Directed momentum current induced by the $\mathcal{PT}$-symmetric driving

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We investigate the directed momentum current in the quantum kicked rotor model with $\mathcal{PT}$-symmetric deriving potential. For the quantum nonresonance case, the values of quasienergy become complex when the strength of the imaginary part of the kicking potential exceeds a threshold value, which demonstrates the appearance of the spontaneous $\mathcal{PT}$ symmetry breaking. In the vicinity of the transition point, the momentum current exhibits a staircase growth with time. Each platform of the momentum current corresponds to the mean momentum of some eigenstates of the Floquet operator whose imaginary parts of the quasienergy are significantly large. Above the transition point, the momentum current increases linearly with time. Interestingly, its acceleration rate exhibits a kind of "quantized" increment with the kicking strength. We propose a modified classical acceleration mode of the kicked rotor model to explain such an intriguing phenomenon. Our theoretical prediction is in good agreement with numerical results.

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I. INTRODUCTION

The directed transport of macroscopic particles has attracted intensive attentions in the past few decades [1–5]. The phenomenon implies the flow of energy or the transmission of information. Therefore, its mechanism is important for the design of quantum heat engines and quantum information. It has also wide applications in the construction of nanoscale devices, such as particle separation and electron pumps, and for the understanding of biological molecular motors [6–8]. Formation of directed current is closely related to the breaking of spatiotemporal symmetry or the topological property of system. These two kinds of features are controllable in the Floquet-driven system by manipulating the external potential. For example, the spatially nonsymmetric driving potential leads to the directed motion of cold atoms in optical lattice [9–12]. A double-kicking extension of a quantum kicked rotor (QKR) model exhibits topological momentum current [13]. More recently, the atom-optics experiments of the QKR model has reported the directed acceleration in momentum space [14]. Indeed, the system of cold atoms driven by time-periodical optical lattice is an ideal platform for investigating the directed transport phenomenon [15–17].

On the other hand, the quantum dynamics with $\mathcal{PT}$ symmetry is fundamentally important [18–21]. This field has been regarded as a significant extension of traditional Hermitian systems [22]. A unique property of $\mathcal{PT}$-symmetric quantum systems is a spontaneous transition, that is, the real energy eigenvalues become complex when the strength of the imaginary part of the complex potential exceeds a threshold value. Interestingly, experimental progress has realized the ultracold atoms in complex optics lattices [23–29], which opens the opportunity for investigating transport behavior in the $\mathcal{PT}$-symmetric potential. Previous studies on wave-packet dynamics of these system show that, with the $\mathcal{PT}$ symmetry being broken, the energy band merging leads to peculiar transport behavior of optic wave and matter wave, such as double refraction, nonreciprocal diffraction, bifurcation, and many others [30,31].

The extension of a Floquet-driven system to a $\mathcal{PT}$ symmetric regime induces exotic physics [32]. A recent study on a QKR model with $\mathcal{PT}$ symmetry shows that the spontaneous $\mathcal{PT}$ transition occurs in the dynamical localization regime, while the $\mathcal{PT}$ symmetry is always broken in the quantum resonance case [33]. Moreover, they discovered that the momentum current is unboundedly accelerated in quantum resonance and that it is suppressed by dynamical localization in condition that the $\mathcal{PT}$ symmetry is preserved. It is known that the Talbot effect of a quantum resonance case causes the periodical revival of the QKR wave packets, and therefore the system experiences a constant force which facilitates the directed current. On the other hand, the transport behavior of matter wave in the quantum nonresonance situation is still an open topic, which needs urgent investigation.

Motivated by these studies, we investigate the directed acceleration of wave packets in the quantum nonresonance case via a QKR model with $\mathcal{PT}$ symmetry. We find that the breaking of the $\mathcal{PT}$ symmetry can induce rich transport behavior in momentum space. Specifically, in the vicinity of transition

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points, the mean momentum exhibits the staircase growth with time. Detailed investigation reveals that the platform of the momentum current is determined by mean momentum of some eigenstates of the Floquet operator with large imaginary parts of quasienergy. Moreover, it is found that the momentum of those eigenstates concentrates on several separate values which are in one-to-one correspondence to the momentum of each platform. Above the transition points, the mean momentum increases linearly with time. More interesting is that the acceleration rate shows a "quantized" increment with the kick strength. The underlying physics behind such an interesting phenomenon is the modification of the classical acceleration mode of kicked rotor model by the gain-or-loss mechanism of the complex kicking potential. Our theoretical prediction of the acceleration rate of such "quantized" momentum current is in perfect consistence with numerical results.

Conventionally, it is believed that in the quantum nonresonance case, the mechanism of dynamical localization suppresses directed motion of microscopic particles. Our finding of the directed momentum current in the dynamical localization regime which is induced by breaking of $\mathcal{PT}$ symmetry may shine a new light on this fundamental problem. The experimental advances in atom optics have made it possible to realize the complex optical lattice. We therefore hope our theoretical results will stimulate future experiments in this field.

