

Voter model on networks and the multivariate beta distribution

Shintaro Mori*

*Department of Mathematics and Physics, Faculty of Science and Technology, Hirosaki University,
Bunkyo-cho 3, Hirosaki, Aomori 036-8561, Japan*

Masato Hisakado†

Nomura Holdings Inc., Otemachi 2-2-2, Chiyoda-ku, Tokyo 100-8130, Japan

Kazuaki Nakayama‡

Department of Mathematical Sciences, Faculty of Science, Shinshu University, Asahi 3-1-1, Matsumoto, Nagano 390-8621, Japan

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In elections, the vote shares or turnout rates show a strong spatial correlation. The logarithmic decay with distance suggests that a two-dimensional (2D) noisy diffusive equation describes the system. Based on the study of U.S. presidential elections data, it was determined that the fluctuations of vote shares also exhibit a strong and long-range spatial correlation. Previously, it was considered difficult to induce strong and long-range spatial correlation of the vote shares without breaking the empirically observed narrow distribution. We demonstrate that a voter model on networks shows such a behavior. In the model, there are many voters in a node who are affected by the agents in the node and by the agents in the linked nodes. A multivariate Wright-Fisher diffusion equation for the joint probability density of the vote shares is derived. The stationary distribution is a multivariate generalization of the beta distribution. In addition, we also estimate the equilibrium values and the covariance matrix of the vote shares and obtain a correspondence with a multivariate normal distribution. This approach largely simplifies the calibration of the parameters in the modeling of elections.

DOI: [10.1103/PhysRevE.99.052307](https://doi.org/10.1103/PhysRevE.99.052307)**I. INTRODUCTION**

Social physics has become an active research field [1–4] and many studies have been devoted to the understanding of social phenomena and interacting human behaviors [5–18]. Opinion dynamics is a central research theme, and empirical studies based on election data have been extensively pursued [19–22]. In these investigations, the correlation between the voters' decisions was evaluated by studying the dependence of the variance of the turnout rate on the number of voters N [20,21]. If the voters' decisions are independent, then the variance of the turnout rate should be proportional to N^{-1} . An empirical study of French election data showed that the voters' decisions were proportional to the power of $N^{-3/4}$. In addition, it was determined that the spatial correlation of the turnout rate in each election exhibited a logarithmic decay with distance that suggested a description based on a two-dimensional (2D) noisy diffusion equation.

A threshold model was introduced for the binary decision of an individual with intension field [20]. If the intension of an individual exceeds a certain threshold, then the decision is 1. When it is below the threshold, the decision is 0. The intension field was decomposed into the sum of a noise which is an instantaneous contribution, a space-dependent “cultural”

field and the influence of the decision of other individuals. Here, “cultural field” encodes all the local, stable features that influence the final decision. Without the noise and the cultural field, the model simplifies to the random field Ising model [23–25]. It was concluded that the long-range spatial correlations cannot be due to the influence of the decision of others, because the interaction cannot induce the empirically observed unimodal and narrow distribution of turnout rates. The long-range spatial correlation was thus attributed to that of the “cultural field.” As a phenomenological model of the cultural field, a 2D noisy diffusion equation was proposed.

The voter model and its noisy extension have been studied extensively in opinion dynamics [2,26–32]. In particular, the validity of the voter model as a model for elections was tested in the U.S. presidential election [22]. In this model, agents move between their living places and their workplaces. In both places, their decisions are affected by other voters. The model is called the social influence recurrent mobility (SIRM) model. Based on the diffusion approximation of the model, a noisy diffusion equation was derived. By balancing the strength of the noise with the voter model's consensus mechanism or the force of conformity, it was concluded that the SIRM model can reproduce the statistical features of the vote share in presidential elections, i.e., the stationarity of the variance of vote-share distributions and the long-range spatial correlation that decays logarithmically with distance. However, the model has a drawback in that under certain circumstances, the noise might break the range of vote shares. This was addressed by introducing the beta distributed noise [33].

*shintaro.mori@hirosaki-u.ac.jp

†hisakadom@yahoo.co.jp

‡nakayama@math.shinshu-u.ac.jp

Furthermore, a generalization to the case of more than two political parties was also proposed in the same framework.

In this report, we study the correlation of the fluctuations of vote shares using theoretical and empirical methods. Based on U.S. presidential election data, we show that the correlation of the fluctuation of the vote shares between the nearest neighbor counties exceed 80% and it is much higher than that of the temporal averages of the vote shares. Furthermore, as with the latter ones, the fluctuation also shows long-range spatial correlation. In the threshold model without the influence of the decisions of others, the fluctuations of the vote shares are independent of each other even if the cultural field shows a strong spatial correlation. The correlation of the cultural field only affects the correlation of the temporal averages of the vote shares. The threshold model with the social influence term is inappropriate for inducing such a strong correlation of the fluctuations because it contradicts the empirical results. Therefore, an alternate model that can incorporate a strong correlation without losing the unimodality of the vote share distribution should be introduced. According to the results of the SIRM model, a voter model should be a good candidate. We show that the vote shares of a voter model on networks obeys a multivariate beta distribution which can incorporate strong correlation without losing the unimodality of the distribution of the vote shares. Furthermore, the distribution is similar to the multivariate normal distribution and the calibration of the model parameters is easy.

The paper is organized into multiple sections. In Sec. II, the U.S. presidential election data is studied and the vote shares are decomposed into the equilibrium values and the fluctuations around them. The cultural field is encoded in the former and both exhibit strong and long-ranged spatial correlation. It is shown that the vote shares approximately obey a multivariate normal distribution. A voter model on networks is introduced in Sec. III. The multivariate Wright-Fisher diffusion equation is then derived for the joint probability density function (pdf) of the vote shares. The stationary distribution is a multivariate beta distribution. We approximate the distribution using a multivariate normal distribution and estimate the covariance matrix of the vote shares. Section IV is devoted to the numerical analysis and verification of the theoretical results. Section V includes the conclusions and discussions of future problems.

II. EMPIRICAL STUDY

U.S. presidential election data from 1980 to 2016 were studied. A total of 10 elections occurred during this interval and they are labeled as $t = 1, 2, \dots, T = 10$, where $t = 1$ corresponds to the election in 1980. The data of 3105 counties was studied and label as $i = 1, 2, \dots, I$. The data consist of the number of votes $N(i, t)$ and the number of votes for the democratic party $n(i, t)$ in county i and election t . The total number of votes cast in election t is calculated as $N_T(t) = \sum_i N(i, t)$. The votes that were not cast for either the Democratic party or the Republican party were excluded, and the votes for the latter party is given by $N(i, t) - n(i, t)$. We denote the vote share for the democratic party as $v(i, t) \equiv n(i, t)/N(i, t)$.

Initially, we detrend the vote share data. The weighted spatial average of $v(i, t)$ is estimated as

$$v(t) \equiv \sum_i n(i, t)/N_T(t).$$

We estimate the temporal average of $v(t)$ as $v_{\text{avg}} \equiv \sum_t v(t)/T$ and obtain the detrended vote share as

$$v_d(i, t) \equiv v(i, t) - [v(t) - v_{\text{avg}}].$$

Based on this process, the weighted spatial average of $v_d(i, t)$ becomes v_{avg} and it does not depend on t . The temporal average of $v_d(i, t)$ is defined as $v_d(i) = \sum_t v_d(i, t)/T$, which is an estimate of the equilibrium values of $v_d(i, t)$ in county i . We interpret $v_d(i)$ as the ‘‘cultural field’’ because it reflects the local and stable features. We denote the fluctuation (deviation) of $v_d(i, t)$ around $v_d(i)$ as $\Delta v_d(i, t) \equiv v_d(i, t) - v_d(i)$. Figure 1(a) shows the distribution of $\Delta v_d(i, t)$. The standard deviation (SD) is approximately 8% and slightly left skewed.

