Dynamic resettlement as a mechanism of phase transitions in urban configurations

Bohdan Slavko, Kirill Glavatskiy, and Mikhail Prokopenko
Complex Systems Research Group and Centre for Complex Systems, Faculty of Engineering and IT, The University of Sydney, Sydney, NSW 2006, Australia

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We study formation and growth of human settlements as thermodynamic phenomena and focus on critical regimes associated with these processes. In doing so we develop a common thermodynamic perspective on modeling urban structures driven by "fast" and "slow" underlying dynamics of human resettlement and infrastructure development. The unifying perspective is illustrated by a comparative analysis of two qualitatively and quantitatively different models of dynamic resettlement. We demonstrate that the considered urban systems undergo phase transitions, irrespective of the particular details of the dynamic models. This suggests that the observed critical behavior is intrinsic to resettlement process.

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I. INTRODUCTION

Development of a tractable quantitative theory explaining and predicting formation and growth of human settlements, as well as the associated critical phenomena, is an active challenge [1,2]. Identifying conditions under which a sudden change (i.e., critical behavior) in residential settlement patterns may occur in response to a redistribution in employment opportunities or labor market is a valuable practical and scientific knowledge. Interpreting these insights in terms of key socioeconomic parameters may inform a more effective urban policy. At the same time, an increasing amount of high-resolution data on modern urban and regional (re)settlement motivates construction of more accurate models which take into account stochastic nature of human interactions [3–6]. Some of these models draw on statistical mechanics in an attempt to produce a rigorous description of coupled social and economic dynamics [7–9].

The apparent need for quantification of social phenomena has led to the emergence of econophysics and sociophysics [8,10], comprising a fairly broad spectrum of rigorous methods and novel insights into quantitative description of social systems, inspired by physics. In resolving several important questions, these fields have expanded beyond the reach of conventional methodologies offered by social sciences [8,10]. Successful applications of the methodology of physics to social sciences include the analysis and modeling of financial market microstructure [11,12], the financial market ecosystem [13,14], the evolution of cooperation [15–20], opinion formation [21,22], traffic flows and pedestrian behavior [23–25], and urban structure dynamics [3,26], among others. All of these examples share similar traits, deviating from the perfect rationality hypothesis, which is normally accepted in the traditional economic models. Instead, typical sociophysical models assume that the individual human decisions resemble the stochastic behavior of particles in physical systems [3,7,14,25]. These decisions, however, are not completely random as they take into account the attractiveness of different alternatives, in order to define the probability distribution over the decision space. In particular, many sociophysical models consider interactions between individuals and assume that these interactions affect the process of decision making. Abstracting away the perfect rationality often enhances the analysis of bubbles and long-term inefficiencies manifested in different financial and commodity markets [7,13].

Considering individual decision making is also valuable in modeling the altruistic human behavior and cooperation, which are explicitly represented in evolutionary game theory [15,17,20]. Some notable examples include modeling vaccination behavior in epidemiological settings [27–29], as well as driving behavior in traffic flows [20,30–32].

Modeling urban structures dates back to the book by von Thünen, *The Isolated State* [33], which demonstrated that purely economical reasons can lead to the emergence of a unique monocentric town surrounded by a rural area. Importantly, such pattern emerges as a result of decentralized self-organization based on the competition between *attractiveness* of the living place with respect to the working place. Further research revealed the importance of spatial interactions between individuals, households, and firms—placing these interactions as a focal point of study in complex systems theory and sociophysics [7]. Some of the prominent examples of the so-called "spatial externalities" which take into account *transportation costs* demonstrated phase transitions and critical behavior (e.g., transition between monocentric and polycentric cities [9,34,35], as well as symmetry breaking [36]).

In this paper we follow the common sociophysical approach and consider the resettlement process as a collective result of human individual decisions, who are driven by the spatial attractiveness of their choice of residence and the transportation cost between the residence and the work place [3,34,35,37]. The exact interpretation of these driving factors in the literature depends on the specific modeling framework, hindering attempts to derive a systematic generic theory. Therefore there is a need to develop a unifying statistical-mechanical perspective to model dynamic resettlement based on residential attractiveness and transportation costs. We address this question and base our approach on two methodologies which analyze human resettlement from different perspectives, examining their respective impacts on urban development and, in particular, on formation of residential suburbs. One of the models combines the equilibrium thermodynamic principle of maximum entropy and the dynamic Lotka-Volterra mechanism [38–40]. The other model is based on the concepts of the driving force in nonequilibrium thermodynamics [21].

The paper is organized as follows. In Sec. II we review the two methodologies, Boltzmann-Lotka-Volterra (BLV) and demographic balancing equation (DBE). We then extend them by providing a unifying framework to model dynamic resettlement for a given spatial structure of the employment. In particular, we assume that the resettlement activity is driven by two factors: attractiveness of the residential location and the cost of commute between the place of residence and employment. In Sec. III we analyze the stationary population of a hypothetical city which consists of two or three suburbs. The results show that irrespective of the particular model used we observe phase transitions between different urban configurations. Finally, in Sec. IV we discuss possible reasons for this universality in the thermodynamic context and their applied significance.

II. METHODOLOGY

Consider an isolated city consisting of N suburbs. We assume that each individual working in the suburb j can freely choose a location to settle in, based on the attractiveness W_i of the location i itself and the travel-to-work cost c_{ij}. At every moment we can identify the number of people working at j and living at i, which is represented by the fraction of the population ρ_{ij} ($\sum_{i,j}\rho_{ij} = 1$). We will also refer to ρ_{ij} as the density distribution. The occupation of the agents is assumed to be fixed (which means that the $n_j \equiv \sum_i \rho_{ij}$, the number of people working in the suburb j, is constant) whereas the place of residence may vary and the agents can change it without any restrictions. While agents may change the place or residence, the matrix c_{ij} of the travel-to-work costs for the entire city remains fixed: the agents optimize their own travel-to-work route by choosing the most efficient location i. The urban configuration is determined by a coevolution of the main variables, attractiveness $W_i(t)$, and transport flow $\rho_{ij}(t)$. Each of the approaches mentioned above provides its own model for this evolution, which is discussed below.

A. BLV model

The Boltzmann-Lotka-Volterra methodology arose from the idea to estimate a wide class of urban socioeconomic dynamics, including transport flows, distribution of retails centers, etc., using the Maximum Entropy principle, coupled with the Lotka-Volterra model of population dynamics [3,38,41] [in this paper we follow the original naming convention [3,40] and call Eq. (6) the Lotka-Volterra component despite the apparent distinction from the classical Lotka-Volterra model [42]. For instance, in the study of retail system redistribution [38], it was assumed that retail centers are growing (or declining) with rates which are proportional to the amount available for investment, while being constrained by the dependence of traffic distribution on attractiveness and travel costs. Although this analysis contained important insights about critical phenomena in urban systems, the technique treated the exterior of the system as an infinitely large reservoir that already reached its equilibrium. Nevertheless, the BLV formalism attracted a lot of attention, including the investigation of the equilibria stability using Poincare-Hopf index theorem [39], which demonstrated a variety of equilibria. Yet, because of the internal complexity of the model, the most interesting results were proven for a two-location setup only. Overall, many qualitative properties of the most general BLV model remain underexplored, except for some particular cases (see, e.g., Ref. [40], where bifurcations were analytically derived for the racetrack economy). Modification of the BLV model can also accommodate stochastic noise in the settlement process [4], which employs stochastic differential equations. This modified model has the advantage that it predicts a distribution of urban configurations, rather than a particular one, which is more convenient for model calibration and analysis of real data. There have been several attempts to connect this approach with real data [43–46]. It was shown that this model can, in general, demonstrate critical behavior depending on the parameters of the rent cost function and level of attractiveness of the retail objects size for consumers, which leads to the phase transitions between (1) retail system with large dispersed retail objects in peripheral locations and (2) small clustered objects located in central areas. Furthermore, it was demonstrated that the BLV model could be useful in optimization of the network structure of bank branches. Apart from that, there were attempts to place the BLV model into the discourse of conventional economic literature [40,47].