The paper is organized as follows. In Sec. II, we describe the system and show the directed momentum current. In Sec. III, we study the "quantized" acceleration mode. A summary is presented in Sec. IV.

II. DIRECTED MOMENTUM CURRENT

We consider the QKR model with the $\mathcal{PT}$ symmetry for which the Schrödinger equation takes the form

$$i\hbar\frac{\partial\psi}{\partial t} = \left\{\frac{\hat{P}^2}{2I} + V_0[\cos(\theta) + i\lambda\sin(\theta)]\delta_T\right\}\psi, \tag{1}$$

where $\hat{P}$ is the angular-momentum operator, θ is the angle coordinate, I is the moment of inertia, V_0 is strength of the kicking potential with λ being the strength of its imaginary component, and $\delta_T = \sum_n \delta(t - nT)$ with T being the kick period. The time evolution of quantum states during one period is governed by the Floquet operator,

$$U = \exp\left[-i\frac{V_0(\theta)}{\hbar}\right]\exp\left(\frac{-i}{\hbar}\frac{\hat{P}^2 T}{2I}\right), \tag{2}$$

where $V_0(\theta) = V_0[\cos(\theta) + i\lambda\sin(\theta)]$ [33].

In the angular-momentum representation, i.e., $\hat{P}|\varphi_n\rangle = n\hbar|\varphi_n\rangle$ with $\langle\theta|\varphi_n\rangle = e^{in\theta}/\sqrt{2\pi}$, the free evolution operator, namely the second term on the right-hand side of Eq. (2), can be written as

$$U_f = \exp\left(-i\frac{n^2\hbar T}{2I}\right). \tag{3}$$

It is evident that the U_f is determined by a dimensionless factor $\hbar T/I$. Hence, for the convenience of investigation, we define it as an effective Planck constant $\hbar_{\text{eff}} = \hbar T/I$, for which the dimensionless angular momentum is $p_n = n\hbar_{\text{eff}}$. Accordingly, we have made the scaling for the angular-momentum operator as $\hat{p} = \hat{P}T/I$, for which the eigenequation is $\hat{p}|\varphi_n\rangle = p_n|\varphi_n\rangle$. Then the free evolution operator can be expressed as

$$U_f = \exp\left(-\frac{i}{\hbar_{\text{eff}}}\frac{\hat{p}^2}{2}\right). \tag{4}$$

With the effective Planck constant, the kicking evolution operator, namely the first term on the right-hand side of Eq. (2), is rewritten as

$$\begin{aligned} U_K &= \exp\left[-i\frac{V_0(\theta)}{\hbar}\right] = \exp\left[-i\frac{TV_0(\theta)}{\hbar_{\text{eff}}I}\right], \\ &= \exp\left[-i\frac{V_K(\theta)}{\hbar_{\text{eff}}}\right], \end{aligned} \tag{5}$$

where $V_K(\theta) = K[\cos(\theta) + i\lambda\sin(\theta)]$ with the dimensionless kicking strength $K = V_0T/I$. The reason for introducing this dimensionless kick strength K is that it is the only parameter controlling the classical dynamics governed by the well-known mapping equation [34]. Traditional investigations on the quantum-classical correspondence of such a chaotic system mainly concentrate on the case where K is a constant for which the classical limit is fixed. With the decrease of $\hbar_{\text{eff}}$, i.e., $K/\hbar_{\text{eff}} \to \infty$, the quantum dynamics will be consistent with its classical counterpart for a long-enough time [35]. In the present work, we indeed find that the spreading of wave packets of the non-Hermitian kicked rotor follows the classical acceleration modes in the condition where $K/\hbar_{\text{eff}} \gg 1$.

By combining Eqs. (4) and (5), the Floquet operator in dimensionless units reads

$$U = \exp\left[-i\frac{V_K(\theta)}{\hbar_{\text{eff}}}\right]\exp\left(-\frac{i}{\hbar_{\text{eff}}}\frac{\hat{p}^2}{2}\right). \tag{6}$$

The eigenequation of the Floquet operator reads

$$U|\psi_\varepsilon\rangle = e^{-i\varepsilon}|\psi_\varepsilon\rangle, \tag{7}$$

where ε indicates the quasienergy. Quantum nonresonance corresponds to irrational values of $\hbar_{\text{eff}}/4\pi$. To numerically investigate the quasienergy and quasieigenstate, we should use the finite truncation to approximate the $U_{m,n}$ matrix of infinite dimension [33]. Such an approximated method is effective, since the matrix $U_{m,n}$ has the band structure [34].