We study the N dependence of the variance of $v_d(i, t)$. The spatial average of $v_d(i)$ is denoted as $v_{d,\text{avg}} = \sum_i v_d(i)/I$. The fluctuation of $v_d(i, t)$ around $v_{d,\text{avg}}$ is decomposed as the sum of the fluctuation of $\Delta v_d(i, t)$ and that of $v_d(i)$,

$$\begin{aligned} V[v_d(i, t)] &= \frac{1}{IT} \sum_{i,t} [v_d(i, t) - v_{d,\text{avg}}]^2 \\ &= \frac{1}{IT} \sum_{i,t} [v_d(i, t) - v_d(i) + v_d(i) - v_{d,\text{avg}}]^2 \\ &= \frac{1}{IT} \sum_{i,t} \{\Delta v_d(i, t)^2 + [v_d(i) - v_{d,\text{avg}}]^2\} \\ &= V[\Delta v_d(i, t)] + V[v_d(i)]. \end{aligned}$$

As $\sum_t \Delta v_d(i, t) = 0$, the cross term vanishes and the third equality holds. The N dependence of the fluctuation of $\Delta v_d(i, t)$ is then investigated. We bin $v_d(i, t)$ according to $N(i) \equiv \sum_t N(i, t)/T$ into 31 classes and each class contains 100 counties, with almost the same number of average votes $N(i)$. As previously discussed [20], if the voters choose independently, then the variance of $\Delta v_d(i, t)$ is proportional to the inverse of $N(i)$ as $v_d(i)[1 - v_d(i)]/N(i)$. Figure 1(b) plots $V[v_d(i, t)]$, $V[v_d(i)]$, and $V[\Delta v_d(i, t)]$ vs. $1/N$. It is evident that $V[\Delta v_d(i, t)]$ is much larger than $1/4N$ in all the bins. One also observes that the decomposition of the variance holds.

Next, we study the spatial correlation of $v_d(i, t)$. There are $I(I - 1)/2 \simeq 4.81 \times 10^6$ pairs of counties (i, j) , $i, j \in \{1, 2, \dots, I\}$. They are sorted according to the distance $r(i, j)$ between county i and county j . The database of the national bureau of economic research is utilized to obtain information on the intercounty distance [34]. The distance is taken as the separation of the centroids. The sorted pairs (i, j) are binned into 481 classes and each class contains 10^4 pairs. We label the bin of the county pairs (i, j) separated by their average distance r as $R(r)$ and $|R(r)| = 10^4$ represents the number of pairs in the bin.

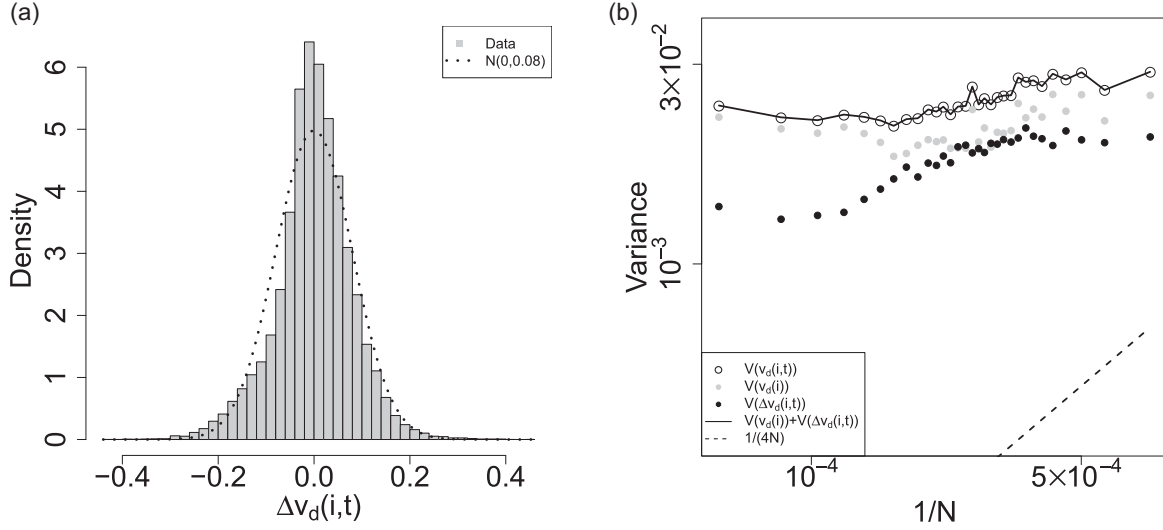


FIG. 1. (a) Plot of the distribution of $\Delta v_d(i, t) = v_d(i, t) - v_d(i)$. The dotted line shows the normal distribution with the same mean and the variance. (b) Plot of the variance of $v_d(i, t)$ and its decomposition to the variance of $v_d(i)$ and $\Delta v_d(i, t)$ vs. the inverse of the average value of $N(i)$. The solid line shows the sum of the two variances. The broken line shows $1/(4N)$.

The covariance of $v_d(i, t)$ and $v_d(j, t)$ of the pairs in $R(r)$ are defined as:

$$\text{Cov}[v_d(i, t), v_d(j, t)|r] = \frac{1}{T} \sum_t \frac{1}{|R(r)|} \sum_{(i, j) \in R(r)} \{v_d(i, t) - v_d[R(r)]\} \{v_d(j, t) - v_d[R(r)]\}.$$

Here $v_d[R(r)]$ is the average value of $v_d(i, t)$, $v_d(j, t)$ in $R(r)$,

$$\begin{aligned} v_d[R(r)] &= \sum_t \sum_{(i, j) \in R(r)} v_d(i, t) / |R(r)| T \\ &= \sum_t \sum_{(i, j) \in R(r)} v_d(j, t) / |R(r)| T. \end{aligned}$$

The covariance is then decomposed using the following identity:

$$\begin{aligned} &\{v_d(i, t) - v_d[R(r)]\} \{v_d(j, t) - v_d[R(r)]\} \\ &= \{v_d(i, t) - v_d(i) + v_d(i) - v_d[R(r)]\} \{v_d(j, t) - v_d(j) \\ &\quad + v_d(j) - v_d[R(r)]\}. \end{aligned}$$

We then obtain the next decomposition of the covariance as the cross term vanishes by the equality $\sum_t [v_d(i, t) - v_d(i)] = 0$,

$$\begin{aligned} \text{Cov}(v_d(i, t), v_d(j, t)|r) &= \text{Cov}[\Delta v_d(i, t), \Delta v_d(j, t)|r] \\ &\quad + \text{Cov}[v_d(i), v_d(j)|r]. \end{aligned}$$

By normalizing the covariances with the variances, we estimate the correlation coefficients ρ for $v_d(i, t)$, $v_d(i)$, and $\Delta v_d(i, t)$. Figure 2 represents the semilogarithmic plot of the covariance and the correlation vs. r .

