtains the master equation which contains the statistical properties of the CTRW. In particular, if we sum the equations (2) and (3) for m from zero to ∞ (with $j_m = 0$ for $m < 0$) we recover the well-known master equation of the CTRW as it appears in the books on the subject [7]:

$$j(x,t) = \int_0^t \int_{-\infty}^{\infty} j(x-z,t-\tau)\varphi(\tau)\Phi(z)dzd\tau + \delta(t)j_0, \quad (4)$$

$$p(x,t) = \int_0^t j(x,t-\tau)\phi(\tau)d\tau. \quad (5)$$

B. Preferential relocation to visited sites

Now we include in the CTRW framework the possibility that the walker can use memory to return to previously visited sites. While the original model with preferential relocation rules was discrete both in time and space [18], a continuous generalization has already been proposed in a different context [26], though a mesoscopic derivation was not provided there as we do in the following. In particular, a generalization within the CTRW framework allows one the possibility to consider in a natural way different waiting time distributions, including those with power-law tails (leading to a Lévy statistics) or with a combination of different dispersal modes, for example.

We will consider that at the end of each waiting time the particle can either (i) decide to do a random jump governed by the kernel $\Phi(x)$ with probability α (in the following, we denote this as the *normal* transport mode) or (ii) use its memory and then *fly* instantaneously to a previously visited site with probability $1-\alpha$ (in the following, this is termed as *memory* transport mode). If we denote the memory kernel (that is, the probability to remember a site visited i jumps ago) as $K_m(i)$, with $\sum_{i=1}^m K_m(i) = 1$ for any m , then we can write

$$j_m(x,t) = \alpha \int_0^t \int_{-\infty}^{\infty} j_{m-1}(x-z,t-\tau)\varphi(\tau)\Phi(z)dzd\tau + (1-\alpha) \sum_{i=1}^m K_m(i)[j_{m-i}(x,t) * \varphi(t)^{*i}], \quad (6)$$

where the asterisk symbol $*$ denotes the time-convolution operator and $\varphi(t)^{*i}$ denotes the time convolution of the distribution $\varphi(t)$ with itself i times.

From now on, for the sake of simplicity we will use a memory kernel that gives the same weight to all previously visited sites, so $K_n(i) = 1/m$, in agreement with the original model [18]. For that case, the previous equation becomes

$$j_m(x, t) = \alpha \int_0^t \int_{-\infty}^{\infty} j_{m-1}(x-z, t-\tau) \varphi(\tau) \Phi(z) dz d\tau + \frac{1-\alpha}{m} \sum_{i=1}^m [j_{m-i}(x, t) * \varphi(t)^{*i}]. \quad (7)$$

We can apply Fourier-Laplace transforms to the previous equation to take advantage of the renewal property of the process in time and space,

$$j_m(k, s) = \alpha j_{m-1}(k, s) \varphi(s) \Phi(k) + \frac{1-\alpha}{m} \sum_{i=1}^m j_{m-i}(k, s) \varphi(s)^i. \quad (8)$$

Note that we use k and s as the Fourier and Laplace arguments, so that $j_m(k, s)$ represents the Fourier-Laplace transform of $j_m(x, t)$, and we will distinguish them just by explicitly writing their variables. Now, due to the explicit dependence of the memory factor on m , it is also convenient to carry out a Z transform on the jump index m such that

$$\hat{j}(\lambda, k, s) = \sum_{m=0}^{\infty} \lambda^m j_m(k, s),$$

with $0 < \lambda < 1$. By applying this on Eq. (8) we obtain

$$\hat{j}(\lambda, k, s) = \alpha \lambda \hat{j}(\lambda, k, s) \varphi(s) \Phi(k) + (1-\alpha) \times \int_0^\lambda \frac{\varphi(s) \hat{j}(u, k, s)}{1-u\varphi(s)} du. \quad (9)$$

Due to the integral that appears now in the equation one needs to rewrite this expression as a differential equation by differentiating with respect to λ and replacing the integral in the resulting equation with the help of Eq. (9). This procedure leads to

$$\frac{d\hat{j}(\lambda, k, s)}{d\lambda} = \hat{j}(\lambda, k, s) \left[\frac{\alpha \varphi(s) \Phi(k)}{1-\alpha \lambda \varphi(s) \Phi(k)} + \frac{(1-\alpha) \varphi(s)}{[1-\lambda \varphi(s)][1-\alpha \lambda \varphi(s) \Phi(k)]} \right]. \quad (10)$$