The broken of $\mathcal{PT}$ symmetry is quantified by the appearance of the complex quasienergy, i.e., $\varepsilon = \varepsilon_{\text{r}} + i\varepsilon_i$. To identify such spontaneous symmetry breaking, we numerically calculate the average value of the imaginary part of the quasienergy

$$|\bar{\varepsilon}_i| = \frac{1}{N}\sum_{j=1}^{N}|\varepsilon_i^j|,$$

where N is the dimension of the Floquet matrix and ε_i^j denotes the imaginary part of the jth the quasienergy [33]. Our numerical results show that the average value $\bar{\varepsilon}_i$ is virtually zero for small λ, and it abruptly increases once the λ exceeds a certain critical threshold, i.e., $\lambda > \lambda_c$ (see Fig. 1). This is clear evidence of the spontaneous $\mathcal{PT}$-symmetry breaking controlled by the parameter λ. In numerical simulations, the truncation of the $U_{m,n}$ matrix is $N = 2048$. In fact, such $\mathcal{PT}$-symmetry breaking for the dynamical localization case has been reported in Ref. [33].

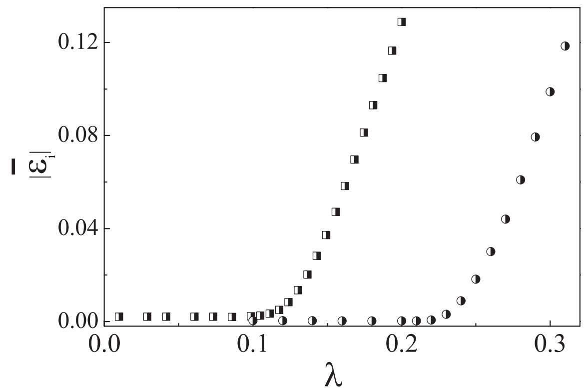

FIG. 1. The average value of the imaginary part of the quasienergy, i.e., $|\bar{\varepsilon}_i|$ versus the strength of the complex potential λ for $\hbar_{\rm eff} = 1.0$ (squares) and 1.5 (circles). The kick strength is $K = 5$.

After studying the properties of eigenvalues of the Floquet operator, we are then to discuss the dynamical behavior of the system, and we focus on the momentum current here. In the basis of $|\varphi_n\rangle$, an arbitrary state can be expressed as $\psi(\theta,t) = \sum_{n=-\infty}^{+\infty} \psi_n(t)\langle\theta|\varphi_n\rangle$, with $\psi_n(t)$ being the wave function in the momentum representation. The momentum current is quantified by the mean momentum

$$\langle p(t)\rangle = \frac{\sum_n p_n|\psi_n(t)|^2}{\mathcal{N}},$$

where $\mathcal{N} = \sum_n |\psi_n(t)|^2$ is the norm of the quantum state [33]. For $\lambda > \lambda_c$, the appearance of the complex quasienergies will lead to the exponentially fast increase of norm with time. The above definition of momentum current drops the contribution from the growth of the norm to the current behavior. We numerically investigate the momentum current for the irrational values of $\hbar_{\rm eff}/4\pi$ with different λ. In our numerical simulations, the initial state is taken as the ground state of the angular-momentum operator, i.e., $\psi(\theta, 0) = 1/\sqrt{2\pi}$. Our numerical results show that, below the transition point, i.e., $\lambda < \lambda_c$ (see Fig. 2 for $\lambda = 0.06$), the momentum current saturates to a small asymptotic value after the growth during the initially short time interval. In fact, in the limit of $\lambda \to 0$, the wave packets spread symmetrically in momentum space, and thus the average momentum is virtually zero. Interestingly, around the threshold value of spontaneous $\mathcal{PT}$-symmetry breaking, i.e., $\lambda \sim \lambda_c$ (e.g., $\lambda = 0.09$ in Fig. 2), there is the staircase growth of the momentum current. Moreover, the jump of the momentum current from the lower stair to the upper one is very sharp, which implies that the system changes to different quantum states with time evolution. Above the transition point, i.e., $\lambda > \lambda_c$ (e.g., $\lambda = 0.2$ in Fig. 2), the momentum current linearly increases with time, i.e., $\langle p(t)\rangle = Dt$, for which the growth rate D is independent on λ if $\lambda \gg \lambda_c$ (e.g., $\lambda = 0.6$ and 0.9 in Fig. 2). Our investigation on momentum current in dynamical localization regime may sight a new light on the understanding of the unidirectional transport phenomenon.