The left figure plots the covariance vs. r . The solid line shows the sum of the covariance of $v_d(i)$ and $\Delta v_d(i, t)$. It is evident that the sum lies on the plot of the covariance of $v_d(i, t)$. The covariance of $\Delta v_d(i, t)$ is larger than that of $v_d(i)$. The right figure plots the correlation coefficient ρ vs. r . In all the three cases, the correlation exhibits a logarithmic

decay with r . The interesting point is that the correlation of $\Delta v_d(i, t)$ is the largest and it is over 80% for the bin of the nearest neighbor county pairs. The correlation of $\Delta v_d(i, t)$ for nearest-neighbor pairs is estimated to be 83.4%. The spatial correlation of $v_d(i)$ implies that the cultural field of two counties are similar when these counties are near each other. The spatial correlation of $\Delta v_d(i, t)$ represents the comovement of the voters' decisions in the two counties. These two correlations have completely different physical origins. For $r \geq 10^3$ km, an unusual behavior is observed. The correlation decays with r and local maxima appear in the correlation of $v_d(i)$ and $v_d(i, t)$. The correlation of $\Delta v_d(k, t)$ shows a monotonically decreasing behavior up to 2000 km.

The results will now be summarized. We decompose I variables $\vec{v}_d(t) = [v_d(1, t), \dots, v_d(I, t)]$ as the sum of $\vec{v}_d = [v_d(1), \dots, v_d(I)]$ and $\Delta \vec{v}_d(t) = [\Delta v_d(1, t), \dots, \Delta v_d(I, t)]$. $\vec{v}_d(t)$ fluctuates around \vec{v}_d . \vec{v}_d is a proxy for the cultural field and $\Delta \vec{v}_d(t)$ shows a stronger spatial correlation than \vec{v}_d . $\vec{v}_d(t)$ approximately obeys a multivariate normal distribution with a mean of \vec{v}_d and the variance-covariance matrix $V(\vec{v}_d) + V[\Delta \vec{v}_d(t)]$,

$$\vec{v}_d(t) = \vec{v}_d + \Delta \vec{v}_d(t) \sim N_I\{\vec{v}_d, V(\vec{v}_d) + V[\Delta \vec{v}_d(t)]\}.$$

It should be noted that the timescale of $\Delta \vec{v}_d(t)$ and that of \vec{v}_d are quite different. The voting habits of the different regions is extremely persistent and the timescale of \vec{v}_d is a century or more [20]. However, $\Delta \vec{v}_d(t)$ fluctuates rapidly and the timescale is short.

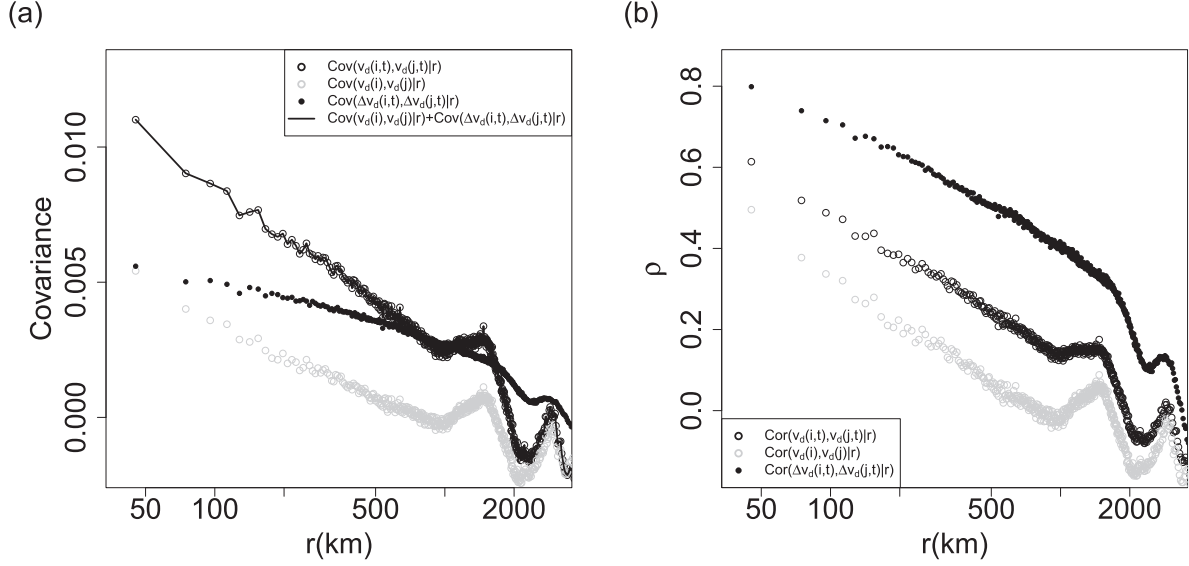


FIG. 2. Semilogarithmic plots of the (a) covariance and (b) correlation of $v_d(i, t)$, $v_d(i)$, and $\Delta v_d(i, t)$ vs. r . The solid line in the left figure shows the sum of the covariance of $v_d(i)$ and $\Delta v_d(i, t)$. According to the decomposition of the covariance, the contribution of $\Delta v_d(i, t)$ is larger than that of $v_d(i)$. The correlation coefficient of $\Delta v_d(i, t)$ is also larger than that of $v_d(i)$.

III. SOCIAL INFLUENCE MODEL ON NETWORKS

We now introduce a voter model on networks. There are I nodes and they are labeled as $i = 1, 2, \dots, I$. The link set $E = \{(i, j)\}$ consists of links that connect node i and j . $J(i) \equiv \{j | (i, j) \in E\}$ denotes the set of nodes that are linked with node i and $|J(i)|$ is the number of nodes linked with node i . In each node, there are N_i agents whose decisions obey the dynamics of the voter model [22,26]. One agent is chosen at random from $N_T = \sum_i N_i$ agents. If the agent is from node i , another agent is chosen from node i or from node $j \in J(i)$, which is connected to node i . n_i and $v_i \equiv n_i/N_i$ denote the number of votes and the vote share of an option. We assume that the intrinsic tendency of the voters in node i to vote for an option is determined by the parameters μ_i and θ_1 . μ_i is the probability that a voter votes for an option and θ_1 is a parameter that controls the variance of the vote share. Intuitively, θ_1 represents the number of voters who are not influenced by other voters. $a_i \equiv \mu_i \theta_1$ and $b_i \equiv (1 - \mu_i) \theta_1$ corresponds to the number of such voters who choose and do not choose the option, respectively. The strength of the influence of the voters in the linked node $j \in J(i)$ is denoted as θ_2 . Figure 3 illustrates the model.

The probability that the number of voters for an option n_i increases by 1 is written as the product of the probabilities of the next two processes. Initially, a voter of node i who does not choose the option is selected. The probability is $(N_i - n_i)/N_T$. Second, an infectious voter who can affect the voter and choose the option is selected. The infectious voters should live in node i or in the linked node j , $j \in J(i)$. θ_1 voters in node i also can affect the voter. The number of infectious voters in the linked node j is set to be θ_2 . This is the simplification of the infectious process. The total number of the infectious voter is $N_i - 1 + \theta_1 + \theta_2 |J(i)|$. Here -1 of $N_i - 1$ indicates that we omit the voter who is chosen in the first process. Among them $n_i + a_i + \theta_2 \sum_{j \in J(i)} v_j$ choose the option. The probability for

the second process is then:

$$\frac{n_i + a_i + \theta_2 \sum_{j \in J(i)} v_j}{N_i - 1 + \theta_1 + \theta_2 |J(i)|}.$$

The probability for $n_i \rightarrow n_i + 1$ is then written as:

$$P(n_i \rightarrow n_i + 1) = \frac{N_i - n_i}{N_T} \frac{n_i + a_i + \theta_2 \sum_{j \in J(i)} v_j}{N_i - 1 + \theta_1 + \theta_2 |J(i)|}.$$

