High-density percolation on the modified Bethe lattice

C. Widder and T. Schilling*
Physikalisches Institut Albert-Ludwigs-Universität Freiburg, Hermann-Herder-Strasse 3, 79104 Freiburg, Germany

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High-density percolation is the formation of a system spanning cluster of vertices with at least m occupied neighbors. We discuss high-density percolation on the modified Bethe lattice in terms of the theory of large random graphs with arbitrary degree distributions. Using the formalism of generating functions, we derive expressions for the cluster size distribution, the percolation threshold, the percolation probability, and the mean size of finite clusters. We show that the critical exponents $\beta = \gamma = 1$. Additionally, numerical solutions and simulation results for the percolation probability and mean size of finite clusters are compared for illustration.

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I. INTRODUCTION

The term *percolation* refers to the formation of a system spanning cluster, which can either sustain a mechanical load ("rigidity percolation") or on which particles can be transported ("connectivity percolation") [1–3]. In this paper we concentrate on connectivity percolation. One branch of connectivity percolation theory deals with the formation of clusters on random graphs (for a comprehensive overview see Ref. [4]). Percolation on random graphs is used as a model in various fields of research, in the context of epidemiology [5], as well as materials science, e.g., for conductive composite systems [6–9], magnetic systems [10–12], and glasses [13,14]. *High-density* percolation is a generalization of the notion of connectivity percolation to clusters of vertices with at least m occupied neighbors. For certain magnetic alloys as well as for glasses, high-density percolation has proven to be a useful model [12,14,15]. Also, in the context of contagion processes, the extension from models that require contact with one infected individual to produce an infection to models that require several contacts has been shown to display interesting dynamics [16].

The high-density percolation problem on the ordinary Bethe lattice has been solved exactly by Reich and Leath by mapping the formation of an m-cluster onto a random walk [15]. Here, we study high-density percolation on the *modified* Bethe lattice, employing the close resemblance between this problem and the properties of large random graphs with arbitrary degree distributions, which have been extensively studied [17–19].

In Sec. II, we remind the reader of the theory of random graphs. This section is based on the insightful paper from Newman, Strogatz, and Watts [17]. In Sec. III, we present solutions for the percolation threshold, the percolation probability, the mean size of finite clusters, as well as critical exponents for high-density percolation on the modified Bethe lattice.

II. THEORY OF RANDOM GRAPHS WITH ARBITRARY DEGREE DISTRIBUTION

Let p_k be the probability distribution of a random variable $k \in \mathrm{N}_0$. Then, the generating function $G(x)$ is defined as

$$G(x) := \sum_k p_k x^k. \tag{1}$$

Therefore, the probabilities p_k can be obtained by

$$p_k = \frac{1}{k!}\frac{d^k}{dx^k}G(x)\big|_{x=0} \tag{2}$$

and the nth moment is given by

$$\langle k^n \rangle = \sum_k k^n p_k = \left[x\frac{d}{dx}\right]^n G(x)\big|_{x=1}. \tag{3}$$

The probability distribution p_k is normalized, if and only if $G(1) = 1$. A useful feature of the generating function $G(x)$ is that the generating function $F(n, x)$ for the sum of n independent realizations of the random variable k can be evaluated according to

$$F(n, x) = [G(x)]^n. \tag{4}$$

We consider a large unipartite and undirected random graph with arbitrary degree distribution p_k. A random graph is generated by the degree sequence which is drawn from the specified degree distribution. The degree k of a vertex denotes the number of its edges. The edges of each vertex are then uniformly connected over all possibilities. In the limit of infinite vertices, the probability of closed loops vanishes.

Let $G_0(x)$ be the generating function for the probabilities p_k. We introduce the excess probability q_k which denotes the probability that a vertex with degree k is reached, when following an arbitrarily chosen edge to one of its ends. The excess probability is proportional to the number of edges; therefore, one obtains

$$q_k = \frac{k p_k}{\langle k \rangle}. \tag{5}$$

*tanja.schilling@physik.uni-freiburg.de

Hence, the corresponding generating function $\tilde{G}_1(x)$ for the probabilities that a vertex has degree k when arriving from an arbitrarily chosen edge is

$$\tilde{G}_1(x) = \sum_{k=0}^{\infty} \frac{kp_k}{\langle k\rangle} x^k.$$

Then, the generating function $G_1(x)$ for the probabilities that a vertex has $k-1$ outgoing edges when arriving from an arbitrarily chosen edge is obtained by shifting the probabilities by 1:

$$G_1(x) = \sum_{k=1}^{\infty} \frac{kp_k}{\langle k\rangle} x^{k-1} = \frac{G_0'(x)}{\langle k\rangle}. \tag{6}$$

Now, we consider generating functions for the finite sizes of components. (These will correspond to the generating functions for the cluster size distribution once we make the transition to the percolation problem.) The generating function for the probabilities that the component at the end of an arbitrarily chosen edge has a certain arbitrary but finite size $s \in \mathrm{N}$ is denoted by $H_1(x)$. In the case where a giant component (component of infinite size) exists, one therefore has $H_1(1) < 1$.