The solution of this first-order differential equation reads

$$\hat{j}(\lambda, k, s) = \left[\frac{1}{1-\alpha \lambda \varphi(s) \Phi(k)} \right]^{\frac{\alpha[1-\Phi(k)]}{1-\alpha\Phi(k)}} \left[\frac{1}{1-\lambda \varphi(s)} \right]^{\frac{1-\alpha}{1-\alpha\Phi(k)}}, \quad (11)$$

where we have taken into account the boundary condition $\hat{j}(\lambda \rightarrow 0^+, k, s) = 1$. Finally, since we are mainly interested in the behavior of $j(k, s)$ (independent of m) we can remove the dependence on the jump index by using $j(k, s) \equiv \hat{j}(\lambda = 1, k, s) = \sum_{m=0}^{\infty} j_m(k, s)$. On setting $\lambda = 1$ into Eq. (11) we

have

$$j(k, s) = \left[\frac{1}{1-\alpha \varphi(s) \Phi(k)} \right]^{\frac{\alpha[1-\Phi(k)]}{1-\alpha\Phi(k)}} \left[\frac{1}{1-\varphi(s)} \right]^{\frac{1-\alpha}{1-\alpha\Phi(k)}}. \quad (12)$$

This expression, together with (5) (which is still valid for the model with memory), represents the CTRW generalization of the random walk with preferential relocation to visited places. Next, we will use this result to derive several properties of the model.

III. DISPERSAL PROPERTIES

First we will explore the dispersal properties of the model in order to check that they agree with those found for the time-discrete version [18]. For this, we will compute the MSD $\langle x^2(t) \rangle$, which is nothing but the second moment of $p(x, t)$ in space. So that, working again in Fourier-Laplace space,

$$\langle x^2(s) \rangle = - \lim_{k \rightarrow 0} \frac{\partial^2 p(k, s)}{\partial k^2} = - \frac{1-\varphi(s)}{s} \lim_{k \rightarrow 0} \frac{\partial^2 j(k, s)}{\partial k^2}, \quad (13)$$

where we have made use of Eq. (5). Performing the second derivative of $j(k, s)$ we find, after some tedious calculations,

$$\langle x^2(s) \rangle = - \frac{\alpha}{1-\alpha} \frac{\Phi''(0)}{s} \ln \left[\frac{1-\alpha \varphi(s)}{1-\varphi(s)} \right], \quad (14)$$

where $\Phi''(0)$ stands for the second derivative of $\Phi(k)$ evaluated at $k=0$. If the waiting time distribution has finite moments then we can make use of the expansion $\varphi(s) \simeq 1 - \tau s + \dots$ for large times, where τ is the mean waiting time. Hence, using this expansion and assuming $s \rightarrow 0$, we obtain

$$\langle x^2(s) \rangle \approx \frac{\alpha}{1-\alpha} \frac{\Phi''(0)}{s} \ln \left(\frac{\tau s}{1-\alpha} \right), \quad (15)$$

which yields

$$\langle x^2(t) \rangle \approx - \frac{\alpha \Phi''(0)}{1-\alpha} \left[\gamma + \ln \left(\frac{t}{\tau(1-\alpha)} \right) \right] \quad \text{for } t \rightarrow \infty \quad (16)$$

after inverting by Laplace, where $\gamma = 0.577\dots$ is the Euler-Mascheroni constant. This result is very general and holds for any dispersal kernel and waiting time distributions with finite moments and predicts an ultraslow diffusion due to the logarithmic growth of the MSD [27,28]. If, for example, the dispersal kernel is Gaussian $\Phi(x) = (2\pi\sigma^2)^{-1/2} \exp(-x^2/2\sigma^2)$ the asymptotic form of the MSD is given by

$$\langle x^2(t) \rangle = \frac{\alpha \sigma^2}{1-\alpha} \left[\gamma + \ln \left(\frac{t}{\tau(1-\alpha)} \right) \right] \quad \text{for } t \rightarrow \infty. \quad (17)$$

In Fig. 1 we confirm this result via direct comparison with extensive simulations of the random walk process described in Sec. II B (points) for different α values. Simulations were carried out in one-dimensional domains for the case of exponential waiting times between jumps (with mean $\tau = 1$) and Gaussian jump lengths with $\sigma^2 = 1$ so a direct comparison to the result in (17) can be performed. We find that not only the ultraslow (logarithmic) character of the dispersal is observed, but the prefactor and the constant term in (17) are

