Nonequilibrium driven by an external torque in the presence of a magnetic field

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We investigate a two-dimensional motion of a colloid in a harmonic trap driven out of equilibrium by an external nonconservative force producing a torque in the presence of a uniform magnetic field applied perpendicular to the plane of motion. We find a circulating steady-state current diagnostic to nonequilibrium. Unlikely in the overdamped limit, inertial motion requires a sufficient central force to reach steady state. The magnetic field can enhance or depress central force depending on its direction. We find that steady state exists only for a proper range of parameters such as mass, viscosity coefficient, stiffness of the harmonic potential, and the magnetic field. We rigorously derive the existence condition for the steady state. We examine the combined influence of nonconservative force and magnetic field on nonequilibrium characteristics. We find non-Boltzmann steady-state probability density function and circulating probability current. We show that nonnegative entropy production is composed of usual heat dissipation and unconventional contribution from velocity-dependence of the Lorentz force. We derive the full list of correlation functions, including position-velocity correlation function originated from nonequilibrium circulation. We finally give rigorous expression for the violation of fluctuation-dissipation relation. We verify our analytical results by using the Monte Carlo simulation.

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I. INTRODUCTION

Stochastic thermodynamics for the nonequilibrium motion of small systems has been an interesting issue since the discovery of the fluctuation theorem (FT). There have been many studies on nonequilibrium fluctuation driven by external nonequilibrium sources such as nonconservative forces and time-dependent protocols which produce work and heat persistently [1–13]. There have been extensive experimental studies, measuring work and confirming the FT [14–23]. Under a particular circumstance, there exists a nonequilibrium steady state (NESS) characterized by non-Boltzmann distribution, nonzero current, nonzero rate of perpetual heat or work production, etc.

The influence of magnetic field on nonequilibrium systems has been an interesting issue. Diffusion under no confining potential is an intrinsic nonequilibrium process and becomes more complicated under a magnetic field, observed in many plasmas. The diffusion under a magnetic field has been studied extensively [24–29]. Nonequilibrium system in a time-varying potential has also been studied in the presence of a constant magnetic field [30–32].

Nonequilibrium driven by a nonconservative force in the presence of a magnetic field has not been considered in many places. The magnetic field does not produce any work so that the system does not undergo any energetic change solely due to the magnetic field. Contrary to deterministic dynamics, a usual circular motion cannot be observed due to thermal fluctuation in stochastic dynamics. The system under a conservative force in the presence of the magnetic field can reach a steady state with the Boltzmann distribution in the absence of any nonequilibrium source. Though the role of the magnetic field is not clear in this seemingly equilibrium situation, the dependence of the time-correlation functions on the magnetic field was found [27] and will be examined more thoroughly in our study. Recently, it was reported that an unconventional entropy is produced by the magnetic field as generally done by a velocity-dependent force [33]. It was also found that the proper overdamped limit cannot be found by neglecting an inertia term but by investigating a colored noise induced by a magnetic field [34].

In our study, we investigate the motion of a charged colloid in a harmonic trap potential in the presence of a magnetic field, which is driven out of equilibrium by a torque-generating nonconservative force. We have investigated the overdamped limit in the absence of the magnetic field [35,36]. To study the effect of the magnetic field rigorously, we investigate the motion in the phase space (position-velocity space). For simplicity, we do not consider magnetic moment of the colloid which generates a rotational motion around the center of mass of the colloid, which has been investigated extensively in recent studies [37–39]. In Sec. II, we present a mathematical setup for our model. In Sec. III, we derive the existence condition for a steady state. In Sec. IV, we find the probability distribution function (PDF) in NESS. As the characteristics of NESS, we find nonequilibrium probability currents in Sec. V, entropy production in Sec. VI. In Sec. VII, we derive two-time correlation functions among the pairs of positions and momenta. In Sec. VIII, we examine the violation of fluctuation-dissipation relation (FDR) caused by the nonconservative force and the magnetic field. We summarize our results in Sec. IX

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II. MODEL

We consider a colloid of mass m and charge $q_{\rm col}$ which is immersed in a two-dimensional liquid between parallel plates, as in an experimental setup. We consider a Brownian motion under a harmonic potential mimicking an optical trap, which is driven out of equilibrium by a torque-generating nonconservative force. Let $\vec{r}=(x,y)$ and $\vec{v}=(v_x,v_y)$ be the position and velocity vectors of the colloid and $V(\vec{r})=(k_1x^2+k_2y^2)/2$ be the trap potential with $k_1,\ k_2>0$. We suppose that a uniform magnetic field $\vec{B}=B\hat{z}$ is applied perpendicular to the plane of the plates. We consider an external force $\vec{f}_{\rm ex}=-\left(\begin{smallmatrix}0 & a_1\\ a_2 & 0\end{smallmatrix}\right)\cdot\vec{r}$ for $a_1\neq a_2$. It is a nonconservative force ($\vec{\nabla}\times\vec{f}_{\rm ex}\neq\vec{0}$) yielding a torque in z direction and driving the colloidal motion out of equilibrium.

Let γ be a viscosity coefficient and β be a fixed inverse temperature of the liquid. Under this condition, the motion of the colloid can be described by the Langevin equation written as $m\dot{\vec{v}}=-\vec{\nabla}V+\vec{f}_{\rm ex}+q_{\rm col}B\vec{v}\times\hat{z}-\gamma\vec{v}+\vec{\eta}(t)$, where $\vec{\eta}(t)=(\eta_x(t),\eta_y(t))$ is a Gaussian noise vector with zero mean and variance given by $\langle\eta_i(t)\eta_i(t')\rangle=2\gamma\beta^{-1}\delta_{ij}\delta(t-t')$ for $i,j=1,2$ denoting x,y. It can be rewritten as

$$m\dot{\vec{v}}=-\mathsf{F}\cdot\vec{r}-\Gamma\cdot\vec{v}+\vec{\eta}(t),\tag{1}$$

where $\mathsf{F}=\left(\begin{smallmatrix}k_1 & a_1\\ a_2 & k_2\end{smallmatrix}\right)$ and $\Gamma=\left(\begin{smallmatrix}\gamma & b\\ -b & \gamma\end{smallmatrix}\right)$ for $b=-q_{\rm col}B$.

Let $\vec{q}=(x,y,v_x,v_y)$ be a state vector in the position-velocity space. Then, combining Eq. (1) and $\dot{\vec{r}}=\vec{v}$, we have the Langevin equation in extended dimensions as

$$\dot{\vec{q}}(t)=-\mathsf{M}\cdot\vec{q}(t)+\vec{\xi}(t),\tag{2}$$

where

$$\mathsf{M}=\begin{pmatrix}\mathsf{0} & -\mathsf{I}\\ \mathsf{F}/m & \Gamma/m\end{pmatrix},\tag{3}$$

where $\vec{\xi}(t)=(0,0,\eta_x(t)/m,\eta_y(t)/m)$. $\mathsf{0}$ and I are 2×2 null and identity matrix, respectively. It belongs to the Ornstein-Ulenbeck process in four dimensions, which can be exactly solvable [35,40]. The Fokker-Planck equation for the PDF $\rho(\vec{q},t)$ in the position-velocity space, called the Kramers equation, is written as

$$\partial_t\rho(\vec{q},t)=-\partial_{\vec{q}}\cdot(-\mathsf{M}\cdot\vec{q}-\mathsf{D}\cdot\partial_{\vec{q}})\rho(\vec{q},t),\tag{4}$$

where ∂_t ($\partial_{\vec{q}}$) denotes partial differentiation with respect to t ($\vec{q}$). D is a 4×4 diffusion matrix defined as $\gamma\beta^{-1}/m^2\left(\begin{smallmatrix}\mathsf{0} & \mathsf{0}\\ \mathsf{0} & \mathsf{I}\end{smallmatrix}\right)$.