Likewise, the probability that n_i decreases by 1 is written as:

$$P(n_i \rightarrow n_i - 1) = \frac{n_i}{N_T} \frac{N_i - n_i + b_i + \theta_2 \sum_{j \in J(i)} (1 - v_j)}{N_i - 1 + \theta_1 + \theta_2 |J(i)|}.$$

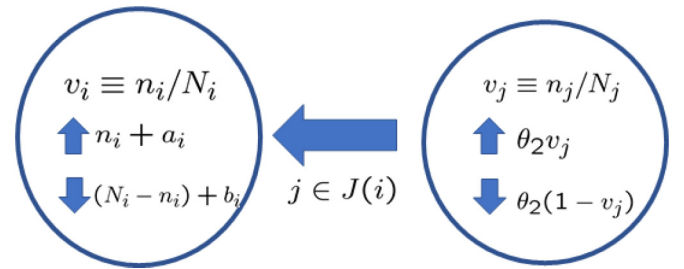


FIG. 3. Illustration of the voter model. The big circles represent node i and node $j \in J(i)$ that is linked with node i . There are N_i and N_j voters in node i and node j , respectively. Among them $n_i(n_j)$ voters (up arrow) and $N_i - n_i(N_j - n_j)$ voters (down arrow) are for and against an option in node $i(j)$. Initially, a susceptible voter is chosen at random from node i . Next, an infectious voter is chosen from node i or node $j \in J(i)$. Node i has $N_i - 1$ infectious voters after a voter is chosen from the node. In addition, there are $\theta_1 = a_i + b_i$ voters who do not change their decisions. Among them, a_i and b_i voters are for and against the option, respectively, $\mu_i = a_i/\theta_1$ is the intrinsic tendency of node i to vote for the option and θ_1 controls the strength of the tendency. Node j has θ_2 infectious voters.

There are two main differences in this model compared to the SIRM model [22,33]. One difference is the node-dependent terms μ_i and θ_1 . If the influence from other voters in the linked nodes $j \in J(i)$ is turned off by setting $\theta_2 = 0$, then the model reduces to Kirman's ant colony model [5,35]. The stationary probability distribution of n_i is the beta binomial distribution. The vote share v_i obeys a beta distribution with shape parameters (a_i, b_i) in the limit $N_i \rightarrow \infty$. The derivation of the beta distribution is given in Appendix B,

$$v_i \sim \text{Beta}(a_i, b_i).$$

The expectation value of v_i is μ_i and the variance of v_i is $\mu_i(1 - \mu_i)/(\theta_1 + 1)$. This variance originates from the interaction between the voters in node i . The correlation of the voters binary choices is $1/(\theta_1 + 1)$ [36]. The second change is the normalization of n_j by N_j and we use $v_j = n_j/N_j$. The mathematical reason for the modification is to avoid the ill-posedness in the original SIRM model [33]. As we shall show shortly, if we normalize as indicated, the noise term becomes proportional to $\sqrt{v_i(1 - v_i)}$ as in the Wright-Fisher diffusion equation and it does not break the condition $v_i \in (0, 1)$ even when v_i approaches 0 or 1. Appendix A shows how to treat the SIRM model in the same framework.

The raising operator $R_i \equiv P(n_i \rightarrow n_i + 1)$ is written using vote shares $\vec{v} = (v_1, v_2, \dots, v_l)$ as:

$$R_i(\vec{v}) = \frac{N_i}{N_T} (1 - v_i) \frac{v_i + a_i/N_i + \theta_2 \sum_{j \in J(i)} v_j/N_i}{1 - 1/N_i + \theta_1/N_i + \theta_2 \sum_{j \in J(i)} v_j/N_i}.$$

The lowering operator $L_i \equiv P(n_i \rightarrow n_i - 1)$ is also written as:

$$L_i(\vec{v}) = \frac{N_i}{N_T} v_i \frac{(1 - v_i) + b_i/N_i + \theta_2 \sum_{j \in J(i)} (1 - v_j)/N_i}{1 - 1/N_i + \theta_1/N_i + \theta_2 \sum_{j \in J(i)} v_j/N_i}.$$

We write $v_{i,c}$ for the average value of the vote shares $\{v_j\}$ of the linked nodes $j \in J(i)$.

$$v_{i,c} = \sum_{j \in J(i)} v_j / |J(i)|.$$

R_i, L_i are then rewritten as

$$R_i(\vec{v}) = \frac{N_i}{N_T} (1 - v_i) \frac{v_i + a_i/N_i + \theta_2 |J(i)| v_{i,c}/N_i}{1 - 1/N_i + \theta_1/N_i + \theta_2 |J(i)|/N_i},$$

$$L_i(\vec{v}) = \frac{N_i}{N_T} v_i \frac{(1 - v_i) + b_i/N_i + \theta_2 |J(i)| (1 - v_{i,c})/N_i}{1 - 1/N_i + \theta_1/N_i + \theta_2 |J(i)|/N_i}.$$

The stochastic differential equation [37] for v_i is written with drift d_i and diffusion D_i as:

$$dv_i = d_i dt + \sqrt{D_i} dW_i(t).$$

Here $dW_i(t)$ is independent and identically distributed white noise or Brownian motion. The drift term d_i is estimated as:

$$d_i = \frac{\delta v_i}{\delta t} (R_i - L_i) = \frac{\delta v_i}{\delta t} \frac{1}{N_T} [a_i - \theta_1 v_i + \theta_2 |J(i)| (v_{i,c} - v_i)].$$

Here we take the limit $N_i \rightarrow \infty$ in the second equality. The diffusion term D_i is estimated as:

$$D_i = \frac{(\delta v_i)^2}{\delta t} (R_i + L_i) = \frac{(\delta v_i)^2}{\delta t} \frac{N_i}{N_T} 2v_i(1 - v_i).$$

If we set $N_i = N$ and $N_T = IN$, then we have $\delta v_i = 1/N$ and $\delta t = 1/IN^2$. d_i and D_i are written as:

$$d_i = [a_i - \theta_1 v_i + \theta_2 |J(i)| (v_{i,c} - v_i)]$$

$$D_i = 2v_i(1 - v_i).$$

The Fokker-Planck equation for the time evolution of the joint probability density function $f(\vec{v}, t)$ is give as:

$$\partial_t f(\vec{v}, t) = - \sum_i \left\{ \partial_i d_i - \frac{1}{2} \partial_i^2 D_i \right\} f. \quad (1)$$

Here we write the derivative by v_i as ∂_i . This is a multivariate Wright-Fisher diffusion process [38].