The generating function $H_1(x)$ fulfills the following self-consistency condition: the size of the component at the end of an arbitrarily chosen edge equals one plus the sum of the sizes of the components following each outgoing edge (which can also be regarded as arbitrarily chosen). To express this self-consistency condition, Eq. (4) is applied

$$H_1(x) = x\sum_{k=1}^{\infty} \frac{kp_k}{\langle k\rangle}[H_1(x)]^{k-1} = xG_1(H_1(x)), \tag{7}$$

where multiplication by x again simply shifts the probabilities by 1 due to the loss of one vertex, when following the outgoing edges.

Similarly, the size of the component of an arbitrarily chosen vertex is one plus the sum of the sizes of the components at the end of every edge of the chosen vertex. Again, the generating function $H_0(x)$ for the probabilities that an arbitrarily chosen vertex is part of a component of size $s \in \mathrm{N}$ is obtained by using Eq. (4) and shifting the probabilities by 1:

$$H_0(x) = x\sum_{k=0}^{\infty} p_k[H_1(x)]^k = xG_0(H_1(x)). \tag{8}$$

Equations (7) and (8) determine the generating function $H_0(x)$ uniquely for $x \in [0, 1)$; for $x = 1$, two solutions may occur.

The percolation threshold corresponds to the *critical point* of a random graph at which the so-called phase transition occurs, which is the appearance of the giant component. Since the right-hand side of Eq. (7) and its derivatives are monotonic functions in $y = H_1(x) \in [0, 1]$, the "graphical solution" implies that there is only one solution for $x \in [0, 1)$. However, for $x = 1$ there may exist two solutions. One solution occurs at $H_1(1) = 1$. In the case where a giant component exists, it is known that $H_1(1) < 1$; therefore, one has to take the minimum of the two solutions. Furthermore, it is known that the critical point occurs when $H_1(1)$ reaches the value 1. This is the case, if and only if $G_1'(1) = 1$ (again, due to monotonicity). Therefore, the condition for the phase transition is given by

$$G_1'(1) = 1 \Leftrightarrow \sum_{k=1}^{\infty} k(k-2)p_k = 0. \tag{9}$$

In the case where a giant component exists, one obtains

$$G_1'(1) > 1 \Leftrightarrow \langle k^2\rangle > 2\langle k\rangle \tag{10}$$

and beyond the phase transition

$$G_1'(1) < 1 \Leftrightarrow \langle k^2\rangle < 2\langle k\rangle. \tag{11}$$

This also implies that in the case where no giant component exists, there is only one solution of Eq. (7) for $x = 1$. Therefore, the full expression which determines $H_1(x)$ for $x \in [0, 1]$ is

$$H_1(x) = \min\{u \in [0, 1] | u = xG_1(u)\}. \tag{12}$$

Consider the *fraction of the giant component S*, which is the probability that a vertex is part of the giant component and corresponds to the percolation probability. It is clear that the probability u that a component at the end of a randomly chosen edge has a finite size is given by

$$u := H_1(1) = \min\{\tilde{u} \in [0, 1] | \tilde{u} = G_1(\tilde{u})\}. \tag{13}$$

Then, the probability that an arbitrarily chosen vertex is part of a finite component is

$$1 - S = \sum_{k=0}^{\infty} p_k u^k = G_0(u). \tag{14}$$

Given the generating function $H_0(x)$, the mean size of finite components $\langle s\rangle_{\text{finite}}$, which leads to the mean size of finite clusters, can be evaluated according to Eqs. (6)–(8):

$$\langle s\rangle_{\text{finite}} := \frac{H_0'(1)}{H_0(1)} = 1 + \frac{\langle k\rangle H_1(1)^2}{H_0(1)[1 - G_1'(H_1(1))]} \tag{15}$$

$$\overset{(13)(14)}{=} 1 + \frac{\langle k\rangle u^2}{[1 - S][1 - G_1'(u)]}. \tag{16}$$

Beyond or at the critical point, we have $u = 1$. Then, the mean size of finite components diverges, if and only if $G_1'(1) = 1$, in agreement with the condition for the phase transition.

III. HIGH-DENSITY PERCOLATION ON THE MODIFIED BETHE LATTICE

In this section we present solutions for high-density percolation on the modified Bethe lattice.

Consider a modified Bethe lattice with arbitrary degree distribution $f(z)$, where z denotes the number of neighbors (see Fig. 1). This is an infinite, acyclic, unipartite, and undirected graph whose vertex degrees are drawn from an arbitrary degree distribution. Here, finite components of the Bethe lattice are explicitly allowed.

