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Local conservation laws in ultracold Fermi systems with time-dependent interaction potential

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In the context of an ultracold Fermi gas, we derive conservation laws for mass, energy and momentum based on a generalized nonlocal Boltzmann equation with gradient corrections in the collision integral. The corrections are expressed in terms of effective collision duration, particle displacement and changes of total momentum and energy. Their origin is in the in-medium T matrix. Using variations of the optical theorem, we show that in the collision integral the particle-hole symmetry can be recast into a form of collision symmetry amenable to semiclassical simulation. Pauli-blocked collisions are distinguished from Bose-stimulated nondissipative ones; the latter are not present in the absence of gradient corrections. Consolidating with the microscopic theory, we extract local conservation laws for a general time-dependent interaction potential, and demonstrate how both types of collisions affect densities and flows of conserving quantities. Comparison is made with the approach of Nozières and Schmitt-Rink in the limit of thermal equilibrium. Under approximations used for normal-state ultracold Fermi gases interacting via Feshbach resonances we demonstrate the effect of the collision delay on the shear viscosity.

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I. INTRODUCTION

The ability to tune interaction strength via Feshbach resonance in ultracold gases, moving the system continuously from the BCS to the BEC regime [1–4], is important for the the study of many-body effects in strongly correlated systems. At low temperatures, where thermal fluctuations are reduced significantly, the quantum nature of particle interactions such as Pauli blocking, pairing in space and momentum space, quantum interference and other virtual processes dramatically affects the thermodynamic properties of the system and characterizes its dynamic responses to external fields.

A. Background

In the unitary region, universal relations between physical properties of Fermi gases with s-wave interaction and the asymptotic behavior of the momentum distribution have been predicted theoretically [5–11]; these relations are observed in the ultracold gas of ^{40}K atoms [12,13], as well as in nuclear reactions [14] in which the spatial density of fermions is 20 orders of magnitude higher. In nuclear matter, the rise of the particle correlation is identified as a precursor of deuterons formed in the Fermi liquid before transit into the dilute deuteron gas; the density of virtual pairs matches the deuteron density at the Mott transition line [15,16]. Particle interactions also determine the density profile which is the key element in obtaining the equation of state in studies of trapped Fermi gases [17–21], nucleus [22,23], neutron stars [24,25] and exploding supernovae [26].

Transport properties are also affected by the quantum nature of particle interactions. For example, the reduction of viscosity upon approaching the phase of condensation from the normal phase is studied in the context of heavy-ion reactions [27–30], large-hadron collider [31] and quark-gluon dynamics [32]. In ultracold gases viscosity is experimentally accessible as it determines the breathing mode and other collective oscillations [33–38] of the trapped gas and its anisotropic expansion [39] after the trap is released. Experiences from ultracold Fermi gases show that linear response theory may be inadequate in many cases [40–47].

Fermi-liquid theory provides a framework to study many-body systems in the weakly interacting BCS and BEC regimes by introducing quasiparticles. Particles are represented by quasiparticles with energy ε having the dominant bare single-particle energy ϵ and a momentum-dependent potential provided by other particles. As long as quasiparticles have sufficiently long lifetimes, the dynamics of the system is well represented by their quasiclassical motion interrupted by binary collisions.

Numerical techniques are often employed in the study of heavy-ion reactions ([48] comparing 18 alternative methods). Standard techniques include the use of molecular-dynamics simulation with Monte Carlo collisions, and more direct numerical solution of the Boltzmann equation.

Between the two well-known weak-interaction regimes is the strongly interacting regime where many-body effects dominate; systems in this regime are no longer accessible in terms of quasiparticles. With increasing interaction strength, particles spend an increasingly higher fraction of time in interacting with others (collision states); as a particle with a given momentum passes others, momentum and energy are exchanged; the chance for the particle to possess energy other than the energy-momentum relation of a quasiparticle is increased. These events are known as *off-shell motions* and cause the reduction of quasiparticle peaks in the spectral function. Experiments have shown how the spectrum evolves with increasing interaction strength, going from the dominant singularity to the fully off-shell form, in atom-photoemission spectroscopy [49]. Spectroscopic studies of ultracold Fermi gases have shown that energy conservation with virtual processes included reveals many-body features which cannot be captured with single-particle spectra [50].

In nuclear physics, the contribution of the off-shell motion has been recognized and discussed for decades. This effect accompanying collision processes is first discussed in the study of phase transitions in the expansion of nuclear matter by Schmide and Röpke in which the correlated density emerging from collisions is identified as a precursor of deuterons formed in the Fermi liquid before transit into the dilute deuteron gas [15,16]. Studies have shown that this more complex motion can be accommodated by a generalized Boltzmann equation featuring gradient corrections in the collision integral [51]. The correlated density counts collision states; it is an effective collision duration weighted with collision rates for both Pauli-blocked and Bose-enhanced binary collisions. In the standard Boltzmann equation collisions occur instantaneously and locally; the generalized transport equation is known as the *nonlocal Boltzmann equation*.

Based on nonequilibrium Green functions, the nonlocal Boltzmann equation is applicable to systems in nonequilibrium states and with interaction potentials of any symmetry. Its derivation [51] justifies its validity for the weakly off-shell systems, where the correlated density is much smaller than the quasiparticle density. Normal nuclear matter satisfies this requirement. The present study is restricted to this regime.

Spatial nonlocality is not a novel concept. The nonzero distance between colliding particles results in an increase of pressure [52,53]; this is known as the excluded volume of the van der Waals equation of state. The delay of collision in time is interpreted as the lifetime of a new particle, in particular in a resonance state [30,54–58].

Compared to intuitive nonlocal corrections, the Green-function approach has advantages. An encounter of two fermions either proceeds as the Pauli-blocked collision event or as the Bose-stimulated correlation. The former is very natural because the Pauli-blocked collisions control dissipation in the system. The latter possibility was made promising by Danielewicz and Pratt [54], in which Bose-stimulated correlations were treated via collisions of a negative duration.

Another advantage of the Green-function approach is that all nonlocal corrections appear in a controlled way; the collision delay, space displacements, and shifts in energy and momentum are mutually consistent, and consistent with the scattering rate and the quasiparticle energy.

B. Consistency and balance

The consistency of all components is essential in the present work. While the nonzero distance between particles can be introduced as the only *ad hoc* correction independent of others, collision duration always emerges in the case of nonlocal corrections. Indeed, shifts in energy and momentum represent the external action on the colliding pair as well as its interaction with the background (the medium). In the presence of time-dependent external fields controlling particle interactions, the collision frequency changes and particle correlations change; dynamics of quasiparticles change correspondingly. Whether this nonlocal Boltzmann equation captures features of the dynamics and, at the same time, preserves the conservation laws of mass, energy, and momentum, as the standard Boltzmann equation does, are issues of reliability.

In the present paper we extract the conservation laws from the nonlocal Boltzmann equation, in the form of familiar-looking balance equations. This is done by incorporating the microscopic theory. Interested in the dynamics of out-of-equilibrium strongly correlated systems, we consider a system consisting of different types of Fermi gas with tunable particle interactions near Feshbach resonances. The interaction strength is restricted to values with a small fraction of correlated density (which excludes the unitary region).

We show that the dynamics of interactions gives rise to the off-shell motion due to which the momentum distribution (directly observed in trap releasing experiments) differs from the quasiparticle distribution by the power-law tail, and their integrals differ by the correlated density.

The system is subject to general external fields at sufficiently low temperatures where particle correlation and relevant quantum effects are pronounced. We assume that the external fields change slowly on the space-time scale of collisions; at the same time these changes may be fast compared with the characteristic scales of thermalization and hydrodynamics. We verify that although quasiparticles, as a subsystem, do not satisfy the conservation laws, correlated particles emerging from the collision events possess different physical properties to compensate the loss.

Deriving the conservation laws is more involved in comparison to the standard Boltzmann equation. Since energy transfers include interaction between collision states and quasiparticles, contributions of the quasiparticle energy (drift terms) and energy shifts in the collision integrals ought to be combined according to interation type.

In Sec. II we introduce the nonlocal Boltzmann equation which emerges as the quasiclassical limit of the nonequilibrium Green function transport equation. Then, using variations of the optical theorem, we arrange it into a form suitable for quasiclassical Monte Carlo simulations.

In Sec. III we set up the machinery required to employ the nonlocal Boltzmann equation in what we call a *scattering-symmetric* calculation; we then discuss the implications for Bose stimulation.

In Sec. IV we derive the local conservation law of particle number (the equation of continuity) from the kinetic equation; the quasiparticles and correlated particles together exhibit the conservation law. Based on microscopic theory, we further discuss the properties of correlated particles in Sec. IV B. We make comparison with the theory of Noziéres and Schmitt-Rink [59] in the limit of thermal equilibrium in Sec. IV C.

In Sec. V the conservation law of energy is derived. We start from the microscopic theory to calculate the total energy density of the system which consists of the contributions from the quasiparticles and the correlated particles.

In Sec. VI we focus on momentum conservation and derive the Cauchy momentum equation. We evaluate the shear viscosity for a gas with the s-wave scattering in Sec. VI B using relaxation time approximation.

Definitions of Green functions and self-energies, general formulas such as Kramers-Kronig relations, and various related optical theorems are included in the Appendixes.

II. THE NONLOCAL KINETIC THEORY

In this section we prepare the necessary methodology to be employed in subsequent sections. The technology here is generally selected from [51] and various references therein. Here we summarize succinctly, in appropriate notation, what will be needed in subsequent sections.

We begin with the standard Boltzmann equation, and exhibit the nonlocal kinetic equation as it appears from quantum statistics. In Sec. III this leads to a new form of the nonlocal Boltzmann equation suitable for Monte Carlo simulations.

A. Local Boltzmann equation

The standard Boltzmann equation is based on local collisions. The distribution (a density in phase space) of a type-a quasiparticle is $f(a, \boldsymbol{k}, \boldsymbol{r}, t)$. Here, a represents an internal particle state; for example a could be the spin, the isospin, or an atomic species and its excitation state.

Considering particle 1, of several in a collision event, the dynamics can be written as

$$\frac{df_1}{dt} = I_1, \tag{1}$$

where I_1 is the collision integral, and the drift term is

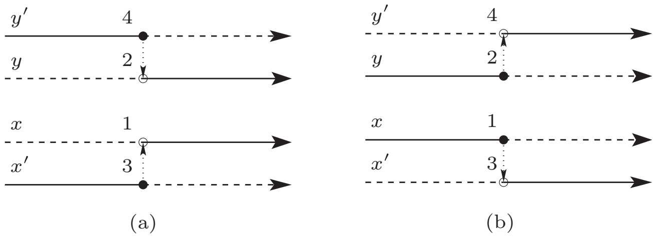
$$\frac{df_1}{dt} = \partial_t f_1 + \nabla_k \varepsilon_1 \cdot \nabla_r f_1 - \nabla_r \varepsilon_1 \cdot \nabla_k f_1. \tag{2}$$

The quasiparticle energy $\varepsilon_1 = \varepsilon(a, \boldsymbol{k}, \boldsymbol{r}, t)$ is defined using the spectral function [seen later in Eq. (46)]; this energy includes kinetic energy, external potentials, and self-energy.

As usual, the self-energy contribution is what leads to the *quasiparticle* concept; the self energy produces a change in the distribution. The influence of other particle types gives rise to a further modification of the distribution; this effect is described by the collision integral. Later it will lead us to the concept of *correlated particles* in the next section.

We separate a collision event into two components;

$$I_1 = I_1^{(\mathrm{in})} - I_1^{(\mathrm{out})}. \tag{3}$$



(a) (b)

FIG. 1. Phase space near a local collision: The horizontal axis represents $\boldsymbol{r}$ and the vertical axis $\boldsymbol{k}$. (a) Particles 3 and 4 switch to new trajectories x and y following the ‘scattering in’ event. Particle 1 leaves from $(\boldsymbol{r}, \boldsymbol{k})$ and particle 2 from $(\boldsymbol{r}, \boldsymbol{k}_2)$ at time t. (b) Particles 1 and 2 switch to new trajectories x' and y' following the ‘scattering out’ event. Particle 3 leaves from $(\boldsymbol{r}, \boldsymbol{k}_3)$ and particle 4 from $(\boldsymbol{r}, \boldsymbol{k}_4)$ at t.

$I^{(\mathrm{in})}$ and $I^{(\mathrm{out})}$ describe ‘scattering in,’ into the intermediate state $(\boldsymbol{k}, \boldsymbol{r})$ and ‘scattering out,’ from the state $(\boldsymbol{k}, \boldsymbol{r})$. The reason for this separation is to accommodate in the following section nonlocal correction to interactions. Each component of I represents a two-particle collision event; in the local approximation we have

$$I_1^{(\mathrm{in})} = \sum_b \int_{\boldsymbol{p},\boldsymbol{q}} \left|\mathcal{T}_{\mathrm{sc}}^R\right|^2 2\pi\delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4) \times (1 - f_1)(1 - f_2) f_3 f_4, \tag{4}$$

$$I_1^{(\mathrm{out})} = \sum_b \int_{\boldsymbol{p},\boldsymbol{q}} \left|\mathcal{T}_{\mathrm{sc}}^R\right|^2 2\pi\delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4) \times (1 - f_3)(1 - f_4) f_1 f_2. \tag{5}$$

We write $\int_k$ or $\int_{\mathbf{k}}$, omitting notation like $d^3k/(2\pi)^3$. b is the particle type.

The specific scattering event shown in Fig. 1 is of two particles of different types. We assume that particle types are invariant during such an event. In the scattering event shown in Fig. 1(b), particle 1 of type a on trajectory x collides with particle 2 of type b, resulting in particles 3 and 4. During the collision, momentum $\boldsymbol{q}$ is transferred between the particles; we write $\boldsymbol{k}_1 = \boldsymbol{k}$, $\boldsymbol{k}_2 = \boldsymbol{p}$, $\boldsymbol{k}_3 = \boldsymbol{k} - \boldsymbol{q}$, and $\boldsymbol{k}_4 = \boldsymbol{p} + \boldsymbol{q}$ since momentum is conserved. A similar event is shown in Fig. 1(a) where an a-type particle 3 and a b-type particle 4 collide and result in particles 1 and 2.

In collisions of identical species, $a = b$, the *direct process* shown in Fig. 1 interferes with the *exchange process* in which $\boldsymbol{k}_1 = \boldsymbol{k}$ and $\boldsymbol{k}_2 = \boldsymbol{p}$ become $\boldsymbol{k}_3 = \boldsymbol{p} + \boldsymbol{q}$ and $\boldsymbol{k}_4 = \boldsymbol{k} - \boldsymbol{q}$. The transferred momentum is now $\boldsymbol{k} - \boldsymbol{p} - \boldsymbol{q}$. The T matrix $\mathcal{T}_{\mathrm{sc}}^R$ includes this interference; details can be found in Appendix A 1.

For any quantity λ, we write for the four particle states in a scattering event, $\lambda_n = \lambda(s_n, \boldsymbol{k}_n, \boldsymbol{r}_n, t_n)$ where $n = 1, 2, 3, 4$, $s_1 = s_3 = a$, and $s_2 = s_4 = b$; for a local collision, $\boldsymbol{r}_n = \boldsymbol{r}$ and $t_n = t$.

$|\mathcal{T}_{\mathrm{sc}}^R|^2$ describes scattering from states 1,2 to states 3,4, or 3,4 to 1,2. Since it always connects all four states involved, we often omit them for notational simplicity.

B. Nonlocal Boltzmann equation

The nonlocal kinetic equation below is obtained using quasiclassical approximation of nonequilibrium Green functions. Including off-shell contributions of strong interactions with collision durations parametrized by Δ, this nonlocal

Boltzmann equation is written as

$$\frac{df_1}{dt} = I_1[f;\Delta]. \tag{6}$$

The collision integral of the quantum-field transport theory and that of the quasiclassical kinetic equation have identical structure; without the off-shell contributions, the former reduces to the latter. Below, we study the properties of the collision duration Δ, and use *derived optical theorems* to recast the equation to a form suitable for Monte Carlo simulations.

The collision duration is in agreement with the first-order off-shell contribution beyond the Landau theory of Fermi liquids.

C. Particle-hole symmetry

It will be important to incorporate shifts of momentum, position, and time into the collision parametrization. Throughout the paper a tilde or bar on a quantity will be used to indicate such a positive or negative shift. The precise shift is dependent on the subscript indicating one of the four particle states. Such a shift involves a momentum change $\boldsymbol{\Delta}_K$, displacement $\boldsymbol{\Delta}_{(i)}$, and collision delay Δ_t; these shifts, due to gradient corrections, are characteristic of the collision and will be discussed below.

Thus for any quantity $\lambda(a,\boldsymbol{k},\boldsymbol{r},t)$ (e.g., ε or f), we define

$$\begin{aligned}
\bar\lambda_1 &:= \lambda(a,\boldsymbol{k},\boldsymbol{r},t),\\
\bar\lambda_2 &:= \lambda(b,\boldsymbol{p},\boldsymbol{r}-\boldsymbol{\Delta}_{(2)},t),\\
\bar\lambda_3 &:= \lambda(a,\boldsymbol{k}-\boldsymbol{q}-\boldsymbol{\Delta}_K,\boldsymbol{r}-\boldsymbol{\Delta}_{(3)},t-\Delta_t),\\
\bar\lambda_4 &:= \lambda(b,\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{\Delta}_K,\boldsymbol{r}-\boldsymbol{\Delta}_{(4)},t-\Delta_t),
\end{aligned} \tag{7}$$

and similarly for $\tilde\lambda$ but with the sign of every Δ shift reversed.

In Eq. (7), the reference point $(\boldsymbol{k},\boldsymbol{r})$ of particle 1 is not shifted, but is included in this definition for notational consistency.

A collision event spanning the time interval from $t-\Delta_t$ to t is said to have a collision-delay time of Δ_t; there are shifts in phase space and energy as a consequence. Each Δ shift is evaluated at E, $\boldsymbol{k}$, $\boldsymbol{p}$, $\boldsymbol{q}$, $\boldsymbol{r}$, and t as given in (12). The $\boldsymbol{\Delta}_{(n)}$, $n=2,3,4$ is the displacement in space for particle state n; the f and ε functions are shifted as in (7) above.