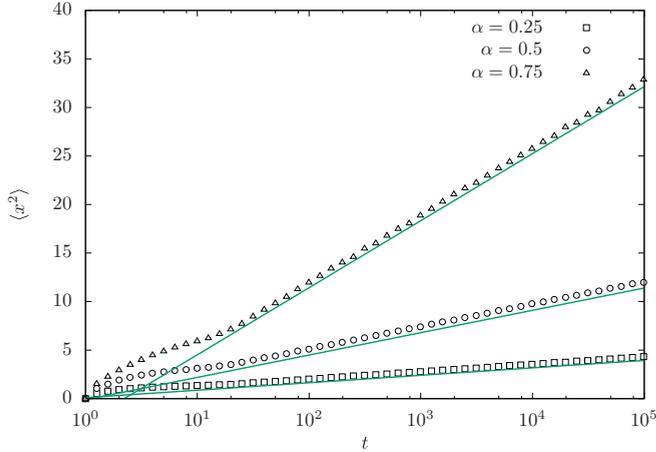


FIG. 1. MSD for the CTRW model with preferential relocation to visited places. Symbols represent the results from Monte Carlo simulations averaged over 2×10^6 realizations, while lines correspond to the asymptotic prediction for a Gaussian kernel with $\sigma^2 = 1$ as in Eq. (17). For the waiting time distribution $\varphi(t)$ an exponential distribution with $\tau = 1$ has been used.

relevant in order to fit properly the behavior obtained from the simulations. The small differences observed in the plot should be due to lower order terms in the expansion, which due to the ultraslow nature of the process may still have some minor effect even at relatively long times. As a whole, the agreement observed confirms the validity of our CTRW approach before we proceed to the analysis of the recurrence statistics in the next section.

IV. RECURRENCE STATISTICS

A. Theoretical framework

First-passage times and the statistics of recurrence times for Markovian random walks can be derived in general for finite domains with the help of renewal properties [29], but this no longer holds for non-Markovian walks [22]. While the problem is analytically unattainable for the model with preferential relocation presented in Sec. II B, some results can still be derived by simplifying the model considerably.

In particular, we will use the assumption that the system is finite (with N denoting the number of accessible sites) and it is spatially unconstrained [that is, the kernel $\Phi(x)$ is such that any site is accessible from any other with the same probability $1/N$]. This is not unreasonable in the context of the preferential relocation model we are considering here: since the *memory* mode can lead the individual to any visited site without restrictions, then it may be plausible to consider that the *normal* transport mode has this spatial capacity, too. In Sec. IV C, however, we will relax this assumption to check how our numerical results change when only jumps of a given maximum length are permitted.

1. Case $n = 1$

To illustrate our method, we first derive a formal expression for the first-passage distribution, following a similar strategy to that in Sec. II B based on discretizing the process

according to the jump index m . We denote by $S_1(t)$ the survival probability through a target site x^* (that is, the probability that after time t that site has not been visited yet; note that we do not consider an explicit dependence on x^* due to the assumption of unconstrained space). Then, this probability can be decomposed through

$$S_1(t) = \sum_{m=0}^{\infty} S_1^{(m)} P(m|t) = \sum_{m=0}^{\infty} S_1^{(m)} [\varphi(t)^{*m} * \phi(t)], \quad (18)$$

where $S_1^{(m)}$ is the survival probability after m jumps and $P(m|t) = \varphi(t)^{*m} * \phi(t)$ is the probability of having performed m jumps at time t . The latter is given by the convolution of m times the waiting time distribution [plus the convolution with $\phi(t)$, which is necessary to assert that the $(m+1)$ th jump has not been done yet].

Since the *memory* transport mode does not contribute to the first-passage time (as the target site has not been visited yet), the survival probability after m jumps is

$$S_1^{(m)} = (1 - \alpha/N)^m, \quad (19)$$

making use of the unconstrained space assumption. This result follows since the probability to find the target at any jump is just α/N . Transforming (18) to the Laplace space one finds

$$S_1(s) = \frac{\phi(s)}{1 - (1 - \alpha/N)\varphi(s)}. \quad (20)$$

Hence one can write as usual the first-passage distribution $f_1(t)$ as the time derivative of $S_1(t)$ or, alternatively, as $f_1(s) = sS_1(s) - 1$ in the Laplace space. The first moment of $f_1(t)$, namely $\langle T_1 \rangle = \int_0^{\infty} t f_1(t) dt$, provides the mean first-passage time, which reads then

$$\langle T_1 \rangle = \lim_{s \rightarrow 0} \frac{df_1(s)}{ds} = \lim_{s \rightarrow 0} \left[s \frac{dS_1(s)}{ds} + S_1(s) \right]. \quad (21)$$

So, if we assume again that $\varphi(t)$ has finite moments [with $\varphi(s) \approx 1 - \tau s + \dots$], then we reach from (20) and (21)

$$\langle T_1 \rangle = \frac{N\tau}{\alpha} \quad (22)$$

and, similarly, for the second order moment

$$\langle T_1^2 \rangle = \int_0^{\infty} t^2 f_1(t) dt = 2 \left(\frac{N\tau}{\alpha} \right)^2. \quad (23)$$

These results are to be expected. In the absence of contributions from the *memory* mode the process is Markovian, so (22) simply represents Wald's identity [30] for a stochastic process with constant rate $\alpha/\tau N$ (which is nothing but the rate at which the target site will be reached through the *normal* mode).