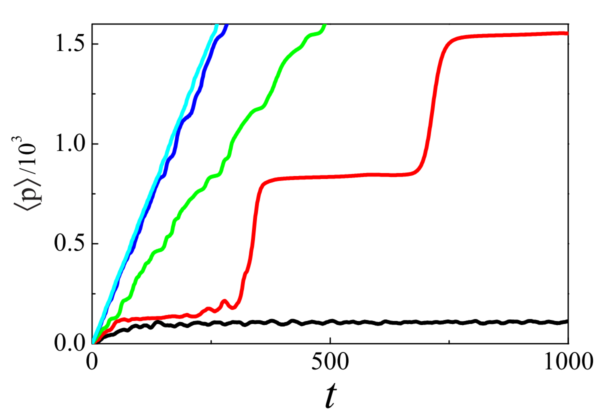

FIG. 2. Time dependence of the momentum current $\langle p\rangle$ with $\hbar_{\rm eff} = 1$ and $K = 5.0$. From bottom to top $\lambda = 0.06$ (black), 0.09 (red), 0.2 (green), 0.6 (blue), and 0.9 (cyan).

The mechanism of the staircase growth of the momentum current can be understood as follows. An arbitrary state can be expanded in the basis of the Floquet eigenstates. At the initial time, the expansion of the quantum state takes the form

$$|\psi(t_0)\rangle = \sum_{\varepsilon} C_\varepsilon|\psi_\varepsilon\rangle, \tag{8}$$

where $|\psi_\varepsilon\rangle$ indicates the eigenstate of the Floquet operator and C_ε is components of the quantum state. After the nth kicks, the quantum state has the expression

$$|\psi(t_n)\rangle = \sum_{\varepsilon} C_\varepsilon e^{-in\varepsilon}|\psi_\varepsilon\rangle. \tag{9}$$

When the $\mathcal{PT}$ symmetry is broken, the quasienergy is complex, i.e., $\varepsilon = \varepsilon_{\rm r} + i\varepsilon_i$. Therefore, the expansion of $|\psi(t_n)\rangle$ can be rewritten as

$$|\psi(t_n)\rangle = \sum_{\varepsilon} C_\varepsilon e^{n\varepsilon_i} e^{-in\varepsilon_{\rm r}}|\psi_\varepsilon\rangle. \tag{10}$$

It is apparent that the components with $\varepsilon_i > 0$ will exponentially grow. Then the mean momentum $\langle p\rangle$ corresponding to these eigenstates contributes mainly to the momentum current. Interestingly, we find that most of the eigenstates with positive ε_i are localized in momentum space [see Fig. 3(a)]. Moreover, the mean momentum of those eigenstates concentrates on several separate values p_m [see Fig. 3(b)]. Detailed observations show that each p_m is in one-to-one correspondence to the platform of the momentum current in Fig. 2. It is evident that, during the appearance of the platform of the momentum current, the quantum state is the eigentate of the largest ε_i with the same $\langle p\rangle$. The transition between the eigenstates of different mean momentum leads to the staircase growth of the momentum current.

The underlying physics of the linear growth of the momentum current for $\lambda > \lambda_c$ is due to the gain or loss mechanism induced by the imaginary part of the kicking potential. Such mechanism happens when the Floquet operator of the non-Hermitian term $U_K^i(\theta) = \exp[K\lambda\sin(\theta)/\hbar_{\rm eff}]$ operates on the quantum state, i.e., $U_K^i\psi(\theta)$. The action can dramatically cause the annihilation of the quantum state in the region of $\theta \in (-\pi, 0)$, since in this region the value of $\sin(\theta)$ is negative. In contrast, the probability of the particle in the region of $(0, \pi)$ will be enhanced as $\sin(\theta)$ takes a positive value in this region. For $K\lambda/\hbar_{\rm eff} \gg 1$, the value of the $U_K^i(\theta)$ is extremely large in the position of $\theta = \pi/2$. Therefore, the action of the Floquet operator U_K^i on a quantum state can effectively

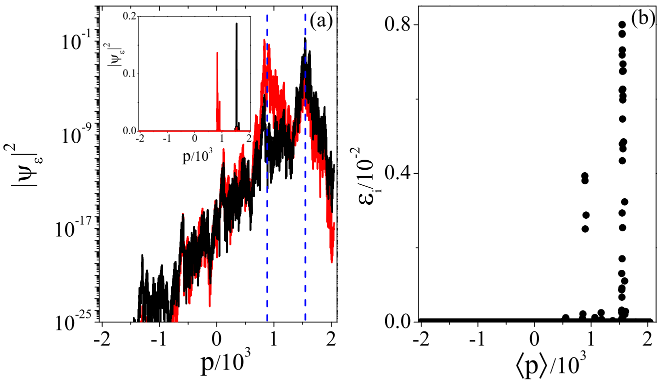

FIG. 3. (a) The Floquet eigenstates (in semilog scale) in momentum space with $\varepsilon_i = 0.00393$ (red) and 0.008 (black) which corresponds to the two peaks of ε_i in (b). The main plot and the inset show the same data in the lin-log and linear scales, respectively. Dashed lines (in blue) mark the center of each Floquet eigenstates. (b) The imaginary part of the quasienergy ε_i versus the mean momentum $\langle p\rangle$ of the corresponding eigenstate. The parameters are $K = 5$, $\hbar_{\rm eff} = 1$, and $\lambda = 0.09$.

generate a quantum particle in $\theta = \pi/2$ if the center of the quantum state is not very far from this position. Indeed, our theoretical analysis proves that the wave function after each kick can be well described by a Gaussion wave packet with the center $\theta_0 = \pi/2$ [36]. In this position, the quantum particle experiences the kicking force of strength K. Therefore, the time growth of the mean momentum is roughly in the form of $\langle p(t)\rangle \propto Kt$.