The shifted collision integral is written as

$$\begin{aligned}
\bar I_1^{(\mathrm{in})} = \sum_b \int_{\boldsymbol{p},\boldsymbol{q}} &\left|\bar{\mathcal{T}}_{\mathrm{sc}}^R\right|^2 2\pi\delta(\bar\varepsilon_1+\bar\varepsilon_2-\bar\varepsilon_3-\bar\varepsilon_4-2\Delta_E)\\
&\times(1-\chi)(1-\bar f_1)(1-\bar f_2)\bar f_3\bar f_4,
\end{aligned} \tag{8}$$

$$\begin{aligned}
\bar I_1^{(\mathrm{out})} = \sum_b \int_{\boldsymbol{p},\boldsymbol{q}} &\left|\bar{\mathcal{T}}_{\mathrm{sc}}^R\right|^2 2\pi\delta(\bar\varepsilon_1+\bar\varepsilon_2-\bar\varepsilon_3-\bar\varepsilon_4-2\Delta_E)\\
&\times(1-\chi)(1-\bar f_3)(1-\bar f_4)\bar f_1\bar f_2.
\end{aligned} \tag{9}$$

Scattering probability is described by

$$\bar{\mathcal{T}}_{\mathrm{sc}}^R = \mathcal{T}_{\mathrm{sc}}^R\left(\bar E,\boldsymbol{k}-\tfrac{1}{2}\boldsymbol{\Delta}_K,\boldsymbol{p}-\tfrac{1}{2}\boldsymbol{\Delta}_K,\boldsymbol{q},\boldsymbol{r}-\boldsymbol{\Delta}_r,t-\tfrac{1}{2}\Delta_t\right) \tag{10}$$

with the two-particle energy $\bar E = \frac{1}{2}(\bar\varepsilon_1+\bar\varepsilon_2+\bar\varepsilon_3+\bar\varepsilon_4)$.

This T matrix has no dependence on individual particle positions; the position dependence is solely on the center of mass of the four particle states. The T matrix can also accommodate the tunable interatomic potential. The two-particle correlation function is important in the consideration of in-medium effects (Appendix A).

All Δ shifts result from the gradient (linear-order) expansion of the T matrix. Throughout this paper we shall use the notation $\approx$ to denote equality up to this order of approximation.

The scattering phase shift is

$$\phi = \mathrm{Im}\ln\mathcal{T}_{\mathrm{sc}}^R(\Omega,\boldsymbol{k},\boldsymbol{p},\boldsymbol{q},\boldsymbol{r},t) \tag{11}$$

and produces the Δ shifts by differentiation:

$$\begin{aligned}
\boldsymbol{\Delta}_{(2)} &= (\boldsymbol{\nabla}_p-\boldsymbol{\nabla}_q-\boldsymbol{\nabla}_k)\phi|_{\Omega=E}, & \boldsymbol{\Delta}_K &= \tfrac{1}{2}\boldsymbol{\nabla}_r\phi\big|_{\Omega=E},\\
\boldsymbol{\Delta}_{(3)} &= -\boldsymbol{\nabla}_k\phi|_{\Omega=E}, & \Delta_t &= \partial_\Omega\phi\big|_{\Omega=E},\\
\boldsymbol{\Delta}_{(4)} &= -(\boldsymbol{\nabla}_q+\boldsymbol{\nabla}_k)\phi|_{\Omega=E}, & \Delta_E &= -\tfrac{1}{2}\partial_t\phi\big|_{\Omega=E}.
\end{aligned} \tag{12}$$

We have defined

$$\boldsymbol{\Delta}_r = \tfrac{1}{4}(\boldsymbol{\Delta}_{(2)}+\boldsymbol{\Delta}_{(3)}+\boldsymbol{\Delta}_{(4)}). \tag{13}$$

Each Δ has the same arguments as the T matrix. The on-shell two-particle energy $E=\frac{1}{2}(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4)$ is used for the substitution.

Due to displacements, the energy-conserving δ function is accompanied by the factor $(1-\chi)$ where

$$\begin{aligned}
\chi = \tfrac{1}{2}\Big\{&\partial_t\partial_\Omega\phi+\left(\partial_\Omega^2\phi\right)\partial_t(\varepsilon_3+\varepsilon_4)\\
&+\boldsymbol{\nabla}_r\cdot[(\boldsymbol{\nabla}_p-\boldsymbol{\nabla}_q-\boldsymbol{\nabla}_k)\phi]\\
&+(\boldsymbol{\nabla}_r\varepsilon_2)\cdot[(\boldsymbol{\nabla}_p-\boldsymbol{\nabla}_q-\boldsymbol{\nabla}_k)\partial_\Omega\phi]\\
&-(\boldsymbol{\nabla}_r\varepsilon_4)\cdot[(\boldsymbol{\nabla}_q+\boldsymbol{\nabla}_k)\partial_\Omega\phi]-(\boldsymbol{\nabla}_r\varepsilon_3)\cdot(\boldsymbol{\nabla}_k\partial_\Omega\phi)\\
&+\tfrac{1}{2}(\boldsymbol{\nabla}_k\varepsilon_3+\boldsymbol{\nabla}_p\varepsilon_4)\cdot(\boldsymbol{\nabla}_r\partial_\Omega\phi)\Big\}_{\Omega=E}.
\end{aligned} \tag{14}$$

To the authors' knowledge this factor has not yet been analyzed in detail. Implementations of the nonlocal kinetic theory based on the Galitskii-Feynman T matrix are restricted to the nuclear matter [51,60,61], and approximate all Δ shifts by constant Δ_t and related displacement of the center of mass. The factor χ is zero in these implementations. The χ is nontrivial in a Fermi gas system, whether due to external modification of the interaction potential or due to effects within the medium.

In the approach summarized here the collision delay and the spatial displacements in (8) and (9) are obtained from the gradients of the two-particle propagator represented by the T matrix. These gradient corrections are different from those used in the study of Ivanov and Voskresensky [62,63] where their off-shell corrections are obtained by gradients of the single-particle propagator. Necessarily, these two forms of gradient corrections are associated with different values of the collision delay and displacements. Here the corrections vanish in the second-order Born approximation $\mathcal{T}_{\mathrm{sc}}^R \approx \mathcal{V}_{\mathrm{sc}}$, while those of Ivanov and Voskresensky do not. Several interpretations of the collision delay are compared by Kolomeitsev and Voskresensky [63] and Texier [64]. The concept used here corresponds to Wigner's collision delay [65].

D. Optical theorem

In the local Boltzmann equation, $I^{(\mathrm{in})}$ and $I^{(\mathrm{out})}$, exhibit particle-hole symmetry: interchanging particles and holes $f \leftrightarrow 1 - f$ causes $I^{(\mathrm{in})} \leftrightarrow I^{(\mathrm{out})}$. Although this particle-hole symmetry remains in the collision integral with gradient corrections, the time ordering of states requires a more careful understanding.

Considering the case $\Delta_t > 0$, $\bar{I}^{(\mathrm{out})}$ describes the a-type particle with $(\boldsymbol{k}, \boldsymbol{r})$ at time t scattering to the phase-space point $(\boldsymbol{k} - \boldsymbol{q} - \boldsymbol{\Delta}_K, \boldsymbol{r} - \boldsymbol{\Delta}_{(3)})$ and earlier time $t - \Delta_t$. To order the times, we can instead introduce an a-type hole with $(\boldsymbol{k} - \boldsymbol{q} - \boldsymbol{\Delta}_K, \boldsymbol{r} - \boldsymbol{\Delta}_{(3)})$ which enters the collision at $t - \Delta_t$; at time t, this hole annihilates the a-type particle with $(\boldsymbol{k}, \boldsymbol{r})$. That is, the 'scattering out' process is equivalent to an event where two holes collide at time $t - \Delta_t$ and annihilate two particles at time t.

Contemporary evolution of particles and holes is natural in quantum field theory, but it is a complication for quasiclassical simulations. It is possible to simulate trajectories of holes along with trajectories of particles, but there are an infinite number of holes while the number of particles is limited. For this reason we want to avoid holes in simulations.

To avoid holes, we need a collision integral which corresponds to a semiclassical picture of two-particle collisions. Let us suppose that in a simulation we generate a single collision event with $\boldsymbol{k}, \boldsymbol{p} \to \boldsymbol{k} - \boldsymbol{q}, \boldsymbol{p} + \boldsymbol{q}$ lasting from t to $t + \Delta_t$. In the kinetic equation it would correspond to four integrals: 'scattering out' of $\boldsymbol{k}$ and $\boldsymbol{p}$ states at time t, and 'scattering in' of $\boldsymbol{k} - \boldsymbol{q}$ and $\boldsymbol{p} + \boldsymbol{q}$ states at time $t + \Delta_t$. The 'scattering in' process has initial states earlier in time than the collision event; 'scattering out' has final states which occur later in time.

If the Pauli blocking reduces the scattering rate, both final states have to be empty. In the above picture of the 'scattering out' at time t one needs to know occupation at $t + \Delta_t$ to decide whether the collision happens or is rejected as Pauli blocked. To employ such information in the future requires a step which is anticausal on the microscopic time scale Δ_t.

The perturbative expansion of nonequilibrium Green functions is strictly causal, therefore it is not explicitly dependent on occupation of states at later times. It is possible, however, to accommodate dependence on later times in the 'scattering out' stage via the anticausal form of the optical theorem. The causal and anticausal optical theorem are discussed in Appendix A 2.

The causal optical theorem $\mathcal{M} = \mathcal{T}^R_{\mathrm{sc}} \cdot \mathcal{A} \cdot \mathcal{T}^A_{\mathrm{sc}}$ requires that the spectral function of the T matrix $\mathcal{M} = i(\mathcal{T}^R_{\mathrm{sc}} - \mathcal{T}^A_{\mathrm{sc}})$ equals the two-particle spectral function $\mathcal{A} = i(\mathcal{G}^R - \mathcal{G}^A)$ weighted with scattering rates. The anticausal version is written $\mathcal{M} = \mathcal{T}^A_{\mathrm{sc}} \cdot \mathcal{A} \cdot \mathcal{T}^R_{\mathrm{sc}}$ and it is this which we will utilize in Sec. III in deriving an appropriate form of the nonlocal Boltzmann equation.

III. SCATTERING-SYMMETRIC NONLOCAL BOLTZMANN EQUATION

In this section we derive a new form of the nonlocal Boltzmann equation which is symmetric in what we term 'scattering in' and 'scattering out' states; it will be seen to be particularly suited to Monte Carlo simulation.

The two-particle spectral function for uncorrelated particles,

$$\mathcal{A}_{34}(\Omega) = \int_\omega \left(G_3^>(\Omega - \omega)G_4^>(\omega) - G_3^<(\Omega - \omega)G_4^<(\omega)\right), \tag{15}$$

corresponds to the occupation factor $(1 - f_3)(1 - f_4) - f_3 f_4$, therefore we subtract

$$\bar{J}_1 = \sum_b \int_{\boldsymbol{p},\boldsymbol{q}} \left|\bar{\mathcal{T}}^R_{\mathrm{sc}}\right|^2 2\pi\delta(\bar{\varepsilon}_1 + \bar{\varepsilon}_2 - \bar{\varepsilon}_3 - \bar{\varepsilon}_4 - 2\Delta_E) \times (1 - \chi)\bar{f}_1\bar{f}_2\bar{f}_3\bar{f}_4 \tag{16}$$

from each collision integral.

From Eq. (9),

$$\begin{aligned}\bar{\mathcal{I}}_1^{(\mathrm{out})} &= \bar{I}_1^{(\mathrm{out})} - \bar{J}_1 \\ &= \sum_b \int_{\boldsymbol{p},\boldsymbol{q}} \left|\bar{\mathcal{T}}^R_{\mathrm{sc}}\right|^2 2\pi\delta(\bar{\varepsilon}_1 + \bar{\varepsilon}_2 - \bar{\varepsilon}_3 - \bar{\varepsilon}_4 - 2\Delta_E) \\ &\quad \times (1 - \chi)\bar{f}_1\bar{f}_2(1 - \bar{f}_3 - \bar{f}_4).\end{aligned} \tag{17}$$

The key at this point is to notice that the factor $1 - \bar{f}_3 - \bar{f}_4$ functions as the occupation for the final state, therefore we can use the equivalence of the causal and anticausal expressions of the optical theorem to interchange retarded and advanced T matrices. This will allow us to express all shifts in a positive temporal direction in a hole-free formulation.

Since $\mathcal{T}^A_{\mathrm{sc}} = \mathcal{T}^{R\dagger}_{\mathrm{sc}}$,

$$\operatorname{Im}\ln\mathcal{T}^A_{\mathrm{sc}}(\Omega, \boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}, t) = -\phi, \tag{18}$$

which changes the sign of gradient corrections. From (17),

$$\tilde{\mathcal{I}}_1^{(\mathrm{out})} = \sum_b \int_{\boldsymbol{p},\boldsymbol{q}} \tilde{W}_{1234}, \tag{19}$$

where $\tilde{W}_{1234}$ is the probability of the collision event $12 \to 34$. For convenience, we write a generic form for such an event $nm \to n'm'$,

$$\begin{aligned}\tilde{W}_{nmn'm'} = {} & \left|\tilde{\mathcal{T}}^R_{\mathrm{sc}}\right|^2 (1 + \chi)\tilde{f}_n\tilde{f}_m(1 - \tilde{f}_{n'} - \tilde{f}_{m'}) \\ & \times 2\pi\delta(\tilde{\varepsilon}_1 + \tilde{\varepsilon}_2 - \tilde{\varepsilon}_3 - \tilde{\varepsilon}_4 + 2\Delta_E),\end{aligned} \tag{20}$$

where $nmn'm'$ is one of the permutations 1234, 2143, 3412, or 4321. The energy and momentum can be modified by external fields or the interaction with the background during each collision; $\tilde{\varepsilon}_1 + \tilde{\varepsilon}_2 - \tilde{\varepsilon}_3 - \tilde{\varepsilon}_4 + 2\Delta_E = 0$ and $\boldsymbol{k}_1 + \boldsymbol{k}_2 - \boldsymbol{k}_3 - \boldsymbol{k}_4 + 2\boldsymbol{\Delta}_K = 0$.

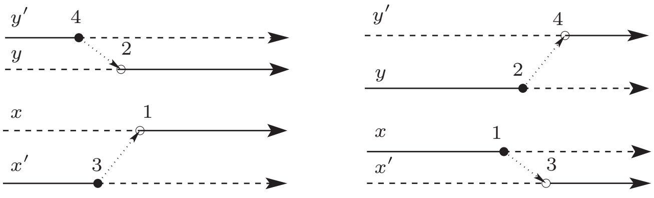
The sign changes between (17) and the equivalent (19) are the reason for our tilde notation. This applies also to the T matrix

$$\tilde{\mathcal{T}}^R_{\mathrm{sc}} = \mathcal{T}^R_{\mathrm{sc}}\left(\tilde{E}, \boldsymbol{k} + \tfrac{1}{2}\boldsymbol{\Delta}_K, \boldsymbol{p} + \tfrac{1}{2}\boldsymbol{\Delta}_K, \boldsymbol{q}, \boldsymbol{r} + \boldsymbol{\Delta}_r, t + \tfrac{1}{2}\Delta_t\right), \tag{21}$$

where the two-particle energy is $\tilde{E} = \frac{1}{2}(\tilde{\varepsilon}_1 + \tilde{\varepsilon}_2 + \tilde{\varepsilon}_3 + \tilde{\varepsilon}_4)$.

Similarly, applying to the 'scattering in' process, we write

$$\bar{\mathcal{I}}_1^{(\mathrm{in})} = \bar{I}_1^{(\mathrm{in})} - \bar{J}_1 = \sum_b \int_{\boldsymbol{p},\boldsymbol{q}} \tilde{W}_{3412} \tag{22}$$



(a) scattering in (b) scattering out

FIG. 2. Phase space near nonlocal collisions: The horizontal axis represents $\boldsymbol{r}$ and the vertical axis $\boldsymbol{k}$. The scattering effects nonlocal shifts in $\boldsymbol{r}$ and $\boldsymbol{k}$.

with

$$\begin{aligned}\bar{W}_{nmn'm'} = {}& \left|\bar{\mathcal{T}}^R_{\rm sc}\right|^2(1-\chi)\bar{f}_n\bar{f}_m(1-\bar{f}_{n'}-\bar{f}_{m'})\\ &\times 2\pi\delta(\bar{\varepsilon}_1+\bar{\varepsilon}_2-\bar{\varepsilon}_3-\bar{\varepsilon}_4-2\Delta_E)\end{aligned} \tag{23}$$

and now $\bar{\varepsilon}_1+\bar{\varepsilon}_2-\bar{\varepsilon}_3-\bar{\varepsilon}_4-2\Delta_E=0$, $\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{k}_3-\boldsymbol{k}_4-2\boldsymbol{\Delta}_K=0$.

We realize that since $\bar{\mathcal{I}}_1^{(\rm out)}=\tilde{\mathcal{I}}_1^{(\rm out)}$, gradient corrections to the integral vanish; all the corrections are in $\bar{\mathcal{I}}_1^{(\rm in)}$ which has no anticausal form. Since the gradient expansion is limited to linear order in gradients, we might replace $\tilde{\mathcal{I}}_1^{(\rm out)}$ with a form like $\mathcal{I}_1^{(\rm out)}$, but defined without gradient corrections. However, to retain the manifest symmetry between initial and final states, we write the kinetic equation in terms of $\tilde{\mathcal{I}}^{(\rm out)}$.

The spatial and temporal ordering of the particles in the scattering event is now amenable to adaptation for numerical simulation. The result is that the nonlocal kinetic equation (6) is now written

$$\frac{df_1}{dt}=\bar{\mathcal{I}}_1^{(\rm in)}-\tilde{\mathcal{I}}_1^{(\rm out)} \tag{24}$$

with the causal form of the ‘scattering in’ and the anticausal form for the ‘scattering out’ events. The corresponding nonlocal picture of a collision is illustrated in Fig. 2.

Pauli blocking and Bose stimulation

Integrals $\bar{I}_1^{(\rm in)}$ and $\tilde{I}_1^{(\rm out)}$ in the collision integral are customary scattering integrals incorporating Pauli blocking with gradient corrections, in the form required for quasiclassical simulations. Their contribution to the dynamics of the Fermi liquid has been studied for heavy-ion reactions [51,60].