2. Case $n > 1$

Now we can extend the procedure above to study subsequent recurrence times (T_2, T_3, \dots , as we defined them in the Introduction) through a site. For this, we introduce the distribution function $f_n(t)$ for the random variable T_n and, equivalently, the joint distribution $f_n(t; m_{n-1})$ for T_n and the random variable m_{n-1} , which is the number of jumps done by the individual (counting since $t = 0$) when it hits the site for

the $(n - 1)$ th time (this is, the number of jumps performed when $\Theta_{n-1} = t$). Then we can write

$$\begin{aligned} \langle T_n \rangle &= \sum_{m_{n-1}=0}^{\infty} \int_0^{\infty} t f_n(t; m_{n-1}) dt \\ &= \sum_{m_{n-1}=0}^{\infty} \int_0^{\infty} t f_n(t|m_{n-1}) q_{n-1}(m_{n-1}) dt \\ &= - \sum_{m_{n-1}=0}^{\infty} q_{n-1}(m_{n-1}) \int_0^{\infty} t \frac{dS_n(t|m_{n-1})}{dt} dt, \end{aligned} \quad (24)$$

where in the second step we have introduced, through the Bayes theorem, the conditional probability $f_n(t|m_{n-1})$ and the probability distribution function $q_{n-1}(m_{n-1})$ for the variable m_{n-1} . Finally, in the last step we have expressed the conditional n -passage distribution $f_n(t|m_{n-1})$ as the derivative of the corresponding survival probability $S_n(t|m_{n-1})$. The reason for writing the mean recurrence time in that way is because for each recurrence process one can provide explicit expressions equivalent to (18) and (19) for the first passage, respectively:

$$S_n(t|m_{n-1}) = \sum_{m=0}^{\infty} S_{n|m_{n-1}}^{(m)} P(m|t) = \sum_{m=0}^{\infty} S_{n|m_{n-1}}^{(m)} [\varphi(t)^{*m} * \phi(t)], \quad (25)$$

where

$$S_{n|m_{n-1}}^{(m)} = \prod_{i=0}^{m-1} \left(1 - \frac{\alpha}{N} - \frac{(1-\alpha)(n-1)}{m_{n-1} + i} \right). \quad (26)$$

Note first of all that both expressions for the survival probabilities reduce to (18) and (19) for $n = 1$ (since $m_0 = 0$ and so the conditional probability is unnecessary in that case). Now, we observe that the *memory* mode explicitly contributes to the survival probability through the last term within the parentheses of (26). So, the probability that the *memory* mode leads the individual to the target is given by $(n - 1)/(m_{n-1} + i)$, where $(m_{n-1} + i)$ is the total number of jumps performed up to date [that is, those done up to the $(n - 1)$ th hitting, m_{n-1} , plus those done afterwards, i].

As a whole, the expressions (24)–(26) provide a recurrent method to determine the statistics of recurrence times as follows. First, once we know the properties of the first-passage time, we can use them to determine $q_1(m_1)$, which by definition satisfies

$$q_1(m_1) = S_1^{(m_1)} - S_1^{(m_1-1)}. \quad (27)$$

This, in combination with (24)–(26) for $n = 2$, will be used to determine $f_2(t; m_1)$ and its mean value $\langle T_2 \rangle$. Then we will be able to determine $q_2(m_2)$ from the general expression

$$q_n(m_n) = \sum_{m=0}^{m_n-1} q_{n-1}(m) [S_{n|m}^{(m_n-m)} - S_{n|m}^{(m_n-m-1)}] \quad (28)$$

and the same idea can be applied recurrently for $n = 3, 4, \dots$. This method, obviously, will become increasingly cumbersome as n increases, so at practice we can only expect it to be of practical utility for n small. In the next subsection we will

illustrate its use for $n = 2$, which is enough for the specific objectives we pursue in this paper.