When an initial PDF at $t=0$ is Gaussian, given as $\rho(\vec{q},0)\propto e^{-\vec{q}\cdot\mathsf{U}(0)\cdot\vec{q}/2}$, the PDF at time t can be written as

$$\rho(\vec{q}(t),t)=\left[\frac{\det\mathsf{U}(t)}{(2\pi)^4}\right]^{1/2}\exp\left[-\frac{1}{2}\vec{q}\cdot\mathsf{U}(t)\cdot\vec{q}\right],\tag{5}$$

where

$$\mathsf{U}(t)^{-1}=\mathsf{U}_{\rm ss}^{-1}+e^{-\mathsf{M}t}\big[\mathsf{U}(0)^{-1}-\mathsf{U}_{\rm ss}^{-1}\big]e^{-\mathsf{M}^{\rm T}t}.\tag{6}$$

Here, the superscript T denotes the transpose of a matrix. $\mathsf{U}_{\rm ss}$ is the kernel of the steady state reached for $t\to\infty$. The formal expression for the steady state kernel is given by

$$\mathsf{U}_{\rm ss}=(\mathsf{D}+\mathsf{Q})^{-1}\mathsf{M}.\tag{7}$$

Q is an antisymmetric matrix satisfying

$$\mathsf{QM}+\mathsf{M}^{\rm T}\mathsf{Q}=\mathsf{DM}-\mathsf{M}^{\rm T}\mathsf{D}.\tag{8}$$

Solving this equation for Q, one can find the PDF at time t [35,40].

III. EXISTENCE OF STEADY STATE

The formula for the PDF in Eqs. (5) and (6) is meaningful only if M is positive-definite; otherwise, the steady-state PDF does not exist. The characteristic equation for the eigenvalue λ of M is given as

$$\begin{aligned}0=\lambda^4-\frac{2\gamma}{m}\lambda^3+\frac{b^2+\gamma^2+m(k_1+k_2)}{m^2}\lambda^2\\ -\frac{b(a_1-a_2)+\gamma(k_1+k_2)}{m^2}\lambda+\frac{k_1k_2-a_1a_2}{m^2}.\end{aligned}\tag{9}$$

Then, existence condition for the steady state is given by the positivity of $\mathrm{Re}(\lambda)$, which guarantees the convergence of $\mathsf{U}(t)$ to $\mathsf{U}_{\rm ss}$ as t increases, as seen in Eq. (6).

A. General criterion

1. In the absence of a magnetic field

We first consider for zero magnetic field ($b=0$). In this simple case, Eq. (9) can be solved as

$$\lambda=\frac{\gamma}{2m}\{1\pm\sqrt{1-\frac{2m}{\gamma^2}[k_1+k_2\pm\sqrt{4a_1a_2+(k_1-k_2)^2}]}\},\tag{10}$$

and the other two eigenvalues are complex conjugates of theses. For brevity we write the eigenvalue in Eq. (10) as $\lambda=\gamma(1\pm\sqrt{\psi})/(2m)$. We find that the existence condition depends on the sign of $4a_1a_2+(k_1-k_2)^2$.

If $4a_1a_2+(k_1-k_2)^2>0$, then ψ is real. Then, the condition for $\mathrm{Re}(\lambda)>0$ is $\psi<1$, which leads to the existence condition

$$k_1k_2-a_1a_2=\det\mathsf{F}>0,\tag{11}$$

where F is defined in Eq. (1). The existence condition does not depend on mass m. Therefore, it can be applied to the overdamped limit for large γ or small m, where Eq. (1) reduces to $\gamma\dot{\vec{r}}=-\mathsf{F}\cdot r+\vec{\eta}(t)$ in the position space. In this limit, $\det\mathsf{F}>0$ is nothing but the condition that $\vec{r}=\vec{0}$ be a stable fixed point.

In the other case for $4a_1a_2+(k_1-k_2)^2<0$, ψ is imaginary. We can write $\psi=L_1\pm iL_2$ where $L_1=1-(2m/\gamma^2)(k_1+k_2)$, $L_2=(2m/\gamma^2)\sqrt{-4a_1a_2-(k_1-k_2)^2}$. Then, the condition that the smallest value of $\mathrm{Re}(\lambda)$ be positive can be found as $1-\sqrt{(\sqrt{L_1^2+L_2^2}+L_1)/2}>0$. Then, we get the existence condition

$$k_1+k_2+\frac{m}{2\gamma^2}[(k_1-k_2)^2+4a_1a_2]>0.\tag{12}$$

For a sufficiently small m/γ^2, the existence of the steady state is always guaranteed, hence this condition is beyond the overdamped limit.

The two-dimensional motion for $4a_1a_2+(k_1-k_2)^2>0$ can be shown to map to the previously studied cases such as a two-dimensional motion subject to different noise sources (heat reservoirs) acting in the two perpendicular directions [41] and a one-dimensional model for two particles interacting via a harmonic force each of which is thermostatted to a different heat reservoir [42]. The latter is also equivalent to an electric circuit with two subcircuits coupled via a capacitor [43]. The stability criterion in Eq. (11) was examined for the heat engine designed from the former model [42]. Throughout the paper in the following, we consider the other case for $4a_1a_2 + (k_1 - k_2)^2 < 0$, which cannot be mapped from the previous studies. Such a force can be generated experimentally by feedback process where an electric field is computed from an instant measurement of the colloid's position. It can also be generated from an electric field induced form a magnetic field varying linearly in time. For the latter, observation is only restricted in a short period of time for which magntic field does not change significantly.

2. In the presence of magnetic field

The solution of the characteristic equation in Eq. (9) for nonzero b can also be solved exactly with the help from Mathematica, but cannot be expressed in a simple form as Eq. (9). However, for small b, we can find the expression for the eigenvalues by using the perturbation expansion. Up to the first order in b, the correction to the zeroth-order value $\lambda^{(0)}$ is found as

$$\lambda^{(1)} = \frac{(a_1 - a_2)b}{(2m\lambda^{(0)} - \gamma)(2m\lambda^{(0)2} - 2\gamma\lambda^{(0)} + k_1 + k_2)}\lambda^{(0)}. \tag{13}$$

After some algebra, we find the positivity condition for the smallest value of $\mathrm{Re}(\lambda^{(0)} + \lambda^{(1)})$ as

$$1 - \sqrt{\frac{\sqrt{L_1^2 + L_2^2} + L_1}{2}} + \frac{2m(a_1 - a_2)b}{L_2\sqrt{L_1^2 + L_2^2}}\sqrt{\frac{\sqrt{L_1^2 + L_2^2} - L_1}{2}} > 0, \tag{14}$$

where $L_{1,2}$ are given in the last subsection. As a result, we have the existence condition for the steady state for a small b as

$$k_1 + k_2 + \frac{m}{2\gamma^2}[4a_1a_2 + (k_1 - k_2)^2] + \frac{b(a_1 - a_2)}{\gamma} > 0, \tag{15}$$

where b is kept up to the first order.