Integrals $\tilde{J}_1$ and $\bar{J}_1$ represent the ‘scattering in’ and ‘scattering out’ processes stimulated by pairs on occupied states. In Bose systems such stimulated processes are common, but such processes have not been explored in the case of composite bosons made of a pair of colliding fermions. As is known from the cases of lasers and superconductors, Bose-stimulated processes are not inherently dissipative, and can lead to coherent dynamics. We define

$$\tilde{J}_1-\bar{J}_1=f_1\Upsilon_1, \tag{25}$$

where $\Upsilon_1=\sum_b\int_{\boldsymbol{p},\boldsymbol{q}}\upsilon_1$ and

$$\begin{aligned}\upsilon_1 = {}& \left|\tilde{\mathcal{T}}^R_{\rm sc}\right|^2(1+\chi)\tilde{f}_2\tilde{f}_3\tilde{f}_4 2\pi\delta(\tilde{\varepsilon}_1+\tilde{\varepsilon}_2-\tilde{\varepsilon}_3-\tilde{\varepsilon}_4+2\Delta_E)\\ &-\left|\bar{\mathcal{T}}^R_{\rm sc}\right|^2(1-\chi)\bar{f}_2\bar{f}_3\bar{f}_4 2\pi\delta(\bar{\varepsilon}_1+\bar{\varepsilon}_2-\bar{\varepsilon}_3-\bar{\varepsilon}_4-2\Delta_E).\end{aligned}$$

Υ_1 is a function of all the Δ shifts, and is nonzero if and only if gradient corrections are nonvanishing. Positive Υ_1 corresponds to insertion of quasiparticles at $(\boldsymbol{k},\boldsymbol{r})$, and negative Υ_1 corresponds to removal.

The authors know of no process which can demonstrate Bose-stimulated collisions in normal Fermi systems; such processes can indeed be found at the interface of a normal metal and a superconductor. In Andreev reflection [66], a quasiparticle with energy in the superconducting gap is reflected back to the normal metal as a hole by the gradient of the Bardeen-Cooper-Schrieffer gap at the interface, while the condensate gains a Cooper pair. Thus a gradient of the occupied pair state acts as a sink for quasiparticles.

IV. CONSERVATION OF MASS

A weakly interacting system can be described using quasiparticles. However, in a strongly interacting system there arises a difference between the density of physical particles and the density of quasiparticles, which in nuclear physics is known as the *correlated density*.

This correlated density can be attributed to collisions, and is identified as a precursor of deuterons formed in the Fermi liquid before transit into the dilute deuteron gas; it matches the deuteron density at the Mott transition line [15,16]. In a generalized Beth-Uhlenbeck theory, it is concluded that the correlated density is a consequence of the noninstantaneous collisions. The correlated density is an off-shell contribution which cannot be described using quasiparticles alone [16].

In this section we derive the local conservation law of mass, a continuity equation, from the nonlocal Boltzmann equation, and show that our correlated density indeed corresponds to the difference of the densities of physical particles and quasiparticles. To show that the correlated density is of off-shell origin, we will use the *extended quasiparticle* approximation of the microscopic theory [67–72].

A. Particle numbers

We begin by deriving a balance equation for particle numbers. Integrating the nonlocal Boltzmann equation (24),

$$\int_{\boldsymbol{k}}\frac{df_1}{dt}=S_a(\boldsymbol{r},t), \tag{26}$$

where we have defined

$$S_a(\boldsymbol{r},t)=\sum_b\int_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}}(\bar{W}_{3412}-\tilde{W}_{1234}). \tag{27}$$

The left side of (26) is the sum of the quasiparticle terms $\partial_t n_a^q$ and $\nabla_r\cdot\boldsymbol{j}_a^q$, where

$$n_a^q(\boldsymbol{r},t)=\int_{\boldsymbol{k}}f_1,\quad \boldsymbol{j}_a^q(\boldsymbol{r},t)=\int_{\boldsymbol{k}}f_1\nabla_k\varepsilon_1. \tag{28}$$

In the standard Boltzmann-equation formulation, an important concept is collision invariance; $S_a(\boldsymbol{r},t)\equiv 0$. The nonlocal version (24) only exhibits collision invariance when all Δ shifts vanish. This can be seen by interchanging the initial and the final states by means of the transformation,

$$X := \begin{cases} \boldsymbol{k} \to \boldsymbol{k} - \boldsymbol{q} \\ \boldsymbol{p} \to \boldsymbol{p} + \boldsymbol{q} \\ \boldsymbol{q} \to -\boldsymbol{q} \end{cases}. \tag{29}$$

Clearly, $X^2 = 1$. The T matrix is X invariant:

$$X\mathcal{T}^R_{\text{sc}} = \mathcal{T}^R_{\text{sc}}(\Omega, \boldsymbol{k} - \boldsymbol{q}, \boldsymbol{p} + \boldsymbol{q}, -\boldsymbol{q}) = \mathcal{T}^R_{\text{sc}}(\Omega, \boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q}), \tag{30}$$

assuming that space and time variations of external fields are small on the scale of collision events. Therefore,

$$XW_{3412} = W_{1234}. \tag{31}$$

As expected, we have $S_a \to 0$ in the limit of vanishing gradient corrections $\Delta \to 0$. In the case of gradient corrections, we can use X to extract the nonlocal contribution.

Although $\phi = X\phi$, the derivatives of ϕ transform as

$$X\boldsymbol{\nabla}_p\phi = \boldsymbol{\nabla}_p\phi, \quad X\boldsymbol{\nabla}_k\phi = \boldsymbol{\nabla}_k\phi,$$
$$X\boldsymbol{\nabla}_q\phi = (\boldsymbol{\nabla}_p - \boldsymbol{\nabla}_k - \boldsymbol{\nabla}_q)\phi. \tag{32}$$

Energy-shell Δ shifts from (12) transform as

$$X\boldsymbol{\Delta}_{(2)} = \boldsymbol{\Delta}_{(3)} - \boldsymbol{\Delta}_{(4)}, \tag{33}$$

$$X\boldsymbol{\Delta}_{(4)} = \boldsymbol{\Delta}_{(3)} - \boldsymbol{\Delta}_{(2)}, \tag{34}$$

$$X\boldsymbol{\Delta}_{r} = \boldsymbol{\Delta}_{(3)} - \boldsymbol{\Delta}_{r}, \tag{35}$$

while the rest of the Δ shifts are X invariant.

To calculate the X transformation up to linear order on the quasiparticle energies and on the distribution functions, in Δ, we introduce an operator

$$\mathcal{L} = 1 - \boldsymbol{\Delta}_{(3)} \cdot \boldsymbol{\nabla}_r - \boldsymbol{\Delta}_K \cdot (\boldsymbol{\nabla}_k + \boldsymbol{\nabla}_p) - \Delta_t\partial_t \tag{36}$$

in terms of which

$$X\bar{\varepsilon}_1 = \mathcal{L}\tilde{\varepsilon}_3 = \varepsilon(a, \boldsymbol{k} - \boldsymbol{q}, \boldsymbol{r}, t),$$
$$X\bar{\varepsilon}_2 = \mathcal{L}\tilde{\varepsilon}_4 = \varepsilon(b, \boldsymbol{p} + \boldsymbol{q}, \boldsymbol{r} - \boldsymbol{\Delta}_{(3)} + \boldsymbol{\Delta}_{(4)}, t),$$
$$X\bar{\varepsilon}_3 = \mathcal{L}\tilde{\varepsilon}_1 = \varepsilon(a, \boldsymbol{k} - \boldsymbol{\Delta}_K, \boldsymbol{r} - \boldsymbol{\Delta}_{(3)}, t - \Delta_t),$$
$$X\bar{\varepsilon}_4 = \mathcal{L}\tilde{\varepsilon}_2 = \varepsilon(b, \boldsymbol{p} - \boldsymbol{\Delta}_K, \boldsymbol{r} - \boldsymbol{\Delta}_{(3)} + \boldsymbol{\Delta}_{(2)}, t - \Delta_t). \tag{37}$$

X exchanges an incoming particle with its outgoing counterpart. Similarly, with the help of (12),

$$X\chi = -\chi + \partial_t\Delta_t + (\boldsymbol{\nabla}_p + \boldsymbol{\nabla}_k) \cdot \boldsymbol{\Delta}_K + \boldsymbol{\nabla}_r \cdot \boldsymbol{\Delta}_{(3)}. \tag{38}$$

Finally,

$$X\tilde{W}_{3412} = \tilde{W}_{1234} - \partial_t(\Delta_t\tilde{W}_{1234}) - \boldsymbol{\nabla}_r \cdot (\boldsymbol{\Delta}_{(3)}\tilde{W}_{1234})$$
$$- (\boldsymbol{\nabla}_k + \boldsymbol{\nabla}_p) \cdot (\boldsymbol{\Delta}_K\tilde{W}_{1234}). \tag{39}$$

The collision contribution (27) is

$$S_a = -\sum_b \int_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}} (\partial_t(\Delta_t W_{1234}) + \boldsymbol{\nabla}_r \cdot (\boldsymbol{\Delta}_{(3)}W_{1234})). \tag{40}$$

We have neglected higher orders in gradients; $\tilde{W}_{1234} \approx W_{1234}$. To compare with the continuity equation, we write

$$S_a = -\partial_t n^c_a - \boldsymbol{\nabla}_r \cdot \boldsymbol{j}^c_a, \tag{41}$$

where the *correlated density* is defined as

$$n^c_a = \sum_b \int_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}} \left|\mathcal{T}^R_{\text{sc}}\right|^2 2\pi\delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4)$$
$$\times \Delta_t f_1 f_2(1 - f_3 - f_4) \tag{42}$$

and the *correlated current density* is defined

$$\boldsymbol{j}^c_a = \sum_b \int_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}} \left|\mathcal{T}^R_{\text{sc}}\right|^2 2\pi\delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4)$$
$$\times \boldsymbol{\Delta}_{(3)} f_1 f_2(1 - f_3 - f_4). \tag{43}$$

The balance equation (26) then yields the expected equation of continuity

$$\partial_t n_a + \boldsymbol{\nabla}_r \cdot \boldsymbol{j}_a = 0, \tag{44}$$

with total density and current

$$n_a = n^q_a + n^c_a, \quad \boldsymbol{j}_a = \boldsymbol{j}^q_a + \boldsymbol{j}^c_a. \tag{45}$$

Considering a Fermi gas at temperature higher than the Fermi energy, for the $\Delta_t > 0$ case, n^c_a can be interpreted as the density of type-a particles undergoing collisions; the probability of finding colliding pairs at given initial and final states is $P = \Delta_t W_{1234}$. Similarly, $\boldsymbol{j}^c_a$ is the current carried by collision states; during collisions the mean velocity of a particle is $\boldsymbol{v}^c_a = \boldsymbol{\Delta}_{(3)}/\Delta_t$, thus $\boldsymbol{\Delta}_{(3)}W_{1234} = \boldsymbol{v}^c_a P$.

At low temperature, the factor $f_1 f_2(1 - f_3 - f_4)$ appearing in (42) and (43) has a dominant negative contribution, since the Pauli-blocked part $f_1 f_2(1 - f_3)(1 - f_4)$ "freezes out" while in superfluid systems the Bose-stimulated part $f_1 f_2 f_3 f_4$ does not [16]. We avoid speculative interpretations of the correlated density in this regime; instead, in the following section we calculate the correlated density from the microscopic theory and show in Sec. IV C that this definition of correlated density is in agreement with the density obtained by Nozières and Schmitt-Rink [59] in the first-order approximation of the off-shell contribution to the self-energy.

B. Correlated density

The Wigner distribution (like f, a density in phase space), also called the *local-momentum distribution*, in addition to the quasiparticle contribution includes contributions from collision processes occurring at short time scales.

Here we evaluate the Wigner distribution from the microscopic theory using nonequilibrium Green functions, and split it into the quasiparticle and off-shell components. After integration over momentum to get particle density, we show that this off-shell contribution is responsible for a reduction of quasiparticle number potentially observable in experiments of ultracold Fermi gases near Feshbach resonances; this type of experiment is reviewed in [73]. Details of the Green functions used below can be found in Appendix A and [51].

The spectral function evaluated for a particle in the interacting system $A(\omega, a, \boldsymbol{k}, \boldsymbol{r}, t)$ has the form

$$A_1(\omega) = \frac{\Gamma_1(\omega)}{[\omega - \epsilon_1 - \Sigma_1(\omega)]^2 + \frac{1}{4}\Gamma_1^2(\omega)} \tag{46}$$

with ϵ_1 equal to $\epsilon(a, \boldsymbol{k}, \boldsymbol{r}, t)$, the bare energy including the kinetic energy of a quasiparticle and also that due to general external fields acting on the particle. $\Sigma_1(\omega)$ and $\Gamma_1(\omega)$ are the real and imaginary parts of the self-energy. Now we approximate, keeping only terms linear in Γ. We have

$$A_1(\omega) \approx [1 + \partial_\omega\Sigma_1(\omega)]2\pi\delta(\omega - \varepsilon_1) + \Gamma_1(\omega)\frac{\wp'}{\omega - \varepsilon_1} \tag{47}$$

with the quasiparticle energy $\varepsilon_1 = \epsilon_1 + \Sigma_1(\varepsilon_1)$. We use the notation

$$\frac{\wp'}{z-\omega} = \frac{\partial}{\partial\omega}\left(\frac{\wp}{z-\omega}\right) = \mathrm{Re}\,\frac{1}{(z-\omega+i\delta)^2}, \tag{48}$$

where the eventual limit $\delta \to 0^+$ is understood.

The self-energy satisfies the Kramers-Kronig relation

$$\Sigma_1(\omega) = \Sigma_1^{\mathrm{HF}} - \int_z \Gamma_1(z)\frac{\wp}{z-\omega}, \tag{49}$$

where Σ_1^{HF} is the Hartree-Fock self-energy. The spectral function (46) satisfies two basic sum rules:

$$\int_\omega A_1(\omega) = 1; \quad \int_\omega \omega A_1(\omega) = \epsilon_1 + \Sigma_1^{\mathrm{HF}}. \tag{50}$$

The approximate expression given in (47) also satisfies both of these sum rules exactly (see Appendix D of [51]).

As in [74], the correlation function is

$$G_1^<(\omega) \approx [1+\partial_\omega\Sigma_1(\omega)]2\pi\delta(\omega-\varepsilon_1)f_1 + \Sigma_1^<(\omega)\frac{\wp'}{\omega-\varepsilon_1}. \tag{51}$$

In the ladder approximation of the T matrix,

$$\Sigma_1^<(\omega) = \sum_b \int_{\boldsymbol{p},\upsilon,\boldsymbol{q},z} \left|\mathcal{T}_{\mathrm{sc}}^R(\upsilon+\omega,\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})\right|^2 \times G_2^>(\upsilon)G_3^<(\omega-z)G_4^<(\upsilon+z), \tag{52}$$

and its hole counterpart $\Sigma^>$ is obtained by interchanging $>$ and $<$. Since the self-energy already contains off-shell corrections, the Green functions within are approximated by $G_2^>(\upsilon) \approx (1-f_2)2\pi\delta(\upsilon-\varepsilon_2)$, $G_3^<(\omega-z) \approx f_3 2\pi\delta(\omega-z-\varepsilon_3)$, and $G_4^<(\upsilon+z) \approx f_4 2\pi\delta(\upsilon+z-\varepsilon_4)$. We then have

$$\Sigma_1^<(\omega) \approx \sum_b \int_{\boldsymbol{p},\boldsymbol{q}} \left|\mathcal{T}_{\mathrm{sc}}^R(\omega+\varepsilon_2,\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})\right|^2 \times (1-f_2)f_3f_4 2\pi\delta(\omega+\varepsilon_2-\varepsilon_3-\varepsilon_4), \tag{53}$$

and similarly

$$\Sigma_1^>(\omega) \approx \sum_b \int_{\boldsymbol{p},\boldsymbol{q}} \left|\mathcal{T}_{\mathrm{sc}}^R(\omega+\varepsilon_2,\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})\right|^2 \times f_2(1-f_3)(1-f_4)2\pi\delta(\omega+\varepsilon_2-\varepsilon_3-\varepsilon_4). \tag{54}$$

The imaginary part $\Gamma = \Sigma^> + \Sigma^<$ is related to the real part Σ by the Kramers-Kronig relation (49).

This approximation is known as the limit of small scattering rates [67–70,72], or the extended quasiparticle approximation [71]. A nonlinear modification is extended by Arshad, Kondratyev, and Siddique [75]. The dominant on-shell term depends on the quasiparticle distribution f_1; the off-shell correction is to be evaluated from the correlation function of the self-energy $\Sigma^>$ and $\Sigma^<$.

Having the singularity on the energy shell $\omega = \varepsilon_1$, the spectral function (47) is applicable on the long time scale in which quasiparticle contribution dominates over quickly decaying off-shell contributions. The sum rules in (50) guarantee correct behavior at short time scales which is important during internal states of scattering processes.

We have seen in the previous section that the particle density n_a can be written as the sum of quasiparticle density and correlated density (45). We will next show that the correlated density defined in (42) as a contribution from collision processes is identified with the off-shell contribution.

In the microscopic theory, the density of particles defined with the Wigner distribution $\rho_1 = \rho(a,\boldsymbol{k},\boldsymbol{r},t)$ is written as

$$n_a(\boldsymbol{r},t) = \int_{\boldsymbol{k}} \rho_1, \tag{55}$$

where

$$\rho_1 = \int_\omega G_1^<(\omega). \tag{56}$$

In the extended quasiparticle approximation (51),

$$\rho_1 \approx f_1 + \int_\omega (\Sigma_1^<(\omega)(1-f_1) - \Sigma_1^>(\omega)f_1)\frac{\wp'}{\omega-\varepsilon_1}. \tag{57}$$

The off-shell part of (57) has an asymptotic k^{-4} momentum “tail,” which can be seen as follows. For a contact interaction, the T matrix is of a simpler form; instead of $\mathcal{T}_{\mathrm{sc}}^R(\omega+\varepsilon_2,\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})$ we write $\mathcal{T}_{\mathrm{sc}}^R(\omega+\varepsilon_2,\boldsymbol{k}+\boldsymbol{p})$. Due to the factor f_3f_4 in self-energy (53), $|\boldsymbol{k}+\boldsymbol{p}| < 2k_{\mathrm{F}}$. In the case $k \gg k_{\mathrm{F}}$ we can thus use $\boldsymbol{p} \approx -\boldsymbol{k}$. The self-energy is nonzero only when $\omega \approx -\varepsilon_2 \approx -\varepsilon_b(-\boldsymbol{k})$. At large k, the Wigner distribution (57) behaves as $\rho_1 \approx [\varepsilon_a(\boldsymbol{k})+\varepsilon_b(-\boldsymbol{k})]^{-2}\int_\omega \Sigma_1^<(\omega) \propto k^{-4}$. This asymptotic behavior, independent of interaction potentials, is sometimes termed Tan’s identity [5].