An m-cluster is a maximal component of occupied vertices with at least m occupied neighbors. An occupied vertex

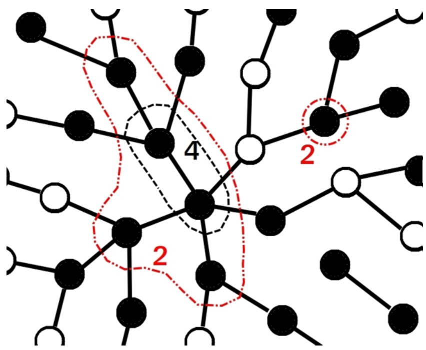

FIG. 1. Section of a modified Bethe lattice. Solid vertices are occupied, open vertices are vacant. The dashed lines surround clusters of occupied vertices. The numbers denote the parameter m.

with fewer than m occupied neighbors is considered to be a cluster of size zero. High-density percolation is the formation of (infinite) m-clusters. The only difference from ordinary percolation is the generalized cluster definition.

The basic idea behind obtaining exact solutions for the high-density percolation problem on the modified Bethe lattice is to identify all m-clusters with size $s > 0$ with a unipartite and undirected random graph in the limit of many vertices (see Figs. 1 and 2). The equations from Sec. II can be directly applied to the set of occupied vertices with at least m occupied neighbors, which is in the following simply called *the* random graph. Hence, the high-density percolation problem is reduced to the task of calculating the generating function for the degree distribution of the random graph. Afterwards, the properties of the random graph can easily be converted into the statements regarding the cluster size distribution, the percolation threshold, the percolation probability, as well as the mean size of finite clusters.

A. Generating functions for m-cluster

Here, we construct the generating function for the degree distribution of the random graph for m-clusters, $G_0(x; p)$, using the binomial distribution and the excess probability.

The binomial distribution with n samples and probability p is denoted by B_n^p. Let k denote the number of neighbors of a cluster vertex that are also part of the cluster. Furthermore, the excess probability on the modified Bethe lattice is

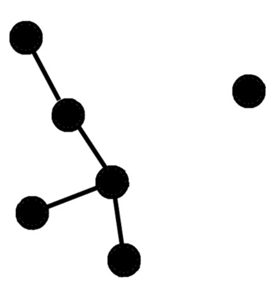

FIG. 2. Components of the random graph for the 2-cluster percolation problem which can be obtained from the section of the modified Bethe lattice (see Fig. 1).

denoted by

$$q_z = \frac{f(z)z}{\langle z\rangle}. \tag{17}$$

Now, the probability $p(z, t, k)$ that a vertex, which is part of the random graph, has degree z, t occupied neighbors, and k neighbors within the cluster is

$$p(z, t, k) = \frac{1}{r_0} f(z) B_z^p(t) B_t^r(k) \tag{18}$$

$$\text{for } (m \leqslant t \leqslant z) \wedge (0 \leqslant k \leqslant t), \tag{19}$$

where p is the occupation probability,

$$r_0 = \sum_{z=m}^{\infty} f(z) \sum_{t=m}^{z} B_z^p(t) \tag{20}$$

is the probability that an occupied vertex has at least m occupied neighbors, and

$$r = \sum_{z=m}^{\infty} \sum_{t=m-1}^{z-1} q_z B_{z-1}^p(t) \tag{21}$$

is the probability that an occupied vertex, which is reached when following an edge of an occupied vertex, has at least m occupied neighbors.

Hence, the generating function for the degree distribution of the random graph, $G_0(x; p)$, is obtained by summing up all probabilities $p(z, t, k)$, multiplied with x^k:

$$G_0(x; p) = \frac{1}{r_0} \sum_{z=m}^{\infty} f(z) \sum_{t=m}^{z} B_z^p(t) \sum_{k=0}^{t} B_t^r(k) x^k \tag{22}$$

$$= \frac{1}{r_0} \sum_{z=m}^{\infty} f(z) \sum_{t=m}^{z} B_z^p(t)[1 + r(x-1)]^t, \tag{23}$$

which can be written as

$$G_0(x; p) = \frac{\sum_{z=m}^{\infty} \sum_{t=m-1}^{z-1} \frac{q_z}{t+1} B_{z-1}^p(t)[1 + r(x-1)]^{t+1}}{\sum_{z=m}^{\infty} \sum_{t=m-1}^{z-1} \frac{q_z}{t+1} B_{z-1}^p(t)}. \tag{24}$$

With the previous equation, one obtains for the generating function $G_1(x; p)$ for the number of outgoing edges of a vertex at the end of an arbitrary chosen edge (within the random graph):

$$G_1(x; p) \overset{(6)}{=} \frac{G_0'(x; p)}{G_0'(1; p)} \tag{25}$$

$$= \frac{1}{r} \sum_{z=m}^{\infty} \sum_{t=m-1}^{z-1} q_z B_{z-1}^p(t)[1 + r(x-1)]^t. \tag{26}$$

B. Percolation threshold

The percolation threshold corresponds to the phase transition of the random graph. It is clear that the mean size of the clusters will be infinite, if and only if the average number of outgoing edges of a cluster vertex within the cluster exceeds the value 1. Hence, with the result for the generating function