B. Isotropic case in the presence of a magnetic field

We consider an isotropic case for $a_1 = -a_2 = a$, $k_1 = k_2 = k$, for which we can find the exact existence condition for steady state nonperturbatively for arbitrary b, while the condition for nonisotropic case can be found numerically. For the isotropic case, the eigenvalue equation in Eq. (9) reduces to

$$0 = \lambda^4 - \frac{2\gamma}{m}\lambda^3 + \frac{b^2 + \gamma^2 + 2mk}{m^2}\lambda^2 - \frac{2ab + 2\gamma k}{m^2}\lambda + \frac{k^2 + a^2}{m^2}. \tag{16}$$

It is convenient to define dimensionless coefficients as follows:

$$A = \frac{ma}{\gamma^2}, \quad B = \frac{b}{\gamma}, \quad K = \frac{mk}{\gamma^2}. \tag{17}$$

Then, the two typical eigenvalues of M can be written as

$$\lambda_{1,2} = \frac{\gamma}{2m}[1 - iB \pm \sqrt{R}e^{i\phi/2}], \tag{18}$$

where

$$R = \sqrt{(1 - B^2 - 4K)^2 + (2B - 4A)^2},$$
$$\phi = \tan^{-1}\frac{2B - 4A}{1 - B^2 - 4K}. \tag{19}$$

The other two eigenvalues are complex conjugates of λ_1 and λ_2. Then, the condition $\mathrm{Re}(\lambda_{1,2}) > 0$ leads to $|\sqrt{R}\cos(\phi/2)| < 1$, leading to

$$1 > \frac{R(1 + \cos\phi)}{2} = \frac{R + 1 - B^2 - 4K}{2}. \tag{20}$$

Simplifying it more, we find the stability condition as $K - AB + A^2 > 0$ or

$$\Omega = k + ab/\gamma - ma^2/\gamma^2 > 0, \tag{21}$$

where we define Ω which frequently appears for other quantities obtained later. Note that it is consistent with Eq. (15) in the isotropic limit. It implies that all the higher-order corrections in b to Eq. (15) vanishes in the isotropic limit, which is nontrivial to show rigorously in the perturbation scheme.

We provide a more physical derivation based on the stability of a fixed point. A deterministic trajectory of the motion generated by Eq. (2) is given by $\dot{\vec{q}}_{\rm d} = -\mathsf{M}\cdot\vec{q}_{\rm d}$. In polar coordinates (r, θ), there is a fixed point at $r = 0$, which is either stable or unstable in the parameter space (m, γ, a, b, k). At the critical boundary in the parameter space, there exists a fixed circular orbit the radius of which depending depends on an initial condition and hence infinitely many circular orbits including $r = 0$. A circular orbit satisfies the two force-balance equations in radial and angular directions, given as $mr\dot{\theta}^2 = kr + br\dot{\theta}$ and $mr\ddot{\theta} = -\gamma r\dot{\theta} + ar = 0$. Eliminating $\dot{\theta}$, we find $mr(a/\gamma)^2 = (k + ab/\gamma)r$ where the right-hand-side is the centripetal force for the circular orbit, hence $\Omega = 0$ from Eq. (21). For $\Omega > 0$, a deterministic trajectory converges to $r = 0$ as time evolves, which comes up with a stable PDF through fluctuation by noise. For $\Omega < 0$, however, any trajectory diverges to $r = \infty$ so that noise cannot produce any stable PDF. Figure 1 shows a circular orbit where harmonic and magnetic forces in radial direction. For $ab > 0$, the two forces are in the same radial direction so as to strengthen centripetal force, and vice versa for $ab < 0$.

The external torque gives an acceleration in angular direction to drive a spiral motion outward from the origin, so it tends to depress the stability, as seen from the last term, $-ma^2/\gamma^2$ in Eq. (21). For $ab > 0$, the magnetic field is in the same direction as the torque $\nabla \times \vec{f}_{\rm ex}$ so that it yields a magnetic force in the same centripetal direction as the harmonic force, and vice versa for $ab < 0$. Therefore, the

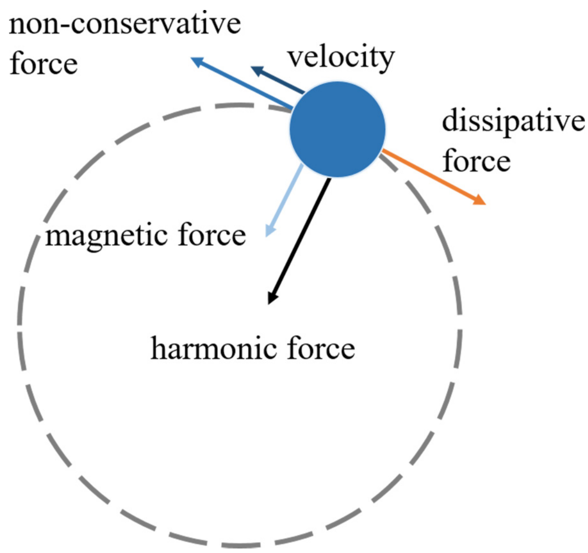


FIG. 1. A circular orbit and involved forces. The dissipative force is $-\gamma\vec{v}$. The figure is drawn for $a,\ b>0$. The harmonic force and magnetic force for $ab>0$ ($ab<0$) are in the same (opposite) direction so that they strengthen (weaken) centripetal force.

magnetic field tends to enhance (depress) the stability for $ab>0$ ($ab<0$), as seen in the second term, ab/γ, in Eq. (21). Figure 2 shows the diagram for the existence of steady state in k-a space for various values of b, where the competing and supplementary tendencies in the influence of a and b on the stability condition is well observed.

IV. NONEQUILIBRIUM STEADY STATE

In the existence region satisfying the condition $\Omega>0$ in Eq. (21), we can find the steady-state PDF and show it explicitly for the isotropic case. First, we solve Eq. (8) for the antisymmetric matrix Q, which can be converted into a set of linear equations for six unknown elements of the matrix. We

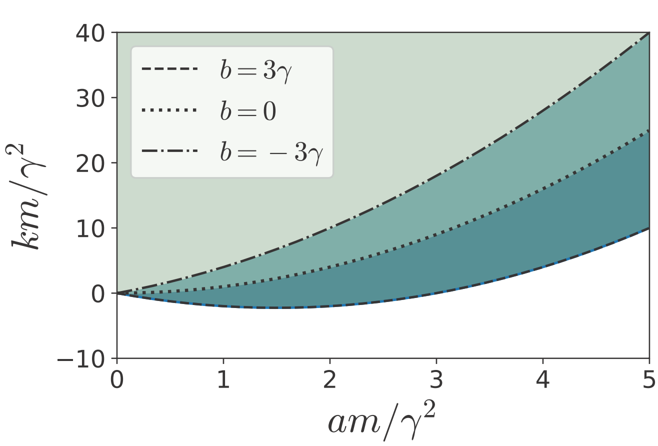


FIG. 2. The existence region for steady state in a parameter space for k and $a>0$. The stable region is above the boundary line. Boundaries are drawn for $b=-3\gamma,\ 0,\ 3\gamma$. The larger b and the smaller a, the more widened the stable region.

find

$$\mathsf{Q}=\frac{1}{\Omega}\begin{pmatrix} 0 & \frac{a}{\gamma} & -\frac{ab+\gamma k}{m\gamma} & 0 \\ -\frac{a}{\gamma} & 0 & 0 & -\frac{ab+\gamma k}{m\gamma} \\ \frac{ab+\gamma k}{m\gamma} & 0 & 0 & \frac{ab^2+\gamma bk+akm}{m^2\gamma} \\ 0 & \frac{ab+\gamma k}{m\gamma} & -\frac{ab^2+\gamma bk+akm}{m^2\gamma} & 0 \end{pmatrix}. \tag{22}$$

From Eq. (7), we have

$$\mathsf{U}_{\rm ss}=\beta\begin{pmatrix} \frac{ab+\gamma k}{\gamma} & 0 & 0 & -\frac{am}{\gamma} \\ 0 & \frac{ab+\gamma k}{\gamma} & \frac{am}{\gamma} & 0 \\ 0 & \frac{am}{\gamma} & m & 0 \\ -\frac{am}{\gamma} & 0 & 0 & m \end{pmatrix}. \tag{23}$$

For $a=0$, $\mathsf{U}_{\rm ss}$ is equal to that for the equilibrium Boltzmann PDF, independent of a magnetic field. It is well explained from the fact that the magnetic field does not work. However, for the transient period for $t<\infty$, a relaxation behavior of the PDF in time toward the Boltzmann PDF is determined by $e^{-\mathsf{M}t}$ and $e^{-\mathsf{M}^{\rm T}t}$, as seen in Eq. (6), and hence various forms of exponential decaying with sinusoidal oscillation as $e^{-(\lambda_i+\lambda_j)t}$ for all possible $i,\ j$. As seen in Eq. (18), even for $a=0$, eigenvalues λ_i's depend on b, so the transient PDF depends on b.