Originally, the BLV approach [3] was applied to the spatial distribution of retail activity. Here we use it to describe a resettlement process. The model consists of two components. The first one defines the traffic flows distribution for a given structure of economic activity, transportation costs, and attractiveness of suburbs. These traffic flows are determined by the maximum entropy argument which results in a Boltzmann distribution. The second component determines the dynamics of the attractiveness of the residential suburbs competing for the attention of the agents using the analogy with biological species competing for the same resource.

The main physical part of the BLV approach is the maximum entropy principle, which states that stationary probability distribution of a thermodynamic system maximizes its entropy under certain constraints. Following the above approaches we view a city with its population as a thermodynamic system, so that the distribution of human settlement satisfies a similar principle. In particular, we assume that the traffic flows ρ_{ij} maximize the "resettlement entropy,"

$$H[\{\rho_{ij}\}] = -\sum_{i=1}^{N}\sum_{j=1}^{N}\rho_{ij}\log\rho_{ij}, \tag{1}$$

subject to the set of constraints

$$\sum_{i=1}^{N}\rho_{ij} = n_j, \tag{2}$$

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\rho_{ij}\log W_i = \log W^{\text{tot}}, \tag{3}$$

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\rho_{ij}c_{ij} = c^{\text{tot}}, \tag{4}$$

where W^{tot} and c^{tot} can be interpreted as the average values of the attractiveness and transportation cost, respectively. Equation (2) is a normalization constraint which ensures that number of people working at suburb j is constant. The second and the third constraints are analogous to the energy constraint in the thermodynamic canonical ensemble.

The problem of maximizing the entropy H given the constraints (2)–(4) results in the Boltzmann distribution. Specifically, the number of people ρ_{ij} working at the zone j and living at the zone i is determined by the attractiveness of the destination W_j and transportation costs c_{ij} as follows:

$$\rho_{ij} = \frac{n_j}{Z_j} W_i^{\alpha} e^{-\beta c_{ij}}, \tag{5}$$

where $Z_j = \sum_{i=1}^{N} W_i^{\alpha} e^{-\beta c_{ij}}$ is the normalizing factor.

The parameters α and β are the Lagrange multipliers of the maximization procedure. In the context of urban configuration they are interpreted as the intensity of attractiveness and the intensity of the transportation costs respectively. The values of these parameters determine the actual urban configuration.

Equation (5) needs to be complemented by an evolution equation for the attractiveness W_i of a suburb. Typically, more populated suburbs have better chances to become more attractive. We adopt the original model [38], which states that the evolution of the attractiveness is governed by Lotka-Volterra-like equations:

$$\frac{dW_i}{dt} = \epsilon(\rho_i - KW_i)W_i, \tag{6}$$

where ϵ and K are the parameters of the evolution and $\rho_i = \sum_{j=1}^{N} \rho_{ij}$ is the total population of the suburb i.

B. Demographic balancing equation approach

An alternative approach is based on a form of demographic balancing equation (DBE) and describes the migration flow explicitly. It is analogous to the mass balance equation (or, in general, the continuity equation) in fluid mechanics [48] and expresses the idea that the corresponding quantity of the entire system (total mass in the case of flowing fluid and the total population in our case) is conserved. It has been adopted and widely used in physics and chemistry [49], in particular, in nonequilibrium thermodynamics [50] and chemical engineering [51].

From the physical perspective, the fluid flow is the result of the concentration inhomogeneity, which is due to the inhomogeneity in the chemical potential. The system tends to the equilibrium state with uniform chemical potential, which results in the net molecular flow from the regions with high chemical potential to the region with low chemical potential. A similar perspective can be adopted in the case of population redistribution. Interpreting people as molecules, we allow their "chaotic" movement, which can be identified with daily home-to-work travels. If these travels are costly, people will tend to redistribute their residential (or work) place to minimize their costs. The state of minimal costs corresponds to the equilibrium state with uniform chemical potential, and residential redistribution corresponds to the net people flow from the regions with high "demographic" potential to the regions with the low "demographic" potential. In this paper we use the notation of "attractivenes," which can be identified with the negative of this "demographic" potential.

The demographic balancing equation formulates the balance of the population for every region: change (increase or decrease) of the population in a region happens due to its migration to and from the other regions. In our approach we do not model births and deaths, so the total population of all regions together is conserved. In the context of population redistribution, this approach has also been used extensively [52]. In particular, Weildlich and Munz [26] introduced a consistent model of an isolated social system consisting of different groups of people engaged in different occupations. The attractiveness of each particular job is a function of population density, so the employees can change their occupation and migrate depending on the profitability. The complexity of the flow structure makes an analytical investigation of these models very difficult, and the research in this direction has been sparse [53]. Numerical simulations demonstrate that the model is able to reproduce such spatial patterns as agglomeration of industrial activities, clustering of industrial towns and attraction of agricultural activities to the boundaries of industrial towns [26]. A microlevel description used in this approach also demonstrates critical behavior and is quite conventional in population dynamics [54–56], as well as in agent-based simulations [57]. Similar techniques were used [58] to provide microlevel justification for the migration processes in the baseline model [59] from the new economic geography.

Our analysis extends the original approach based on the transition rates [26,60,61]. Unlike the original model, we distinguish between the locations of employment and residence. In particular, we assume that each individual chooses their residence randomly but the probability to relocate from a suburb k to a suburb i depends on i's relative attractiveness and its advantage in terms of travel-to-work cost. Following the physical analogy, we assume that resettlement described by the dynamics of population structure $\rho_{ij}(t)$ follows the deterministic transition rate equation:

$$\frac{d\rho_{ij}}{dt} = \sum_{k\neq i} [r_{k\to i,j}\rho_{kj} - r_{i\to k,j}\rho_{ij}]. \tag{7}$$

This means that the net change of transport flow $(d\rho_{ij}/dt)$ between the suburbs i and j is represented by the difference between the overall migration rate to $(r_{k\to i,j})$ and from $(r_{i\to k,j})$ the residential suburb i among people who work in j.

Unlike the BLV approach, in this model we assume that attractiveness of a suburb i depends explicitly on the population of this suburb $\rho_i = \sum_{j=1}^{N} \rho_{ij}$. It reflects the idea that people avoid both underpopulated (which usually means poor infrastructure) and overpopulated (which usually means overloaded infrastructure) areas [61]. Formally, this implies that there exists a certain population of a suburb at which its attractiveness reaches its maximum value. The particular shape of the function $W_i(\rho_i)$ is not important for this study, and we choose this function following the original approach [61] to be a Gaussian function, centered around a certain nonzero value ρ_0 of the population (which is the same for all suburbs):

$$W_i = \exp[-(\rho_i - \rho_0)^2]. \tag{8}$$

Furthermore, we assume that people working at the suburb j and living at the suburb k move to another suburb i to resettle with intensity $r_{k\to i,j}$ which depends on the difference in transportation costs and relative attractiveness of the suburb i over the suburb k:

$$r_{k\to i,j} = \mu\left(\frac{W_i}{W_k}\right)^{\alpha} e^{-\beta(c_{ij}-c_{kj})}. \tag{9}$$

The parameters α and β have the same meaning as in the BLV model, although here they do not come from a maximization procedure. If the transportation costs is zero, the transition rate in Eq. (9) has exactly the same functional form with respect to the population levels as the one in the original model [61].