Given that the drift and diffusion terms do not explicitly depend on t , the stochastic system is a statistically stationary process and the solution of the Fokker-Planck equation converges to a stationary distribution [37],

$$f_{\text{st}}(\vec{v}) = \lim_{t \rightarrow \infty} f(\vec{v}, t).$$

We define $J_i \equiv d_i f - \frac{1}{2} \partial_i D_i f$ and Eq. (1) can be written as:

$$\partial_t f(\vec{v}, t) = - \sum_i \partial_i J_i.$$

We obtain f_{st} by solving $J_i = 0$ and we have:

$$[d_i - \frac{1}{2} (\partial_i D_i)] f_{\text{st}} = \frac{1}{2} D_i \partial_i f_{\text{st}}.$$

The equation can be rewritten as

$$Z_i \equiv \frac{(2d_i - \partial_i D_i)}{D_i} = \partial_i \ln f_{\text{st}}.$$

We see Z_i should satisfy $\partial_j Z_i = \partial_i Z_j$. A potential solution $f_{\text{st}}(\vec{v}) = e^{-\phi(\vec{v})}$ of Eq. (1) exists [37,39] if ϕ satisfy

$$-\partial_i \phi = \partial_i \ln f_{\text{st}} = \frac{2d_i - \partial_i D_i}{D_i} \equiv Z_i.$$

From the constraint $\partial_j Z_i = \partial_i Z_j$, we obtain

$$\frac{\theta_2}{v_i(1 - v_i)} = \frac{\theta_2}{v_j(1 - v_j)}.$$

If we set $\theta_2 = 0$, then the stationary solution $f_{\text{st}}^0(\vec{v})$ becomes the direct product of the beta distribution $f_{\text{Beta}}(v_i | a_i, b_i)$,

$$f_{\text{st}}^0(\vec{v}) = \prod_i f_{\text{Beta}}(v_i | a_i, b_i).$$

The expectation value of \vec{v} is $\vec{\mu} = (\mu_1, \mu_2, \dots, \mu_l)$ and the covariance matrix Σ^0 of \vec{v} is given by:

$$\Sigma_{i,j}^0 = \text{Cov}(\vec{v})_{i,j} = \delta_{i,j} v_i(1 - v_i) \frac{1}{\theta_1 + 1}.$$

When $\theta_1 \gg 1$, the joint probability function $f_{\text{st}}^0(\vec{v})$ can be approximated by the multivariate normal distribution as:

$$\vec{v} \sim N_l(\vec{\mu}, \Sigma^0).$$

When $\theta_2 \neq 0$, the potential solution does not exist. \vec{v} fluctuates around their equilibrium values \vec{v}^* and v_i^* is determined by the condition that $d_i = 0$,

$$v_i^* = \mu_i + (\theta_2 |J(i)| / \theta_1) (v_{i,c}^* - v_i^*).$$

Here $v_{i,c}^*$ is the average value of v_j^* , $j \in J(i)$.

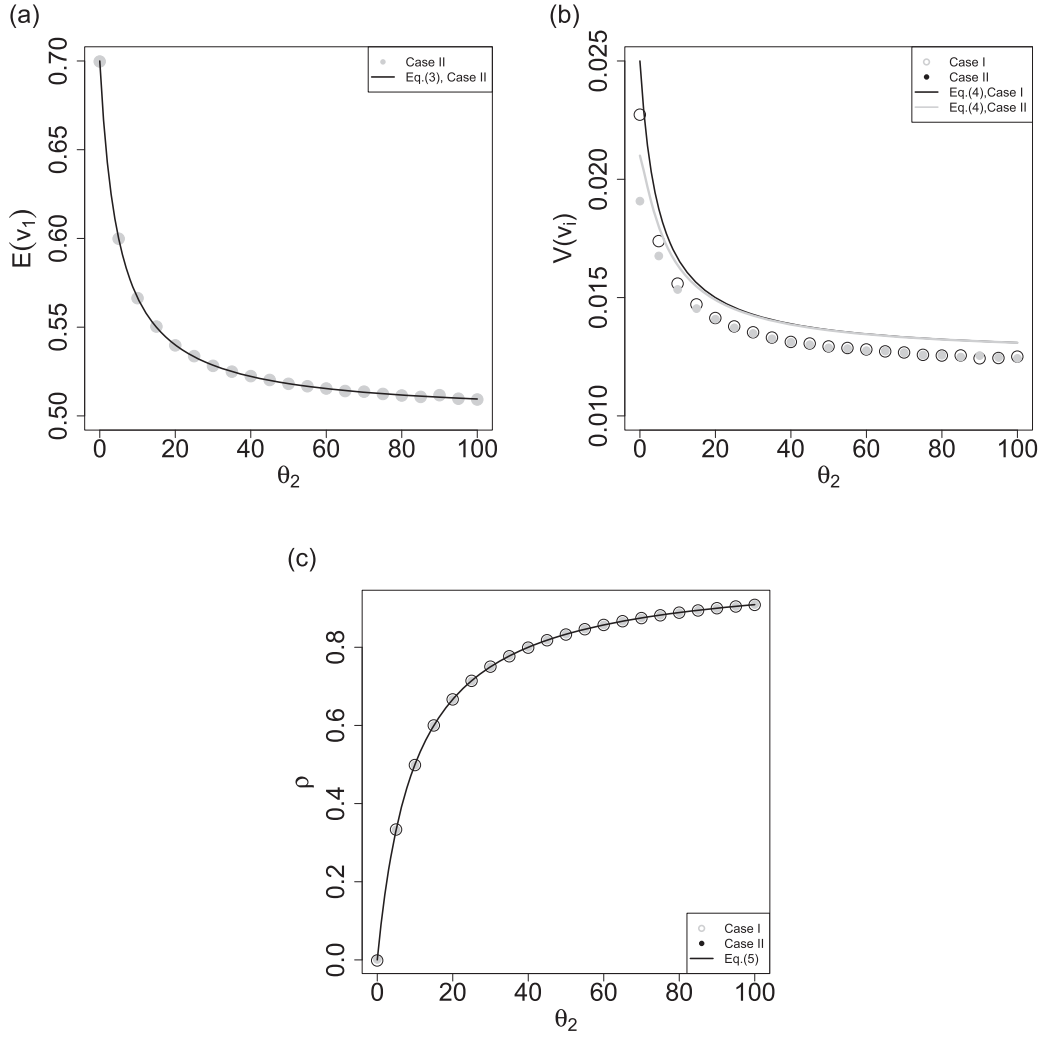


FIG. 4. (a) Plots of $E(v_1)$ vs. θ_2 for case II. Solid line plots of Eq. (3) vs. θ_2 . (b) Plots of $V(v_1)$ vs. θ_2 for case I (gray circle), and case II (hollow black circle). The theoretical results for Eq. (4) are plotted using solid curves in black and gray for case I and case II, respectively. (c) Plots of ρ vs. θ_c for case I (gray circle), case II (hollow black circle) and the theoretical results (solid curves) of Eq. (5).

When $\theta_2 \gg \theta_1$, $v_i^* \simeq v_{i,c}^*$ holds and v_i^* is equal to the average value of μ_i , $\mu_{\text{avg}} \equiv \sum_i \mu_i / I$. It is assumed that the fluctuation of \vec{v} around μ_{avg} is small and approximates $D_i = 2v_i(1 - v_i)$ as $D_i = D = 2\mu_{\text{avg}}(1 - \mu_{\text{avg}})$. In this case, the potential condition is satisfied. We call the approximation ‘‘Gaussian approximation,’’ because the Wright-Fisher diffusion equation has a solution that can be approximated as a Gaussian distribution. We obtain $\ln f_{\text{st}}$ as:

$$\begin{aligned} \ln f_{\text{st}} &= \int^{\vec{v}} \sum_i Z_i dv_i \\ &= \frac{1}{\mu_{\text{avg}}(1 - \mu_{\text{avg}})} \left[\theta_1 \bar{\mu} \vec{v} - \mu_{\text{avg}}(1 - \mu_{\text{avg}}) \frac{1}{2} {}^t \vec{v} \Sigma^{-1} \vec{v} \right]. \end{aligned}$$