For $a\neq 0$ in nonequilibrium, the steady-state PDF ($\rho_{\rm ss}$) depends on b as well as the transient one. We can observe that positions and velocities are coupled in the PDF, as seen from the off-diagonal elements of $\mathsf{U}_{\rm ss}$, which gives rise to a non-Maxwellian distribution as an important characteristics of nonequilibrium steady state (NESS). One can observe $\beta^{-1}\ln\rho_{\rm ss}=-(m/2)[(v_x+ay/\gamma)^2+(v_y-ax/\gamma)^2]+\cdots$. Then, we have a nonzero average velocity at a fixed position, given as

$$\langle\vec{v}\rangle_{\vec{v}}=-\frac{a}{\gamma}\mathsf{A}\cdot\vec{r}, \tag{24}$$

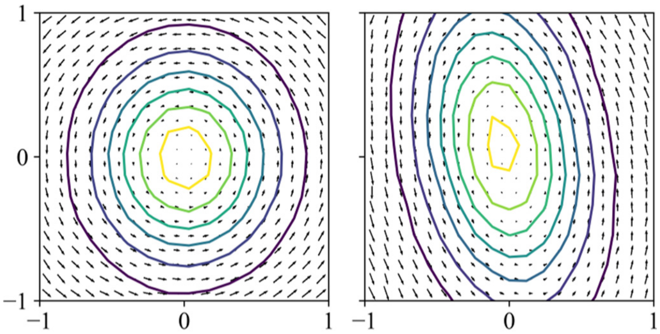
where $\langle\cdots\rangle_{\vec{v}}=\int d\vec{v}\rho_{\rm ss}(\vec{r},\vec{v})(\cdots)$ denotes the average of the given quantity over $\vec{v}$ for a fixed position. $\mathsf{A}=\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)$ is an antisymmetric matrix. It manifests a nonzero probability current, also known as an important property of NESS. This property is more examined in the next section.

We can find second moments in the steady state as

$$\langle\vec{q}\vec{q}\rangle=\mathsf{U}_{\rm ss}^{-1}=\frac{1}{\beta\Omega}\begin{pmatrix} 1 & 0 & 0 & \frac{a}{\gamma} \\ 0 & 1 & -\frac{a}{\gamma} & 0 \\ 0 & -\frac{a}{\gamma} & \frac{ab+k\gamma}{m\gamma} & 0 \\ \frac{a}{\gamma} & 0 & 0 & \frac{ab+k\gamma}{m\gamma} \end{pmatrix}, \tag{25}$$

where $(\vec{q}\vec{q})_{ij}=q_iq_j$ is a 4×4 dyad (outer product) of a state vector in the position-velocity space.

Figure 3 shows the contour lines for the steady state PDF and the velocity field lines. We mainly present analytical results for the isotropic case. In the figure, we also present the plot for a nonisotropic case where contour lines are elliptic. For both cases, velocity field lines circulate along contour lines, which was shown to be a special feature for linear drift,



(a) (b)

FIG. 3. Contour plots of the probability density function and velocity fields for (a) isotropic and (b) nonisotropic cases. The vertical and horizontal axes represent x and y in unit of $\gamma/\sqrt{mk_BT}$. Velocity field lines are shown to circulate around the contour lines for both cases.

$-\mathsf{M}\cdot\vec{q}$, in the Langevin equation in Eq. (2) [40]. We further investigate the velocity field in the next section.

V. NONEQUILIBRIUM PROBABILITY CURRENTS

NESS is characterized by a nonzero irreversible current in the variable space. We follow a well-established formalism in the textbook by Risken [44]. The Fokker-Planck Eq. (4) can be rewritten as

$$\partial_t\rho(\vec{q},t) = -\partial_{q_i}(f_i(\vec{q}) - D_{ij}\partial_{q_j})\rho(\vec{q},t). \tag{26}$$

Parity ϵ_i in time reversal: $t\to\tau-t$ for a final time τ is either $+1$ for position coordinates ($i=1,2$) or -1 for velocity coordinates ($i=3,4$). Then, the drift terms f_i's are decomposed into reversible and irreversible parts as

$$f_i^{\mathrm{rev}} = \frac{f_i(q_j) - \epsilon_i f_i(\epsilon_j q_j)}{2}, \quad f_i^{\mathrm{irr}} = \frac{f_i(q_j) + \epsilon_i f_i(\epsilon_j q_j)}{2}. \tag{27}$$

The Fokker-Planck equation can be written as $\partial_t\rho = -\partial_{\vec{q}}\cdot\vec{j}_{\vec{q}}$ in terms of the probability current $\vec{j}_{\vec{q}}$, which can also be decomposed into the reversible and irreversible parts as

$$j_i^{\mathrm{rev}} = f_i^{\mathrm{rev}}\rho, \quad j_i^{\mathrm{irr}} = \left(f_i^{\mathrm{irr}} - D_{ij}\partial_{q_j}\right)\rho. \tag{28}$$

Note that j_i^{irr} exists only in the velocity space, i.e., $j=3,4$. We use $\vec{j}_{\vec{r}} = (j_1, j_2)$ and $\vec{j}_{\vec{v}} = (j_3, j_4)$. In a usual convention, the magnetic field is to flip ($\vec{B}\to-\vec{B}$) in time reversal. In this study, however, we use a different rule without flipping $\vec{B}$ in time reversal to investigate irreversibility in dynamics under a given magnetic field. Then, we have

$$\vec{j}_{\vec{r}}^{\mathrm{rev}} = \vec{v}\rho, \quad \vec{j}_{\vec{v}}^{\mathrm{rev}} = -\frac{1}{m}\mathsf{F}\cdot\vec{r}\rho,$$
$$\vec{j}_{\vec{v}}^{\mathrm{irr}} = \left[-\frac{1}{m}\Gamma\cdot\vec{v} - \mathsf{D}_{\mathrm{red}}\cdot\partial_{\vec{v}}\right]\rho, \tag{29}$$

where $\mathsf{D}_{\mathrm{red}} = (\gamma\beta^{-1}/m^2)\mathsf{I}$ for 2×2 identity matrix I.

The current in the velocity space, $\vec{j}_{\vec{v}} = \vec{j}_{\vec{v}}^{\mathrm{rev}} + \vec{j}_{\vec{v}}^{\mathrm{irr}}$, is the sum of forces per mass times PDF. We call $-m\mathsf{D}_{\mathrm{red}}\cdot\partial_{\vec{v}}\ln\rho$ stochastic force, which originates from noise in the Langevin dynamics. As seen in Eq. (29), any position-dependent force belongs to $\vec{j}_{\vec{v}}^{\mathrm{rev}}$. However, the dissipative force $(-\gamma\vec{v})$, the stochastic force, and the magnetic force belong to $\vec{j}_{\vec{v}}^{\mathrm{irr}}$. The dissipative and stochastic forces in $\vec{j}_{\vec{v}}^{\mathrm{irr}}$ contribute to the production of heat, which is consistent with the definition of heat production rate in the system: $[-\gamma\vec{v} + \vec{\eta}(t)]\circ\vec{v}$ for $\circ$ denoting the Stratonovich convention. The role of the magnetic force in the irreversible current is intriguing because it costs no energy, which will be discussed in this and the following section.