Similarly to the BLV model, the concurrent evolution of $\rho_{ij}(t)$ and $W_i(t)$ is defined by two equations: a dynamic equation (7) and a static equation (8). The dynamic equation can also be formulated in terms of W_i:

$$\frac{dW_i}{dt} = \frac{\partial W_i}{\partial \rho_i}\sum_{j=1}^{N}\sum_{k\neq i}[r_{k\to i,j}\rho_{kj} - r_{i\to k,j}\rho_{ij}]. \tag{10}$$

However, unlike (6), in general case, the dynamics of W_i depends on population distribution ρ_{ij} but not on the total number of residents ρ_i (i.e., the occupation of residents has an impact on W_i which is not the case in the BLV model).

C. Common perspective

From the perspective of statistical mechanics each of these two approaches can be interpreted as a combination of "fast" and "slow" dynamics of the underlying human resettlement. However, the interpretation of which process is fast and which one is slow is not the same in these approaches. One of the purposes of this paper is to reveal the role of "fast" and "slow" processes in the statistical mechanical description of the dynamic evolution of human resettlement.

In particular, the use of the maximum entropy principle together with the Lotka-Volterra dynamics can be interpreted as a combination of "fast" and "slow" dynamics in the following way. Building transportation infrastructure is typically a slow process, which happens over years. The evolution of the infrastructure depends both on the existing infrastructure and the population distribution. This process is described by the Lotka-Volterra model and is responsible for the "slow" dynamics. In contrast, people choose the place of living based on the apparent attractiveness of the place and transportation infrastructure. Because of their mobility they can accommodate the changes in the attractiveness and transportation and relocate almost immediately. It can be said that at every moment in time the population distribution is such that it maximizes some gain function, following the maximum entropy principle. As this happens almost instantaneously, the maximum entropy principle can be said to be responsible for the "fast" dynamics.

The use of a demographic balancing equation brings different perspective on the "fast" and "slow" processes. In particular, human reaction to economic incentives is considered to be instantaneous, and the attractiveness of a place for living is determined by its population. It can be argued that the development of the infrastructure is governed by "fast" dynamics. In contrast, the actual migration of population is determined by the relative differences within the infrastructure and takes a longer time. It makes the process of resettlement analogous to a relaxation process, which is governed by "slow" dynamics.

Below we identify common elements of the mathematical framework, so that they could be analyzed under the same perspective. We next reveal that our approach leads to existence of different phases of urban configuration. In particular, depending on the combination of the attractiveness of a living place and costs of transportation between the living and the working place, the population density can be either low or high. In particular, a small change of the control parameters may lead to abrupt changes in the density configuration, which is an indication of a phase transition. This can be seen as a surprising result, since the underlying "fast" processes of each of the approaches do not imply phase transitions. In particular, the principle of maximum entropy in physics is formulated for homogeneous systems only. It suggests that it is the combination of "fast" and "slow" dynamics, which leads to existence of different phases. Further discussion on these aspects concludes the paper.

To illustrate our comparison we use the stability diagrams for a typical suburb of an infinitely large city presented in Fig. 1 in the original paper [38]. Here we use them as an illustration only and consider the most interesting case with multiple equilibria only, but the same analysis could be performed for the other cases as well. Our comparison is conducted for the case of zero transportation costs only and is depicted in Fig. 1. The solid curve represents the state of the suburb, which is described by a relation between W and ρ for each suburb [Eq. (5) in the BLV model, Eq. (8) in the DBE model]. Curve (5) has a vertical asymptote $\rho_i = 1$ [38]. The dashed curve represents the stationary condition, which is described by

$$dW_i/dt = 0 \tag{11}$$

in the BLV (see the original paper [38] for explanation of the shape) and $d\rho_i/dt = 0$ in the DBE [which is derived in the following paragraph; see Eq. (14)].

Assuming that $\beta = 0$, i.e., the occupation of residents does not matter, Eq. (7) is reduced to

$$\frac{d\rho_i}{dt} = \mu\sum_{k\neq i}\left[\rho_k\left(\frac{W_i}{W_k}\right)^{\alpha} - \rho_i\left(\frac{W_k}{W_i}\right)^{\alpha}\right]. \tag{12}$$

Assuming, in addition, that the number of suburbs is very large and the impact of each particular suburb is negligible, Eq. (12) can be written as

$$\frac{d\rho_i}{dt} = \mu\left[\rho_\infty\left(\frac{W_i}{W_\infty}\right)^{\alpha} - \rho_i\left(\frac{W_\infty}{W_i}\right)^{\alpha}\right], \tag{13}$$

where W_∞ and ρ_∞ are in general functionals of the entire distribution $\{\rho_k\}$, which may be considered constant for sufficiently large systems. It yields that the steady-state solution satisfies set of equations $\rho_\infty(W_i/W_\infty)^\alpha = \rho_i(W_\infty/W_i)^\alpha$. Since ρ_∞ and W_∞ are the same for each i, it is straightforward to obtain an equation describing the set of possible steady states with respect to both ρ_i and W_i:

$$\rho_i = \frac{1}{Z}W_i^{2\alpha}, \tag{14}$$

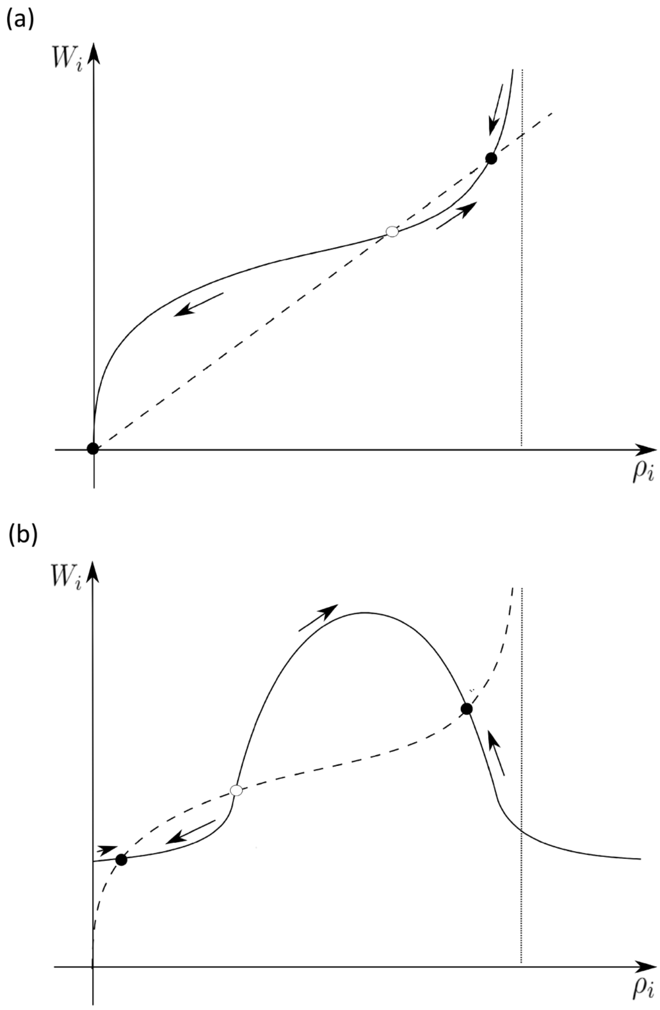

FIG. 1. Stability diagram for (a) the BLV [38] and (b) the DBE models with zero transportation costs. The solid lines are defined by Eqs. (5) and (8), respectively. The dashed lines are defined by the steady-state condition $dW_i/dt = 0$ in the BLV model and $d\rho_i/dt = 0$ in the DBE model. The dotted lines are vertical asymptotes $\rho_i = 1$.

where $Z = \sum_{k=0}^{N} W_k^{2\alpha} = W_\infty^{2\alpha}/\rho_\infty$ is the normalizing factor. Hence, the curve $d\rho_i/dt = 0$ has the same shape as the solid curve (5) in the BLV model whose shape is explained in the original paper [38].