III. "QUANTIZED" ACCELERATION MODE

Further, we numerically investigate the acceleration rate of the momentum current, i.e., $D = \lim_{t\to t_f}\langle p^2(t)\rangle/t_f$ for $\lambda \gg \lambda_c$, where t_f is the total time one can track the time evolution. Due to the linear growth of $\langle p(t)\rangle$, the t_f of a scale of hundreds of kicking periods can ensure the precise acceleration rate. Interestingly, we find that the acceleration rate exhibits the "quantized" increment with increasing K (see Fig. 4), namely

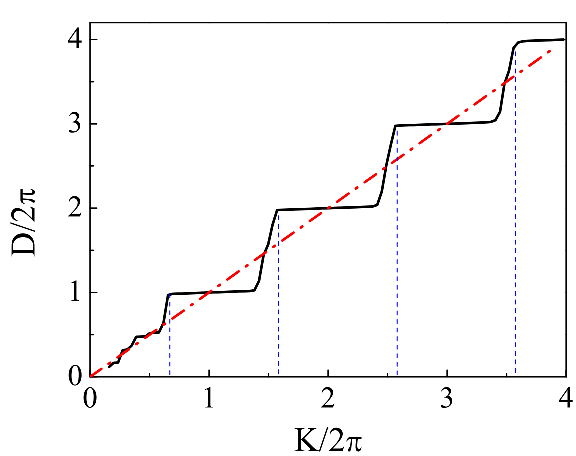

FIG. 4. The acceleration rate D versus K for $\hbar_{\rm eff} = 0.1$. Dashed-dotted line (in red) indicates the function of the form $D(K) = K$. Dashed lines (in blue) mark the transition points.

$D = 2n\pi$ for $K \in [2n\pi - \Delta_0, 2n\pi + \Delta_0]$ with $\Delta_0 \approx \pi$ and $n \geqslant 1$. The mechanism of such an intriguing phenomenon is due to the coexistence of the classical acceleration mode and the "gain-or-loss" effects of the non-Hermitian potential. Remember that for $K\lambda/\hbar_{\rm eff} \gg 1$ the action of the Floquet operator U_λ on a quantum state can effectively generate a particle in the position of $\theta = \pi/2$. Indeed, our analytic analysis proves that, after the action of U_λ, the wave packets can be well described by a Gaussian function with minimum uncertainty $\delta\theta\delta p = \hbar_{\rm eff}/2$, centered at $(\bar\theta = \pi/2, \bar p = 2n\pi)$ [36]. Then, we can regard it as a classical particle and analyze the acceleration mode of its classical trajectory.

We consider the classical acceleration mode of kicked rotor, which is governed by the classical mapping equation [37]

$$\begin{aligned} p(t_{j+1}) &= p(t_j) + K\sin[\theta(t_j)] \\ \theta(t_{j+1}) &= \theta(t_j) + p(t_{j+1}), \end{aligned} \tag{11}$$

where $\theta(t_j)$ and $p(t_j)$ denote the angle coordinate and the angular momentum after the jth kick. It is easy to see that a classical trajectory with $[\theta(t_0) = \pi/2, p(t_0) = 2m\pi]$ will be accelerated linearly as time evolves, i.e., $[\theta(t_j) = \pi/2, p(t_j) = p_0 + jK]$ if $K = 2n\pi$, where m and n are all integers. In our model, we should also consider the effects of the imaginary part of the kicking potential on the time evolution of a classical trajectory.