$G_1(x;p)$, Eq. (26), and the condition for the phase transition, Eq. (9), the percolation threshold is determined by

$$G_1'(1;p)\big|_{p_c} = \sum_{z=m}^{\infty}\sum_{t=m-1}^{z-1} q_z t \binom{z-1}{t} p_c^t (1-p_c)^{z-1-t} \overset{!}{=} 1. \tag{27}$$

For $f(\tilde{z}) = \delta_{z,\tilde{z}}$, this expression is simplified to the condition for the percolation threshold for m-clusters on the Bethe lattice with degree z, derived by Reich and Leath [15]:

$$\sum_{l=m-1}^{z-1} l \binom{z-1}{l} p_c^l (1-p_c)^{z-1-l} = 1. \tag{28}$$

For $m = 1$ and $m = 2$, Eq. (27) yields

$$p_c = \frac{\langle z\rangle}{\langle z^2\rangle - \langle z\rangle}, \tag{29}$$

in agreement with the result for ordinary percolation [6,20,21].

The derivation remains valid if the parameter m is a function of the vertex degree z. For example, the percolation threshold for a cluster, where every neighbor is occupied ($m(z) = z$), is a solution of

$$\left\langle (z^2 - z)p_c^{z-1}\right\rangle = \langle z\rangle. \tag{30}$$

Furthermore, the probability p may be a function of the vertex degree z.

C. Cluster size distribution, percolation probability, and mean size of finite clusters

The cluster size distribution returns the probability that an occupied vertex is part of a cluster of any finite size $s \in \mathrm{N}_0$. Given the generating function of the random graph, $G_0(x;p)$, Eq. (23), the generating function for the cluster size distribution, $G(x;p)$, is almost given by the generating function for the sizes of finite components of the random graph, $H_0(x;p)$, Eqs. (8) and (12). However, following the definition given in Ref. [15], we include clusters of size zero, using the probability r_0, Eq. (20),

$$G(x;p) = (1-r_0) + r_0 H_0(x;p), \tag{31}$$

where $1 - r_0$ is the probability that an occupied vertex is an m-cluster of size zero.

Now, the cluster size distribution can be evaluated using Cauchy's formula [17]. The probability P_s that an occupied vertex is in a cluster of size $s \in \mathrm{N}_0$ is given by

$$P_s = \frac{1}{2\pi i}\oint \frac{G(z;p)}{z^{s+1}}dz. \tag{32}$$

The percolation probability P_m^∞ is defined as the probability that an occupied vertex is part of an infinite m-cluster [15]. Given the generating function $G(x;p)$, Eq. (31), and the fraction of the giant component S, Eq. (14), one obtains

$$1 - P_m^\infty = G(1;p) \Leftrightarrow P_m^\infty = r_0 S. \tag{33}$$

The mean size of finite clusters $\langle s_0\rangle_{\text{finite}}$ is determined according to Eqs. (14), (16), and (31),

$$\langle s_0\rangle_{\text{finite}} := G'(1;p) = r_0(1-S)\langle s\rangle_{\text{finite}}, \tag{34}$$

considering each vertex of the infinite cluster to be part of a cluster of size zero.

D. Bond percolation

For high-density *bond* percolation [22], where every edge is occupied with probability p, one obtains almost the same solutions as for site percolation.

We consider a maximal component of vertices with at least m occupied edges to be an m-cluster and a vertex with fewer than m occupied edges to be an m-cluster of size one. Then, the probability $p(k)$ that a vertex has k occupied edges in the cluster is given by

$$p(k) = \sum_{z=m}^{\infty}\sum_{t=m}^{z} f(z)B_z^p(t)B_t^r(k), \quad \text{for } (1 \leqslant k \leqslant t), \tag{35}$$

$$p(0) = (1-r_0) + \sum_{z=m}^{\infty}\sum_{t=m}^{z} f(z)B_z^p(t)B_t^r(0), \tag{36}$$

where r_0, Eq. (20), is the probability that a randomly chosen vertex has at least m occupied edges and r, Eq. (21), is the probability that a vertex, which is reached when following an occupied edge of an arbitrarily chosen vertex, has at least m occupied edges.

Now, the generating function $G_0(x;p)$ is defined as

$$G_0(x;p) = \sum_{k=0}^{\infty} p(k)x^k. \tag{37}$$

Similarly to site percolation, we obtain for the generating function for the cluster size distribution, the percolation threshold, the percolation probability, and the mean size of finite clusters

$$G(x;p) = H_0(x;p), \tag{38}$$

$$G_1'(1;p_c) = 1, \tag{39}$$

$$P_m^\infty = 1 - G(1;p) = S, \tag{40}$$

$$\langle s_0\rangle_{\text{finite}} = (1-S)\langle s\rangle_{\text{finite}}, \tag{41}$$

where the definitions of these quantities are the same as for site percolation, only the probabilities refer to arbitrarily chosen vertices instead of occupied vertices.

The generating function $G_1(x;p) = G_0'(x;p)/G_0'(1;p)$ is the same as for site percolation. Thus, the percolation threshold is the same.