As the suburb evolves, the point, which represents its state, moves along the state curve. The direction of this motion is shown by the arrows and is determined by the sign of the right-hand side of the equation representing the "slow" dynamics [Eq. (6) for BLV and Eq. (7) for DBE]. In particular, for the BLV system, if the state curve is above the stationary curve, then the suburb moves down, and otherwise moves up. Similarly, for the DBE system, if the state curve is above the stationary curve, then the suburb moves to the right, and otherwise to the left.

The points where the state curve crosses the stationary curve are the equilibrium points. Depending on relative position of the curves each model may have either one or two stable equilibria. In the BLV model the amount of equilibria depends on α and K [38]: if $\alpha < 1$ there are two steady states (only one of them is stable); if $\alpha > 1$ and K is below the certain threshold, there are three equilibria (two of which are stable and one is unstable); and if $\alpha = 1$ and K is below the certain threshold there are two steady states (only one of them is stable). There might be a degenerate situation with a single trivial steady state $\rho_i = 0$, $W_i = 0$ if the curves do not intersect: in this case this analysis is not applicable as it means that there exists a particular suburb i whose influence on others must not be neglected, which contradicts the main assumption of this analysis. In the DBE model the amount of equilibria depends on ρ_0: if ρ_0 is moderate, there are three equilibria (two of them are stable). Otherwise, for sufficiently small or large ρ_0, there is only one stable equilibrium. The exact range of conditions depends, in turn, on the value of α. Here we consider the most interesting case with multiple equilibria (Fig. 1) although our comparison remains legitimate for the single equilibrium case.

The case of three steady states is depicted in Fig. 1 for both models. Two of those three states are stable (full circles) and one (empty circle) is unstable. Despite this similarity, there are two significant differences between the qualitative behavior of BLV and DBE systems. First, nonmonotonicity of attractiveness curve (8) in the DBE model allows decrease of the population when the attractiveness of the suburb increases, which is not the case in the BLV model. This happens because the steady-state condition for the BLV model, $dW_i/dt = 0$, could be reduced to a linear equation with respect to ρ_i and W_i and, hence, Eq. (6) does not accommodate saturation effect captured by the curve (8) in the DBE model. Second, in the BLV model the first stable steady state is $\rho_i = 0$ and $W_i = 0$ regardless of the value of α and other model parameters, which is not the case in the DBE model. This is the case for any number of suburbs, in particular, when there are only two suburbs as will be evident in Sec. III A.

The differences in qualitative behavior of the BLV and DBE models suggest that these two approaches are not reducible to each other. Despite these differences, the above analysis suggests that it is still possible to see a common perspective in the BLV and DBE models. In particular, both models comprise the static, or "fast" [Eq. (5) in the BLV model and (8) in the DBE model], as well as dynamic, or "slow" [Eq. (6) in the BLV model and (7) in the DBE model] components. In particular, we see that both the "fast" dynamics in the BLV approach and the "slow" dynamics in the DBE approach have a thermodynamic meaning. As we will see in the following section, the results produced by the two approaches are nevertheless comparable. This suggests that both approaches are parts of a larger framework, which considers thermodynamics of the urban evolution at different levels.

III. RESULTS

Here we present the numerical solutions of the models for different values of the control parameters. These are the stationary values of the population densities, which solve the corresponding system of differential equations when $t \to \infty$. The system to solve is the set of equations for each suburb: Eq. (6) together with Eq. (5) for the BLV model, and Eq. (7) together with Eq. (8) for the DBE model. To analyze the typical behavior we start with the number of suburbs $N = 2$, extending further the analysis to $N = 3$.

As we will see below, even in this simplest case of a heterogeneous city, we observe several phases of urban structure. For the city of two suburbs we have two densities ρ_1 and ρ_2, which satisfy the normalization condition $\rho_1 + \rho_2 = 1$, so it is sufficient to trace only one independent density $\rho \equiv \rho_1$.

In all simulations (except in Sec. III C) the working configuration was fixed as $n_1 = 0.6$, $n_2 = 0.4$, so that the first suburb has 60% of all work places and the second suburb has 40% of all work places. For the BLV model we used $K = 1$ and $\epsilon = 1$, while for the DBE model we used $\rho_0 = 1$ and $\mu = 1$.

A. Typical urban configurations

We illustrate typical dependencies of the stationary population ρ_i on the attractiveness intensity α in Figs. 2 and 3 for the BLV and DBE models respectively. The three subfigures on each of the figure differ in the values of transportation cost intensity β and the initial living configuration with respect to working configuration. In particular, despite the working configuration being fixed, the relative living configuration determines the effect of transportation costs.

A qualitative behavior of the solution can be illustrated in the case of zero transportation costs [case (a) in both figures], when we look at the variation of the stationary attractiveness or density with respect to the variation of the attractiveness intensity α, provided the transportation cost intensity $\beta = 0$. If α is less than some critical value α^*, the solution is symmetric, $\rho_1 = \rho_2 = 1/2$, so the density distribution is homogeneous. For large α, the symmetry breaks, and the stationary population density ρ_1 depends on the value of α. In the BLV model the critical level of $\alpha^* = 1$ [39]. In the DBE model the critical value α^* depends on the saturation parameter ρ_0 measured in the units of the total population as

$$\alpha^* = \frac{1}{2\rho_0 - 1}. \quad (15)$$

In other words, for low values of α, the neighborhoods are moving towards the same level of population and attractiveness. But if α is large, the symmetry between suburbs breaks and the initially large suburb becomes larger while initially small suburb becomes smaller. It is worth noting that BLV model suggests that the smaller suburb becomes completely depopulated, while according to the the DBE model the smaller suburb retains nonzero population. This behavior reflects the qualitative analysis in Sec. II C. There we also saw that one of the stable solutions of the BLV model implies zero population of one of the suburbs, while the corresponding stable solution of the DBE model is nonzero and depends on the values of the parameters. Because of this when $\alpha = \alpha^*$ the stationary population experiences a jump in the BLV model, while it remains continuous in the DBE model.