Without loss of generality, we assume that at the time $t = t_j$ the position of a classical trajectory is $(\theta = \pi/2, p = 0)$. We consider the case that the kick strength has some deviation from the ideal value, i.e., $K = 2n\pi + \Delta$ with $|\Delta| < \pi$. According to Eq. (11), after one kick period, the trajectory changes to

$$p(t_{j+1}^-) = 2n\pi + \Delta \tag{12}$$

and

$$\theta(t_{j+1}^-) = \frac{\pi}{2} + 2n\pi + \Delta, \tag{13}$$

where the superscript "–" indicates the time immediately before the action of imaginary part of the kicking potential, i.e., the U_K^i operator. Equation (13) reveals that the value of Δ can be regarded as the distance between the center of the wave packets and the position of $\theta = \pi/2 + 2n\pi$ which is essentially equal to $\pi/2$ due to the periodical boundary condition. Since the action of the U_K^i operator on the wave packets greatly enhances the probability of a particle in $\theta = \pi/2 + 2n\pi$ if the value of Δ is smaller than a threshold value, i.e., Δ_0, it is reasonable to believe that, after the action of U_K^i, the particle moves to the position of $\theta(t_{j+1}) = \pi/2 + 2n\pi$. Accordingly, its momentum becomes $p(t_{n+1}) = 2n\pi$ for which the actual increment of the momentum during one-period evolution is $D = 2n\pi$. A rough estimation of Δ_0 is a half of the width of the region $[2(n-1)\pi + \pi/2, 2n\pi + \pi/2]$, i.e., $\Delta_0 \approx \pi$, which is confirmed by our numerical results in Fig. 4.

IV. SUMMARY

In this work, we investigate the directed current of the quantum kicked rotor model whose kicking potential satisfies the $\mathcal{PT}$-symmetric condition. We find that in the vicinity of transition point, i.e., $\lambda \approx \lambda_c$, the eigenstates is well localized in momentum space. Moreover, the mean momentum eigenstates with positive real imaginary parts of the quasienergy concentrates on several separate values. Such property leads to the staircase growth of the momentum current $\langle p(t)\rangle$ with time. When the parameter λ is larger than the transition point, i.e., $\lambda > \lambda_c$, the momentum current linearly increases with time, i.e., $\langle p(t)\rangle = Dt$. We make extensive investigations on the acceleration rate D for $\lambda \gg \lambda_c$. Interestingly, we find that, for $K\lambda/\hbar_{\rm eff} \gg 1$, the acceleration rate exhibits the "quantized" increment with the increase of K, i.e., $D = 2n\pi$ for $K \in [2n\pi - \Delta_0, 2n\pi + \Delta_0]$ with $\Delta_0 \approx \pi$ and $n \geqslant 1$. For $K\lambda/\hbar_{\rm eff} \gg 1$, our analytic analysis proves that, at any time $t = t_n$, the wave packets can be well described by the Gaussian function with a center ($\bar\theta = \pi/2$, $\bar p = 2n\pi$). The motion of the wave packet in phase space follows the classical acceleration mode of the trajectory of the kicked rotor model. The theory of the modified acceleration mode of the classical particle by the gain-or-loss mechanism of the complex kicking potential can successfully explain such "quantized" phenomenon of momentum current. It is known that cold atoms driven by time-periodical optical lattice, with rich and complex physics, such as the Butterfly spectrum [38], the exponentially fast diffusion [39], is an ideal platform for investigating the directed transport phenomenon. Our results may also be useful in the quantum control of the directed transport of matter waves.

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APPENDIX: THE ACCELERATION OF MOMENTUM CURRENT

The Floquet operator of the non-Hermitian QKR reads

$$U = U_f U_K^{\rm r} U_K^i, \tag{A1}$$

with the free evolution operator $U_f = \exp(-ip^2/2\hbar_{\rm eff})$, the evolution operator of the real part of the kicking potential

$$U_K^{\rm r} = \exp\left[-i\frac{K}{\hbar_{\rm eff}}\cos(\theta)\right], \tag{A2}$$

and that of the imaginary part

$$U_K^i = \exp\left[\frac{\lambda K}{\hbar_{\rm eff}}\sin(\theta)\right]. \tag{A3}$$

The maximum value of U_K^i corresponds to $\theta_0 = \pi/2$. In condition that $\hbar_{\rm eff} \to 0$ (with $\lambda K/\hbar_{\rm eff} \gg 1$ and $K/\hbar_{\rm eff} \gg 1$), the expansions of first order for U_K^i and $U_K^{\rm r}$ around θ_0 take the form

$$U_K^i \approx \exp\left[-\frac{\lambda K(\theta-\theta_0)^2}{2\hbar_{\rm eff}}\right]\exp\left(\frac{\lambda K}{\hbar_{\rm eff}}\right) \tag{A4}$$

and

$$U_K^{\rm r} \approx \exp\left[\frac{iK(\theta-\theta_0)}{\hbar_{\rm eff}}\right]. \tag{A5}$$

As a further step, we consider the time evolution of a quantum state under the action of above operators.