In the steady state, we find the irreversible current by using Eq. (23) as

$$\vec{j}_{\vec{v}}^{\mathrm{irr}} = \left[\frac{a}{m}\mathsf{A}\cdot\vec{r} - \frac{b}{m}\mathsf{A}\cdot\vec{v}\right]\rho_{\mathrm{ss}}, \tag{30}$$

The first term in this equation is exactly equal to minus the nonconservative force per mass in $\vec{j}_{\vec{v}}^{\mathrm{rev}}$. This means that the heat produced by this force exactly cancels the work produced by the nonconservative force, so the system can stay in the steady state. The total remaining force is given as

$$m\rho_{\mathrm{ss}}^{-1}\left(\vec{j}_{\vec{v}}^{\mathrm{rev}} + \vec{j}_{\vec{v}}^{\mathrm{irr}}\right) = -k\vec{r} - b\mathsf{A}\vec{v}. \tag{31}$$

However, the reversible current $\vec{j}_{\vec{r}}^{\mathrm{rev}} = \vec{v}\rho_{\mathrm{ss}}$ in the position space is random in $\vec{v}$. We find the average current in position space as

$$\left\langle\vec{j}_{\vec{r}}^{\mathrm{rev}}\right\rangle_{\vec{v}} = \int \mathrm{d}\vec{v}\,\vec{v}\rho_{\mathrm{ss}}(\vec{q}) = -\frac{a}{\gamma}\mathsf{A}\cdot\vec{r}\tilde{\rho}_{\mathrm{ss}}(\vec{r}), \tag{32}$$

where $\tilde{\rho}_{\mathrm{ss}}(\vec{r}) = [\beta\Omega/(2\pi)]e^{-\beta\Omega r^2/2}$ is the reduced steady-state PDF for $\vec{r}$. This average current circulates in the position space and the remaining current in the velocity space in Eq. (31) provides a centripetal force necessary for such circulation. For a more rigorous proof, we write the PDF in polar coordinates as

$$\rho_{\mathrm{ss}}(v_r, v_\theta, r, \theta) = \frac{\beta^2 mr\Omega}{(2\pi)^2}\exp\left[-\frac{\beta m}{2}\left[v_r^2 + \left(v_\theta - \frac{a}{\gamma}r\right)^2\right] - \frac{\beta\Omega}{2}r^2\right], \tag{33}$$

where r in the normalization factor is the Jacobian for the variable-change: $(x,y,v_x,v_y)\to(r,\theta,v_r,v_\theta)$. The existence of the average circular current requires the condition: $\langle k\vec{r} + b\mathsf{A}\cdot\vec{v}\rangle = \langle mv_\theta^2\hat{r}/r\rangle$. The left- and right-hand side of this condition are found as $\langle(k+ab/\gamma)r\rangle\hat{r}$ and $\langle 1/(\beta r) + (ma^2/\gamma^2)r\rangle\hat{r}$, respectively. The two sides are found to be the same by using $\langle 1/r\rangle = \sqrt{\beta\Omega\pi/2}$ and $\langle r\rangle = \sqrt{\pi/(2\beta\Omega)}$ given from Eq. (33).

The magnetic field is shown to be a source for the circulating current in the position space in addition to the torque-generating nonconservative force. It is interesting that the circular current could be possible even for $k=0$ if $ab>0$, rigorously for $ab > ma^2/\gamma$ ($\Omega>0$).

The detailed balance (DB) characterizes dynamical reversibility, for which the condition is given as

$$\langle\vec{q}'|e^{-\mathcal{H}_{\mathrm{FP}}\Delta t}|\vec{q}\rangle\rho_{\mathrm{ss}}(\vec{q}) = \langle\epsilon\vec{q}|e^{-\mathcal{H}_{\mathrm{FP}}\Delta t}|\epsilon\vec{q}'\rangle\rho_{\mathrm{ss}}(\epsilon\vec{q}'), \tag{34}$$

where $(\epsilon\vec{q})_i = \epsilon_i q_i$, $\mathcal{H}_{\mathrm{FP}} = \partial_{q_i}(f_i(\vec{q}) - D_{ij}\partial_{q_j})$ is a non-Hermitiain Fokker-Planck operator, and Δt is the time taken for the transition between the two states. It is shown [33,44,45] that the DB holds only if

$$\rho_{ss}(\vec{q}) = \rho_{ss}(\epsilon\vec{q}),\ \vec{j}_{\vec{v}}^{\rm irr} = \vec{0}. \tag{35}$$

In our case, the DB is found to be broken. First, we clearly see $\rho_{ss}(\vec{r},\vec{v}) \neq \rho_{ss}(\vec{r},-\vec{v})$ from position-velocity coupling in Eq. (23). Second, we find a nonzero irreversible current in Eq. (30). The magnetic field as a part of the irreversible current is partly responsible for the dynamical irreversibility manifested by the circulation in the position space besides its own contribution to the irreversibility in the velocity space.

VI. ENTROPY PRODUCTION

The total entropy ΔS produced for $0 < t \leqslant t_f$ in the system and bath can be regarded as a quantity to measure dynamic irreversibility. It is known to be found from the ratio of two path probabilities, given as

$$\Delta S = \ln\frac{\rho(\vec{q}_0,0)\Pi[\vec{q}(t)|\vec{q}_0;\lambda(t)]}{\rho(\vec{q}_f,t_f)\Pi[\epsilon\vec{q}(t_f-t)|\epsilon\vec{q}_f;\lambda(t_f-t)]}, \tag{36}$$

where $\Pi[\vec{q}(t)|\vec{q}_0;\lambda(t)]$ ($\Pi[\epsilon\vec{q}(t_f-t)|\epsilon\vec{q}_f;\lambda(t_f-t)]$) is the conditional probability of the system evolving along a path $\vec{q}(t)$ (time-reverse path $\epsilon\vec{q}(t_f-t)$) for given $\vec{q}_0$ ($\epsilon\vec{q}_f$) at $t=0$ for $0<t\leqslant t_f$. $\lambda(t)$ is a time-dependent protocol not considered in our study. ΔS satisfies the fluctuation theorem: $\langle e^{-\Delta S}\rangle = 1$ [11–13] and has a nonnegative average as a corollary: $\langle\Delta S\rangle \geqslant 0$. In the absence of any velocity-dependent force, ΔS turns out to be equal to the sum of the Shannon entropy change, $-\Delta\ln\rho$, and the dissipated heat production Q divided by temperature. Then, the dynamical irreversiblity accompanies energetic irreversibility in heat production. In particular, the two kinds of irreversibility are equivalent in the steady state with no Shannon entropy change.