In the case where transportation costs do matter [cases (b) and (c) in each figure] the quantitative pattern becomes more complicated. First, we note that no homogeneous distribution of the population is possible: one suburb will always be larger than the other one, regardless of the attractiveness intensity α. This is a consequence of the "interactions" between the suburbs by means of nonzero transportation cost. Second, we observe two different modes of the stationary density distribution. In case (b), the distribution of the initial attractiveness (or the population) of the residential suburbs correlates with

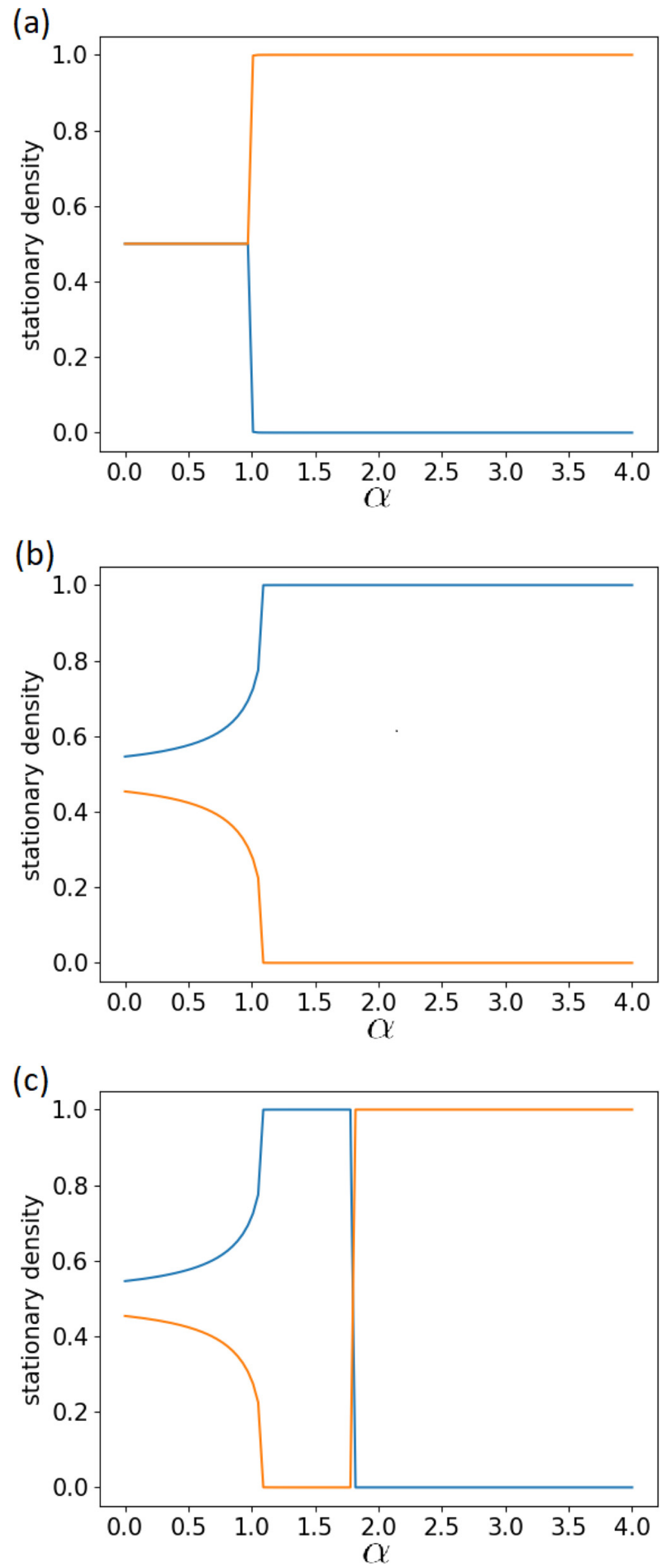

FIG. 2. Density of both suburbs as a function of the attractiveness intensity, as produced by the the BLV model for different initial configurations: (a) $W(0) = (0.6, 0.4)$, $\beta = 0$; (b)$W(0) = (0.6, 0.4)$, $\beta = 1$; (c) $W(0) = (0.4, 0.6)$, $\beta = 1$.

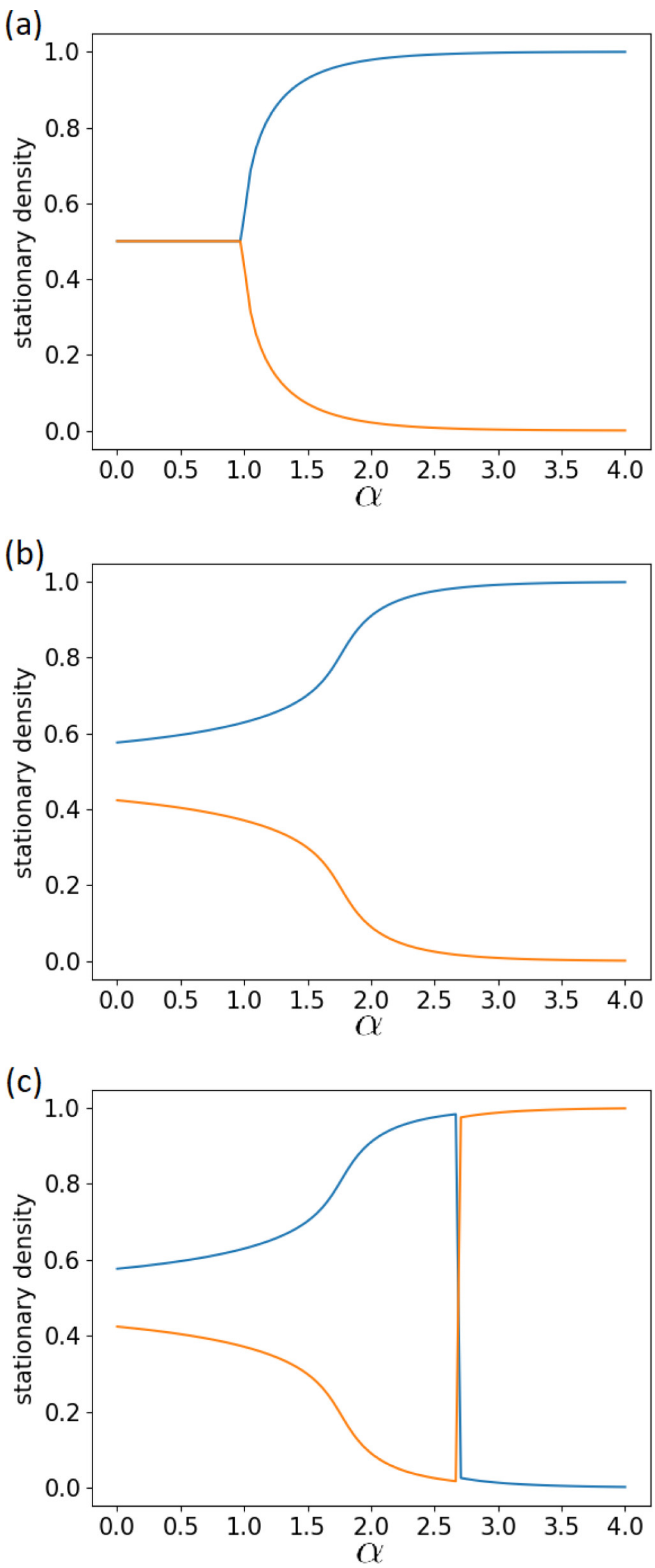

FIG. 3. Density of both suburbs as a function of the attractiveness intensity, as produced by the the DBE model for different initial configurations: (a) $\rho(0) = (0.4, 0.6)$, $\beta = 0$; (b) $\rho(0) = (0.6, 0.4)$, $\beta = 1.0$; (c) $\rho(0) = (0.4, 0.6)$, $\beta = 1.0$.

the distribution of the work place. In contrast, in case (c), the distribution of the initial attractiveness (or the population) of the residential suburbs anticorrelates with the distribution of the work place. If the distributions correlate, then the urban configuration reminds the one observed in case (a): the initially large suburb becomes larger while the initially small suburb becomes smaller. The value of α determines how far the populations of the suburbs diverge: for $\alpha < \alpha^*$ the divergence is small, while for $\alpha > \alpha^*$ it is large. Similarly to case (a), the behavior of the solution is different at $\alpha = \alpha^*$: $d\rho_i/d\alpha$ experience jumps at his point in the BLV model, while they are continuous in the DBE model. Similarly to case (a), the reason for this is the fact that one of the stable solutions in the BLV model is exactly zero when $\alpha \geqslant \alpha^*$.