Without loss of generality, we assume that the initial state is a Gaussian wave packet

$$\psi(\theta, t_0) = \frac{1}{(\sigma^2\pi)^4}\exp\left(-\frac{\theta^2}{2\sigma^2} + \frac{ip_0\theta}{\hbar_{\rm eff}}\right), \tag{A6}$$

for which the uncertainty relation is such that $\delta\theta\delta p = \hbar_{\rm eff}/2$. The evolution of quantum states from t_0 to $t_1 = t_0 + 1$ is governed by $|\psi(t_1)\rangle = U|\psi(t_0)\rangle$. The action of U_K^i on $\psi(\theta, t_0)$ yields that

$$\widetilde{\psi}(\theta, t_0) \propto \exp\left[-\frac{\lambda K(\theta-\theta_0)^2}{2\hbar_{\rm eff}} - \frac{\theta^2}{2\sigma^2} + \frac{ip_0\theta}{\hbar_{\rm eff}}\right]. \tag{A7}$$

In condition that $\lambda K/\hbar_{\rm eff} \gg 1/\sigma^2$, we can neglect the contribution of the second term in right-hand side of the above equation. Then, the quantum state can be approximated as

$$\widetilde{\psi}(\theta, t_0) \propto \exp\left[-\frac{\lambda K(\theta-\theta_0)^2}{2\hbar_{\rm eff}} + \frac{ip_0\theta}{\hbar_{\rm eff}}\right]. \tag{A8}$$

For the convenience of analysis, hereafter, we use this state as the initial state for the time evolution. That is, the Floquet operator is redefined as

$$U = U_K^i U_f U_K^{\rm r}. \tag{A9}$$

By comparison Eq. (A9) with Eq. (A1), one can see that the sequence of the action of U_K^i and U_f in time evolution is rearranged, which actually has no effect on physical results.

Consider the action of $U_K^{\rm r}$ in Eq. (A5) on $\widetilde{\psi}(\theta, t_0)$,

$$\begin{aligned}\psi(\theta, t_0^+) &= U_K^{\rm r}\widetilde{\psi}(\theta, t_0)\\ &\propto \exp\left[-\frac{\lambda K}{2\hbar_{\rm eff}}(\theta-\theta_0)^2 + \frac{iK}{\hbar_{\rm eff}}(\theta-\theta_0)\right]\\ &\quad\times \exp\left(\frac{i}{\hbar_{\rm eff}}p_0\theta\right),\end{aligned} \tag{A10}$$

where the superscript "+" indicates the time immediately after the action of the real part of the kicking potential. Next step is the action of the free evolution operator U_f on the quantum state. Before that we should transform the state to momentum space,

$$\begin{aligned}\psi(p, t_0^+) &= \int_{-\pi}^{\pi}\psi(\theta, t_0^+)\exp(-ip\theta/\hbar_{\rm eff})d\theta\\ &\propto \exp\left[-\frac{(p-p_K)^2}{2\hbar_{\rm eff}\lambda K} - \frac{ip\theta_0}{\hbar_{\rm eff}}\right],\end{aligned} \tag{A11}$$

where $p_K = p_0 + K$. Then the action of U_f on $\psi(p, t_0^+)$ yields

$$\begin{aligned}\psi(p, t_1^-) &= U_f\psi(p, t_0^+)\\ &\propto \exp\left[-\frac{(p-p_K)^2}{2\hbar_{\rm eff}\lambda K} - \frac{ip(p+2\theta_0)}{2\hbar_{\rm eff}}\right],\end{aligned} \tag{A12}$$

where the superscript "–" indicates the time immediately before the action of the kicking operator U_K^i.

The corresponding quantum state in real space is

$$\psi(\theta, t_1^-) \propto \int_{-\infty}^{\infty} dp\psi(p, t_1^-)\exp(ip\theta/\hbar_{\text{eff}})$$
$$\propto \exp\left[-\frac{(\theta-\theta_0-p_K)^2}{2\hbar_{\text{eff}}\lambda K}\right]$$
$$\times \exp\left[\frac{i\left(\theta-\theta_0+\frac{p_K}{\lambda^2K^2}\right)^2}{2\hbar_{\text{eff}}}\right]. \tag{A13}$$

We assume that

$$p_K = 2n_0\pi + \Delta, \tag{A14}$$

with $-\pi < \Delta < \pi$. Then the quantum state in Eq. (A13) is rewritten as

$$\psi(\theta, t_1^-) \propto \exp\left\{-\frac{[\theta-(\theta_0+2n_0\pi)-\Delta]^2}{2\hbar_{\text{eff}}\lambda K}\right\}$$
$$\times \exp\left[\frac{i\left(\theta-\theta_0+\frac{p_K}{\lambda^2K^2}\right)^2}{2\hbar_{\text{eff}}}\right]. \tag{A15}$$