E. Critical exponents β, γ

The critical exponents β and γ are defined by the following equations which determine the behavior of the percolation probability and mean size of finite clusters at the percolation threshold $p \to p_c$:

$$P^\infty \propto (p - p_c)^\beta, \quad \text{for } p > p_c, \tag{42}$$

$$\langle s_0 \rangle \propto |p - p_c|^{-\gamma}. \tag{43}$$

We prove that

$$\langle z^3 \rangle < \infty \Leftrightarrow \beta = 1, \tag{44}$$

$$\langle z^2 \rangle < \infty \Leftrightarrow \gamma = 1. \tag{45}$$

In the following we assume $0 < G_1''(1;p) < \infty$ for $p > 0$.

1. Exponent β

First, consider the quantity $u = H_1(1;p)$ just above the percolation threshold $p > p_c$. The Taylor expansion of $u \overset{(12)}{=} G_1(u;p)$ at $u = 1$ gives the following approximation for $p \to p_c$:

$$u = G_1(1;p) + G_1'(1;p)(u-1) + \frac{1}{2}G_1''(1;p)(u-1)^2 \tag{46}$$

$$\Rightarrow \quad u = \frac{2 - 2G_1'(1;p) + G_1''(1;p)}{G_1''(1;p)}. \tag{47}$$

This equation can now be used to show that the *directional* derivative $u'(p_c)$ exists and is negative:

$$u'(p_c) = \lim_{\epsilon \to 0} \frac{1}{\epsilon}[u(p_c + \epsilon) - u(p_c)] \tag{48}$$

$$= \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left[\frac{2 - 2G_1'(1;p_c+\epsilon)}{G_1''(1;p_c+\epsilon)}\right] = -2\frac{[\partial_p G_1'(1;p)]_{p_c}}{G_1''(1;p_c)}. \tag{49}$$

Therefore, u has for $p \to p_c$ the following form:

$$u = 1 + u'(p_c)(p - p_c). \tag{50}$$

Thus, with $H_0(1;p) \overset{(8)}{=} G_0(u(p);p)$ one obtains

$$H_0(1;p) = 1 + A(p_c)(p - p_c), \tag{51}$$

where

$$A(p_c) = G_0'(u(p_c);p_c)u'(p_c) + [\partial_p G_0(u(p);p)]_{p_c} \tag{52}$$

$$= G_0'(1;p_c)u'(p_c). \tag{53}$$

Hence, for the fraction of the giant component S [Eq. (14)] and the percolation probability [Eqs. (33) and (40)], one obtains for $p \to p_c$

$$S = \begin{cases} 0 & \text{if } p < p_c \\ -A(p_c)(p - p_c) & \text{if } p > p_c, \end{cases} \tag{54}$$

$$P^\infty = \begin{cases} S, & \text{bond percolation} \\ r_0(p_c)S, & \text{site percolation.} \end{cases} \tag{55}$$

Therefore, the critical exponent β equals 1, if $0 < G_1''(1;p) < \infty$ for $p > 0$.

2. Exponent γ

Consider the limit $p \nearrow p_c$. Then the mean size of finite components, $\langle s \rangle_{\text{finite}}$, is given by

$$\langle s \rangle_{\text{finite}} \overset{(16)}{=} \frac{\langle k \rangle_{p_c}}{1 - G_1'(1;p)}. \tag{56}$$

With

$$G_1'(1;p) = 1 + [\partial_p G_1'(1;p)]_{p_c}(p - p_c) \tag{57}$$

one obtains

$$\langle s \rangle_{\text{finite}} = \frac{\langle k \rangle_{p_c}}{[\partial_p G_1'(1;p)]_{p_c}}(p_c - p)^{-1}. \tag{58}$$

Now, consider $p \searrow p_c$. Then,

$$\langle s \rangle_{\text{finite}} \overset{(16)}{=} \frac{\langle k \rangle_{p_c}}{1 - G_1'(u(p);p)}. \tag{59}$$

Furthermore,

$$G_1'(u(p);p) = 1 + \left[\frac{d}{dp}G_1'(u(p);p)\right]_{p_c}(p - p_c) \tag{60}$$

$$= 1 + [u'(p_c)G_1''(1;p_c) + \partial_p G_1'(1;p)]_{p_c} \times (p - p_c) \tag{61}$$

$$\overset{(49)}{=} 1 - [\partial_p G_1'(1;p)]_{p_c}(p - p_c) \tag{62}$$

yields

$$\langle s \rangle_{\text{finite}} = \frac{\langle k \rangle_{p_c}}{[\partial_p G_1'(1;p)]_{p_c}}(p - p_c)^{-1}. \tag{63}$$

Thus, the final result for $p \to p_c$ is given by

$$\langle s \rangle_{\text{finite}} = \frac{\langle k \rangle_{p_c}}{[\partial_p G_1'(1;p)]_{p_c}}|p - p_c|^{-1}, \tag{64}$$

$$\langle s_0 \rangle_{\text{finite}} \overset{(34)(41)}{=} \begin{cases} \langle s \rangle_{\text{finite}}, & \text{bond percolation} \\ r_0(p_c)\langle s \rangle_{\text{finite}}, & \text{site percolation.} \end{cases} \tag{65}$$

Hence, the critical exponent γ equals 1, if $0 < G_1''(1;p) < \infty$ for $p > 0$.