The most interesting behavior happens in case (c). Here we observe the second threshold value α^{**}, which determines which particular suburb will get most of the population. For $\alpha < \alpha^{**}$ the largest residential suburb is the one with more jobs, while for $\alpha > \alpha^{**}$ the largest residential suburb is the one which was larger initially. This illustrates the competition between the residential attractiveness and the cost of transportation from living to working place.

B. "Attractiveness-transportation" phase diagram

We next investigate the urban configuration when both the attractiveness intensity and the transportation cost intensity vary. The results are presented in Figs. 4–7 in the form of a phase diagram, where the color indicates the density of the first suburb. The cases studied in the previous section correspond to the vertical lines on the diagrams.

The diagrams are built for the initial configurations, which anticorrelated with the fixed work configuration, i.e., for the BLV model $W(0) = (0.4, 0.6)$ and for the DBE model $\rho(0) = (0.4, 0.6)$. In this case we can distinguish three different phases with respect to density configuration.

For the BLV model these phases are (1) nearly homogeneous distribution, (2) monocentric residence with center in suburb 1, and (3) monocentric residence with center in suburb 2. The transitions between these phases are sharp and correspond to the threshold values α^* and α^{**}. We point out that while $\alpha^* \approx 1$ and is independent of β, the value of α^{**} varies with β. This is expected because the transition at α^{**} is a result of the competition between attractiveness and transportation cost intensity, with the latter being controlled by β.

For the DBE model the three phases are (1) nearly homogeneous distribution, (2) intermediate distribution, and (3) monocentric residence with center in suburb 2. We observe that the transition between phase 1 and phase 2 in this case is smooth, while the transition between phase 2 and phase 3 is still sharp.

If the initial configurations correlate with the fixed work configuration, we observe only two phases (not shown here), which correspond to phases 1 and 2 in the case of anticorrelated initial configuration.

C. Correlation between residential and working configuration

Here we investigate how the residential activity varies with respect the employment activity, varying the proportion of work places in the suburbs. As we have only two suburbs, this variation is controlled by a single parameter n_1, referring

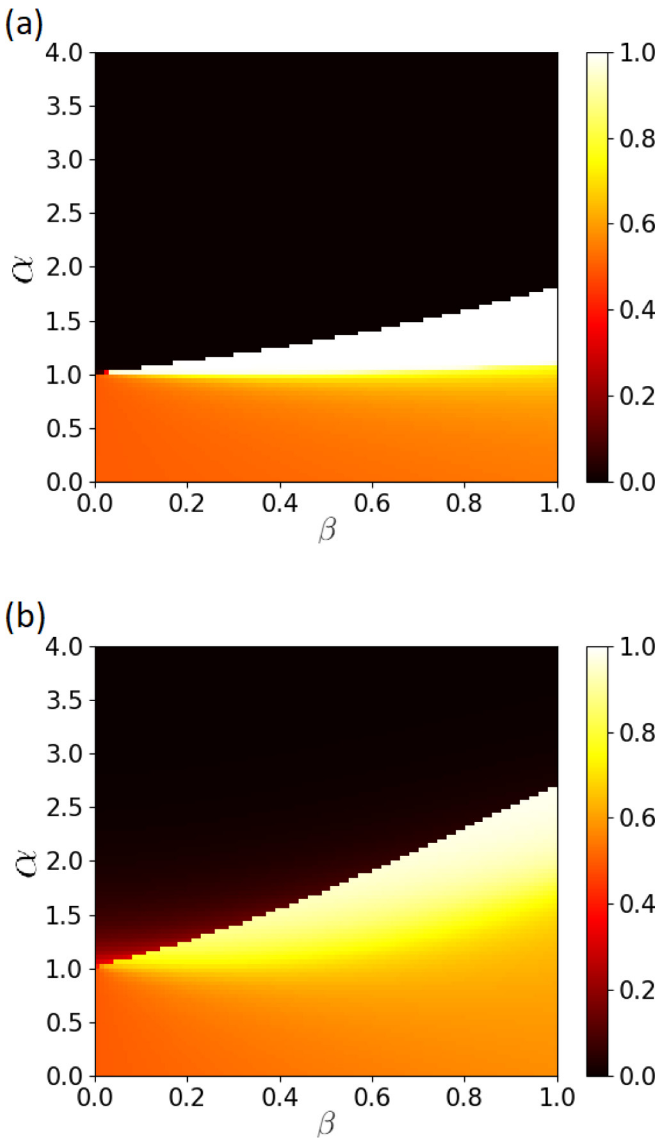

FIG. 4. Density of the first suburb as a function of the attractiveness intensity and the transportation cost intensity, as produced by (a) the BLV model and (b) the DBE model.

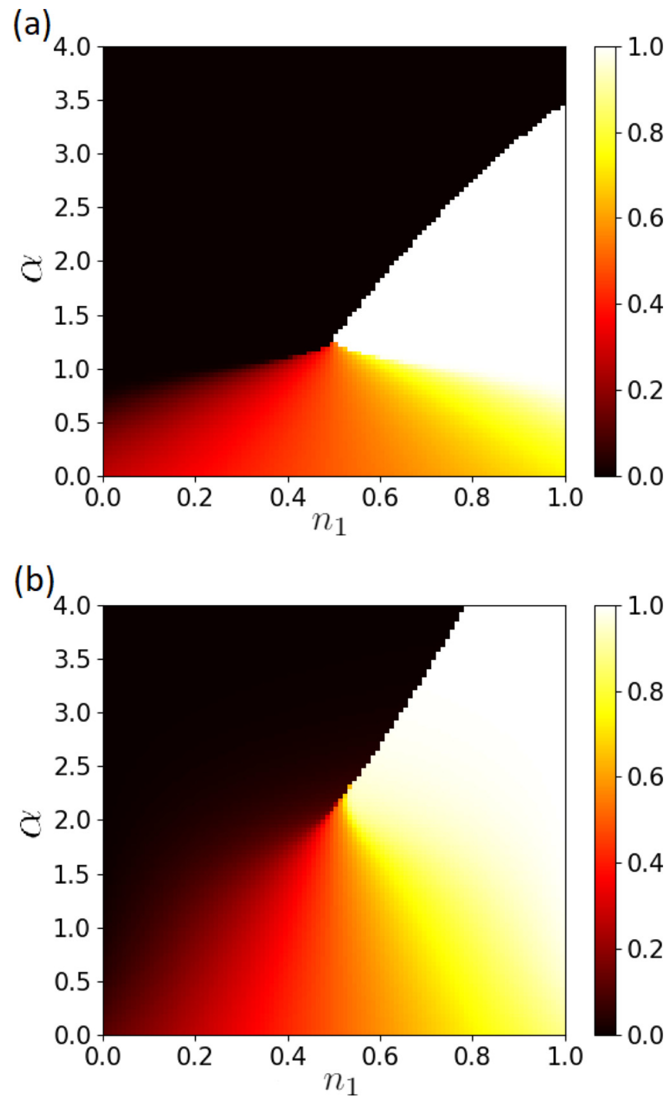

FIG. 5. Stationary population of suburb 1 in (a) BLV model and (b) DBE model as a function attractiveness intensity and the fraction of population working in suburb 1; $\beta = 1$.

to the fraction of work places in suburb 1. In both models the critical transition happens only if the initial residential distribution anticorrelate with the fixed distribution of employment activity. This means that if $n_1 > n_2$ then $\rho_1(0) < \rho_2(0)$ and vice versa, if if $n_1 < n_2$ then $\rho_1(0) > \rho_2(0)$. The phase diagrams for both BLV and DBE models are shown in Fig. 5, and both reveal a critical point. Furthermore, both diagrams feature a phase transition between the phases of monocentric residence in suburb 1 and suburb 2 (either black or white areas) for high values of attractiveness intensity. The discontinuous jump between tho phases is present only if $n_1 > n_2$ [and correspondingly $\rho_1(0) < \rho_2(0)$]. The point $n_1 = n_2$ is the critical point.