The inverse of the covariance matrix Σ^{-1} is

$$\mu_{\text{avg}}(1 - \mu_{\text{avg}})(\Sigma^{-1})_{i,j} = \begin{cases} \theta_1 + \theta_2 |J(i)| & i = j \\ -\theta_2 & j \in J(i) \\ 0 & i \neq j, j \notin J(i) \end{cases}.$$

The multivariate normal approximation of \vec{v} is

$$\vec{v} \sim N_I(\vec{v}^*, \Sigma). \quad (2)$$

IV. NUMERICAL STUDY

We then numerically verify the validity of the normal distribution approximation. The conditional probability density function for v_i with $v_{i,c}$ fixed is a beta distribution with the shape parameters $a_i(v_{i,c}) = \theta_1 \mu_i + \theta_2 |J(i)| v_{i,c}$, $b_i(v_{i,c}) = \theta_1(1 - \mu_i) + \theta_2 |J(i)|(1 - v_{i,c})$,

$$v_i \sim \text{Beta}[a_i(v_{i,c}), b_i(v_{i,c})].$$

We set the initial values for \vec{v} as $v_i \sim \text{Beta}(a_i, b_i)$ with $v_{i,c} = 0.5$. Thereafter, we choose a node i at random and calculate the shape parameters $a_i(v_i^c)$, $b_i(v_i^c)$ and generate new v_i according to $v_i \sim \text{Beta}[a_i(v_i^c), b_i(v_i^c)]$. The process is repeated for I times (1 MCS) and we obtain a sample $\vec{v}(1)$. The procedure is repeated with the initial condition $\vec{v}(t)$, $t = 1, \dots$ and we obtain a sample $\vec{v}(t + 1)$. The length (MCS) of the sample sequence T is set as 10^6 . In a 2D system, we set $T = 2 \times 10^5$.

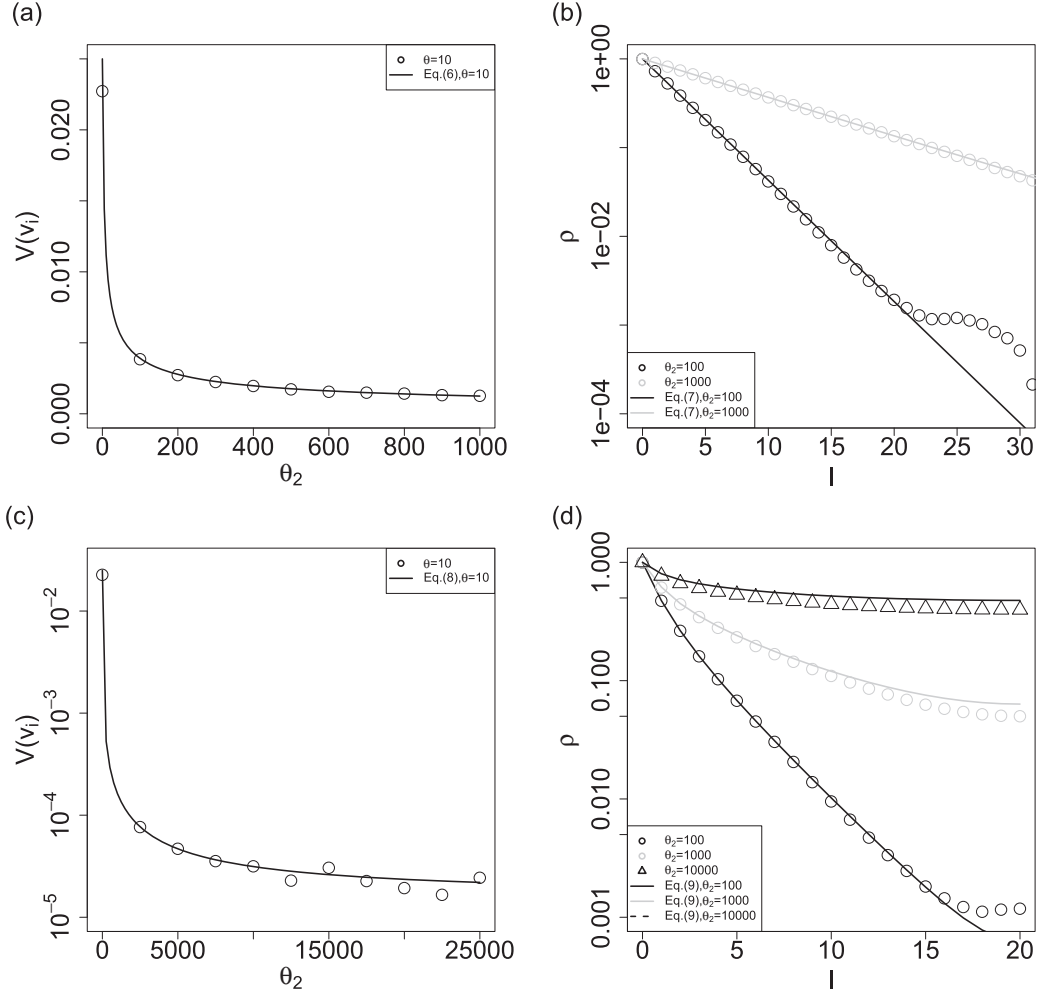


FIG. 5. (a) Plots of $V(v_i)$ vs. θ_2 for a 1D lattice with $L = 10^2$. (b) Plots of ρ vs. l for a 1D lattice with $L = 10^2$. (c) Plots of $V[v_{(i,j)}]$ vs. θ_2 for a 2D lattice with $L = 40$. (d) Semilogarithmic plot of ρ vs. l for a 2D with $L = 40$. We adopt $\theta = 10$ and $\mu_i = \mu_{(i,j)} = \mu_{\text{avg}} = 1/2$. In (a) and (c), solid line plots Eq. (6) vs. θ_2 for a 1D and Eq. (8) vs. θ_2 for a 2D. We adopt $\theta_2 = 10^2$ and 10^3 for a 1D lattice in (b) and $\theta_2 = 10^2$, 10^3 and $\theta_2 = 10^4$ for a 2D lattice in (d). In (b) and (d), we plot the theoretical results from Eq. (7) for 1D and those from Eq. (9) for 2D by solid and broken curves.

A. Two nodes ($I = 2$) case

At first, we consider the $I = 2$ case. We adopt μ_1, μ_2 so that $\mu_{\text{avg}} = 0.5$. We then set $\theta_1 = 10$ and $a_i = b_i = 5$, $\mu_i = 0.5$ in case I. In case II, we set $\theta_1 = 10$ and $a_1 = b_2 = 7$, $a_2 = b_1 = 3$, $\mu_1 = 0.7$, $\mu_2 = 0.3$. Based on the symmetry of the system, v_i^* is estimated as:

$$v_{1,2}^* = \frac{1}{2} \frac{a_1 + a_2}{\theta_1} \pm \frac{a_1 - a_2}{\theta_1 + 2\theta_2}. \quad (3)$$

The variance of v_i is

$$V(v_i) = \mu_{\text{avg}}(1 - \mu_{\text{avg}}) \frac{\theta_1 + \theta_2}{\theta_1^2 + 2\theta_1\theta_2}. \quad (4)$$

The correlation coefficient ρ of v_1 and v_2 is

$$\rho \equiv \frac{\text{Cov}(v_1, v_2)}{\sqrt{V(v_1)V(v_2)}} = \frac{\theta_2}{\theta_1 + \theta_2}. \quad (5)$$

Figure 4 shows the results of the MC studies. The numerical data are plotted with symbols and the Gaussian approximation results are presented with lines. Figure 4(a) shows $E(v_i)$

and v_i^* in Eq. (3) vs. θ_2 for case II. Figure 4(b) shows $V(v_i)$ and Eq. (4) vs. θ_2 . Figure 4(c) shows the correlation coefficient ρ and Eq. (5) vs. θ_2 . There is some discrepancy in the estimation of the variance, which originates from the diffusion approximation. We see that the Gaussian approximation works well.