3. Fat-tailed distributions

In some important cases, e.g., for power-law distributed graphs, the third or even the second moment of the degree distribution diverges. We have

$$\langle z^3 \rangle = \infty \quad \Leftrightarrow \quad \forall_{p \in (0,1]}\, G_1''(1;p) = \infty, \tag{66}$$

$$\langle z^2 \rangle = \infty \quad \Leftrightarrow \quad \forall_{p \in (0,1]}\, G_1'(1;p) = \infty. \tag{67}$$

In order to examine if the critical exponents are different in these cases, we enforce convergence by means of the following replacement,

$$f(z) \to \frac{f(z)e^{-\tau z}}{\sum_{z'} f(z')e^{-\tau z'}}, \tag{68}$$

and take the limit for $\tau \to 0$.

The asymptotic form of $\langle s_0 \rangle_{\text{finite}}$, Eqs. (64) and (65), does not depend on $G_1''(1;p)$. Hence, for $\langle z^2 \rangle < \infty$ and $\langle z^3 \rangle = \infty$, we still obtain $\gamma = 1$. However, $\langle z^2 \rangle = \infty$ yields $\gamma < 1$.

Similarly, for $\langle z^2\rangle < \infty$ and $\langle z^3\rangle = \infty$, we obtain $\beta > 1$. Hence, also for $\langle z^2\rangle = \infty$, we have $\beta > 1$. For completeness, consider $G_1''(1; p) \equiv 0$. For $p_c \in [0, 1]$, this is only true for $f(z) = \delta_{z,2} \wedge p_c = 1$. In this case, the exponent β does not exist, but still $\gamma = 1$, using similar arguments. In summary, we obtain the criteria in Eqs. (44) and (45) for the exponents β and γ.

For power-law distributed graphs $f(z) \propto z^{-\lambda}$, we obtain

$$\gamma = 1 \quad \Leftrightarrow \quad \lambda > 3, \tag{69}$$

$$\beta = 1 \quad \Leftrightarrow \quad \lambda > 4, \tag{70}$$

in agreement with previous results [23].

F. Special cases

1. Ordinary percolation

The general solution for high-density site percolation also works for $m = 1$; however, it is simpler to consider an occupied vertex that does not have an occupied neighbor as a cluster of size 1. Then, the solutions appear to be the same as for *bond* percolation and can also be obtained as the special case $m = 1$ of the solutions given in Sec. III D. This solution was previously derived for bond percolation in the context of the spread of epidemic diseases on networks [5].

The probability $p(k)$ that an occupied vertex has k occupied neighbors is

$$p(k) = \sum_{z\geqslant k} f(z)\binom{z}{k} p^k(1-p)^{z-k}.$$

Thus, the generating function $G_0(x; p)$ for the number of occupied neighbors of an occupied vertex is given by

$$G_0(x; p) = \sum_{k=0}^{\infty} p(k)x^k = \sum_{z=0}^{\infty} f(z)[1 + p(x-1)]^z,$$

which corresponds to the generating function of the degree distribution of the random graph, consisting of all clusters (or all occupied vertices).

The generating function $G_1(x; p)$ for the number of outgoing edges of a vertex, which is reached when following an arbitrarily chosen edge (within the random graph), is given by

$$G_1(x; p) \overset{(6)}{=} \sum_{z=0}^{\infty} \frac{zf(z)}{\langle z\rangle}[1 + p(x-1)]^{z-1}. \tag{71}$$

Now, the solutions are given according to Eqs. (38)–(41).

2. High-density percolation on the Bethe lattice

For the ordinary Bethe lattice, one can simply choose the degree distribution to be $f(z_0) = \delta_{z,z_0}$, for $z \geqslant m$, where z denotes the number of neighbors of each vertex. (In the following, consistency of the generating function $G_1(x; p)$ with previous results from Reich and Leath is ensured.)

The generating function $G_1(x; p)$ can be easily obtained from $\psi(x)$ (Eq. (18) from Ref. [15]), where $\psi(x)$ is the generating function for the step size of the corresponding random walk problem. Imagine constructing a cluster step by step. Then, the probability of executing a certain step size $l - 1$ is equal to the probability that l neighbors are added to a vertex from the cluster that also become part of the cluster. Therefore, $G_1(x; p)$ is given by

$$G_1(x; p) \overset{!}{=} x\psi(x), \tag{72}$$

where $\psi(x)$ is given by

$$\psi(x) = \frac{1}{x\tilde{r}} \sum_{l=m-1}^{z-1} \binom{z-1}{l}[p(1 - \tilde{r}[1-x])]^l(1-p)^{z-1-l}, \tag{73}$$