The density from the off-shell part is given by

$$n_a^{(1)} = \int_{\boldsymbol{k},\omega}(\rho_1 - f_1) \tag{58}$$

$$= \int_{\boldsymbol{k},\omega}[\Sigma_1^<(\omega)(1-f_1) - \Sigma_1^>(\omega)f_1]\frac{\wp'}{\omega-\varepsilon_1}. \tag{59}$$

Substitution of the self-energy (53) and (54) gives

$$n_a^{(1)} = \sum_b \int_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}} \left|\mathcal{T}_{\mathrm{sc}}^R(\varepsilon_3+\varepsilon_4)\right|^2 \frac{\wp'}{\varepsilon_3+\varepsilon_4-\varepsilon_1-\varepsilon_2} \times [(1-f_1-f_2)f_3f_4 - f_1f_2(1-f_3-f_4)]. \tag{60}$$

To bring this formula into a two-particle form, we write

$$n_c^{(1)} = \sum_{a,b}\int_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}} \frac{1}{2}(\delta_{ac}+\delta_{bc})\left|\mathcal{T}_{\mathrm{sc}}^R(\varepsilon_3+\varepsilon_4)\right|^2 \times [(1-f_1-f_2)f_3f_4 - f_1f_2(1-f_3-f_4)] \times \frac{\wp'}{\varepsilon_3+\varepsilon_4-\varepsilon_1-\varepsilon_2} \tag{61}$$

so that we can read off the two-particle representation

$$n_c^{(1)} = \int_{\Omega,\Omega',\boldsymbol{q}} \big\{\mathrm{Tr}\big[\mathcal{T}_{\mathrm{sc}}^R(\Omega)\cdot\mathcal{G}^<(\Omega)\cdot\mathcal{T}_{\mathrm{sc}}^A(\Omega)\cdot\mathcal{A}(\Omega')\cdot\mathcal{C}\big] - \mathrm{Tr}\big[\mathcal{T}_{\mathrm{sc}}^R(\Omega)\cdot\mathcal{A}(\Omega)\cdot\mathcal{T}_{\mathrm{sc}}^A(\Omega)\cdot\mathcal{G}^<(\Omega')\cdot\mathcal{C}\big]\big\}\frac{\wp'}{\Omega-\Omega'}, \tag{62}$$

where we have defined

$$\mathrm{Tr}[\lambda] = \sum_{a,b}\int_{\boldsymbol{k},\boldsymbol{p}} \lambda. \tag{63}$$

Here the dot notation indicates a "matrix product" with $\sum_a \int_{\boldsymbol{k}}$. The two-particle functions in (62) now depend on the particle types a and b. We have

$$\begin{aligned}\mathcal{G}^<(\Omega) &\approx 2\pi\delta(\Omega-\varepsilon_3-\varepsilon_4)f_3f_4,\\ \mathcal{G}^<(\Omega') &\approx 2\pi\delta(\Omega'-\varepsilon_1-\varepsilon_2)f_1f_2,\\ \mathcal{A}(\Omega) &\approx 2\pi\delta(\Omega-\varepsilon_3-\varepsilon_4)(1-f_3-f_4),\\ \mathcal{A}(\Omega') &\approx 2\pi\delta(\Omega'-\varepsilon_1-\varepsilon_2)(1-f_1-f_2),\end{aligned} \tag{64}$$

and the operator for the presence of particle type c in the two-particle propagator is

$$\mathcal{C} = \tfrac{1}{2}(\delta_{ac}+\delta_{bc}). \tag{65}$$

Using the optical theorem $\mathcal{M}_{\rm ex} = \mathcal{T}_{\rm sc}^R \cdot \mathcal{A} \cdot \mathcal{T}_{\rm sc}^A$ from (A6), the Kramers-Kronig relation for $\mathcal{T}_{\rm ex}$ and $\mathcal{M}_{\rm ex}$ is

$$\int_\Omega \Omega \frac{\wp}{\Omega-\Omega'}\mathcal{M}_{\rm ex}(\Omega) = \mathcal{V}_{\rm ex} - \mathcal{T}_{\rm ex}(\Omega') - \Omega'\frac{\partial\mathcal{T}_{\rm ex}}{\partial\Omega'}, \tag{66}$$

and the Kramers-Kronig relation between $\mathcal{G}$ and $\mathcal{A}$ is

$$\int_{\Omega'} \frac{\wp}{\Omega-\Omega'}\mathcal{A}(\Omega') = -\frac{\partial\mathcal{G}}{\partial\Omega}. \tag{67}$$

We can now simplify (62),

$$n_c^{(1)} = \int_{\Omega,\boldsymbol{q}} \mathrm{Tr}\left[\mathcal{G}^< \cdot \left(\mathcal{C}\cdot\frac{\partial\mathcal{T}_{\rm ex}}{\partial\Omega} - \mathcal{T}_{\rm sc}^A\cdot\frac{\partial\mathcal{G}}{\partial\Omega}\cdot\mathcal{C}\cdot\mathcal{T}_{\rm sc}^R\right)\right]. \tag{68}$$

In this case $\mathcal{C}$ commutes with the other factors present; simplification using the *anticausal optical theorem* (A20) yields

$$n_c^{(1)} = \mathrm{Im}\int_{\Omega,\boldsymbol{q}} \mathrm{Tr}\left[\mathcal{C}\cdot\mathcal{G}^<\cdot\mathcal{T}_{\rm sc}^A\cdot\mathcal{A}\cdot\frac{\partial\mathcal{T}_{\rm sc}^R}{\partial\Omega}\right]. \tag{69}$$

In terms of the collision delay,

$$\begin{aligned}n_c^{(1)} = \sum_{a,b}\int_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}} &\left|\mathcal{T}_{\rm sc}^R(E,\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})\right|^2 2\pi\delta(\varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4)\\ &\times\frac{1}{2}(\delta_{ac}+\delta_{bc})\Delta_t f_1f_2(1-f_3-f_4).\end{aligned} \tag{70}$$

This is equal to the correlated density (42) in the case $c=a$.

C. Correlated density of Nozières and Schmitt-Rink

In the ground state, the existence of correlated density implies that the k-volume under the Fermi momentum is not equal to the number of particles. This is in conflict with Luttinger's theorem [76], shown recently by Pieri and Strinati [77] for any Φ-derivable approximation; those authors discuss the T-matrix approximation for which they obtain zero correlated density provided that the ground state is the normal Fermi liquid. They also comment on the formula of Nozières and Schmitt-Rink [59], arguing that the nonzero correlated density might be a consequence of the non-self-consistent Green functions used in the expansion. Nevertheless, the T-matrix approximation becomes unstable at the superconducting critical temperature, therefore conclusions based on the "normal" ground state are potentially erroneous, as discussed by Dave and Phillips [78].

The long-persisting discussion about the correlated density can now be embellished with experimental input. There are already two methods to measure the momentum distribution: time-of-flight expansion followed by absorption imaging [79] and sudden photoexcitation of atoms from the interacting state into a noninteracting state [80]. To support such experimental effort we have derived the momentum distribution (57). Using (53) and (54) it becomes a functional of the quasiparticle distribution.

Here we show that the correlated density (42) agrees with the Nozieres and Schmitt-Rink (NS) formula. In thermal equilibrium, the particle density obtained by NS [59] is

$$n_a^{\rm NS} = -2\int_{\boldsymbol{k},\omega} f_{\rm FD}\mathrm{Im}\left(\hat{G}_1^R + \hat{G}_1^R\Sigma_1^R\hat{G}_1^R\right), \tag{71}$$

where $\hat{G}_1^R(\omega) = 1/(\omega-\epsilon_1+i\delta)$ is the retarded free-particle Green function, $f_{\rm FD}(\omega) = 1/\{1+\exp[(\omega-\mu)/T]\}$ and $\Sigma^R = \Sigma - \frac{i}{2}\Gamma$ is the retarded self-energy.

The formula of NS is thus linear in the self-energy for both on- and off-shell contributions. We can compare it with the linear approximation in the off-shell motion (51). With the help of the free-particle spectral function $(-2)\mathrm{Im}\,\hat{G}_1^R(\omega) = 2\pi\delta(\omega-\epsilon_1)$ we arrive at

$$\begin{aligned}n_a^{\rm NS} = &\int_{\boldsymbol{k}} f_{\rm FD}(\epsilon_1) + \int_{\boldsymbol{k},\omega}\Sigma_1^<(\omega)\,\mathrm{Re}\,\frac{1}{(\omega-\epsilon_1+i\delta)^2}\\ &-\int_{\boldsymbol{k},\omega} f_{\rm FD}(\omega)\Sigma_1(\omega)\,2\mathrm{Im}\,\frac{1}{(\omega-\epsilon_1+i\delta)^2},\end{aligned} \tag{72}$$

where we have used the equilibrium correlation part of self-energy $\Sigma_1^<(\omega) = (-2)\mathrm{Im}\,\Sigma_1^R(\omega)f_{\rm FD}(\omega)$. The second term parallels the off-shell contribution of the extended quasiparticle approximation (51). The extended quasiparticle approximation has the quasiparticle energy ε in the denominator, while the NS expansion has the bare energy ϵ. This difference is at second order in Σ^R. The last line can be rearranged using

$$\begin{aligned}2\mathrm{Im}\,\frac{1}{(\omega-\epsilon_1+i\delta)^2} &= \frac{\partial}{\partial\omega}(-2)\mathrm{Im}\,\frac{1}{\omega-\epsilon_1+i\delta}\\ &= \frac{\partial}{\partial\omega}2\pi\delta(\omega-\epsilon_1).\end{aligned} \tag{73}$$

After performing the integration by parts, we obtain

$$\begin{aligned}n_a^{\rm NS} = &\int_{\boldsymbol{k}}\{f_{\rm FD}(\epsilon_1)[1+\partial_\omega\Sigma_1(\omega)] + \Sigma_1(\epsilon_1)\partial_\omega f_{\rm FD}(\omega)\}_{\omega=\epsilon_1}\\ &+\int_{\boldsymbol{k},\omega}\Sigma_1^<(\omega)\,\mathrm{Re}\,\frac{1}{(\omega-\epsilon_1+i\delta)^2}.\end{aligned} \tag{74}$$

The first line represents a linear expansion of the on-shell part of the extended quasiparticle approximation (51) in Σ. That is, in the limit of thermal equilibrium, our result reduces to the Nozières and Schmitt-Rink formula in the linear order of the self-energy.

In this section, we have seen that the kinetic equation with nonlocal collision integral includes consistently the particle interactions up to the first order of the off-shell contributions.

During collisions the correlated density emerges at the expense of quasiparticle density. We anticipate a corresponding role of the correlated density in energy balance and investigate this in Sec. V.

V. LOCAL ENERGY BALANCE

In the Boltzmann-equation formulation, two consequences of interactions are slow momentum changes in quasiparticle drift and abrupt changes of momentum in collisions. In the presence of time-dependent interactions, changes of the interaction energy are transformed to quasiparticle kinetic energy in the same two ways; between and during collisions. The drift contribution is described already by Landau's concept of quasiparticles; here we require a nonlocal collision integral, in particular with nonzero collision duration.

The drift mechanism can be described by use of effective forces and time-dependent effective mass and vector potentials, along the lines of a framework known as *Berry connections*. These concepts are closely tied to quasiparticle energy, and their use is widespread in nuclear physics and condensed matter physics, for example in [81–83].

In the present context, it is desirable clearly to understand the collision mechanism; to this end it is useful to think initially in classical terms.

The interaction strength has a time dependence, affecting how kinetic energy is distributed during collision processes. When the change in interaction is on a short time scale, the subsequent asymptotic "tail" of the momentum distribution of particles is pronounced. This effect has been observed in the ultracold gas of ^{40}K atoms [12,13]. In the present case we discuss only changes slow on the time scale of collisions [14].

In the nonlocal approach energy is only conserved when both drift and collision processes are taken into account; the Landau picture is insufficient as quasiparticle collisions do not contribute to the energy balance.

We begin by considering two-particle interaction and derive the energy density from the microscopic theory. We show energy conservation from the kinetic equation, making use of relations obtained via the time derivative of this energy density.

A. Energy density

We calculate the energy density using the *extended quasiparticle approximation* discussed in Sec. IV B. By definition, the total energy density of the system is the expectation value of its Hamiltonian,

$$\mathcal{E}(\boldsymbol{r},t)=\sum_a\int_{\boldsymbol{k}}\rho_1\epsilon_1+\frac{1}{2}\sum_{a,b}\int_{\boldsymbol{k},\boldsymbol{p}}\rho_{12}\mathcal{V}_{12}. \tag{75}$$

ρ_{12} is a two-particle density matrix for particles 1 and 2. The interaction potential between particles $\mathcal{V}_{12}$ is tunable via time-dependent external fields. The integral over the interaction potential can be evaluated from the energy-dependence of the Green function [84],

$$\mathcal{E}=\frac{1}{2}\sum_a\int_{\boldsymbol{k},\omega}(\omega+\epsilon_1)G_1^<(\omega). \tag{76}$$

At linear order in the off-shell motion, we substitute $G_1^<$ with (51) and obtain

$$\begin{aligned}\mathcal{E}=\frac{1}{2}\sum_a\int_{\boldsymbol{k}}\bigg\{&(\varepsilon_1+\epsilon_1)f_1\\&-\int_\omega[\Sigma_1^>(\omega)+\Sigma_1^<(\omega)](\varepsilon_1-\omega)f_1\frac{\wp}{\omega-\varepsilon_1}\\&+\int_\omega[\Sigma_1^<(\omega)(1-f_1)-\Sigma_1^>(\omega)f_1](\omega+\epsilon_1)\frac{\wp}{\omega-\varepsilon_1}\bigg\}.\end{aligned} \tag{77}$$

The first two rows can be combined by using the Kramers-Kronig relation for the self-energy (49);

$$\begin{aligned}\mathcal{E}=\sum_a\int_{\boldsymbol{k}}\left(\epsilon_1+\frac{1}{2}\Sigma_1^{\mathrm{HF}}\right)f_1+\frac{1}{2}\sum_a\int_{\boldsymbol{k},\omega}&[\Sigma_1^<(1-f_1)\\&-\Sigma_1^>f_1](\omega+\epsilon_1)\frac{\wp}{\omega-\varepsilon_1}.\end{aligned} \tag{78}$$

The Hartree-Fock self-energy Σ_1^{HF} is proportional to the Wigner distribution; it can be written as a sum of the on-shell and off-shell parts,

$$\Sigma_1^{\mathrm{HF}}=\mathcal{V}_{\mathrm{ex}}\cdot\rho=\mathcal{V}_{\mathrm{ex}}\cdot f+\mathcal{V}_{\mathrm{ex}}\cdot(\rho-f). \tag{79}$$

The interaction potential is antisymmetrized with respect to the exchange of identical particles,

$$\mathcal{V}_{\mathrm{ex}}(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})=\mathcal{V}(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})-\delta_{ab}\mathcal{V}(\boldsymbol{k},\boldsymbol{p},\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q}). \tag{80}$$

Since the convoluted potential $\mathcal{V}_{\mathrm{ex}}\cdot\rho$ is instantaneous, Σ_1^{HF} does not depend on ω.

The total energy density to linear order in the off-shell expansion is

$$\mathcal{E}\approx\mathcal{E}^{(0)}+\mathcal{E}^{(1)}. \tag{81}$$

The on-shell part is

$$\mathcal{E}^{(0)}=\sum_a\int_{\boldsymbol{k}}\epsilon f_1+\frac{1}{2}\sum_{a,b}\int_{\boldsymbol{k},\boldsymbol{p}}\mathcal{V}_{\mathrm{ex}}(\boldsymbol{k},\boldsymbol{p},\boldsymbol{0})f_1f_2 \tag{82}$$

and the off-shell part is

$$\mathcal{E}^{(1)}=\frac{1}{2}\sum_a\int_{\boldsymbol{k},\omega}[\Sigma_1^<(1-f_1)-\Sigma_1^>f_1](\omega+\varepsilon_1)\frac{\wp}{\omega-\varepsilon_1}. \tag{83}$$

As we will see below, the on-shell part comes from quasiparticle motion; the off-shell part contains collision contributions of quasiparticles and correlated particles.

Substitution of $\Sigma_1^<$ and $\Sigma_1^>$ using (53) and (54) gives

$$\begin{aligned}\mathcal{E}^{(1)}=\frac{1}{2}\sum_{a,b}\int_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q},\Omega}\bigg\{&\left|\mathcal{T}_{\mathrm{sc}}^R(\Omega)\right|^2 2\pi\delta(\Omega-\varepsilon_3-\varepsilon_4)\\&\times[(1-f_1-f_2)f_3f_4-f_1f_2(1-f_3-f_4)]\\&\times(\Omega+\varepsilon_1-\varepsilon_2)\frac{\wp}{\Omega-\varepsilon_1-\varepsilon_2}\bigg\}.\end{aligned} \tag{84}$$

$\mathcal{E}^{(1)}$ is symmetric with respect to $1\leftrightarrow 2$ combined with $3\leftrightarrow 4$; the term associated with $\varepsilon_1-\varepsilon_2$ must vanish. Thus, it is convenient to write $\mathcal{E}^{(1)}$ in the manifestly symmetric form

$$\begin{aligned}\mathcal{E}^{(1)} = \frac{1}{2}\sum_{a,b}\int_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q},\Omega,\Omega'} &\left|\mathcal{T}^R_{\rm sc}(\Omega)\right|^2 2\pi\delta(\Omega-\varepsilon_3-\varepsilon_4)\\ &\times[(1-f_1-f_2)f_3f_4 - f_1f_2(1-f_3-f_4)]\\ &\times 2\pi\delta(\Omega'-\varepsilon_1-\varepsilon_2)\Omega\frac{\wp'}{\Omega-\Omega'}. \qquad (85)\end{aligned}$$

In terms of the two-particle functions $\mathcal{G}$ and $\mathcal{A}$ given in (64), we write

$$\begin{aligned}\mathcal{E}^{(1)} = \frac{1}{2}\int_{\Omega,\Omega'} &\mathrm{Tr}\big[\mathcal{A}(\Omega')\mathcal{T}^R_{\rm sc}(\Omega)\mathcal{G}^<(\Omega)\mathcal{T}^A_{\rm sc}(\Omega)\\ &-\mathcal{G}^<(\Omega')\mathcal{M}_{\rm ex}(\Omega)\big]\Omega\frac{\wp'}{\Omega-\Omega'}. \qquad (86)\end{aligned}$$

The second term inside the trace has been simplified using the optical theorem $\mathcal{M}_{\rm ex} = \mathcal{T}^R_{\rm sc}\cdot\mathcal{A}\cdot\mathcal{T}^A_{\rm sc}$.