B. Recurrence time for $n = 2$

Although the method described above could be applied to any waiting time distribution, to keep notation and results manageable we will focus here on the case of exponential waiting times, $\varphi(t) = \tau^{-1} e^{-t/\tau}$. Note that in this case the random walk would become Markovian in the absence of the preferential relocation rule (that is, for $\alpha = 1$). We already know that the mean first-passage time is determined by (22). Also, introducing (19) into (27) one has

$$q_1(m_1) = \frac{\alpha}{N} \left(1 - \frac{\alpha}{N} \right)^{m_1-1}. \quad (29)$$

On the other side, the combination of Eqs. (25) and (26) yields for this case

$$S_2(t|m_1) = \sum_{m=0}^{\infty} B_{m,m_1} \left(\frac{t}{\tau} \right)^m \frac{e^{-t/\tau}}{\Gamma(m+1)}, \quad (30)$$

where we have defined

$$B_{m,m_1} \equiv \prod_{i=0}^{m-1} \left(1 - \frac{\alpha}{N} - \frac{1-\alpha}{m_1+i} \right). \quad (31)$$

So, we finally insert (29) and (30) into (24) to determine the mean recurrence time after first hitting, T_2 . By doing this and performing the integral in t we get

$$\langle T_2 \rangle = \sum_{m_1=0}^{\infty} \frac{\alpha\tau}{N} \left(1 - \frac{\alpha}{N} \right)^{m_1-1} \sum_{m=0}^{\infty} B_{m,m_1}. \quad (32)$$

This expression cannot be further simplified due to the product within the definition of B_{m,m_1} , but we can reach useful approximations in the large-domain limit $N \gg 1$, so the number of jumps required for the first passage, m_1 , is also large in average. Using such an approximation, one can provide an analytical approximation for $\langle T_2 \rangle$ which can be expressed as a combination of exponential integral (or, alternatively, confluent hypergeometric) functions (see the discussion in the Appendix for the details).

Also, note that the same idea can be applied to higher order or cross moments (e.g., $\langle T_2^2 \rangle$ or $\langle T_1 T_2 \rangle$) (see the Appendix). This allows us to provide an estimation for the correlation coefficient between first-passage and subsequent recurrence times

$$\text{Cor}(T_1, T_2) \equiv \frac{\langle T_1 T_2 \rangle - \langle T_1 \rangle \langle T_2 \rangle}{\sigma_1 \sigma_2}, \quad (33)$$

where σ_i denotes the standard deviation of T_i . The goodness of that estimation will be checked through the comparison with results from numerical simulations in the next section.

C. Numerical results

All the simulations of the stochastic process presented in this section have been carried out for a discrete domain (a lattice of unit lattice size) with $N = 1024$. While this value of N may seem small for a normal transport mode, one should take into account that the ultraslow dynamics of the process makes

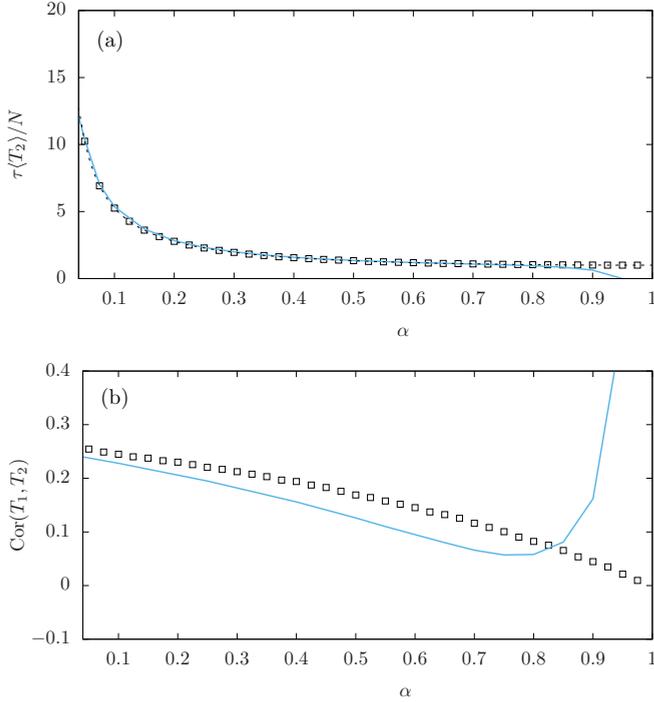


FIG. 2. Comparison between numerical results (symbols) and analytical approximations (lines) for (a) the nondimensional mean recurrence time after first hitting and (b) the correlation coefficient between first hitting and first recurrence times as a function of the memory parameter α . Results from simulations correspond to averages over 10^6 realizations. All results are for the case without spatial constraints and an exponential waiting time distribution $\varphi(t) = \tau^{-1}e^{-t/\tau}$ with $\tau = 1$. The dotted line represents the heuristic approximation $\langle T_2 \rangle = \tau N / \alpha(2 - \alpha)$.

first-passage times through a site increase exponentially with N , so very large values of N are unattainable. In agreement with our approximations above, exponential waiting times between jumps (with $\tau = 1$) have been taken in all cases.