$$\tilde{r} = \sum_{l=m-1}^{z-1} \binom{z-1}{l} p^l(1-p)^{z-1-l}. \tag{74}$$

Indeed, one obtains for $G_1(x; p)$ from Eqs. (26) and (21) for $f(z_0) = \delta_{z,z_0}$

$$G_1(x; p) = \frac{1}{r} \sum_{l=m-1}^{z-1} B^p_{z-1}(l)[1 - r(1-x)]^l, \tag{75}$$

$$r = \tilde{r}, \tag{76}$$

$$\Rightarrow \quad G_1(x; p) = x\psi(x). \tag{77}$$

G. Numerical solutions and simulation results

In order to validate the solutions given in the previous sections, we evaluated the percolation probability and the mean size of finite clusters for $f(z) = B^{0.2}_{20}(z-2)$ and $m = 1, \ldots, 9$ and compared them with the simulation results. We simulated the "growth" of clusters within the given modified Bethe lattice 10^5 to 10^6 times (the number of samples for each case are given in the captions of the respective graphs) and we evaluated the percolation probability and mean size of the finite clusters by calculating the average over all samples. For $p \geqslant p_c$, we set a maximum cluster size, which we considered to correspond to the infinite cluster. The growth of clusters was simulated starting with an occupied root, for which a number of neighbors was drawn from the degree distribution $f(z)$. Then, the neighbors were occupied with probability p.

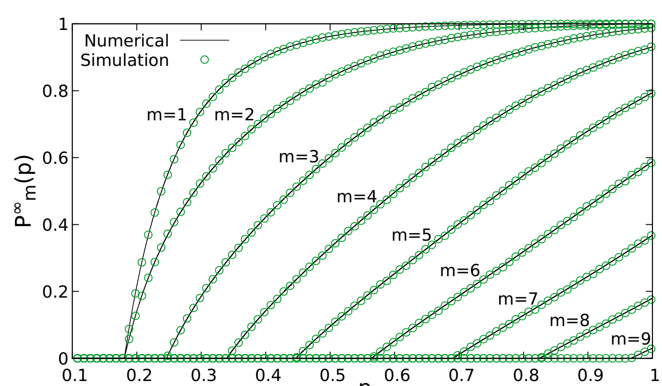

FIG. 3. Numerical solutions (lines, black online) and simulation results (circles, green online) for the percolation probability P^∞_m for $m = 1, \ldots, 9$ as a function of the occupation probability p for the degree distribution $f(z) = B^{0.2}_{20}(z-2)$. For each value of p, 10^5 samples and a maximum cluster size of 10^4 vertices is used.

Afterwards, if the root had m occupied neighbors, for each occupied neighbor the number of its neighbors was drawn from the excess probability distribution $q_z = \frac{f(z)z}{\bar{z}}$ and, again, each new neighbor was occupied with probability p. This procedure was repeated recursively if the occupied vertices had m occupied neighbors, until the cluster was complete or reached the maximum cluster size.

This method is efficient for $p \lesssim p_c$ but gets computationally expensive for $p \gtrsim p_c$, because a fraction of the simulated clusters will reach the maximum cluster size. Figure 3 shows P_m^∞ for $m = 1, \dots, 9$ as a function of the occupation probability p for the degree distribution $f(z) = B_{20}^{0.2}(z-2)$. The numerical solution of Eq. (33) and the simulation data collapse perfectly. (Note: If there exists a fraction of vertices with fewer than m neighbors, the percolation probability cannot reach 1, even for the fully occupied lattice.) Figure 4 shows the mean size of finite clusters. Again, Eq. (34) and the simulation data coincide.

Additionally we calculated numerical solutions for the power-law distribution with a cutoff $f(z) \propto z^{-\tau}$ for $z = 1, \dots, 50$. Figures 5 and 6 show the mean size of finite clusters and the percolation probability for $\tau = 2.5$ and $\tau = 3$ for several m. The cutoff at $z_{\max} = 50$ changes the critical behavior dramatically, since $p_c \to 0$ for $z_{\max} \to \infty$ and $m = 1, 2$.