$\mathcal{E}^{(1)}$ is further simplified using the Kramers-Kronig relations and the anticausal derived optical theorem as was done for $n_a^{(1)}$ in Sec. IV. Using the Kramers-Kronig relations (67) and (66) we write $\mathcal{E}^{(1)}$ as a sum of two terms,

$$\begin{aligned}\mathcal{E}^{(1)} = &\frac{1}{2}\int_{\Omega,\boldsymbol{q}}\mathrm{Tr}[\mathcal{G}^<\cdot(\mathcal{T}_{\rm ex}-\mathcal{V}_{\rm ex})]\\ &+\frac{1}{2}\int_{\Omega,\boldsymbol{q}}\Omega\mathrm{Tr}\left[\mathcal{G}^<\cdot\left(\frac{\partial\mathcal{T}_{\rm ex}}{\partial\Omega}-\mathcal{T}^A_{\rm sc}\cdot\frac{\partial\mathcal{G}}{\partial\Omega}\cdot\mathcal{T}^R_{\rm sc}\right)\right], \qquad (87)\end{aligned}$$

where the second line can be simplified with the anticausal derived optical theorem (A20). Consequently, we have

$$\begin{aligned}\mathcal{E}^{(1)} = &\frac{1}{2}\int_{\Omega,\boldsymbol{q}}\mathrm{Tr}\big[\mathcal{G}^<\cdot(\mathcal{T}_{\rm ex}-\mathcal{V}_{\rm ex})\big]\\ &+\mathrm{Im}\int_{\Omega,\boldsymbol{q}}\Omega\mathrm{Tr}\left[\mathcal{G}^<\cdot\mathcal{T}^A_{\rm sc}\cdot\mathcal{A}\cdot\frac{\partial\mathcal{T}^R_{\rm sc}}{\partial\Omega}\right]. \qquad (88)\end{aligned}$$

Compared with (69), we see that the second term corresponds to the correlated density.

At linear order of the off-shell correction,

$$\begin{aligned}\mathcal{E}^{(1)} = &\frac{1}{2}\sum_{a,b}\int_{\boldsymbol{k},\boldsymbol{p}} f_1f_2[\mathcal{T}_{\rm ex}(\varepsilon_1+\varepsilon_2,\boldsymbol{k},\boldsymbol{p},\boldsymbol{0})-\mathcal{V}_{\rm ex}(\boldsymbol{k},\boldsymbol{p},\boldsymbol{0})]\\ &+\frac{1}{2}\sum_{a,b}\int_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}}2\pi\delta(\varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4)Ef_1f_2\\ &\times(1-f_3-f_4)\mathrm{Im}\left(\mathcal{T}^A_{\rm sc}(\Omega)\cdot\frac{\partial\mathcal{T}^R_{\rm sc}}{\partial\Omega}\right)\bigg|_{\Omega=E}, \qquad (89)\end{aligned}$$

where $E = (\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4)/2$. The T matrix with $\boldsymbol{q}=\boldsymbol{0}$ in the first line denotes interaction processes with zero deflection angle of the relative velocity of the two colliding particles. The second integral of (89), as we expected, is the energy due to noninstantaneous collisions; in terms of collision delay,

$$\begin{aligned}\mathcal{E}_c = \frac{1}{2}\sum_{a,b}\int_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}}&\left|\mathcal{T}^R_{\rm sc}\right|^2 2\pi\delta(\varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4)\\ &\times Ef_1f_2(1-f_3-f_4)\Delta_t. \qquad (90)\end{aligned}$$

This term is analogous to the correlated density; it is the *correlated energy density*. This is the energy contribution from the colliding pairs during the effective collision time Δ_t.

The quasiparticle energy density is the sum of $\mathcal{E}^{(0)}$ given in (82) and the $\mathcal{T}_{\rm ex}-\mathcal{V}_{\rm ex}$ part of $\mathcal{E}^{(1)}$ which is the first line of (89). Thus,

$$\mathcal{E}_q = \sum_a\int_{\boldsymbol{k}} f_1\epsilon_1 + \frac{1}{2}\sum_{a,b}\int_{\boldsymbol{k},\boldsymbol{p}} f_1f_2\mathcal{T}_{\rm ex}(\varepsilon_1+\varepsilon_2,\boldsymbol{k},\boldsymbol{p},\boldsymbol{0}). \qquad (91)$$

The T matrix within accounts for correlations with zero momentum transfer; the interaction time dependence is reflected in the off-shell dynamics. $\mathcal{E}_q$ contains all the energy due to quasiparticle dynamics, including the interaction with other collision pairs as will be discussed in detail subsequently.

The total energy density of the system contains the dominant part $\mathcal{E}_q$ and a correction from collisions $\mathcal{E}_c$,

$$\mathcal{E} = \mathcal{E}_q + \mathcal{E}_c. \qquad (92)$$

In nuclear matter, for instance, the collision contribution is about 10% for the symmetric matter of normal density [71]. In ultracold gases, it is tunable via interaction potentials. In the presence of interactions, the difference between $\mathcal{E}$, $\mathcal{E}_q$, and $\langle\varepsilon\rangle := \int_{\boldsymbol{k}}\varepsilon f$ rise as we will see in the following.

The quasiparticle energy defined from the singularity of the spectral function is $\varepsilon_1 = \epsilon_1 + \Sigma_1(\varepsilon_1)$ where the self-energy Σ_1 is given by the convolution of the T matrix with the real part of single-particle Green function G_1, which we denote as $\Sigma_1 = \mathcal{T}_{\rm ex}\cdot G_1^< - \mathcal{T}^<_{\rm ex}\cdot G_1$.

The self-energy of reference particle 1 is

$$\Sigma_1(\omega) \approx \Sigma_1^{\rm mf}(\omega) + \Sigma_1^{\rm HF'} + \Sigma_1^{\rm pair}(\omega), \qquad (93)$$

which includes the mean-field, off-shell Hartree-Fock and pair contributions.

The on-shell part of $G_1^<$ gives

$$\Sigma_1^{\rm mf}(\omega) = \sum_b\int_{\boldsymbol{p}}\mathcal{T}_{\rm ex}(\omega+\varepsilon_2,\boldsymbol{k},\boldsymbol{p},\boldsymbol{0})f_2. \qquad (94)$$

This term contains part of the off-shell propagation of interacting pairs included in the T matrix.

The off-shell part of $G_1^<$ leads to the interaction with background off-shell states,

$$\begin{aligned}\Sigma_1^{\rm HF'} &= \sum_b\int_{\boldsymbol{p},\omega'}\mathcal{T}_{\rm ex}(\omega+\omega',\boldsymbol{k},\boldsymbol{p},\boldsymbol{0})\\ &\quad\times[G_2^<(\omega')-2\pi\delta(\omega'-\varepsilon_2)f_2]\\ &\approx\sum_b\int_{\boldsymbol{p}}\mathcal{V}_{\rm ex}(\boldsymbol{k},\boldsymbol{p},\boldsymbol{0})(\rho_2-f_2). \qquad (95)\end{aligned}$$

Since $\Sigma_1^{\rm HF'}$ is a first-order correction of off-shell contribution $\rho - f$, we use $\mathcal{T}_{\rm ex}\approx\mathcal{V}_{\rm ex}$ as in the Hartree-Fock approximation.

The pairing self-energy contains a contribution from interaction with other colliding pairs,

$$\Sigma_1^{\rm pair}(\omega) = -\sum_b\int_{\boldsymbol{p},\Omega}\mathcal{T}^<_{\rm ex}(\Omega,\boldsymbol{k},\boldsymbol{p},\boldsymbol{0})\frac{\wp}{\Omega-\omega-\varepsilon_2}, \qquad (96)$$

where $\mathcal{T}^<_{\rm ex} = \mathcal{T}^R_{\rm sc}\cdot\mathcal{G}^<\cdot\mathcal{T}^A_{\rm sc}$. We use

$$\mathcal{T}^<_{\rm ex}(\Omega,\boldsymbol{k},\boldsymbol{p},\boldsymbol{0})\approx\int_{\boldsymbol{q}}\left|\mathcal{T}^R_{\rm sc}(\Omega,\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})\right|^2 f_3f_4 2\pi\delta(\Omega-\varepsilon_3-\varepsilon_4). \qquad (97)$$

Since we do not assume bounded pairs, this self-energy includes only binary interactions.

The Landau quasiparticle energy ε_L is the change in total energy when adding a quasiparticle; for a time-independent Hamiltonian it implies $\partial_t\mathcal{E}=\sum_a\int_{\boldsymbol{k}}\varepsilon_L\partial_t f$ so that $\partial_t\mathcal{E}$ can be directly evaluated from the kinetic equation; obtaining the energy balance equation is trivial. However, in the strongly interacting case, the spectral function is not of the δ-function form; more than only the quasiparticle peak must be considered. That is, inclusion of a Landau-type contribution

$$\partial_t\mathcal{E}_q|_L=\sum_a\int_{\boldsymbol{k}}\varepsilon_1\partial_t f_1 \tag{98}$$

does not conserve the total energy; $\partial_t\mathcal{E}_q|_L\neq\partial_t\mathcal{E}$. The difference $\varepsilon_L-\varepsilon$ known as the *rearrangement term* [85,86] in nuclear physics, thus deforms quasiparticle trajectories in order to mimic hydrodynamic consequences of nonlocal collisions.

In order to describe these hydrodynamical effects without deforming quasiparticle trajectories, the energy and momentum balance must be reconsidered, taking into account correlated particles as we show below.

In what follows, we will show that at linear order the power of the quasiparticle subsystem can be attributed to three sources;

$$\partial_t\mathcal{E}_q\approx\partial_t\mathcal{E}_q|_L+\partial_t\mathcal{E}_q|_c+\partial_t\mathcal{E}_q|_e. \tag{99}$$

The terms represent the Landau-type contribution, the collision energy gain, and the external-field energy gain. The collision energy gain $\partial_t\mathcal{E}_q|_c$ is the energy transferred from the collision processes to quasiparticles. The external-field energy gain $\partial_t\mathcal{E}_q|_e$ corresponds to the response of free quasiparticle to external fields, and the correlated effect to the interaction between particles. Like the self-energy (93), the Landau-type contribution is expected to contain three terms:

$$\partial_t\mathcal{E}_q|_L\approx\partial_t\mathcal{E}_q|_{L;\mathrm{mf}}+\partial_t\mathcal{E}_q|_{L;\mathrm{HF}'}+\partial_t\mathcal{E}_q|_{L;\mathrm{pair}}. \tag{100}$$

We calculate $\partial_t\mathcal{E}_q$ using (91), and will identify each contribution listed in (99):

$$\begin{aligned}\partial_t\mathcal{E}_q=&\sum_a\int_{\boldsymbol{k}}f_1\partial_t\epsilon_1 &(101\mathrm{a})\\ &+\sum_a\int_{\boldsymbol{k}}\frac{\partial f_1}{\partial t}\left(\epsilon_1+\sum_b\int_{\boldsymbol{p}}\mathcal{T}_{\mathrm{ex}}f_2\right) &(101\mathrm{b})\\ &+\frac{1}{2}\sum_{a,b}\int_{\boldsymbol{k},\boldsymbol{p}}f_1f_2\frac{\partial\mathcal{T}_{\mathrm{ex}}}{\partial t}\bigg|_{\Omega=\varepsilon_1+\varepsilon_2} &(101\mathrm{c})\\ &+\frac{1}{2}\sum_{a,b}\int_{\boldsymbol{k},\boldsymbol{p}}f_1f_2\frac{\partial\mathcal{T}_{\mathrm{ex}}}{\partial\Omega}\bigg|_{\Omega=\varepsilon_1+\varepsilon_2}(\partial_t\varepsilon_1+\partial_t\varepsilon_2). &(101\mathrm{d})\end{aligned}$$

In the following we shall refer to the four rows on the right side of (101) as expressions (101a), (101b), (101c), and (101d), and we shall also use similar notation in referring to other expressions.

Expression (101a) is part of $\partial_t\mathcal{E}_q|_e$; it represents the single-particle kinetics in response to external fields excluding the particle-particle interactions. The effect from the dynamics of interaction between particles is encoded in the T matrix coming from (101c) and (101d).

Expression (101b) represents the mean-field part of the Landau-type contribution;

$$\partial_t\mathcal{E}_q\big|_{L;\mathrm{mf}}=\sum_a\int_{\boldsymbol{k}}\left(\epsilon_1+\Sigma_1^{\mathrm{mf}}(\varepsilon_1)\right)\partial_t f_1. \tag{102}$$

More complicated is expression (101c); to identify the contributions within, we use the time-derived optical theorem

$$\begin{aligned}\frac{\partial\mathcal{T}_{\mathrm{ex}}}{\partial t}=&\,\mathcal{T}_{\mathrm{sc}}^R\cdot\left(\frac{\partial\mathcal{G}}{\partial t}-\frac{\partial\mathcal{V}^{-1}}{\partial t}\right)\cdot\mathcal{T}_{\mathrm{sc}}^A\\ &+\frac{i}{2}\left(\mathcal{T}_{\mathrm{sc}}^R\cdot\mathcal{A}\cdot\frac{\partial\mathcal{T}_{\mathrm{sc}}^A}{\partial t}-\frac{\partial\mathcal{T}_{\mathrm{sc}}^R}{\partial t}\cdot\mathcal{A}\cdot\mathcal{T}_{\mathrm{sc}}^A\right),\end{aligned} \tag{103}$$

where $\mathcal{V}$ accounts for time-dependent interactions, as shown in Appendix A 3. In terms of the T-matrix element used in the collision integral,

$$\begin{aligned}\frac{\partial\mathcal{T}_{\mathrm{ex}}}{\partial t}(\Omega,\boldsymbol{k},\boldsymbol{p},\boldsymbol{0})=&-\int_{\boldsymbol{q}}\left|\mathcal{T}_{\mathrm{sc}}^R\right|^2\partial_t\mathcal{V}^{-1} &(104\mathrm{a})\\ &-\int_{\boldsymbol{q}}\left|\mathcal{T}_{\mathrm{sc}}^R\right|^2 2\pi\delta(\Omega-\varepsilon_3-\varepsilon_4)2\Delta_E(1-f_3-f_4) &(104\mathrm{b})\\ &+\int_{\boldsymbol{q}}\left|\mathcal{T}_{\mathrm{sc}}^R\right|^2(1-f_3-f_4)(\partial_t\varepsilon_3+\partial_t\varepsilon_4)\frac{\wp}{\Omega-\varepsilon_3-\varepsilon_4} &(104\mathrm{c})\\ &-\int_{\boldsymbol{q}}\left|\mathcal{T}_{\mathrm{sc}}^R\right|^2(\partial_t f_3+\partial_t f_4)\frac{\wp}{\Omega-\varepsilon_3-\varepsilon_4}. &(104\mathrm{d})\end{aligned}$$

Expression (104a) enters expression (101c) and gives the dominant contribution of the interactions between particles in $\partial_t\mathcal{E}_q|_e$.

Expression (104b) corresponds to a phase shift of the T matrix transferring energy Δ_E from collision processes to quasiparticles;

$$\begin{aligned}\partial_t\mathcal{E}_q|_c=&-\sum_{a,b}\int_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}}|\mathcal{T}_{\mathrm{sc}}|^2 2\pi\delta(\Omega-\varepsilon_3-\varepsilon_4)\\ &\times\Delta_E f_1f_2(1-f_3-f_4).\end{aligned} \tag{105}$$

The two contributions (104c) and (104d) come from the time derivative of $\mathcal{G}$ which includes the effect of interactions between quasiparticles and colliding pairs. Using expression (104d), we are able to identify the $\partial_t\mathcal{E}_q|_{L;\mathrm{pair}}$ contribution in (101);

$$\begin{aligned}\partial_t\mathcal{E}_q\big|_{L;\mathrm{pair}}=&-\frac{1}{2}\sum_{a,b}\int_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q},\Omega}\left|\mathcal{T}_{\mathrm{sc}}^R\right|^2 2\pi\delta(\Omega-\varepsilon_1-\varepsilon_2)\\ &\times f_1f_2(\partial_t f_3+\partial_t f_4)\frac{\wp}{\Omega-\varepsilon_3-\varepsilon_4}\\ =&\sum_a\int_{\boldsymbol{k}}\Sigma_1^{\mathrm{pair}}(\varepsilon_1)\partial_t f_1.\end{aligned} \tag{106}$$

The last equality can be obtained by renaming variables $1\leftrightarrow3$ and $2\leftrightarrow4$ for the term with $\partial_t f_3$, and $1\leftrightarrow4$ and $2\leftrightarrow3$ for the term with $\partial_t f_4$. This pairing energy is the contribution of the colliding pairs coming from the correlated function of the T matrix.

Making use of expression (104c) in (101c) and combining with (101d) we can identify the off-shell contribution

$$F = \sum_a \int_{\boldsymbol{k}} (\rho_1 - f_1)\partial_t \varepsilon_1. \tag{107}$$

Here we have used relations between two-particle correlators derived in Appendix B. We use $\varepsilon \approx \epsilon + \mathcal{V}_{\mathrm{ex}} f$ to decompose the quantity (107) into three parts:

$$\begin{aligned} F \approx{}& \sum_a \int_{\boldsymbol{k}} (\rho_1 - f_1)\partial_t \epsilon_1 && \text{(108a)}\\ &+ \sum_{a,b} \int_{\boldsymbol{k},\boldsymbol{p}} (\rho_1 - f_1)\mathcal{V}_{\mathrm{ex}}\partial_t f_2 && \text{(108b)}\\ &+ \sum_{a,b} \int_{\boldsymbol{k},\boldsymbol{p}} (\rho_1 - f_1) f_2 \partial_t \mathcal{V}_{\mathrm{ex}}. && \text{(108c)} \end{aligned}$$

Now (108a) and (108c) are attributed to $\partial_t \mathcal{E}_q|_e$ and (108b) is a Landau-type contribution, containing the off-shell Hartree-Fock self-energy. After exchanging $1 \leftrightarrow 2$, we have

$$\partial_t \mathcal{E}_q|_{L;\mathrm{HF}'} = \sum_a \int_{\boldsymbol{k}} \Sigma_1^{\mathrm{HF}'} \partial_t f_1. \tag{109}$$

At this stage, we have shown that the Landau-type contribution is the sum of (102), (106), and (109).