The first results we show try to verify the validity of the approximations carried out in the previous section. In Fig. 2 we see that our analytical approximations for the nondimensional recurrence time $\langle T_2 \rangle / \tau N$ and $\text{Cor}(T_1, T_2)$ (solid lines) fits rather well the results for α small (strong memory regime) but the approximation fails clearly as long as we approach the limit without memory, $\alpha = 1$. Note that our approximation is valid as long as m_1 (and so $\langle T_1 \rangle$) are large enough, and this will be satisfied in particular in the limit $\alpha \rightarrow 0$, so our method is specially suited for explaining the regime of strong memory effects.

Surprisingly, we find that a simple heuristic approximation is able to fit the behavior for $\langle T_2 \rangle$ accurately (dotted lines in Fig. 2). Such approximation follows from assuming that the probability of reaching the target at each particular jump is constant and equals $\alpha/N + (1 - \alpha)\alpha/N = \alpha(2 - \alpha)/N$. Here, the two terms in the sum represent the contribution from the *normal* and *memory* modes, respectively. For the latter we are assuming that the individual revisits the target with probability $(1 - \alpha)/\langle m \rangle$, where $\langle m \rangle$ is the mean number of jumps done previously, N/α (as discussed in Sec. IV A).

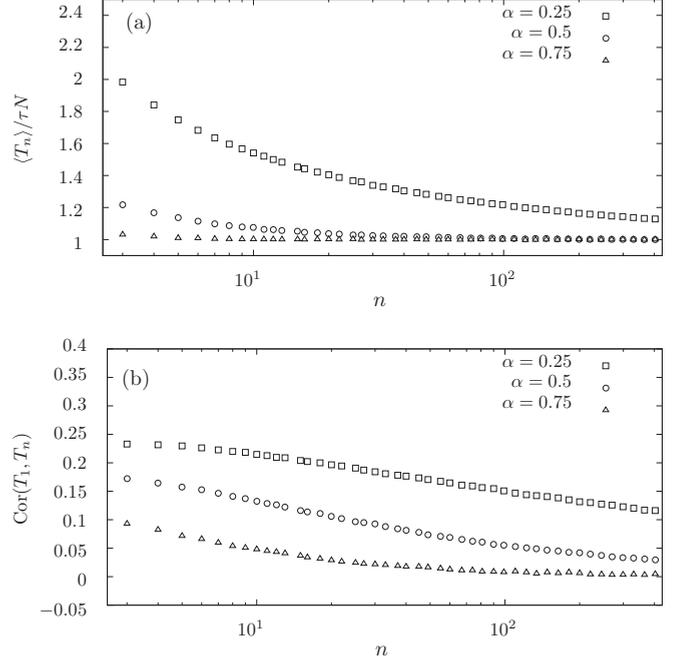


FIG. 3. Results for simulations of the stochastic process (symbols) for the (a) mean recurrence time after first hitting and (b) correlation coefficient between first-passage and subsequent recurrence times as a function of n , and for different values of the memory parameter α . Averages over 10^6 realizations are shown in all cases. The solid line in the lower panel is just a visual cue to illustrate the logarithmic nature of the decay.

The rate at which the target is found will be then τ times the inverse of the overall probability, leading to the estimation $\langle T_2 \rangle = \tau N / \alpha(2 - \alpha)$. The agreement found in Fig. 2 between this approximation (dotted line) and the numerical results is clear. However, this kind of approximation cannot be extended to cross moments (e.g., $\langle T_1 T_2 \rangle$) and so it is not helpful to obtain an estimation of the correlation coefficient, as we pursue here.

Regarding the correlation coefficient, Fig. 2 shows that the tendency predicted by our analytical approach is approximately correct (except, again, in the limit $\alpha \rightarrow 1$), but the accumulated error in the estimation of $\langle T_2 \rangle$, $\langle T_2^2 \rangle$, and $\langle T_1 T_2 \rangle$ makes the quantitative agreement not completely satisfactory. In particular, we have checked that our approximation works much better for predicting first-order moment of the recurrence times than higher-order moments. This should be due to problems of convergence of the series expansion proposed (see the Appendix) but we have not been able to figure out the details. In any case, this plot confirms the main idea of the present work, which is the fact that consecutive hitting times become positively correlated as a consequence of the preferential relocation rule.