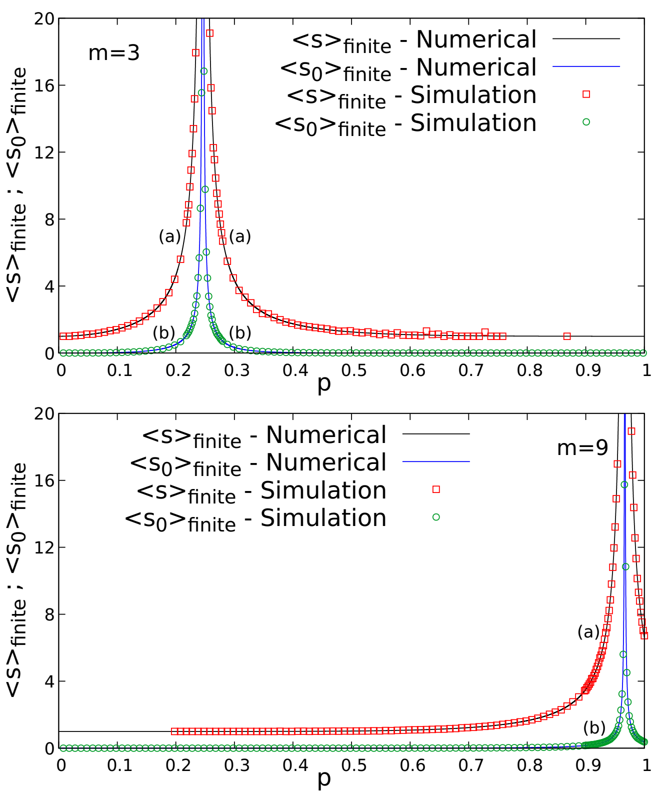

FIG. 4. Numerical solutions for the mean size of finite clusters (a) $\langle s\rangle_{\rm finite}$ and (b) $\langle s_0\rangle_{\rm finite}$ in comparison with the simulation results (squares and circles) for $m = 3$ (top) and $m = 9$ (bottom) as a function of the occupation probability p for the degree distribution $f(z) = B_{20}^{0.2}(z-2)$. For $m = 3$, 10^5 samples and a maximum cluster size of 10^4 vertices are used, and for $m = 9$, 10^6 samples and a maximum cluster size of 10^5 vertices are used.

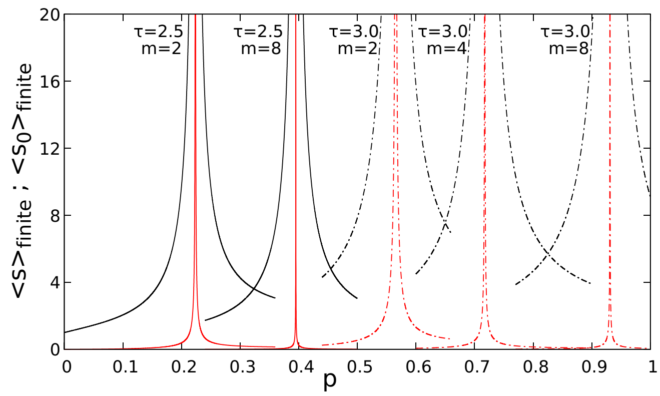

FIG. 5. Numerical solutions for the mean size of finite clusters $\langle s\rangle_{\rm finite}$ (black) and $\langle s_0\rangle_{\rm finite}$ (gray, red online) for the degree distribution $f(z) \propto z^{-\tau}$ for $z = 1, \dots, 50$ with $\tau = 2.5$ and $m = 2, 4$ (solid line) as well as $\tau = 3.0$ and $m = 2, 4, 8$ (dashed line) as a function of the occupation probability p.

In summary, we have presented a study of high-density percolation on the modified Bethe lattice. Transferring methods from the theory of random graphs, we have derived expressions for the cluster size distribution, the percolation threshold, the percolation probability, and the mean size of finite clusters and the critical exponents β and γ.

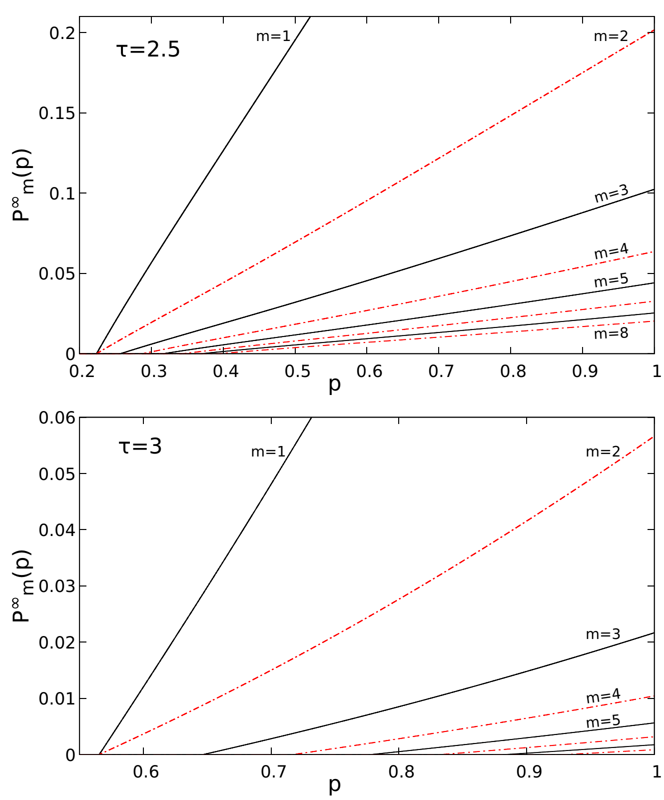

FIG. 6. Numerical solutions for the percolation probability P_m^∞ for $m = 1, \dots, 8$ (solid lines for odd m, dashed lines for even m) for the degree distribution $f(z) \propto z^{-\tau}$ for $z = 1, \dots, 50$ with $\tau = 2.5$ (top) and $\tau = 3.0$ (bottom) as a function of the occupation probability p.

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