Finally, all the terms related directly to the external fields can be collected:

$$\partial_t \mathcal{E}_q|_e = \sum_a \int_{\boldsymbol{k}} \rho_1 \partial_t \epsilon_1 + \frac{1}{2}\sum_{a,b}\int_{\boldsymbol{k},\boldsymbol{p}} \rho_{12}\partial_t \mathcal{V}, \tag{110}$$

where the two-particle density matrix is

$$\rho_{12} = \int_{\boldsymbol{q}} \left\{\hat{\mathcal{U}}_{\mathrm{sc}}[f_2(\rho_1 - f_1) + f_1(\rho_2 - f_2)]\hat{\mathcal{U}}_{\mathrm{sc}} + \left|\mathcal{T}^R_{\mathrm{sc}}\mathcal{V}^{-1}\right|^2 f_3 f_4\right\}. \tag{111}$$

The ρ_1 term of (110) is the sum of (101a) and (108a). The ρ_{12} term is obtained from (101c), using (104) and (108c). The off-shell contribution with antisymmetrization operators $\hat{\mathcal{U}}_{\mathrm{sc}}$ is transferred from the interaction potential to the two-particle density matrix, $\hat{\mathcal{U}}_{\mathrm{sc}}\mathcal{T}^R_{\mathrm{sc}} = \mathcal{T}^R_{\mathrm{ex}}$. Further details can be found in Appendix C.

We have shown that the time derivative of the quasiparticle energy density is indeed the form given in (99). The Landau-type contribution $\partial_t \mathcal{E}_q|_L$ follows from the drift of quasiparticles. $\partial_t \mathcal{E}_q|_c$ is the energy gain during collisions, the summation of small energy changes in collision processes. $\partial_t \mathcal{E}_q|_e$ has the expected structure of the energy of the external sources, as we can see by comparing (110) with (75). As we expect, the time derivative of the Hamiltonian is the only external contribution to the change of the total energy density $\partial_t \mathcal{E}$.

B. Energy balance

The balance equation for energy can be obtained from the nonlocal kinetic equation (24);

$$\sum_a \int_{\boldsymbol{k}} \varepsilon_1 \frac{df_1}{dt} = S_\varepsilon(\boldsymbol{r}, t). \tag{112}$$

The left side can be written with the Landau-type contribution (98) where $\boldsymbol{j}^q_\varepsilon$ is the energy flux of quasiparticles,

$$\boldsymbol{j}^q_\varepsilon = \sum_a \int_{\boldsymbol{k}} \varepsilon_1 f_1 \boldsymbol{\nabla}_k \varepsilon_1. \tag{113}$$

The right side of the balance equation is the energy contribution from collision events,

$$S_\varepsilon(\boldsymbol{r}, t) = \sum_{a,b}\int_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}} (\varepsilon_1 \bar{W}_{3412} - \varepsilon_1 \tilde{W}_{1234}). \tag{114}$$

The method of interchanging initial and final states has been discussed in Sec. IV A using the transformation X defined in (29), and we can thereby calculate the energy contribution from the collision processes, obtaining

$$\begin{aligned} S_\varepsilon = \sum_{a,b}\int_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}} &\{(\tilde{\varepsilon}_3 - \tilde{\varepsilon}_1)\tilde{W}_{1234} - \partial_t(\Delta_t \tilde{\varepsilon}_3 \tilde{W}_{1234})\\ &- \boldsymbol{\nabla}_r \cdot (\boldsymbol{\Delta}_{(3)}\tilde{\varepsilon}_3 \tilde{W}_{1234})\}. \end{aligned} \tag{115}$$

Comparing with the collision contribution of the particle number (40), we see that the additional term with $\tilde{\varepsilon}_3 - \tilde{\varepsilon}_1$ appears as the energy input from the external fields.

We can see from the form of the collision integral that energy is conserved among particles save for the Δ_E correction.

By interchanging particles under the summation of a and b, we will show explicitly this compensation in the energy balance equation of the nonlocal kinetic equation. This interchange is

$$Y := \begin{cases} a \leftrightarrow b\\ \boldsymbol{k} \leftrightarrow \boldsymbol{p}.\\ \boldsymbol{q} \to -\boldsymbol{q} \end{cases} \tag{116}$$

Now, we require a symmetric T matrix;

$$Y\mathcal{T}^R_{\mathrm{sc}} = \mathcal{T}^R_{\mathrm{sc}}. \tag{117}$$

Accordingly, the scattering phase shift $\phi = Y\phi$, and the derivatives of the phase is

$$Y\boldsymbol{\nabla}_k\phi = \boldsymbol{\nabla}_p\phi, \quad Y\boldsymbol{\nabla}_p\phi = \boldsymbol{\nabla}_k\phi, \quad Y\boldsymbol{\nabla}_q\phi = -\boldsymbol{\nabla}_q\phi. \tag{118}$$

On the energy shell these derivatives imply

$$\begin{aligned} &Y\boldsymbol{\Delta}_{(2)} = -\boldsymbol{\Delta}_{(2)}, \quad Y\boldsymbol{\Delta}_{(3)} = \boldsymbol{\Delta}_{(4)} - \boldsymbol{\Delta}_{(2)},\\ &Y\boldsymbol{\Delta}_{(4)} = \boldsymbol{\Delta}_{(3)} - \boldsymbol{\Delta}_{(2)}. \end{aligned} \tag{119}$$

The Δ_t, Δ_E, and $\boldsymbol{\Delta}_K$ remain unchanged under Y transformation. χ is transformed to

$$Y\chi = \chi - \boldsymbol{\nabla}_r \cdot \boldsymbol{\Delta}_{(2)}. \tag{120}$$

At linear order in gradient correction,

$$\begin{aligned} Y\tilde{\varepsilon}_1 &= \mathcal{P}\tilde{\varepsilon}_2 = \varepsilon(b, \boldsymbol{p}, \boldsymbol{r}, t),\\ Y\tilde{\varepsilon}_2 &= \mathcal{P}\tilde{\varepsilon}_1 = \varepsilon(a, \boldsymbol{k}, \boldsymbol{r} - \boldsymbol{\Delta}_{(2)}, t),\\ Y\tilde{\varepsilon}_3 &= \mathcal{P}\tilde{\varepsilon}_4 = \varepsilon(b, \boldsymbol{k} + \boldsymbol{p} + \boldsymbol{\Delta}_K, \boldsymbol{r} + \boldsymbol{\Delta}_{(4)} - \boldsymbol{\Delta}_{(2)}, t + \Delta_t),\\ Y\tilde{\varepsilon}_4 &= \mathcal{P}\tilde{\varepsilon}_3 = \varepsilon(a, \boldsymbol{k} - \boldsymbol{q} + \boldsymbol{\Delta}_K, \boldsymbol{r} + \boldsymbol{\Delta}_{(3)} - \boldsymbol{\Delta}_{(2)}, t + \Delta_t), \end{aligned}$$

where we have used a linear operator for the shift

$$\mathcal{P} = 1 - \boldsymbol{\Delta}_{(2)} \cdot \boldsymbol{\nabla}_r; \tag{121}$$

$Y\tilde{f}_n$ is of similar form. Since the T matrix is centered between initial and final states,

$$Y\tilde{\mathcal{T}}^R_{\rm sc} = \mathcal{P}\tilde{\mathcal{T}}^R_{\rm sc}. \qquad (122)$$

Next we calculate

$$\begin{aligned} Y[(\tilde{\varepsilon}_3 - \tilde{\varepsilon}_1)\tilde{W}_{1234}] = {} & (\tilde{\varepsilon}_4 - \tilde{\varepsilon}_2)\tilde{W}_{2143} \\ & - \nabla_r \cdot [\mathbf{\Delta}_{(2)}(\tilde{\varepsilon}_4 - \tilde{\varepsilon}_2)\tilde{W}_{2143}]. \end{aligned} \qquad (123)$$

In obtaining this expression, divergence of $\mathbf{\Delta}_{(2)}$ follows from χ while other gradients follow from $\mathcal{P}$.

Using (115) and (123) to express S_ε,

$$S_\varepsilon \approx \frac{1}{2}\sum_{a,b}\int_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}} \{(\tilde{\varepsilon}_3 + \tilde{\varepsilon}_4 - \tilde{\varepsilon}_1 - \tilde{\varepsilon}_2)\tilde{W}_{1234} \qquad (124a)$$

$$- \nabla_r \cdot [(\varepsilon_3\mathbf{\Delta}_{(3)} + \varepsilon_4\mathbf{\Delta}_{(4)} - \varepsilon_2\mathbf{\Delta}_{(2)})\tilde{W}_{1234}] \qquad (124b)$$

$$- \partial_t[(\varepsilon_3 + \varepsilon_4)\Delta_t\tilde{W}_{1234}]\}. \qquad (124c)$$

Looking first at (124a), we use

$$\tfrac{1}{2}(\tilde{\varepsilon}_3 + \tilde{\varepsilon}_4 - \tilde{\varepsilon}_1 - \tilde{\varepsilon}_2)\tilde{W}_{1234} = \Delta_E\tilde{W}_{1234}; \qquad (125)$$

to identify the integral as the $-\partial_t\mathcal{E}_q|_c$ term defined in (105). Unlike the corresponding term in the collision contribution of the particle number, which is zero, this term arises due to energy exchange of colliding pairs with the background. This contribution is affected by time-dependent external fields, in particular it can be enhanced by increasing the strength of interactions and the rate of change.

Expressions (124b) and (124c) are of linear order in the Δ shifts; we can approximate $\tilde{W}_{1234} = W_{1234}$. The divergence term in (124b) can be written as $-\nabla_r \cdot \boldsymbol{j}^c_\varepsilon$ where

$$\begin{aligned} \boldsymbol{j}^c_\varepsilon = {} & \frac{1}{2}\sum_{a,b}\int_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}} \left|\mathcal{T}^R_{\rm sc}\right|^2 2\pi\delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4) \\ & \times (\varepsilon_3\mathbf{\Delta}_{(3)} + \varepsilon_4\mathbf{\Delta}_{(4)} - \varepsilon_2\mathbf{\Delta}_{(2)})f_1f_2(1 - f_3 - f_4). \end{aligned} \qquad (126)$$

We write $\varepsilon_3 + \varepsilon_4 \approx E$, so that the time derivative term in (124) can be written as $-\partial_t\mathcal{E}_c$, in terms of $\mathcal{E}_c$ defined in (90).

Consequently the energy balance (112) takes the form

$$\partial_t\mathcal{E}_q|_L + \nabla_r \cdot \boldsymbol{j}^q_\varepsilon = -\partial_t\mathcal{E}_q|_c - \partial_t\mathcal{E}_c - \nabla_r \cdot \boldsymbol{j}^c_\varepsilon. \qquad (127)$$

To obtain the familiar form of conservation laws, we write it in terms of the total energy density $\mathcal{E} = \mathcal{E}_q + \mathcal{E}_c$ discussed in Sec. V A and the total energy flux defined as

$$\boldsymbol{j}_\varepsilon = \boldsymbol{j}^q_\varepsilon + \boldsymbol{j}^c_\varepsilon. \qquad (128)$$

We thereby obtain the concise energy conservation law

$$\partial_t\mathcal{E} + \nabla_r \cdot \boldsymbol{j}_\varepsilon = \partial_t\mathcal{E}_q|_e, \qquad (129)$$

which is our main result of this section.

The conservation law (129) features a driven term on the right side which is the externally driven quasiparticle energy given by (110). In the case of time-independent Hamiltonian, the driven term vanishes and the law reduces to a usual continuity equation.

The energy density described by this local conservation law corresponds to the value obtained from Green functions in the T-matrix approximation and linear approximation of off-shell motion. The off-shell contribution is the first step beyond the standard quasiparticle approximation based on Feynman diagrams.

As we have seen from (105) and expression (124a), quasiparticles can exchange energy with colliding pairs, but the total energy balance must be satisfied. The quasiparticle energy must therefore be consistent with gradient corrections to the collision integral. Briefly, in the nonlocal theory the scattering rates and the quasiparticle energies are no longer independent parameters.

For the case of time-dependent Hamiltonian the driven term acts as a source term, transferring energy into the system. While usual effects of external scalar and vector potentials are well understood, less is known about effects due to such a field controlling the interaction; for example one such effect might be the temperature evolution given a slow ramp of the magnetic field producing increasing interaction strength. The nonlocal Boltzmann equation allows one to study such adiabatically slow processes.

VI. HYDRODYNAMICS

To complete our set of hydrodynamical equations, we derive the Cauchy momentum equation, and subsequently convert this to the Navier-Stokes equation. Since we do not employ phenomenological relations for the static pressure, we keep the stress tensor in a compact form. Accordingly, we derive, at linear approximation, the constitutive equations which relate the stress tensor to the pressure and velocity for a Newtonian liquid.

The diagonal components of the stress tensor are proportional to the pressure, and contain the compressibility determining the density profile of trapped Fermi systems [17–21], nuclear matters in nucleus [22,23], neutron stars [24,25], and supernova [26]. The off-diagonal components of the stress tensor are shear viscosity components.

The viscous effects that lead to the shear stresses are of recent interest in nuclear physics [17,27–32]. In [87] it is conjectured that, determined by quantum considerations, the minimum value of the ratio of viscosity to entropy is located at the transition point of condensation. This reduction of viscosity of normal liquid when approaching the transition point is also drawing attention in the field of ultracold atoms [33–38].

To demonstrate how nonlocal corrections affect the shear viscosity, in Sec. VI B we limit our attention to small departures from local equilibrium and derive the shear viscosity to linear order in this perturbation. In the relaxation-time approximation for s-wave scattering we show that the effect of the collision delay can be recast as an effective relaxation time.

A. Momentum balance

Using the nonlocal kinetic equation (24), we write the balance equation for momentum,

$$\sum_a\int_{\boldsymbol{k}} \boldsymbol{k}\frac{df_1}{dt} = \boldsymbol{S}_K(\boldsymbol{r},t), \qquad (130)$$

where the collision contribution $\boldsymbol{S}_K$ is

$$\boldsymbol{S}_K = \sum_a \int_{\boldsymbol{k}} \boldsymbol{k} I_1[f;\Delta]. \tag{131}$$

The drift part of (130) contains time- and space-derivative terms; as we expect, this is similar to the other two balance equations discussed above. In the present case there is an additional force term:

$$\sum_a \int_{\boldsymbol{k}} \boldsymbol{k} \frac{df_1}{dt} = \partial_t \boldsymbol{Q}_q + \nabla_r \cdot \mathcal{J}_q - \boldsymbol{f}. \tag{132}$$

Here the momentum density of quasiparticles is

$$\boldsymbol{Q}_q = \sum_a \int_{\boldsymbol{k}} \boldsymbol{k} f_1, \tag{133}$$

and the momentum flux of quasiparticles is

$$\mathcal{J}_{qij} = \sum_a \int_{\boldsymbol{k}} k_j f_1 \frac{\partial \varepsilon_1}{\partial k_i} \tag{134}$$

in the standard form from the Boltzmann theory. The remaining force term is not in the form of a divergence:

$$\boldsymbol{f} = \sum_a \int_{\boldsymbol{k}} f_1(-\nabla_r \varepsilon_1). \tag{135}$$

This term behaves as a force accelerating a quasiparticle of energy ε_1.

As in the previous section in the case of energy conservation, we decompose the force term:

$$\boldsymbol{f} = -\nabla_r \langle \varepsilon \rangle + \nabla_r \mathcal{E}_q\big|_L. \tag{136}$$

The first term is an average of quasiparticle energy defined as

$$\langle \varepsilon \rangle = \sum_a \int_{\boldsymbol{k}} \varepsilon_1 f_1; \tag{137}$$

the second term is the Landau-type quasiparticle contribution

$$\nabla_r \mathcal{E}_q\big|_L = \sum_a \int_{\boldsymbol{k}} \varepsilon_1 \nabla_r f_1, \tag{138}$$

which is a contribution to the total force on quasiparticles

$$\nabla_r \mathcal{E}_q = \nabla_r \mathcal{E}_q|_L + \nabla_r \mathcal{E}_q|_c + \nabla_r \mathcal{E}_q|_e. \tag{139}$$

(139) is obtained by following the derivation of $\partial_t \mathcal{E}_q$ shown in Sec. V A. We anticipate that the force contributions of the drift appear together with the collision contributions and only their combination yields the momentum conservation law.

As we have seen, the collision contribution to the momentum balance must be entirely due to gradient corrections. To demonstrate cancellation of the nongradient parts of collision terms (131), we again make use of our X and Y transformations, using

$$X(\boldsymbol{k}\bar{W}_{3412}) = (\boldsymbol{k} - \boldsymbol{q})\bar{W}_{1234}. \tag{140}$$

Substituting into $\boldsymbol{S}_K$, we have

$$\begin{aligned} \boldsymbol{S}_K = {} & \sum_{a,b} \int_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}} \tilde{W}_{1234}(\boldsymbol{\Delta}_K - \boldsymbol{q}) \\ & - \nabla_r \cdot \sum_{a,b} \int_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}} \boldsymbol{\Delta}_{(3)} \tilde{W}_{1234}(\boldsymbol{k} - \boldsymbol{q}) \\ & - \frac{\partial}{\partial t} \sum_{a,b} \int_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}} \Delta_t \tilde{W}_{1234}(\boldsymbol{k} - \boldsymbol{q}). \end{aligned} \tag{141}$$

As before, to achieve a more symmetric expression we apply Y; then

$$\boldsymbol{S}_K \approx -\partial_t \boldsymbol{Q}_c - \nabla_r \cdot \mathcal{J}_c + \nabla_r \mathcal{E}_q|_c. \tag{142}$$

The momentum density due to the nonlocal corrections is

$$\begin{aligned} \boldsymbol{Q}_c = {} & \frac{1}{2} \sum_{a,b} \int_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}} \left|\mathcal{T}^R_{\rm sc}\right|^2 2\pi \delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4) \\ & \times (\boldsymbol{k} + \boldsymbol{p}) \Delta_t f_1 f_2 (1 - f_3 - f_4) \end{aligned} \tag{143}$$

representing the momentum carried by colliding particles. The corresponding correlated-particle momentum flux has elements

$$\begin{aligned} \mathcal{J}_{cij} = {} & \frac{1}{2} \sum_{a,b} \int_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}} \big\{ \left|\mathcal{T}^R_{\rm sc}\right|^2 2\pi \delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4) \\ & \times [(k_j - q_j)\Delta_{(3)i} + (p_j + q_j)\Delta_{(4)i} - p_j \Delta_{(2)i}] \\ & \times f_1 f_2 (1 - f_3 - f_4) \big\}. \end{aligned} \tag{144}$$

This describes the transfer of momentum during the collision. The momentum $\boldsymbol{k}$ is carried by the type-a particle over distance $\boldsymbol{\Delta}_{(3)}$, while the $\boldsymbol{p}$ is carried by the type-b particle over $\boldsymbol{\Delta}_{(4)} - \boldsymbol{\Delta}_{(2)}$. During the interaction, momentum $\boldsymbol{q}$ is transferred over distance $\boldsymbol{\Delta}_{(4)} - \boldsymbol{\Delta}_{(3)}$ from the type-a to the type-b particle. The collision momentum gain

$$\begin{aligned} \nabla_r \mathcal{E}_q|_c = {} & \sum_{a,b} \int_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}} \left|\mathcal{T}^R_{\rm sc}\right|^2 2\pi \delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4) \\ & \times \boldsymbol{\Delta}_K f_1 f_2 (1 - f_3 - f_4) \end{aligned} \tag{145}$$

collects small changes in total momentum of colliding quasiparticles. These might result from external and background forces acting on single-particle states during collision, from external effect on the interaction potential, and from background effects on two-particle correlation in internal states of the collision.