Apparently, Eq. (A15) indicates a Gaussian wave packets with the center $\bar{\theta} = \theta_0 + 2n\pi + \Delta$. If the distance between $\bar{\theta}$ and $\theta_0 + 2n\pi$ is smaller than a threshold value Δ_0, then the action of U_K^i [in Eq. (A4)] on the quantum state $\psi(\theta, t_1^-)$ can effectively enhance the probability of a particle in the position of $\bar{\theta} = \theta_0 + 2n\pi$. Then, we get

$$\psi(\theta, t_1^+) = U_K^i\psi(\theta, t_1^-) \propto \exp\left\{-\frac{[\theta-(\theta_0+2n_0\pi)]^2}{2\hbar_{\text{eff}}\lambda K}\right\}$$
$$\times \exp\left[\frac{i\left(\theta-\theta_0+\frac{p_K}{\lambda^2K^2}\right)^2}{2\hbar_{\text{eff}}}\right], \tag{A16}$$

where the superscript "+" indicates the time immediately after the action of U_K^i. A rough estimation of Δ_0 is a half of the width of the region $[2(n-1)\pi + \theta_0, 2n\pi + \theta_0]$, i.e., $\Delta_0 \approx \pi$. Now the time evolution of a quantum state during an entire kicking period ends.

It is apparent that the center of this wave packet in real space is

$$\bar{\theta}_{t_1} = 2n_0\pi + \theta_0 \tag{A17}$$

with the second moment

$$\delta\theta = \sqrt{\overline{\theta^2} - (\bar{\theta})^2} = \sqrt{\frac{\hbar_{\text{eff}}}{2\lambda K}}. \tag{A18}$$

To obtain the momentum center of the wave packet $\psi(\theta, t_1^+)$, we should transform it to momentum space,

$$\psi(p, t_1^+) \propto \int d\theta e^{-ip\theta/\hbar}\psi(\theta, t_1^+)$$
$$\propto \exp\left[\frac{(p-2n_0\pi)^2}{2\hbar\lambda K} + if(p)\right], \tag{A19}$$

where $f(p)$ is an unimportant function of momentum. The function $f(p)$ does not determine the mean momentum, and hence there is no need to know the specific form of $f(p)$. From the above quantum state, we can get the mean momentum,

$$\bar{p}_{t_1} = 2n_0\pi, \tag{A20}$$

with the second moment

$$\delta p = \sqrt{\overline{p^2} - (\bar{p})^2} = \sqrt{\frac{\hbar_{\text{eff}}\lambda K}{2}}. \tag{A21}$$

One can find that the quantum state $|\psi(t_1)\rangle$ satisfies the uncertainty relation

$$\delta\theta\delta p = \frac{\hbar_{\text{eff}}}{2}, \tag{A22}$$

which is same as that of the initial wave packet.

As a brief summary, the quantum state after the evolution of first kicking period can be well described by a Gaussian wave packet,

$$\psi(\theta, t_1) \simeq \exp\left[-\frac{\lambda K(\theta-\bar{\theta}_{t_1})^2}{2\hbar_{\text{eff}}} + \frac{i\bar{p}_{t_1}\theta}{\hbar_{\text{eff}}}\right], \tag{A23}$$

where

$$\bar{\theta}_{t_1} = 2n_0\pi + \theta_0$$
$$\bar{p}_{t_1} = 2n_0\pi,$$

with $\theta_0 = \pi/2$. The time evolution of quantum state for $t > t_1$ just repeats the above procedure. Accordingly, the quantum state at any time $t = t_n$ can be well approximated by a Gaussian wave packet. Moreover, we can get the its center $(\bar{\theta}_{t_n}, \bar{p}_{t_n})$ by using the iterative method.

First, we can get the center $(\bar{\theta}_{t_2}, \bar{p}_{t_2})$ at the time $t = t_2$. Taking into account $\bar{p}_{t_1} = 2n_0\pi$ [see Eq. (A20)], for $K = 2n\pi + \Delta\,(-\pi < \Delta < \pi)$, then we arrive at $\bar{p}_{t_1} + K = 2(n_0 + n)\pi + \Delta$. Note that the value of Δ is smaller than a threshold value Δ_0. By repeating the procedure of the derivation for $(\bar{\theta}_{t_1}, \bar{p}_{t_1})$, one obtains

$$\begin{aligned}\bar{\theta}_{t_2} &= 2(n_0+n)\pi + \theta_0\\ \bar{p}_{t_2} &= 2(n_0+n)\pi.\end{aligned} \tag{A24}$$

Using the same method, one can find

$$\begin{aligned}\bar{\theta}_{t_j} &= 2[n_0+(j-1)n]\pi + \theta_0\\ \bar{p}_{t_j} &= 2[n_0+(j-1)n]\pi.\end{aligned} \tag{A25}$$

It is evident that the acceleration rate is

$$D = \bar{p}_{t_j} - \bar{p}_{t_{j-1}} = 2n\pi \tag{A26}$$

for $K \in [2n\pi - \Delta_0, 2n\pi + \Delta_0]$ with $\Delta_0 \approx \pi$. Our analytic analysis is confirmed by numerical results.

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