In the presence of a velocity-dependent force, however, ΔS is found to have an unconventional contribution, $\Delta S_{\rm uc}$, resulting in a modified expression $\Delta S = -\Delta\ln\rho + Q/T + \Delta S_{\rm uc}$ [33]. Various types of velocity-dependent forces have been considered in active matters [46–54] and a magnetic force is the only natural one. The rate of the entropy production is given as

$$\dot{S} = -\frac{d}{dt}\ln\rho + \frac{\dot{Q}}{T} + \dot{S}_{\rm uc}, \tag{37}$$

where

$$\dot{Q} = \dot{W} - \frac{dE}{dt} = -a\vec{v}\cdot\mathsf{A}\cdot\vec{r} - \frac{d}{dt}\left(\frac{m\vec{v}^2}{2} + \frac{k\vec{r}^2}{2}\right), \tag{38}$$

$$\dot{S}_{uc} = \frac{m}{\gamma}(\vec{f}^{\rm irr}+\gamma\vec{v})\cdot(\vec{f}^{\rm irr}-\gamma\vec{v}) - \frac{1}{m}\partial_{\vec{v}}\cdot(\vec{f}^{\rm rev}-\vec{f}^{\rm irr}-\gamma\vec{v}), \tag{39}$$

where $\dot{W}$ is the rate of work done by the nonconservative force. In obtaining $\dot{S}_{\rm uc}$ from Eq. (36), we change the sign of velocity in a time-reverse path, but not the protocol (coefficient) for the velocity-dependent force for the purpose to investigate the irreversibility under a fixed protocol. In fact, we do fix the direction of $\vec{B}$ in a time-reverse path. We are interested in a local irreversibility of the system under a fixed protocol provided from an external agent.

From the previous study [12,13,33], we have

$$\langle\dot{S}\rangle = \int d\vec{q}\,\frac{\vec{j}_{\vec{v}}^{\rm irr}\cdot\mathsf{D}_{\rm red}^{-1}\cdot\vec{j}_{\vec{v}}^{\rm irr}}{\rho} \geqslant 0, \tag{40}$$

which is certainly nonnegative. It explicitly shows the second law of thermodynamics in the presence of a velocity-dependent (magnetic) force. Interestingly, only the irreversible current contributes to the irreversibility appearing in a nonequilibrium process.

For our case, $\vec{f}^{\rm irr} = -\Gamma\cdot\vec{v}/m$ and $\vec{f}^{\rm rev} = -\mathsf{F}\cdot\vec{r}/m$. We find $\dot{S}_{\rm uc} = (\beta b^2/\gamma)v^2$, which is nonzero even when there is no nonconservative force. In the steady state, we find the average values of the components of $\dot{S}$ by using Eq. (25) as

$$\beta\langle\dot{Q}\rangle = -\beta a\langle v_x y - v_y x\rangle = \frac{2a^2}{\gamma\Omega}, \tag{41}$$

$$\langle\dot{S}_{\rm uc}\rangle = \frac{\beta b^2}{\gamma}\langle v_x^2+v_y^2\rangle = \frac{2b^2}{m\gamma\Omega}\left(k+\frac{ab}{\gamma}\right). \tag{42}$$

The total irreversibility quantified by $\langle\dot{S}\rangle$ has contributions from the two components, the nonconservative and the magnetic force, as seen in the irreversible current in Eq. (30). The heat dissipation rate in Eq. (41) has the contribution from the first component and the unconventional entropy production rate in Eq. (42) has the combined contribution from the both components, as seen from the dependence on b^2 and ab, respectively. Note that the magnetic force can have influence on the circulation current in the position space only by being accompanied by the nonconservative force. For $a=0$, there is no such circulation and heat production, but the irreversibility due to helicity, which is a tendency of circulation, is still present, which is measured by $\langle\dot{S}_{\rm uc}\rangle$.

VII. TWO-TIME CORRELATION FUNCTIONS

Correlation functions between position and velocity coordinates at different times are found by using the formula [35], given as

$$\mathsf{C}(t,t') = \langle\vec{q}(t)\vec{q}(t')\rangle = \begin{cases} e^{-(t-t')\mathsf{M}}\mathsf{U}^{-1}(t'), & t>t', \\ \mathsf{U}^{-1}(t')e^{-(t'-t)\mathsf{M}^{\rm T}}, & t<t', \end{cases} \tag{43}$$

where $\mathsf{U}(t)$ is the kernel for the PDF at time t, given in Eq. (6). We consider the correlation functions in the steady state, so $\mathsf{U}(t) = \mathsf{U}_{\rm ss}$. As a result, the two-time correlation functions only depend on the difference of two times. The equal-time correlation functions are found from $\mathsf{U}_{\rm ss}^{-1}$ in Eq. (25).

For the isotropic case, rotational symmetry yields

$$\langle x(t)x(t')\rangle = \langle y(t)y(t')\rangle, \quad \langle x(t)y(t')\rangle = -\langle y(t)x(t')\rangle. \tag{44}$$

Finding $\langle x(t)x(t')\rangle$ and $\langle x(t)y(t')\rangle$, all other correlation functions can be generated by differentiating with respect to one of two times or by exchanging x and y components with minus sign. For example, $\partial_t\langle x(t)y(t')\rangle = \langle v_x(t)y(t')\rangle = -\langle v_y(t)x(t')\rangle$.

For $\tau = t-t'$, we write

$$e^{-\mathsf{M}\tau} = \sum_{i=1}^{4} e^{-\lambda_i\tau}|i\rangle\langle i|, \tag{45}$$

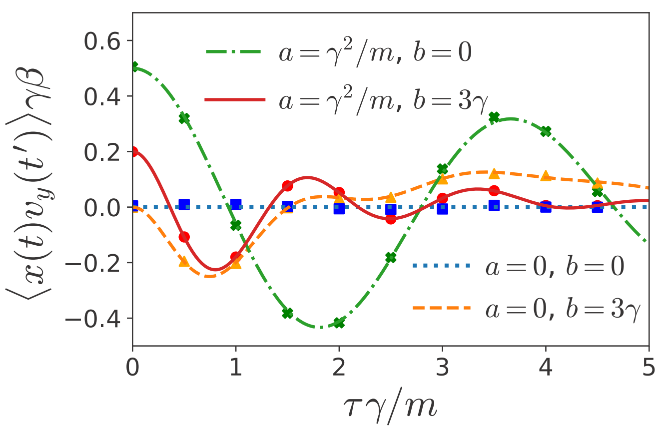


FIG. 4. The plot of $C_{xv_y}(\tau)$ for various $a,\ b>0$ with fixed $k=3\gamma^2/m$. There is a nonvanishing correlation for $b\neq 0$ and $a=0$, which is distinguishable from normal equilibrium. The amplitude of the correlation decreases as b increases. Four lines represent analytic results. We verify the analytic results with Monte Carlo simulation. Dots, squares, triangular, and cross represent the simulation result.

where $|i\rangle$ ($\langle i|$) is an orthonormal right (left) eigenvector for an eigenvalue λ_i for M, i.e., $\langle i|j\rangle=\delta_{ij}$. Using the definition of R and ϕ in Eq. (19), the two kinds of time-dependent terms in $e^{-\mathsf{M}t}$ are found as

$$c_i(\tau)=e^{-\mathrm{Re}(\lambda_i)\tau}\cos[\mathrm{Im}(\lambda_i)\tau+\phi/2], \tag{46}$$

$$s_i(\tau)=e^{-\mathrm{Re}(\lambda_i)\tau}\sin[\mathrm{Im}(\lambda_i)\tau+\phi/2], \tag{47}$$

where λ_i's for $i=1,2$ are the two typical eigenvalues in Eq. (18). Writing $\mathsf{C}(t,t')=\mathsf{C}(\tau)$, we have

$$C_{xx}(\tau)=\frac{1}{\beta\Omega}\sum_{i=1,2}[\alpha_i c_i(\tau)+\beta_i s_i(\tau)],$$
$$C_{xy}(\tau)=\frac{1}{\beta\Omega}\sum_{i=1,2}[\alpha_i s_i(\tau)-\beta_i c_i(\tau)], \tag{48}$$

with

$$\alpha_{1,2}=\frac{1}{2}\left[\cos(\phi/2)\pm\frac{1}{\sqrt{R}}\right],$$
$$\beta_{1,2}=\frac{1}{2}\left[\sin(\phi/2)\pm\frac{2A-B}{\sqrt{R}}\right], \tag{49}$$

where the upper (lower) sign is for the subscript 1 (2). The parameters used are defined in Eqs. (17) and (19). The other correlation functions derivable from Eq. (48) are given in Appendix A.