The collision momentum gain (145) and the Landau-type term (138) can be merged together, leaving the mean external force transferring momentum into the system via coupling with quasiparticles,

$$\nabla_r \mathcal{E}_q|_e = \sum_a \int_{\boldsymbol{k}} (\rho_1 \nabla_r \epsilon_1) + \frac{1}{2} \sum_{a,b} \int_{\boldsymbol{k},\boldsymbol{p}} (\rho_{12} \nabla_r \mathcal{V}). \tag{146}$$

The Wigner distribution ρ_1 and the two-particle reduced density matrix ρ_{12} are taken to the proper order to keep $\nabla_r \mathcal{E}_q|_e$ consistent in the linear order of the gradient corrections.

From (139) and (130) we obtain the momentum balance

$$\partial_t \boldsymbol{Q} + \nabla_r \cdot \mathcal{J} = -\nabla_r \mathcal{E}_q|_e. \tag{147}$$

The total momentum density $\boldsymbol{Q}$ is composed of parts from quasiparticles and collision states,

$$\boldsymbol{Q} = \boldsymbol{Q}_q + \boldsymbol{Q}_c. \tag{148}$$

The total flux of momentum has components

$$\mathcal{J}_{ij} = \mathcal{J}_{cij} + \mathcal{J}_{qij} + (\langle \varepsilon \rangle - \mathcal{E}_q)\delta_{ij}. \tag{149}$$

The major contribution of the $\langle \varepsilon \rangle - \mathcal{E}_q$ term is from the Hartree-Fock self-energy; this term is zero in the absence of interaction. The pressure is $p = -\frac{1}{3}(\mathcal{J}_{11} + \mathcal{J}_{22} + \mathcal{J}_{33})$.

It has been shown that nonlocal theory with collision duration gives results in agreement with the second order viral expansion [54]. The off-diagonal components, for a Newtonian liquid, are proportional to the spatial derivatives of velocity; we will calculate the shear viscosity for an isotropic system in the next section.

B. Shear viscosity

We consider a system near a local equilibrium with a steady-state velocity field $\boldsymbol{u}(\boldsymbol{r})$. We make an expansion in terms of $\nabla_r \boldsymbol{u}$ which we assume to be small. With a standard approach for small perturbations,

$$\bar{f}_n = \bar{f}_n^0 + \delta f_n \tag{150}$$

for the distribution of a colliding particle n close to the local equilibrium

$$\bar{f}_n^0 = \left[\exp\left(\frac{\bar{\varepsilon}_n - \bar{\boldsymbol{k}}_n \cdot \bar{\boldsymbol{u}}_n - \bar{\mu}_n}{\bar{T}_n} \right) + 1 \right]^{-1}. \tag{151}$$

The shift notation $\bar{\lambda}$ is as in (7), and $\bar{\boldsymbol{u}}_n = \boldsymbol{u}(\bar{\boldsymbol{r}}_n)$ and $\bar{\mu}_n = \mu(\bar{\boldsymbol{r}}_n)$. A plausible form of δf_n near the equilibrium is

$$\delta f_n = \bar{f}_n^0 (1 - \bar{f}_n^0) \Phi_n, \tag{152}$$

where Φ_n is proportional to $\nabla_r \boldsymbol{u}$.

To avoid the Bernoulli term, at linear approximation, we use a coordinate system such that $\boldsymbol{u}(\boldsymbol{r}) = \boldsymbol{0}$ at the coordinate position $\boldsymbol{r}$ of particle 1 (which therefore does not change); this comoving frame is an approximately inertial coordinate choice in our case, since the time scale of the velocity field is much smaller than the time scales associated with collisions. The formalism developed in earlier sections can be applied in the comoving frame. This can also be expressed by noting that the leading correction to the Bernoulli term is at quadratic order in the velocity gradient. The local velocity at particle 3 is $\boldsymbol{u}(\boldsymbol{r} - \boldsymbol{\Delta}_{(3)}) = -(\nabla_r \boldsymbol{u})\boldsymbol{\Delta}_{(3)} \neq \boldsymbol{0}$. We approximate temperature $\bar{T}_n$ and chemical potential $\bar{\mu}_n$ with constant T and μ respectively.

Insertion of these ingredients into the linearized kinetic equation, we have, in steady state,

$$(\nabla_k \varepsilon_1) \cdot (\nabla_r f_1^0) - (\nabla_r \varepsilon_1) \cdot (\nabla_k f_1^0) = I_1[f; \Delta], \tag{153}$$

where the collision integral $I_1 = \bar{I}_1^{(\text{in})} - \bar{I}_1^{(\text{out})}$ is divided into

$$\begin{aligned}\bar{I}_1^{(\text{in})} = \sum_b \int_{\boldsymbol{p},\boldsymbol{q}} \Big\{ \big|\mathcal{T}_{\text{sc}}^R\big|^2 &(1-\chi) 2\pi \delta(\bar{\varepsilon}_1 + \bar{\varepsilon}_2 - \bar{\varepsilon}_3 - \bar{\varepsilon}_4) \\ &\times \big[1 - \bar{f}_1^0 \Phi_1 - \bar{f}_2^0 \Phi_2 + \big(1 - \bar{f}_3^0\big)\Phi_3 + \big(1 - \bar{f}_4^0\big)\Phi_4\big] \\ &\times \big(1 - \bar{f}_1^0\big)\big(1 - \bar{f}_2^0\big) \bar{f}_3^0 \bar{f}_4^0 \Big\}\end{aligned} \tag{154}$$

and

$$\begin{aligned}\bar{I}_1^{(\text{out})} = \sum_b \int_{\boldsymbol{p},\boldsymbol{q}} \Big\{ \big|\mathcal{T}_{\text{sc}}^R\big|^2 &(1-\chi) 2\pi \delta(\bar{\varepsilon}_1 + \bar{\varepsilon}_2 - \bar{\varepsilon}_3 - \bar{\varepsilon}_4) \\ &\times \big[1 - \bar{f}_3^0 \Phi_3 - \bar{f}_4^0 \Phi_4 + \big(1 - \bar{f}_1^0\big)\Phi_1 + \big(1 - \bar{f}_2^0\big)\Phi_2\big] \\ &\times \big(1 - \bar{f}_3^0\big)\big(1 - \bar{f}_4^0\big) \bar{f}_1^0 \bar{f}_2^0 \Big\},\end{aligned} \tag{155}$$

where $\Delta_E = 0$.

At first order in gradient corrections, the Pauli factor in the two scattering events has a simple relation,

$$\big(1 - \bar{f}_1^0\big)\big(1 - \bar{f}_2^0\big)\bar{f}_3^0 \bar{f}_4^0 = \big(1 - \bar{f}_3^0\big)\big(1 - \bar{f}_4^0\big)\bar{f}_1^0 \bar{f}_2^0 (1 - \zeta), \tag{156}$$

with the correction from nonlocality,

$$\begin{aligned}\zeta &= 1 - \exp\left[-\frac{\bar{\boldsymbol{k}}_1 \cdot \bar{\boldsymbol{u}}_1 + \bar{\boldsymbol{k}}_2 \cdot \bar{\boldsymbol{u}}_2 - \bar{\boldsymbol{k}}_3 \cdot \bar{\boldsymbol{u}}_3 - \bar{\boldsymbol{k}}_4 \cdot \bar{\boldsymbol{u}}_4}{T} \right] \\ &\approx \frac{\nabla_r \boldsymbol{u}}{T} \cdot [\boldsymbol{p} \cdot \boldsymbol{\Delta}_{(2)} - (\boldsymbol{k} - \boldsymbol{q}) \cdot \boldsymbol{\Delta}_{(3)} - (\boldsymbol{p} + \boldsymbol{q}) \cdot \boldsymbol{\Delta}_{(4)}].\end{aligned} \tag{157}$$

Here we have used that local velocity is zero so that $\boldsymbol{\Delta}_K \cdot \boldsymbol{u} = 0$. We realize from

$$\begin{aligned}I_1[f; \Delta] = -\sum_b \int_{\boldsymbol{p},\boldsymbol{q}} &\big|\mathcal{T}_{\text{sc}}^R\big|^2 2\pi \delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4) \\ &\times f_1^0 f_2^0 \big(1 - f_3^0\big)\big(1 - f_4^0\big) \\ &\times (\Phi_1 + \Phi_2 - \Phi_3 - \Phi_4 + \zeta)\end{aligned} \tag{158}$$

that the collision integral is of the form $I[\Phi] + I[\zeta]$. The linearized kinetic equation (153) reduces to

$$I_1[\Phi] = \frac{1}{T} f_1^0 \big(1 - f_1^0\big) \boldsymbol{k} \cdot \nabla_r \boldsymbol{u} \cdot \nabla_k \varepsilon_1 - I_1[\zeta]. \tag{159}$$

The system is driven by the gradient of local velocity in two ways: drift of quasiparticles and the nonlocal character of collisions (within term $I[\zeta]$).

The collision stress tensor $\mathcal{J}_c$ and the part of the quasiparticle $\mathcal{J}_q$ driven by the collision source $I[\zeta]$ make together a collision contribution to the shear viscosity. To demonstrate how they combine we adopt approximations, following [33], which allow for an analytic solution in an isotropic system with s-wave scattering.

At low temperatures the effective mass m^* at the Fermi velocity is defined as $\nabla_k \varepsilon_1 = \boldsymbol{k}/m^*$, by which (159) is written as

$$I_1[\Phi] = \frac{f_1^0 \big(1 - f_1^0\big)}{m^* T} \boldsymbol{k} \cdot \nabla_r \boldsymbol{u} \cdot \boldsymbol{k} - I_1[\zeta]. \tag{160}$$

In the relaxation-time approximation [33],

$$I_1[\Phi] = -\frac{\delta f_1}{\tau_1} = -\frac{\Phi_1}{\tau_1} f_1^0 \big(1 - f_1^0\big) \tag{161}$$

with the relaxation time

$$\begin{aligned}\frac{1}{\tau_1} = \sum_b \int_{\boldsymbol{p},\boldsymbol{q}} &\big|\mathcal{T}_{\text{sc}}^R\big|^2 2\pi \delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4) \\ &\times f_2^0 \big(1 - f_3^0\big)\big(1 - f_4^0\big).\end{aligned} \tag{162}$$

Here we assumed that integrals over Φ_n for $n = 2, 3, 4$ lead to thermalization and thus their contributions are negligible. The kinetic equation (160) differs from the one used in [33]; we include the corrections due to interactions in the quasiparticle velocity and the collision integral.

In the case of s-wave scattering, the Δ shifts reduce to $\boldsymbol{\Delta}_{(2)} = \mathbf{0}$ and $\boldsymbol{\Delta}_{(4)} = \boldsymbol{\Delta}_{(3)} = (\boldsymbol{k}+\boldsymbol{p})\Delta_t/(2m^*)$, and the correction (157) simplifies to

$$\zeta = -\frac{\Delta_t}{2m^*T}(\boldsymbol{k}+\boldsymbol{p})\cdot\nabla_r\boldsymbol{u}\cdot(\boldsymbol{k}+\boldsymbol{p}). \tag{163}$$

The off-diagonal components of the shear $\nabla_r\boldsymbol{u}$ are all nonzero; terms proportional to p_x^2, p_y^2, and p_z^2 are zero in ζ.

Contributions proportional to $\boldsymbol{p}$ thus vanish under integration in (158), and we have

$$I_1[\zeta] = \kappa_1\frac{f_1^0\big(1-f_1^0\big)}{m^*T}\boldsymbol{k}\cdot\nabla_r\boldsymbol{u}\cdot\boldsymbol{k} \tag{164}$$

with

$$\kappa_1 = \frac{1}{2\big(1-f_1^0\big)}\sum_b\int_{\boldsymbol{p},\boldsymbol{q}}\big|\mathcal{T}_{\rm sc}^R\big|^2 2\pi\delta(\varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4) \times\Delta_t f_2^0\big(1-f_3^0\big)\big(1-f_4^0\big). \tag{165}$$

We obtain from (160) the perturbation as

$$\Phi_1 = -(1-\kappa_1)\frac{\tau_1}{m^*T}\boldsymbol{k}\cdot\nabla_r\boldsymbol{u}\cdot\boldsymbol{k}, \tag{166}$$

which includes a local and a nonlocal contribution; the nonlocal correction is proportional to Δ_t.

For simplicity we consider a steady flow near $\boldsymbol{r} = \mathbf{0}$. $\boldsymbol{u}(\boldsymbol{r}) = (\alpha y, 0, 0)$ so the perturbation is

$$\Phi_1 = -(1-\kappa_1)\frac{\alpha\tau_1}{m^*T}k_xk_y. \tag{167}$$

In the following we calculate the total momentum flux $\mathcal{J} = \mathcal{J}_q + \mathcal{J}_c$ due to the perturbation near the equilibrium. In the linear approximation, we see from (144) that the diagonal components of $\mathcal{J}_c$ come from the local equilibrium distributions, while the off-diagonal components come from the nonequilibrium distribution,

$$\mathcal{J}_{cxy} = \frac{1}{4m^*}\sum_{a,b}\int_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}}\big|\mathcal{T}_{\rm sc}^R\big|^2 2\pi\delta(\varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4) \times(k_y+p_y)(k_x+p_x)\Delta_t f_1f_2(1-f_3-f_4). \tag{168}$$

Using X and Y transformation, the off-diagonal components corresponding to (167) reduce to

$$\mathcal{J}_{cxy} = \frac{\alpha}{2m^{*2}T}\sum_{a,b}\int_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}}\big|\mathcal{T}_{\rm sc}^R\big|^2 2\pi\delta(\varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4) \times\tau_1k_x^2k_y^2\Delta_t f_1^0\big(1-f_1^0\big)\big[f_3^0f_4^0-f_2^0\big(1-f_3^0-f_4^0\big)\big]. \tag{169}$$

Similarly, from (134), the corresponding off-diagonal component of $\mathcal{J}_q$ is dependent on the nonequilibrium distribution

$$\mathcal{J}_{q_{xy}} = \frac{1}{m^*}\sum_a\int_{\boldsymbol{k}}k_yk_xf_1 \tag{170}$$

so that

$$\mathcal{J}_{q_{xy}} = -\frac{\alpha}{m^{*2}T}\sum_a\int_{\boldsymbol{k}}k_y^2k_x^2f_1^0\big(1-f_1^0\big)\tau_1(1-\kappa_1). \tag{171}$$

The total off-diagonal stress is

$$\mathcal{J}_{xy} = -\frac{\alpha}{m^{*2}T}\sum_a\int_{\boldsymbol{k}}k_x^2k_y^2f_1^0\big(1-f_1^0\big)\tau_1(1-\mathcal{K}_1). \tag{172}$$

Here, the final result appears as a correction due to particle interactions which produces the collision duration,

$$\mathcal{K}_1 = \sum_b\int_{\boldsymbol{p},\boldsymbol{q}}\Delta_t\big|\mathcal{T}_{\rm sc}^R\big|^2 2\pi\delta(\varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4)f_2^0f_3^0f_4^0. \tag{173}$$

Alternatively, this term can be interpreted as an effective quasiparticle lifetime $\tau_{\rm ef} = \tau(1-\mathcal{K})$ reminiscent of the relation between the relaxation time of viscosity and the mean-free time of quasiparticle studied in [30]. However, the present result is rather different from the viscosity affected by the finite range interaction studied in [88]. For a positive collision delay, $\mathcal{K}$ is positive and the viscosity is reduced. For a negative collision delay, which can be used as an approximation of the nonlocality in hard-sphere-type collisions [54], $\mathcal{K}$ is negative and viscosity increases in agreement with Enskog theory of hard-sphere gas [89].

The stress tensor (149) also combines Landau and nonlocal terms. The drift stress as well as all diagonal components are known from the Landau theory. One must expect that a different definition of the quasiparticle energy might lead to different quantitative results, but this difference likely becomes invisible for simulations with a fitted interaction. More observable is our genuinely nonlocal collision stress tensor (144) which modifies the material properties like the shear viscosity. As demonstrated in Sec. VI B, the collision stress and nonlocal effect together affect the quasiparticle response to the external fields or boundary conditions.

We have written down a term representing the force acting on colliding pairs via space-modulated binary interaction contained in the second term of (146). In terms of only system variables, this force does not respect Newton's law of action and reaction, but it is a result of the nontrivial interaction which arises with external fields.

The problem of compressibility and viscosity has been discussed extensively within the nuclear physics community. Access to nuclear matter is restricted either by the very small size of nuclei or by very large distances to astronomical objects. There is a hope that studies of ultracold Fermi gases can shed some light on the relation between dynamics of highly nonequilibrium states and thermodynamic properties. The regime of interest in such cases is not the extreme unitary limit but one of interaction of moderate strength; in such a regime the quasiparticle picture with corrections for the off-shell motion is permissible.

VII. DISCUSSION

We have shown that the nonlocal Boltzmann equation with nonzero collision duration in the collision integral, consolidated with the microscopic theory at linear order in the off-shell contribution of the spectral function, does not violate conservation laws of mass, energy, or momentum. Along with the usual dressing of particles as quasiparticles, these off-shell contributions include a genuine two-particle motion which appears in internal dynamics of collisions.

As a result of noninstantaneous collision, the Δ shifts in time, space, momentum, and energy account for particle interactions in terms of the phase shift of the T matrix. Correlated particles, which are particles in collisions with kinetics different from that of quasiparticles, emerge with increasing collision rate resulting in the reduction in quasiparticles. Consequently, the subsystem of quasiparticles alone does not satisfy conservation laws; the conserved quantities consist of the contributions from correlated particles and quasiparticles.