Another perspective on the phase transitions is shown in Fig. 6. First, a discontinuous transition happens only if the initial residential attractiveness distribution does not correlate with the distribution of the business activity. Second, the threshold value α^{**} goes to infinity as the initial residential attractiveness is homogeneously distributed. Third, the asymptotic behavior at the vicinity of zero is different and if in the BLV model the initial attractiveness of the neighborhood is small, the population tends to zero regardless of the level of α which is not the case in the DBE model.

D. "Attractiveness-total population" phase diagram

Another important distinction between the models is shown in Fig. 7. Since DBE model takes into account saturation effect in the relation between ρ_i and W_i, it is fairly intuitive that the DBE solution demonstrates certain symmetry around the saturation point, which is not the case in the BLV model. This is basically the main difference in qualitative results of the considered model. However, it is not a feature of the modeling framework but is a consequence of the difference in the domain-specific assumptions.

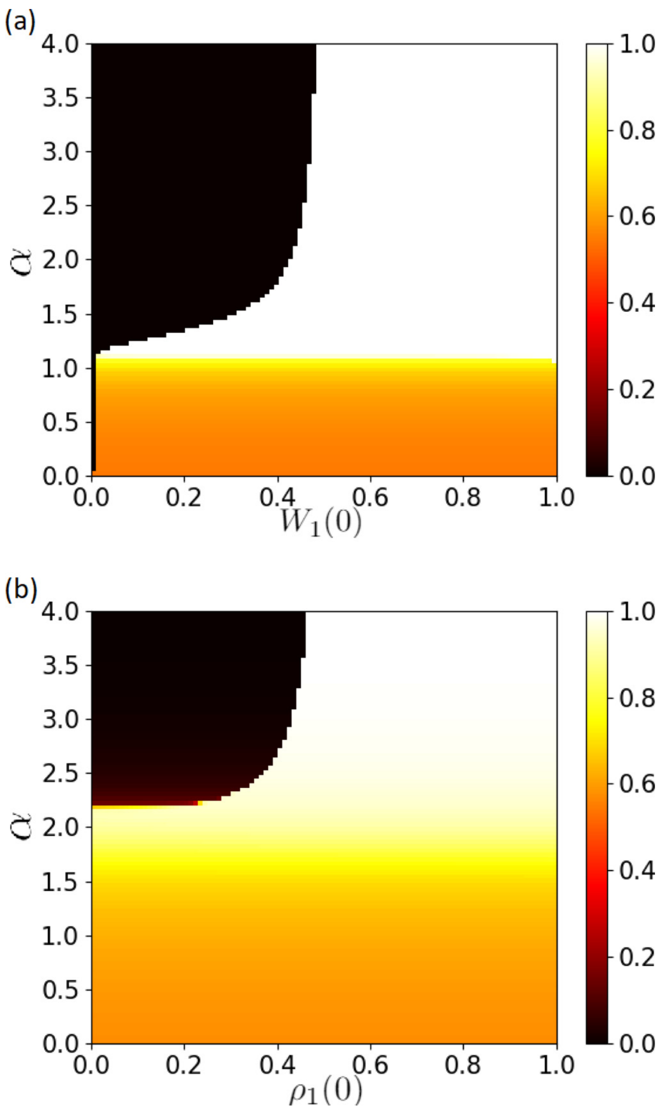

FIG. 6. Stationary population of of suburb 1 in (a) BLV model and (b) DBE model as a function of attractiveness intensity and the initial residential configuration.

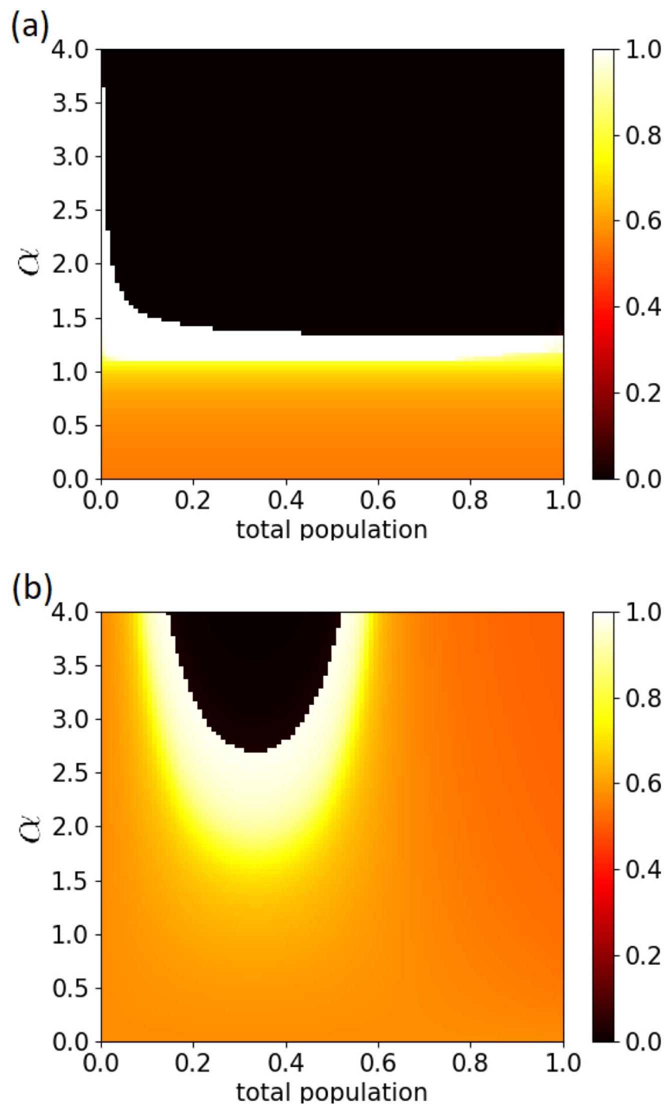

FIG. 7. Stationary population of suburb 1 in (a) BLV model and (b) DBE model as a function of attractiveness intensity and the total population.

E. Three suburbs

Although the outcomes of both models are qualitatively very similar in two-dimensional setup, it is not the case anymore if the number of suburbs is larger than two. In particular, in the case of three suburbs the DBE model may have multiple critical transitions (depending on the initial conditions) whereas the BLV model does not. An example is given in Fig. 8 where the heat maps depict stationary population of the first neighborhood in the DBE and BLV models with respect to changes in α and β.