There is circulating probability current in the position space. It is manifested in a strong correlation between position and velocity in perpendicular directions to each other. We plot a correlation function $C_{xv_y}(\tau)=\langle x(\tau)v_y(0)\rangle$ in Fig. 4. It is interesting that it is nonzero even for $a=0$ and $b\neq 0$, which signals a tendency of helicity around the direction of the magnetic field. Interestingly, all the correlation functions have the same factor Ω in the denominator. Therefore, the nearer is the parameter set from the existence boundary (the larger Ω), the smaller is the amplitude of the correlation function. In Figure 4 drawn for $a,\ b>0$, the correlation function for $b=0$ has a larger amplitude than that for $b>0$ with a larger value of Ω.

VIII. VIOLATION OF FLUCTUATION-DISSIPATION RELATION

The FDR is known to hold for equilibrium. Recently, the violation of FDR was found to be related with the heat produced during a nonequilibrium process [55,56]. The FDR was found to hold in the presence of a magnetic field where there is no nonequilibrium source to produce heat [57]. We examine the FDR in our case where both a nonconservative and a magnetic force are present.

Under an arbitrarily small perturbative force $\vec{h}(t)$, the Langenvin equation in Eq. (1) is written as $m\dot{\vec{v}}=-\mathsf{F}\cdot\vec{r}-\Gamma\cdot\vec{v}+\vec{h}(t)+\vec{\eta}(t)$. The response function for $\langle\vec{q}(t)\rangle$ with respect to variation $\delta\vec{h}(t')$ is defined as

$$\mathsf{R}(t,t')=\frac{\delta}{\delta\vec{h}(t')}\langle\vec{q}(t)\rangle\bigg|_{\vec{h}\to\vec{0}}. \tag{50}$$

The stochastic average over paths is needed to compute the response function. In a discrete-time representation for $t_i=i\Delta t$ in $\Delta t\to 0$ limit, the weight functional of a path is given as proportional to $\exp[-(4\beta^{-1}\gamma\Delta t)^{-1}\sum_i[\Delta\vec{W}_i]^2]$, where $\Delta\vec{W}_i$ is the Wiener process defined as $\int_{t_i}^{t_{i+1}}ds\vec{\eta}(s)$. From the Langevin equation, $\Delta\vec{W}_i=m(\vec{v}_{i+1}-\vec{v}_i)+(\mathsf{F}\cdot\vec{r}_i+\Gamma\cdot\vec{v}_i-\vec{h}_i)\Delta t$, where subscript i denotes a value at t_i. It is basically the Onsager-Machlup formalism [58]. One can replace $\delta/\delta\vec{h}(t')|_{\vec{h}\to\vec{0}}$ at $t'=t_i$ with the multiplication of $(2\beta^{-1}\gamma)^{-1}\Delta\vec{W}_i/\Delta t$ to $\vec{q}(t)$. Taking the continuous-time limit again,

$$\begin{aligned}\beta^{-1}\mathsf{R}(t,t')&=\frac{1}{2\gamma\Delta t}\langle\vec{q}(t)\Delta\vec{W}(t')\rangle\\&=\frac{1}{2}\langle\vec{q}(t)\vec{v}(t')\rangle+\frac{1}{2\gamma\Delta t}\langle\vec{q}(t)[\Delta\vec{W}(t')\\&\quad-\gamma\vec{v}(t')\Delta t]\rangle.\end{aligned} \tag{51}$$

$\mathsf{R}(t,t')=0$ for $t<t'$ because the Wiener process cannot have any influence on $\vec{q}$ at an earlier time, which is known as causality.

The FDR can be examined from $\mathsf{V}(t,t')=\langle\vec{q}(t)\vec{v}(t')\rangle-\beta^{-1}\mathsf{R}(t,t')$ for $t>t'$ [57]. In the following, we use a notation for 4×2 matrices: $[\mathsf{C}]_{\vec{q}\vec{r}}=(C_{i,j})$ and $[\mathsf{C}]_{\vec{q}\vec{v}}=(C_{i,j+2})$ for $1\leqslant i\leqslant 4$ and $j=1,2$. For example, $\mathsf{C}_{\vec{q}\vec{r}}(t,t')=\langle\vec{q}(t)\vec{r}(t')\rangle$ and $\mathsf{C}_{\vec{q}\vec{v}}(t,t')=\langle\vec{q}(t)\vec{v}(t')\rangle$. We also let $\mathsf{V}_{\vec{r}\vec{v}}$ and $\mathsf{V}_{\vec{v}\vec{v}}$ be the upper and the lower block of V, respectively. We can get

$$\begin{aligned}\mathsf{V}(t,t')&=\frac{1}{2}\mathsf{C}_{\vec{q}\vec{v}}(t,t')\\&\quad-\frac{1}{2\gamma}\left\langle\vec{q}(t)\left[m\frac{\Delta\vec{v}(t')}{\Delta t}+\mathsf{F}\cdot\vec{r}(t')+b\mathsf{A}\cdot v(t')\right]\right\rangle,\end{aligned} \tag{52}$$

where $\Delta\vec{v}(t')=\vec{v}(t'+\Delta t)-\vec{v}(t')$. Note that the term in the square bracket in the above equation is $\Delta\vec{W}(t')/\Delta t-\gamma\vec{v}(t')$, which is the force exerted by the heat bath. $\mathsf{V}_{\vec{r}\vec{v}}$ corresponds to the FDR for the position basis [57] and $\mathsf{V}_{\vec{v}\vec{v}}$ for the velocity

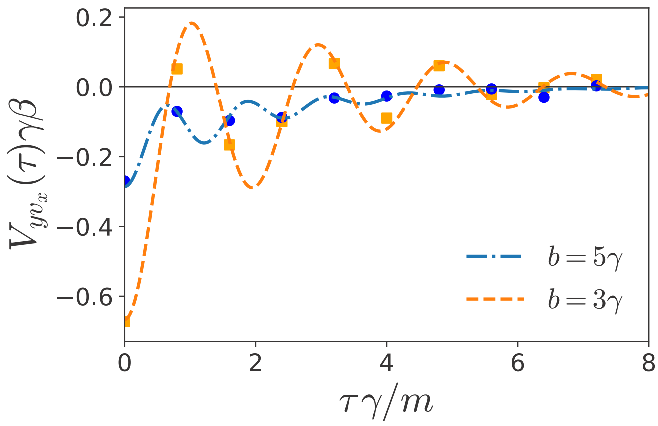


FIG. 5. The Violation of the FDR in position basis: $V_{yv_x}(\tau)$ for $k = \gamma^2/m$ and $a = 2\gamma^2/m$. The fluctuation from the guide line is more amplified for smaller b (smaller Ω). Dots and squares represent the points obtained from the simulation with 10^4 ensembles.

basis [55,56]. In the equal-time limit with $t = t' + \Delta t$, we find

$$\mathsf{V}_{\vec{v}\vec{v}}(t,t) = -\frac{1}{\gamma}\left\langle \frac{\vec{v}(t)+\vec{v}(t')}{2}\left[\frac{\Delta\vec{W}(t')}{\Delta t} - \gamma\vec{v}(t)\right]\right\rangle, \tag{53}$$

for which $\langle \vec{q}(t)\Delta\vec{W}(t')\rangle = 0$ is used. It is the Stratonovich representation for the product $\vec{v}(t)\circ(\vec{\eta}(t)-\gamma\vec{v}(t))$. Then, $\gamma\mathrm{Tr}\mathsf{V}_{\vec{v}\vec{v}}(t,t)$ is minus the rate of work done by the reservoir force, which is indeed the rate of heat dissipation. One can also see $\partial_t\mathsf{V}_{\vec{r}\vec{v}}(t,t') = \mathsf{V}_{\vec{v}\vec{v}}(t,t')$.