In the three balance equations, the Δ shift in time shows up in the relevant densities of correlated particles; the Δ shifts in space show up in the relevant flux quantities. We have shown that the kinetic equation describes the external power supply in a manner consistent with general expectations inherent in the quantum statistical approach. The momentum and energy conservation laws are valid only when the quasiparticle energy includes effects of colliding pairs. Unlike in all previous kinetic equations of the Boltzmann type, in our nonlocal kinetic equation the quasiparticle energies and the collisions (rates and the Δ shifts) cannot be handled as independent parameters subject to fitting.

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APPENDIX A: OPTICAL THEOREM

We assume that $\mathcal{T}^R$ includes only direct processes, defined as those allowed for distinct particles, and we describe the exchange in the momentum representation.

1. Exchange processes

Inside the T matrix, $\boldsymbol{k}$, $\boldsymbol{p}$, and $\boldsymbol{q}$ represent that particles initially in states $\boldsymbol{k}$ and $\boldsymbol{p}$ end in states $\boldsymbol{k}-\boldsymbol{q}$ and $\boldsymbol{p}+\boldsymbol{q}$. If $a=b$, their exchange $\boldsymbol{k}\to\boldsymbol{p}+\boldsymbol{q}$ and $\boldsymbol{p}\to\boldsymbol{k}-\boldsymbol{q}$ leads to a state which interferes with the direct process when the transferred momentum depends on the momentum difference of initial states,

$$\mathcal{T}^R_{\rm ex}(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}) = \mathcal{T}^R(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}) - \delta_{ab}\mathcal{T}^R(\boldsymbol{k},\boldsymbol{p},\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q}), \quad \text{(A1)}$$

where $\mathcal{T}^R$ includes direct processes allowed for distinct particles. The T matrix also depends on Ω, $\boldsymbol{r}$, and t, but as these are unimportant in the present discussion, we write the exchange channel as

$$\overleftrightarrow{\mathcal{T}}^R(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}) = \mathcal{T}^R(\boldsymbol{k},\boldsymbol{p},\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q}). \quad \text{(A2)}$$

For the direct channel the scattering rate depends on the product $\mathcal{T}^R\mathcal{T}^A = |\mathcal{T}^R|^2$. The exchange can be made either in the retarded or the advanced T matrix, but exchange in both leads to double counting in the direct as well as the exchange channel. To make the scattering rate symmetric with respect to the retarded and advanced T matrices, one defines

$$\mathcal{T}^R_{\rm sc} = (1-\delta_{ab})\mathcal{T}^R + \frac{1}{\sqrt{2}}\delta_{ab}(\mathcal{T}^R - \overleftrightarrow{\mathcal{T}}^R). \quad \text{(A3)}$$

We express antisymmetrization of $\mathcal{T}$ using an operator $\hat{\mathcal{U}}_{\rm sc}$ which we shall write concretely in (A15),

$$\mathcal{T}^R_{\rm sc} = \hat{\mathcal{U}}_{\rm sc}\mathcal{T}^R = \mathcal{T}^R\hat{\mathcal{U}}_{\rm sc}. \quad \text{(A4)}$$

Using this operator we can obtain the exchange T matrix

$$\begin{aligned}\hat{\mathcal{U}}_{\rm sc}\hat{\mathcal{U}}_{\rm sc}\mathcal{T}^R &= \hat{\mathcal{U}}_{\rm sc}\mathcal{T}^R_{\rm sc}\\ &= (1-\delta_{ab})\mathcal{T}^R_{\rm sc} + \frac{1}{\sqrt{2}}\delta_{ab}\big(\mathcal{T}^R_{\rm sc} - \overleftrightarrow{\mathcal{T}}^R_{\rm sc}\big)\\ &= \mathcal{T}^R_{\rm ex}, \end{aligned} \quad \text{(A5)}$$

where we have used $\overleftrightarrow{\overleftrightarrow{\mathcal{T}}}^R = \mathcal{T}^R$.

2. Causal and anticausal optical theorem

Particle-number conservation in the present case is guaranteed by the optical theorem

$$\mathcal{M}_{\rm ex} = \mathcal{T}^R_{\rm sc}\cdot\mathcal{A}\cdot\mathcal{T}^A_{\rm sc}. \quad \text{(A6)}$$

The imaginary part of the T matrix $\mathcal{M} = i(\mathcal{T}^R - \mathcal{T}^A)$ represents the scattering out process; the square of the scattering T matrix and the two-particle spectral function $\mathcal{A} = i(\mathcal{G}^R - \mathcal{G}^A)$ have the structure associated with scattering in. In the momentum representation the advanced T matrix $\mathcal{T}^A$ is the complex conjugate of $\mathcal{T}^R$; this is true for all such advanced functions.

For the ladder approximation of the T matrix,

$$\mathcal{T}^R = \mathcal{V} + \mathcal{V}\cdot\mathcal{G}^R\cdot\mathcal{T}^R, \quad \text{(A7)}$$

the derivation is straightforward. From $\mathcal{T}_R^{-1} = \mathcal{V}^{-1} - \mathcal{G}^R$ it follows that $i(\mathcal{T}_R^{-1} - \mathcal{T}_A^{-1}) = -\mathcal{A}$. Therefore

$$\mathcal{M} = -i\mathcal{T}^R\cdot\big(\mathcal{T}_R^{-1} - \mathcal{T}_A^{-1}\big)\cdot\mathcal{T}^A = \mathcal{T}^R\cdot\mathcal{A}\cdot\mathcal{T}^A. \quad \text{(A8)}$$

Finally we multiply (A8) from left and right with operators of antisymmetrization arriving at the optical theorem (A6).

The anticausal optical theorem results from analogous steps

$$\mathcal{M} = -i\mathcal{T}^A\cdot\big(\mathcal{T}_R^{-1} - \mathcal{T}_A^{-1}\big)\cdot\mathcal{T}^R = \mathcal{T}^A\cdot\mathcal{A}\cdot\mathcal{T}^R. \quad \text{(A9)}$$

As above we multiply Eq. (A9) by antisymmetrizing operators,

$$\mathcal{M}_{\rm ex} = \mathcal{T}^A_{\rm sc}\cdot\mathcal{A}\cdot\mathcal{T}^R_{\rm sc}. \quad \text{(A10)}$$

The equivalence of the causal and anticausal versions of the optical theorem implies that linear gradient corrections to the optical theorem vanish.

3. Derived optical theorems

The real part of the T matrix $\mathcal{T} = \frac{1}{2}(\mathcal{T}^R + \mathcal{T}^A)$ satisfies

$$\mathcal{T}_{\rm ex} = \mathcal{T}^R_{\rm sc}\cdot(\mathcal{V}^{-1} - \mathcal{G})\cdot\mathcal{T}^A_{\rm sc}, \quad \text{(A11)}$$

where $\mathcal{G} = \frac{1}{2}(\mathcal{G}^R + \mathcal{G}^A)$ is the real part of the two-particle Green function. The derivation is analogous to the above,

$$\begin{aligned}\mathcal{T} &= \tfrac{1}{2}(\mathcal{T}^R + \mathcal{T}^A) = \mathcal{T}^R \cdot \tfrac{1}{2}\big(\mathcal{T}_R^{-1} + \mathcal{T}_A^{-1}\big) \cdot \mathcal{T}^A \\ &= \mathcal{T}^R \cdot (\mathcal{V}^{-1} - \mathcal{G}) \cdot \mathcal{T}^A,\end{aligned} \tag{A12}$$

and antisymmetrization gives the identity (A11).

The identity (A11) has no straightforward physical meaning and there is no common name for it. As far as the authors know, the identity (A11) itself is rarely used. Its energy derivative, however, has been presented by a number of authors under different names. We use the name *derived optical theorem* introduced by Zimmermann and Stolz [69].

Let us take the energy derivative of identity (A11),

$$\begin{aligned}\frac{\partial \mathcal{T}_{\rm ex}}{\partial \Omega} = {} & \frac{\partial \mathcal{T}^R_{\rm sc}}{\partial \Omega} \cdot (\mathcal{V}^{-1} - \mathcal{G}) \cdot \mathcal{T}^A_{\rm sc} + \mathcal{T}^R_{\rm sc} \cdot (\mathcal{V}^{-1} - \mathcal{G}) \cdot \frac{\partial \mathcal{T}^A_{\rm sc}}{\partial \Omega} \\ & - \mathcal{T}^R_{\rm sc} \cdot \frac{\partial \mathcal{G}}{\partial \Omega} \cdot \mathcal{T}^A_{\rm sc}.\end{aligned} \tag{A13}$$

To rearrange the first and second terms, we write

$$\mathcal{V}^{-1} - \mathcal{G} = \mathcal{V}^{-1} - \mathcal{G}^A + \frac{i}{2}\mathcal{A} = \mathcal{V}^{-1} - \mathcal{G}^R - \frac{i}{2}\mathcal{A}, \tag{A14}$$

and use the concrete representation

$$\mathcal{T}^R_{\rm sc} \cdot (\mathcal{V}^{-1} - \mathcal{G}^R) = \hat{\mathcal{U}}_{\rm sc}, \quad (\mathcal{V}^{-1} - \mathcal{G}^A) \cdot \mathcal{T}^A_{\rm sc} = \hat{\mathcal{U}}_{\rm sc}. \tag{A15}$$

Since $\hat{\mathcal{U}}_{\rm sc}$ does not depend on the energy Ω, from antisymmetrization (A5) directly follows

$$\frac{\partial \mathcal{T}^R_{\rm sc}}{\partial \Omega} \cdot \hat{\mathcal{U}}_{\rm sc} = \frac{\partial \mathcal{T}^R_{\rm ex}}{\partial \Omega}, \quad \hat{\mathcal{U}}_{\rm sc} \cdot \frac{\partial \mathcal{T}^A_{\rm sc}}{\partial \Omega} = \frac{\partial \mathcal{T}^A_{\rm ex}}{\partial \Omega}. \tag{A16}$$

Making use of (A14)–(A16), the first and the second terms of (A13) can be rearranged as

$$\begin{aligned}& \frac{\partial \mathcal{T}^R_{\rm sc}}{\partial \Omega} \cdot (\mathcal{V}^{-1} - \mathcal{G}) \cdot \mathcal{T}^A_{\rm sc} + \mathcal{T}^R_{\rm sc} \cdot (\mathcal{V}^{-1} - \mathcal{G}) \cdot \frac{\partial \mathcal{T}^A_{\rm sc}}{\partial \Omega} \\ &= 2\frac{\partial \mathcal{T}_{\rm ex}}{\partial \Omega} + \frac{i}{2}\frac{\partial \mathcal{T}^R_{\rm sc}}{\partial \Omega} \cdot \mathcal{A} \cdot \mathcal{T}^A_{\rm sc} - \frac{i}{2}\mathcal{T}^R_{\rm sc} \cdot \frac{\partial \mathcal{T}^A_{\rm sc}}{\partial \Omega}.\end{aligned} \tag{A17}$$

Substituting from (A17) into (A13) we immediately obtain the causal *derived optical theorem*

$$\begin{aligned}& \frac{\partial \mathcal{T}_{\rm ex}}{\partial \Omega} - \mathcal{T}^R_{\rm sc} \cdot \frac{\partial \mathcal{G}}{\partial \Omega} \cdot \mathcal{T}^A_{\rm sc} \\ &= -\frac{i}{2}\frac{\partial \mathcal{T}^R_{\rm sc}}{\partial \Omega} \cdot \mathcal{A} \cdot \mathcal{T}^A_{\rm sc} + \frac{i}{2}\mathcal{T}^R_{\rm sc} \cdot \mathcal{A} \cdot \frac{\partial \mathcal{T}^A_{\rm sc}}{\partial \Omega}.\end{aligned} \tag{A18}$$

Proofs of identity (A18) can also be found in [69] and [16].

As before, in (A12) we can interchange retarded and advanced T matrices arriving at the anticausal expression for the real part of the T matrix,

$$\begin{aligned}\mathcal{T} &= \tfrac{1}{2}(\mathcal{T}^R + \mathcal{T}^A) = \mathcal{T}^A \tfrac{1}{2} \cdot \big(\mathcal{T}_R^{-1} + \mathcal{T}_A^{-1}\big) \cdot \mathcal{T}^R \\ &= \mathcal{T}^A \cdot (\mathcal{V}^{-1} - \mathcal{G}) \cdot \mathcal{T}^R,\end{aligned} \tag{A19}$$

which yields the anticausal *derived optical theorem*

$$\frac{\partial \mathcal{T}_{\rm ex}}{\partial \Omega} - \mathcal{T}^A_{\rm sc} \cdot \frac{\partial \mathcal{G}}{\partial \Omega} \cdot \mathcal{T}^R_{\rm sc} = \frac{i}{2}\frac{\partial \mathcal{T}^A_{\rm sc}}{\partial \Omega} \cdot \mathcal{A} \cdot \mathcal{T}^R_{\rm sc} - \frac{i}{2}\mathcal{T}^A_{\rm sc} \cdot \mathcal{A} \cdot \frac{\partial \mathcal{T}^R_{\rm sc}}{\partial \Omega}. \tag{A20}$$

Comparing to the causal form (A18), the opposite sign on the right side is a consequence of the antisymmetry of the spectral function $\mathcal{A}$ with respect to retarded and advanced functions. Of course the two forms of the *derived optical theorem* yield the same linear gradient corrections.

Similarly, the time derivative of the identity (A11) gives the *time-derived optical theorem*. It parallels the *derived optical theorem* (A18) and (A20) in which ∂_Ω stands in the place of ∂_t and the second term on the left side is simplified via $\partial_\Omega \mathcal{V} = 0$. The derivations of the two time-derived optical theorems follow identical steps; we have the causal form

$$\begin{aligned}& \frac{\partial \mathcal{T}_{\rm ex}}{\partial t} + \mathcal{T}^R_{\rm sc} \cdot \frac{\partial (\mathcal{V}^{-1} - \mathcal{G})}{\partial t} \cdot \mathcal{T}^A_{\rm sc} \\ &= \frac{i}{2}\mathcal{T}^R_{\rm sc} \cdot \mathcal{A} \cdot \frac{\partial \mathcal{T}^A_{\rm sc}}{\partial t} - \frac{i}{2}\frac{\partial \mathcal{T}^R_{\rm sc}}{\partial t} \cdot \mathcal{A} \cdot \mathcal{T}^A_{\rm sc},\end{aligned} \tag{A21}$$

and the anticausal form

$$\begin{aligned}& \frac{\partial \mathcal{T}_{\rm ex}}{\partial t} + \mathcal{T}^A_{\rm sc} \cdot \frac{\partial (\mathcal{V}^{-1} - \mathcal{G})}{\partial t} \cdot \mathcal{T}^R_{\rm sc} \\ &= \frac{i}{2}\frac{\partial \mathcal{T}^A_{\rm sc}}{\partial t} \cdot \mathcal{A} \cdot \mathcal{T}^R_{\rm sc} - \frac{i}{2}\mathcal{T}^A_{\rm sc} \cdot \mathcal{A} \cdot \frac{\partial \mathcal{T}^R_{\rm sc}}{\partial t}.\end{aligned} \tag{A22}$$

APPENDIX B: OFF-SHELL CONTRIBUTION TO DERIVATIVES

Here we show that (107) can be rearranged into the sum of expressions (101d) and (104c) in (101c). The Wigner distribution $\rho_1 = \int_\omega G_1^<(\omega)$ in the *extended quasiparticle approximation* yields

$$F = \sum_a \int_{\boldsymbol{k},\omega} \frac{\partial \varepsilon_1}{\partial t}\left(2\pi\delta(\omega - \varepsilon_1) f_1 \frac{\partial \Sigma_1(\omega)}{\partial \omega} + \Sigma_1^<(\omega)\frac{\wp}{\omega - \varepsilon_1}\right). \tag{B1}$$

The integrand differs from the integrand of the correlated density (58) by the factor $\partial_t \varepsilon_1$. Following the same steps as in Sec. IV B, we arrive at

$$F = \int_\Omega {\rm Tr}\left[\mathcal{C} \cdot \frac{\partial \mathcal{T}_{\rm ex}}{\partial \Omega} \cdot \mathcal{G}^< - \mathcal{G}^< \cdot \mathcal{T}^A_{\rm sc} \cdot \frac{\partial \mathcal{G}}{\partial \Omega} \cdot \mathcal{C} \cdot \mathcal{T}^R_{\rm sc}\right], \tag{B2}$$

with

$$\mathcal{C}_{ij} = \frac{1}{2}\left(\frac{\partial \varepsilon_i}{\partial t} + \frac{\partial \varepsilon_j}{\partial t}\right). \tag{B3}$$

Using $\mathcal{G}^<_{12}(z) = 2\pi\delta(z - \varepsilon_1 - \varepsilon_2) f_1 f_2$ one finds that the first term in the trace in (B2) yields expression (101d). From $\mathcal{A}_{34}(z) = 2\pi\delta(z - \varepsilon_3 - \varepsilon_4)(1 - f_3 - f_4)$ and the Kramers-Kronig relation (67) follows the real part of the Green function $\partial_\Omega \mathcal{G}_{34} = -(1 - f_3 - f_4)\wp/(\Omega - \varepsilon_3 - \varepsilon_4)$ so that the second term in the trace in (B2) produces the contribution of (104c) in (101c).

APPENDIX C: TWO-PARTICLE DENSITY MATRIX

In the T-matrix approximation the two-particle density matrix $\rho_{12} = \int_\Omega G^<_{12}(\Omega)$ is the energy integral of the two-particle Green function,

$$\begin{aligned}\mathcal{G}^<(\Omega) = {} & \hat{\mathcal{U}}_{\rm sc} \cdot [1 + \mathcal{G}^R(\Omega) \cdot \mathcal{T}^R(\Omega)] \int_\omega G^<(\Omega - \omega) \cdot G^<(\omega) \\ & \cdot [1 + \mathcal{T}^A(\Omega) \cdot \mathcal{G}^A(\Omega)] \cdot \hat{\mathcal{U}}_{\rm sc}.\end{aligned} \tag{C1}$$

Taking the on-shell approximation of correlation functions $G^<$ one obtains the first term of the two-particle reduced density matrix (111). For off-shell corrections to $G^<$ we can neglect the correlations represented by the ladder, writing $1+\mathcal{T}^A(\Omega)\mathcal{G}^A(\Omega)\sim 1$. This yields the second and third terms of (111).

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