B. Lattice case

Next, we investigate the 1D lattice and 2D square lattice cases. We are interested in the r dependence of the correlation. We consider L sites for a 1D lattice and $L \times L$ sites for a 2D lattice. The periodic boundary condition is imposed in both cases. The nodes are indexed by $i \in \{1, \dots, L\}$ for the 1D lattice and (i, j) , $i, j \in \{1, \dots, L\}$ for the 2D lattice, respectively. Nodes are linked with their nearest neighbors and $|J(i)| = 2(4)$ for a 1D (2D) lattice. We set $\mu_i = \mu_{(i,j)} = \mu_{\text{avg}} = 1/2$ and $\theta_1 = 10$.

\vec{v} obeys a multivariate normal distribution for 1D lattice case. The inverse of the covariance matrix Σ^{-1} for the 1D lattice is

$$(\Sigma^{-1})_{i,j} = (\theta_1 + 2\theta_2)\delta_{i,j} + \theta_2(\delta_{i+1,j} + \delta_{i-1,j}).$$

For a 2D lattice, the inverse of the covariance matrix Σ^{-1} is

$$(\Sigma^{-1})_{(i,j),(k,l)} = (\theta + 4\theta_2)\delta_{i,k}\delta_{j,l} + \theta_2(\delta_{i+1,k}\delta_{j,l} + \delta_{i-1,k}\delta_{j,l} + \delta_{i,k}\delta_{j+1,l} + \delta_{i,k}\delta_{j-1,l}).$$

The variance of v_i is given by:

$$V(v_i) = \mu_{\text{avg}}(1 - \mu_{\text{avg}})\Sigma_{i,i}. \quad (6)$$

The correlation between v_i and v_{i+l} is

$$\rho(l) = \Sigma_{1,1+l}/\Sigma_{1,1}. \quad (7)$$

For a 2D lattice case, we obtain similar equations by replacing $\Sigma_{i,j}$ with $\Sigma_{(i,j),(k,l)}$.

$$V[v_{(i,j)}] = \mu_{\text{avg}}(1 - \mu_{\text{avg}})\Sigma_{(i,j),(i,j)}. \quad (8)$$

The correlation between $v_{(i,j)}$ and $v_{(i+l,j)}$ is

$$\rho(l) = \Sigma_{(1,1),(1+l,1)}/\Sigma_{(1,1),(1,1)}. \quad (9)$$

Figure 5 shows a comparison of the MC data with the results of the Gaussian approximation. It is evident that the multivariate normal distribution describes the joint probability function of \vec{v} quite well. Furthermore, we can confirm that the l dependence of the correlation $\rho(l)$ decays exponentially with r for the 1D lattice case. For the 2D case, we also observe the exponential decay for $\theta_2 = 10^2$. For large $\theta_2 = 10^3, 10^4$, the l dependence does not obey an exponential decay. The correlation length becomes comparable with the system size $L = 40$ and the exponential decay is not observed for the limited system size.

C. U.S. county network case

Here we calibrate the model parameters θ_1, θ_2 using the U.S. presidential election data in Sec. II and the Gaussian approximation of the model. We construct an artificial county network where 3105 counties constitute nodes of the network and the counties with their nearest z neighbors are connected as links. Here, z neighbors are determined based on the geodesic distance of the separation of the centroids. If $j \in J(i)$ and $i \notin J(j)$, then we add i in $J(j)$. The number of neighbors depend on i . We adopt $z \in \{3, 4, 5\}$ and all the nodes are included in the largest components. We then set θ_1, θ_2 so that SD of v_i is approximately 8% and the correlation ρ between the nearest neighbor counties becomes approximately 83%, which are the empirical values in Sec. II. We adopt $\theta_1 = \{0.044, 0.034, 0.03\}$ and $\theta_2 = \{73, 50, 40\}$ for $z = \{3, 4, 5\}$, respectively. Given that $\theta_2 \gg \theta_1$, the equilibrium values v_i^* are almost equal. This suggests that the cultural field cannot be encoded in the model parameters μ_i . This point is discussed in the last section. Here we adopt $\mu_i = 1/2$.

Figure 6 shows the results. The correlation ρ of $\vec{v}(t)$ is plotted as a function of r for the three cases $z = 3$ (solid black), $z = 4$ (solid, gray), and $z = 5$ (broken black). When we set $\mu_{\text{avg}} = 0.5, v_i^* = 0.5$ and the correlation of $\vec{v}(t)$ are the same as that of $\Delta\vec{v}(t)$. They start from the same value of 83% of the nearest-neighbor correlation and decay monotonically with r . As z increases, the decay rate becomes small and the model shows a longer spatial correlation. We also plot the empirical results of the correlation of $\Delta\vec{v}_d(t)$ as a function of r using the symbols \circ . The $z = 5$ case best fits the empirical behavior of the correlation of $\Delta\vec{v}_d(t)$ among the three cases.

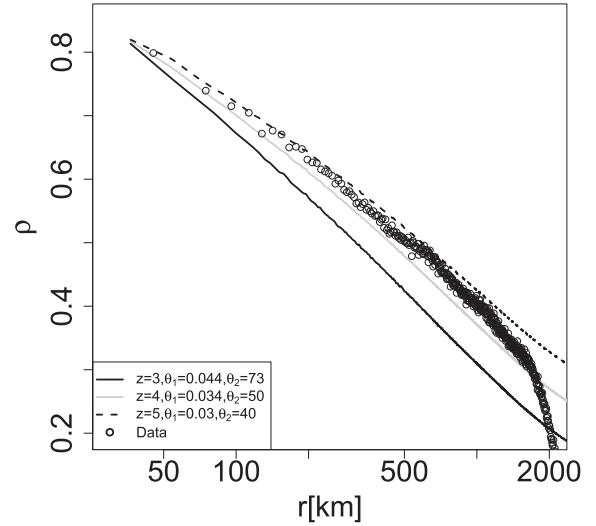


FIG. 6. Semilogarithmic plots of the correlation of v_i (model) vs. r . We adopt $\theta_1 = \{0.044, 0.034, 0.03\}$ and $\theta_2 = \{73, 50, 40\}$ for $z = \{3, 4, 5\}$, respectively. The Gaussian approximation is used to estimate the correlation of the models. The symbols (\circ) is used to plot the correlation of Δv_d vs. r .

V. CONCLUSIONS

In this report, we study the fluctuation of vote share in U.S. presidential election data. Compared with the temporal average of the vote shares in each county, the fluctuation shows a stronger and long-range correlation. In order to describe the behavior, we propose a voter model on networks. There are many voters in each node and they choose another voter at random and copy the another voter's choice as in the case of the voter model. Another voter is selected from the same node where the voter lives or from a neighboring node. Each node has an intrinsic parameter $\bar{\mu}$ that determines the preference for an option. In addition, θ_1 and θ_2 incorporate the influence from the voters in the nodes where the voters live and from the voters in the linked nodes, respectively. We derive the multivariate Wright-Fisher diffusion equation for the joint pdf of the vote shares. The pdf is a multivariate generalization of the beta distribution. We approximate the pdf using the multivariate normal distribution and estimate the variance and the correlation coefficient of the vote shares. The results were then checked numerically.