In the steady state, $\mathsf{V}(t,t')$ depends only on $\tau = t - t'$ as correlation functions, so written as $\mathsf{V}(\tau)$. Using Eqs. (43) and (52), $\mathsf{V}(\tau)$ can be further simplified as

$$\mathsf{V}(\tau) = \frac{1}{2}\mathsf{C}_{\vec{q}\vec{v}}(\tau) - \frac{1}{2\gamma}[m[\mathsf{MC}(\tau)]_{\vec{q}\vec{v}} + \mathsf{C}_{\vec{q}\vec{r}}(\tau)\mathsf{F}^{\mathrm{T}} - b\mathsf{C}_{\vec{q}\vec{v}}(\tau)\mathsf{A}]. \tag{54}$$

In particular, we find

$$\mathsf{V}(0) = \frac{a}{\beta\gamma\Omega}\begin{pmatrix} 0 & 1 \\ -1 & 0 \\ a/\gamma & 0 \\ 0 & a/\gamma \end{pmatrix}. \tag{55}$$

$\mathsf{V}(0) = 0$ if $a = 0$, independent of b, which was found in the previous study [57]. The heat production rate $\langle\dot{Q}\rangle = \gamma\mathrm{Tr}\mathsf{V}_{\vec{v}\vec{v}}(0) = 2a^2\beta^{-1}/(\gamma\Omega)$, which is consistent with Eq. (41).

$\mathsf{V}(\tau)$ in the steady state is given in detail in Appendix B. As for the equal-time case in Eq. (55), $\mathsf{V}(\tau)$ has a multiplicative factor a/Ω. This means the FDR holds only if $a = 0$, independent of b. It is a quite nontrivial result because $\mathsf{V}(\tau)$ in Eq. (54) strongly depends on the correlation functions which differ from those for $b = 0$. The DB is violated in this case because $\vec{j}^{\mathrm{irr}} \neq 0$; see Eqs. (29) and (35). This result was derived and demonstrated as an example for the exclusiveness of the FDR and the DB, which was derived for a general velocity-dependent force [57]. For nonzero a, both the FDR and the DB are violated. Figure. 5 shows $V_{xv_y}(\tau)$ for given a and increasing b.

IX. SUMMARY

Magnetic field does not work so that the steady-state distribution remains Boltzmann in the absence of a nonconservative force. However, accompanied by nonconservative force, the magnetic field is found to have a nontrivial influence on the motion out of equilibrium.

Torque of magnitude a overall tends to accelerate the colloid outwards from the center of harmonic potential and depress the possibility of steady state. However, the magnetic field of magnitude b can enhance it for $ab > 0$ (same circulation as the torque) or depress it for $ab < 0$ (opposite circulation to the torque). Two competing tendencies are signified in the existence condition for steady state as $k + ab/\gamma - ma^2/\gamma^2 > 0$. We find the irreversible current in the velocity space, that is a measure of nonequilibrium, to be composed of magnetic and (minus) nonconservative forces, which is found to come up with the stable circulation of current in the position space and the irreversible production of heat/work. We find the total entropy change ΔS to have an unconventional contribution ΔS_{uc} originated from velocity-dependence of the magnetic force. Although the magnetic force does produce any energy dissipation, its entropic contribution ΔS_{uc} measures the component due to magnetic field in dynamical irreversibility manifested by circulating current. Nonequilibrium characteristics is also found from two-time correlation functions. In particular, correlation between position and velocity in perpendicular directions to each other is diagnostic to circulating current in nonequilibrium steady state. We show the combined influence of the nonconservative force and the magnetic field on the violation of the FDR. We carry out the Monte Carlo simulation to confirm complicated expressions for analytical results.

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APPENDIX A: TWO-TIME CORRELATION FUNCTIONS IN THE STEADY STATE

From the rotational symmetry for the isotropic case, we can use Eq. (44), $C_{xx}(\tau) = C_{yy}(\tau)$, $C_{xy}(\tau) = -C_{yx}(\tau)$, and also $\partial_\tau C_{xy}(\tau) = C_{v_xy}(\tau) = -C_{v_yx}(\tau)$. For a short notation, we introduce

$$\chi = \frac{1}{\beta\sqrt{R}\Omega}, \quad w + iu = \sqrt{R}e^{i\phi/2}, \tag{A1}$$

where R and ϕ are defined in Eqs. (18) and (19). Then, from the two basic correlation functions in Eq. (48), we can find remaining correlation functions as follows:

$$\chi^{-1}C_{xv_x}(\tau) = -(2K+AB-Au)c_1(\tau) - A(w-1)s_1(\tau) + (2K+AB+Au)c_2(\tau) + A(w+1)s_2(\tau), \tag{A2}$$

$$\chi^{-1}C_{yv_x}(\tau) = -A(w-1)c_1(\tau) + (2K+AB-Au)s_1(\tau) - A(w+1)c_2(\tau) - (2K+AB+Au)s_2(\tau), \tag{A3}$$

$$\begin{aligned}\chi^{-1}C_{v_xv_x}(\tau) = &-2[K(w+1)+Aw(B-u)]c_1(\tau) - [A(w^2-1)-(B-u)(2K+AB-Au)]s_1(\tau)\\ &-2[K(w-1)+Aw(B+u)]c_2(t) - [-A(w^2-1)+(B+u)(2K+AB+Au)]s_2(t),\end{aligned} \tag{A4}$$

$$\begin{aligned}\chi^{-1}C_{v_xv_y}(\tau) = &[(B-u)(2K+AB-Au)-A(w^2-1)]c_1(\tau) + 2[K(w+1)+Aw(B-u)]s_1(\tau)\\ &-[(B+u)(2K+AB+Au)-A(w^2-1)]c_2(\tau) + 2[K(w-1)+A(B+u)]s_2(\tau).\end{aligned} \tag{A5}$$

APPENDIX B: THE VIOLATION OF FDR IN THE STEADY STATE

We introduce a common factor for the FDR matrix as

$$\kappa = \frac{a}{2\Omega}. \tag{B1}$$

Then, the elements of $\mathsf{V}(\tau)$ are found as

$$\kappa^{-1}V_{xv_x}(\tau) = -(2A-B-u)c_1(\tau) - (w-1)s_1(\tau) + (2A-B+u)c_2(\tau) - (w+1)s_2(\tau), \tag{B2}$$

$$\kappa^{-1}V_{yv_x}(\tau) = (2A-B-u)s_1(\tau) - (w-1)c_1(\tau) - (2A-B+u)s_2(\tau) - (w+1)c_2(\tau), \tag{B3}$$

$$\kappa^{-1}V_{v_xv_x}(\tau) = -(2K+AB-Au)s_1(\tau) + A(w-1)c_1(\tau) + (2K+AB+Au)s_2(\tau) + A(w+1)c_2(\tau), \tag{B4}$$

$$\kappa^{-1}V_{v_yv_x}(\tau) = -(2K+AB-Au)c_1(\tau) - A(w-1)s_1(\tau) + (2K+AB+Au)c_2(\tau) - A(w+1)s_2(\tau). \tag{B5}$$

The FDR is always violated for $a = 0$, irrespective of b due to the common multiplicative factor κ.

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