Most of the conclusions above for $n = 2$ can be extended to subsequent recurrence times ($n > 2$), as reported in Fig. 3. There we observe that the mean value of T_n becomes progressively reduced as a function of n due to the accumulated effect of memory, and also it can be checked that its behavior as a function of the memory parameter α is qualitatively the same as for T_2 (not shown). Furthermore, Fig. 3(b) yields a very

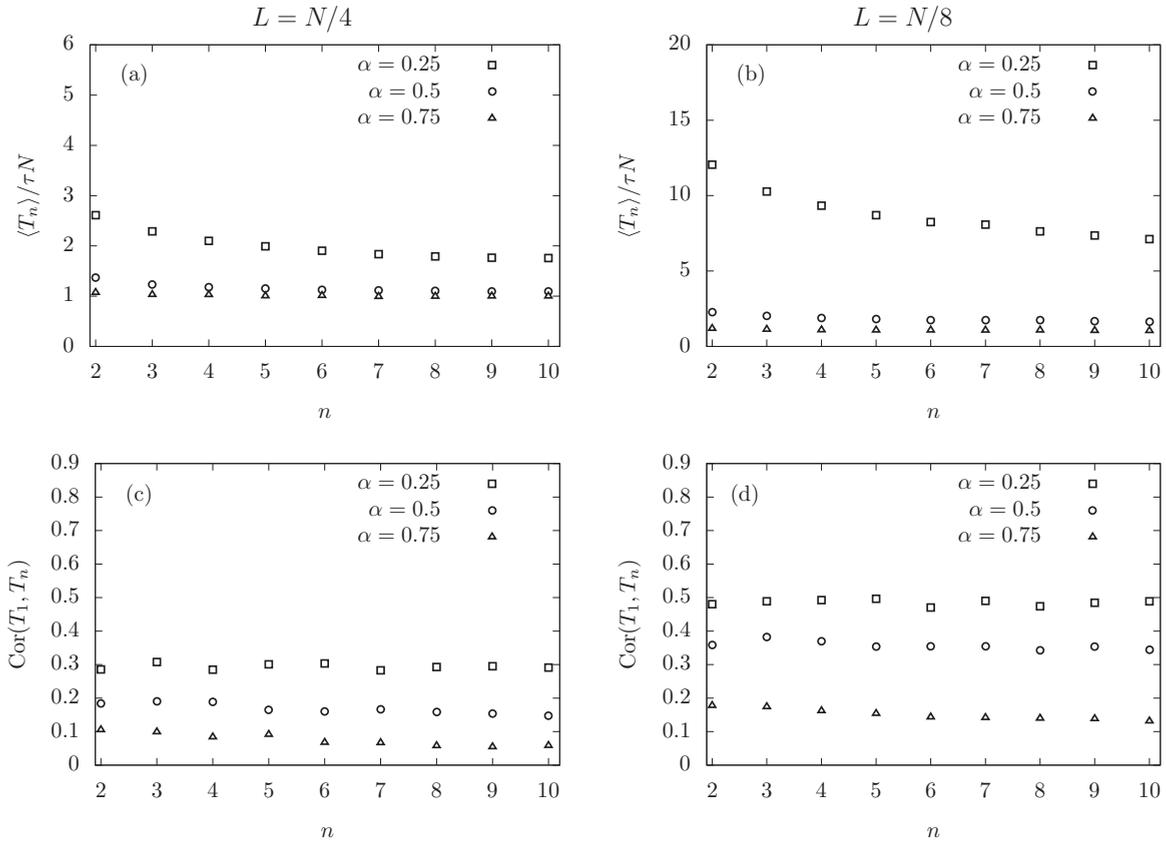


FIG. 4. Results for the (a),(b) mean recurrence time after first hitting and (c),(d) correlation coefficient between first-passage and subsequent recurrence times as a function of n , obtained from simulations of the stochastic process for the case where the *normal* mode is constrained by a maximum jump length of size L . Two values of L and different values of the memory parameter α are reported.

interesting result, as is the fact that correlations between first-passage and subsequent recurrence times T_n show an ultraslow (logarithmic, as for the MSD) relaxation to zero as a function of n . This tells us that the signature of strong memory induced by the preferential relocation rule does not only emerge at the level of dispersal (as was already known from previous works [18,26]) but also on the recurrence statistics.

Finally, to complete the numerical analysis we provide results for the case where individuals are not allowed to jump freely from any site to another, but only short jumps are to be expected, so that we reformulate the dispersal process by assuming that the individual can only do jumps up to a maximum distance L (this rule only applies to the *normal* mode, while the *memory* mode is kept unchanged and without spatial constraints). The results for $L = N/4$ and $L = N/8$ are provided in Fig. 4, where the initial position is chosen at random at each realization of the Monte Carlo simulation.

Note that the spatial constraint increases considerably the values of the first-passage and recurrence times (since now further sites become increasingly difficult to be reached due to the ultraslow dispersal properties of the model). So, low values of L become computationally very costly. Apart from that, we observe that the spatial constraint does not modify qualitatively the picture found in Figures 2 and 3. In particular, positive correlations between first-passage and recurrence times are still present and actually become enforced.