There are a few unresolved problems for future study. For example, the statistical modeling of elections and estimation of the model parameters, $\bar{\mu}, \theta_1$, and θ_2 that can fit the empirical nature of the election data need to be investigated further. The estimation should be compatible with the long-range nature of the correlation with distance r . We think it is necessary to generalize the model by incorporating several types of voters. As the equilibrium values v_i^* becomes approximately equal to μ_{avg} when $\theta_2 \gg \theta_1$, $\{v_d(i)\}$ cannot be encoded in $\{v_i^*\}$. The SIRM model avoids this problem by introducing a noise term [22].

In order to realize \vec{v}_d in our model, which is a proxy of the cultural field, the assumption that all voters are model voters is too simple. Some voters do not change their choices even if they interact with many other voters of different

choices. This possibility was previously considered in the modeling of Japan’s parliament election with three political parties [40]. We assumed that there are two types of voters, the fixed supporter of each political party and the floating voter. The probability function then becomes the combination of the multinomial distribution of the fixed supporter and the Dirichlet distribution of the floating voters. If we take into account the network structure of the social influence, then we have a combination of the multinomial distribution and a multivariate generalization of the Dirichlet distribution. Using this idea, it is possible to incorporate \vec{v}_a in the model. It is also worthwhile to solve the Wright-Fisher diffusion equation for the multivariate beta distribution [Eq. (1)]. Another type of multivariate beta distribution has been derived for the inference in a statistical control process [41]. The multivariate beta distribution for the voter model on networks should be derived since it is the natural multivariate extension of a beta distribution based on the similarity with the multivariate normal distribution.

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APPENDIX A: WRIGHT-FISHER DIFFUSION FOR SIRM MODEL

In the SIRM model, voters who live in node i move to node j for work [22]. They interact with other agents of node i and those of node j in addition to the agents on the link (i, j) [33]. We denote the number of voters on link (i, j) as $N_{i,j}$ and the number of votes for an option as $n_{i,j}$. As in the social influence model on a network in the main text, we introduce the parameters $\vec{\mu}$ that determine the intrinsic preference for an option of node i . We also introduce θ_1 and θ_2 which control the variance of the vote share and correlation of the votes shares between nodes, respectively. In the SIRM model, there is a parameter α which controls the strength of social influence from the node where a voter lives and works. $N_i \equiv \sum_j N_{i,j}$ and $n_i \equiv \sum_j n_{i,j}$ indicate the number of voters who live in node i and the number of votes for an option attributed to them. $N_j \equiv \sum_i N_{i,j}$ and $n_j \equiv \sum_i n_{i,j}$ indicate the number of voters who work in node j and the number of votes for an option due to them. Likewise, we also write v_i, v_j which represents the vote shares of the voters who live in node i and those who work in node j . We then write $N_T = \sum_{i,j} N_{i,j}$ for the total number of voters.

We write the probabilities for $n_{i,j} \rightarrow n_{i,j} + 1$ and $n_{i,j} \rightarrow n_{i,j} - 1$ as:

$$P(n_{i,j} \rightarrow n_{i,j} + 1) = \frac{N_{i,j} - n_{i,j}}{N_T} \frac{n_{i,j} + \alpha(a_i + \theta_2 v_i) + (1 - \alpha)(a_j + \theta_2 v_j)}{N_{i,j} - 1 + \theta_1 + \theta_2}$$

$$P(n_{i,j} \rightarrow n_{i,j} - 1) = \frac{n_{i,j}}{N_T} \frac{N_{i,j} - n_{i,j} + \alpha[b_i + \theta_2(1 - v_i)] + (1 - \alpha)[b_j + \theta_2(1 - v_j)]}{N_{i,j} - 1 + \theta_1 + \theta_2}.$$

Here $a_i \equiv \theta_1 \mu_i, b_i \equiv \theta_1(1 - \mu_i)$ are defined as before. The raising operator $R_{i,j}(\vec{v})$ and the lowering operator $L_{i,j}(\vec{v})$ are then defined as:

$$R_{i,j}(\vec{v}) = \frac{N_{i,j}}{N_T} (1 - v_{i,j}) \frac{v_{i,j} + \alpha(a_i + \theta_2 v_i)/N_{i,j} + (1 - \alpha)(a_j + \theta_2 v_j)/N_{i,j}}{1 - 1/N_{i,j} + \theta_1/N_{i,j} + \theta_2/N_{i,j}}$$

$$L_{i,j}(\vec{v}) = \frac{N_{i,j}}{N_T} v_{i,j} \frac{v_{i,j} + \alpha(a_i + \theta_2 v_i)/N_{i,j} + (1 - \alpha)(a_j + \theta_2 v_j)/N_{i,j}}{1 - 1/N_{i,j} + \theta_1/N_{i,j} + \theta_2/N_{i,j}}$$

The stochastic differential equation for $v_{i,j}$ is written as:

$$dv_{i,j} = d_{i,j} dt + \sqrt{D_{i,j}} W_{i,j}(t).$$

Here $W_{i,j}(t)$ is an independent and identically distributed Wiener process. The drift term $d_{i,j}$ and diffusion terms $D_{i,j}$ are written as

$$d_{i,j} = \alpha a_i + (1 - \alpha) a_j - \theta v_{i,j} + \theta_2 [\alpha v_i + (1 - \alpha) v_j], \tag{A1}$$

$$D_{i,j} = 2v_{i,j}(1 - v_{i,j}). \tag{A2}$$

APPENDIX B: VOTER MODEL ON COMPLETE GRAPH AND BETA BINOMIAL DISTRIBUTION

There are N voters and the number of voters who vote for an option is denoted as n . The probability for $n \rightarrow n + 1$ is written as

$$P(n \rightarrow n + 1) = \frac{N - n}{N} \frac{n + a}{N - 1 + \theta}.$$

The probability for $n \rightarrow n - 1$ is

$$P(n \rightarrow n - 1) = \frac{n}{N} \frac{(N - n) + b}{N - 1 + \theta}.$$

Here $\theta = a + b$. In the stationary state, the condition of the detailed balance between $P(n)$ and $P(n + 1)$ is

$$P(n)\dot{P}(n \rightarrow n + 1) = P(n + 1)P(n + 1 \rightarrow n).$$

We obtain the next recursive relation for $P(n)$ and $P(n + 1)$,

$$P(n + 1) = \frac{(N - n)(n + a)}{(n + 1)(N - (n + 1) + b)}P(n).$$

The solution with the normalization $\sum_n P(n) = 1$ is the beta binomial distribution.

$$P(n) = {}_N C_n \frac{(a)^n (b)^{N-n}}{(\theta)^N} = {}_N C_n \frac{B(n + a, N - n + b)}{B(a, b)}.$$

Here $(x)^n \equiv \prod_{k=1}^n (x + k - 1)$ is the rising factorial. Using the definition of the beta function, we write $P(n)$ as:

$$P(n) = {}_N C_n \int_0^1 p^n (1 - p)^{N-n} \frac{p^{a-1} (1 - p)^{b-1}}{B(a, b)} dp.$$

In the continuum limit, the pdf for $v \equiv \lim_{N \rightarrow \infty} n/N$ is the beta distribution with shape parameters (a, b) .

$$\lim_{N \rightarrow \infty} NP(n = Nv) = f_{\text{Beta}}(v|a, b) \equiv \frac{v^{a-1} (1 - v)^{b-1}}{B(a, b)}.$$

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