IV. DISCUSSION

In general, the significance of the presented results is threefold. First, we have developed a common perspective on two different models formalizing an evolution of urban structure. Although both models are based on thermodynamic arguments, they have not been shown to be mutually interchangeable, focusing on different aspects of the urban evolution. We demonstrated an analogy between the two models and identified the thermodynamic roles of their components. Second, we highlighted the importance of the dynamics in the urban structure. In particular, we discussed the role of the slow and fast processes which contribute to the evolution of the residential structure. Finally, we have shown that the models developed under the common thermodynamic perspective may produce abrupt changes in the residential structure, i.e., phase transitions. Specifically, small changes in the intensity of either residential attractiveness or transportation cost may result in a large swing of the actual residence. Both these intensities represent the city as a whole rather than at the level of specific suburbs.

The two considered approaches produced similar results in terms of the temporal evolution of the urban structure as well as its phase diagram. In particular, the numerical simulations demonstrated that both models exhibit similar

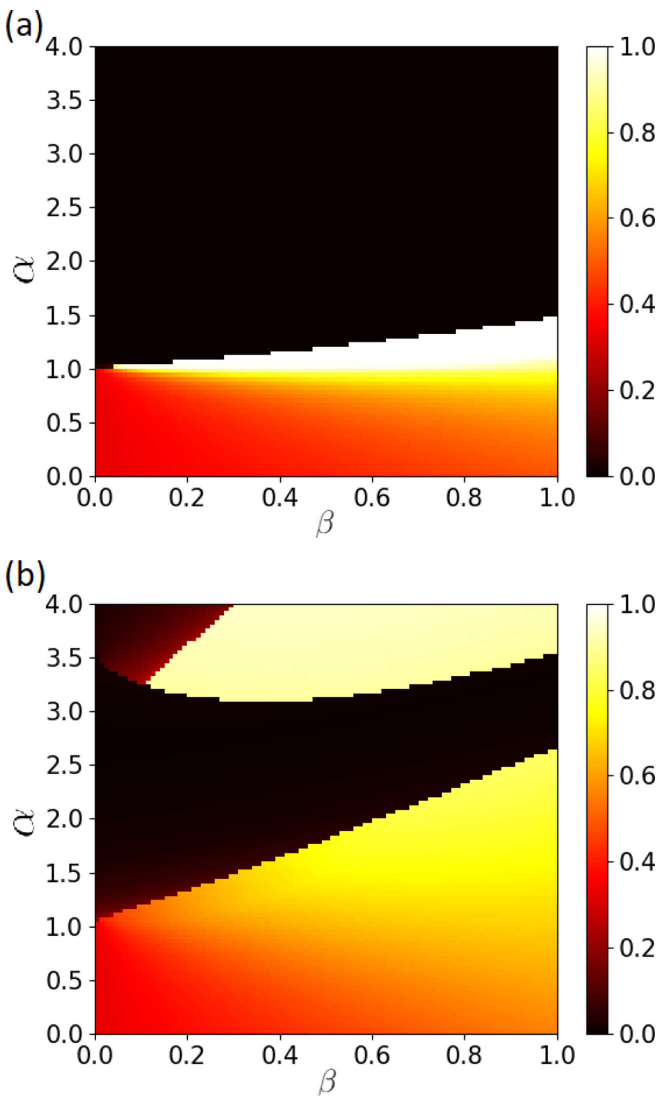

FIG. 8. Stationary population of the first neighborhood in (a) BLV model and (b) DBE model for different levels of α and β: (a) $n = (0.667, 0.167, 0.167)$, $W(0) = (0.2, 0.6, 1.0)$, $K = 1$, $\epsilon = 1$; (b) $n = (0.667, 0.167, 0.167)$, $\rho_1(0) = 0.2$, $\rho_2(0) = 0.6$, $\rho_3(0) = 1.0$, $\rho_{ij}(0) = n_j \rho_i(0)$. Distance matrix is formed as follows: $c_{ii} = 0$, $cij = 1$ if $i \neq j$.

phase diagrams, despite being not reducible to each other and the reliance on different domain assumptions. This suggests that the evolution of a real city may be governed by fundamental sociophysical principles which are model-independent. A proper validation of this hypothesis is the subject of further research.

Both models contain two different concepts of equilibrium, a thermodynamic one and an economic one, and both equilibria are reached if system is in its steady state. In other words, if the system is in its steady state, it does not matter whether the thermodynamic evolution is slow and the economic one is fast or vice versa. It then can be argued that thermodynamics is essential for both models.

Another important observation is that the discontinuous transition in both models is generated in presence of a dynamic component (or "slow" dynamics). In both models the fast dynamics generate a manifold matching attractiveness, transportation costs, and population, while the slow dynamics generate a movement along this manifold. Crucially, the direction of motion (as well as the point of convergence) depends on the combination of suburb attractiveness and transportation cost between the suburbs. In particular, when the attractiveness of a suburb is large, the preferential suburb or residence depends mainly on the attractiveness. In contrast, when the transportation cost is high, people tend to reside close to work, minimizing the transportation cost, as expected. An interesting dependence is also observed with respect to the distribution of the initial population, since the population depends on the attractiveness of the suburb. In particular, if the initial population distribution correlates with the employment distributions, then depending on the transportation costs intensity the stationary population distribution may correlate or anticorrelate with the employment.

It is worth noting that similar phenomena were described in Ref. [35] in the context of polycentric transition. In our case, we could interpret a configuration as polycentric if the residents are evenly distributed among the suburbs, and monocentric if only one neighborhood is populated. Such a transition appears in both of the considered models as well, but characteristics of those transitions do not coincide: in Ref. [35] the number of activity subcenters increases as the total population goes up, which is not the case either in the BLV model or in the DBE model (see Fig. 7). In the BLV model monocentricity or policentricity is independent of the total population, while in the DBE model, if α is sufficiently large, the configuration is duocentric for small and large values of the total population, and monocentric if the total population is at the vicinity of saturation point. However, it is natural to hypothesize that this contradiction is not caused by the difference in the modeling framework but by the difference in the assumptions made about attractiveness and transportation costs. While the model in Ref. [35] reflects the saturation with respect to a particular traffic flow, the DBE model reflects the saturation with respect to the population in a particular suburb, and in the BLV model both phenomena are neglected. Resolving this hypothesis remains a subject of future research.

The presented results may have applied significance. Despite the fact that a typical city has a rich structure, consisting of multiple suburbs and complex transportation network, in many cases it can be viewed as an agglomeration of two large regions, which are separated due to geographical or historical reasons. The examples of such pairs include central business district (or downtown) and all residential suburbs (Sydney, Melbourne, Los Angeles, Oslo), left and right banks of a city divided by a river (Sydney, Budapest, Paris, London, Seoul), and island and mainland (Hong Kong, New York, San Francisco). The numerical simulation performed in this paper shows that abrupt changes in the urban population structure, given small changes in preferences of the residents of such cities, may happen even in a simple two-region layout (for some important examples of similar vulnerabilities in other real ecological, technological, and socioeconomic systems, see Refs. [62–65]). Furthermore, our modeling suggests a clear insight into how to make such a transition smooth, while avoiding negative consequences of the abrupt transition such as an overloaded transport system, abandoned objects of infrastructure, housing crises. For example, a possible response to a critical regime could be to make the areas with low business activity more attractive for doing business. This insight is a straightforward consequence of Fig. 5: if urban planners expect a decline in residential attractiveness intensity (parameter α in the model), the best response is to change the employment structure by increasing the number of jobs in one suburb (parameter n_2 in the model) and decreasing them in the other suburb (parameter n_1 in the model) using available incentivizing mechanisms. Such decisions could help to avoid depopulation of the residential neighborhoods and preserve local infrastructure.

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