V. CONCLUSIONS

Summarizing the ideas reported in the work, we have confirmed that one of the essential signatures of random walks with memory (in this case, we have just focused on the model with preferential relocation to visited places) is the existence of correlations between first-passage and subsequent recurrence times through a site or, equivalently, between consecutive recurrence times. Furthermore, we have been able to characterize such correlations not only numerically but also through an approximated analytical study, at least for the case $n = 2$.

Anyway, the most interesting result we obtain is probably (Fig. 3) that the decay of these correlations with time for the case of walks with preferential relocation exhibits an ultraslow (logarithmic) behavior, in accordance with the dispersal properties of the model which were already known from previous works [18,25]. Since memory effects persist in the model for arbitrarily long times, such long-range dynamics is also present in the hitting and recurrence statistics. This suggests that such correlations capture adequately the memory dynamics in the model and so can be used as a proxy to identify what kind of memory rules govern the process. This could be of great interest, for example, in the analysis of trajectories of real organisms as a method to understand how memory has been employed during the underlying dynamics.

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APPENDIX: DERIVATION OF THE MOMENTS OF T_2

We start from the definition of the coefficients B_{m,m_1} , given implicitly in (30), and express them in a more convenient form:

$$B_{m,m_1} = [\gamma(m_1)]^m \frac{(1 + \frac{1}{m_1 - \beta}) \dots (1 + \frac{m-1}{m_1 - \beta})}{(1 + \frac{1}{m_1}) \dots (1 + \frac{m-1}{m_1})}, \quad (A1)$$

where we have defined $\beta \equiv (1 - \alpha)/(1 - \alpha/N)$ and $\gamma(m_1) \equiv (1 - \alpha/N)(1 - \beta/m_1)$. In this way it is clear that we can propose an expansion for $m_1 \gg 1$ in the form

$$B_{m,m_1} = [\gamma(m_1)]^m \left(1 + \sum_{i=1}^{m-1} \frac{v_i(m)}{(m_1 - \beta)^i} \right) \left(1 - \sum_{i=1}^{m-1} \frac{w_i(m)}{(m_1)^i} \right), \quad (A2)$$

with the first coefficients v_i, w_i given by

$$\begin{aligned} v_1(m) &= w_1(m) = \frac{m(m-1)}{2}, \\ v_2(m) &= \frac{m(m-1)(m-2)(3m-1)}{72}, \\ w_2(m) &= v_2(m) - v_1^2(m), \\ &\dots \end{aligned} \quad (A3)$$

Leading this expansion up to second order (that is, up to powers of order m_1^{-2}) and summing over all values of $m = 0$

we obtain, after some lengthy algebra,

$$\begin{aligned} \sum_{m=0}^{\infty} B_{m,m_1} &= \frac{1}{1 - \gamma(m_1)} + \frac{(1 - \alpha)\gamma(m_1)}{m_1^2 [1 - \gamma(m_1)]^3} \\ &\quad - \frac{\beta[1 + \gamma(m_1)]}{m_1^3 [1 - \gamma(m_1)]^4} + \frac{\beta^2[1 + 2\gamma(m_1)]}{m_1^4 [1 - \gamma(m_1)]^5} + \dots \end{aligned} \quad (A4)$$

In order to apply (32) we use the fact that in the limit of large media size ($N \gg 1, m_1 \gg 1$) the sum over m_1 can be adequately approximated by an integral

$$\begin{aligned} \langle T_2 \rangle &= \sum_{m_1=0}^{\infty} \frac{\alpha\tau}{N} \left(1 - \frac{\alpha}{N} \right)^{m_1-1} \left(\sum_{m=0}^{\infty} B_{m,m_1} \right) \\ &\approx \frac{\alpha\tau}{N} \int_0^{\infty} e^{-\frac{\alpha}{N}u} \left(\sum_{m=0}^{\infty} B_{m,u} \right) du. \end{aligned} \quad (A5)$$

So, by introducing (A4) into (A5) one finally obtains an approximated expression for $\langle T_2 \rangle$. This result can be expressed as a combination of algebraic and exponential integral functions (the resulting expression is too long to be reproduced here).

Similarly, the second order moment of $\langle T_2 \rangle$ can be approximated using exactly the same procedure and leading to

$$\langle T_2^2 \rangle \approx \frac{2\alpha\tau^2}{N} \int_0^{\infty} e^{-\frac{\alpha}{N}u} \left(\sum_{m=0}^{\infty} (1+m)B_{m,u} \right) du \quad (A6)$$

and the same for the cross moment,

$$\langle T_1 T_2 \rangle \approx \frac{\alpha\tau^2}{N} \int_0^{\infty} u e^{-\frac{\alpha}{N}u} \left(\sum_{m=0}^{\infty} B_{m,u} \right) du, \quad (A7)$$

so the value of the correlation coefficient between T_1 and T_2 , as defined in (33), can be estimated.

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