Anomalous diffusion behavior in parliamentary presence

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Concepts of statistical mechanics as well as other typical tools of physics have been largely used in the analysis of several aspects of social systems, for instance, in politics. In this work, we examine parliamentary presence utilizing data from the sessions of the 49th–54th Brazilian Chambers of Deputies (24 years, 1991–2015). For each federal deputy, we construct a random walk by considering their presence in a session as a step of unitary length and their absence as one of zero length. By using this approach, we put in evidence a quantitative description of the dynamics of the system. More specifically, we identify an anomalous diffusive process that corresponds to a robust superdiffusion, well identified with a ballistic regime. In addition, for each legislature and encompassing all its sessions, the system is modeled by a beta probability distribution, where the parliamentary presence scales with the number of sessions.

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I. INTRODUCTION

Ordered structures in several levels are common in natural and social systems. Considering the latter, quantitative aspects of such structures have been investigated via typical tools of physics. Notably, a growing number of studies that deal with data have been facilitated by the increasing ease of accessing databases. Politics is an example of a social context that contains large databases and manifests a hierarchical complex structure, as parties, elections, and executions of activities by those elected. For instance, allometric (power-law) behaviors have been used to analyze party affiliations [1], candidature for council and mayor [2], and financial support to political campaign [3].

Focusing on parliamentary elections in several countries, a series of advances related to distribution of votes have been punctuated, including power laws, generalized Zipf's law, and log-normal distributions [4–10]. In this context, models such as Sznajd [11–15] and entropic and correlation aspects [16–19] were also studied.

Vote distributions were also investigated concerning polls [20–22] and presidential [23–25] and mayoral [26] elections. Other aspects of plurality voting were conducted using turnout rate statistics [27]. Referenda and government approval ratings have also been analyzed [28–30]. Moreover, considering political scandals, empirical investigations were made to identify fraud in elections [31–33] and networks of corruption [34].

Concerning the executions of activities in politics, several analysis can be conducted. In this work, we are going to consider one basic feature of a democratic state: the presence in parliamentary sessions, where laws and projects are discussed and voted on. Here, we employ a random walk approach to investigate empirical diffusive-like aspects of the parliamentary presence in the sessions of the Brazilian Chamber of Deputies.

The organization of the article is as follows: Section II gives details about the data on parliamentary presence, Sec. III explains the random walk constructed from the data, Sec. IV shows the results of our analysis, and, finally, our conclusions are given in Sec. V.

II. DATABASE

The National Congress of Brazil is bicameral, composed of the Chamber of Deputies (513 seats) and the Federal Senate (81 seats). Federal deputies are elected to four-year terms, coinciding with a legislature, whereas senators are elected every eight years. In order to investigate diffusive aspects of the parliamentary presence, we restrict our attention to the Brazilian Chamber of Deputies. We focus on six legislatures (49th–54th), corresponding to 24 years (1991–2015). The data were freely obtained from the official site of the Brazilian Chamber of Deputies [35]. In this investigation, for each parliamentary session, the presence or absence of each federal deputy is computed. Figure 1(a) illustrates the number of deputies that were present in each one of the 659 sessions of the 53rd legislature. Instead of employing the dates to refer to the sessions along our analysis, we use the ordering they occurred over time to enumerate them.

It is common that several federal deputies discontinue their participation in a legislature. Reasons for this include leaving to take up executive positions, cessation, and resignation of the mandate. In such cases, federal deputies are replaced to maintain all the 513 seats occupied. Thus, the number of federal deputies exceeds 513. The 49th–54th legislatures had respectively 396, 470, 426, 585, 659, and 707 sessions and 595, 725, 720, 616, 608, and 734 distinct federal deputies.

Figure 1(b) illustrates the accumulated presence of some federal deputies in the sessions of the 53rd legislature. Each

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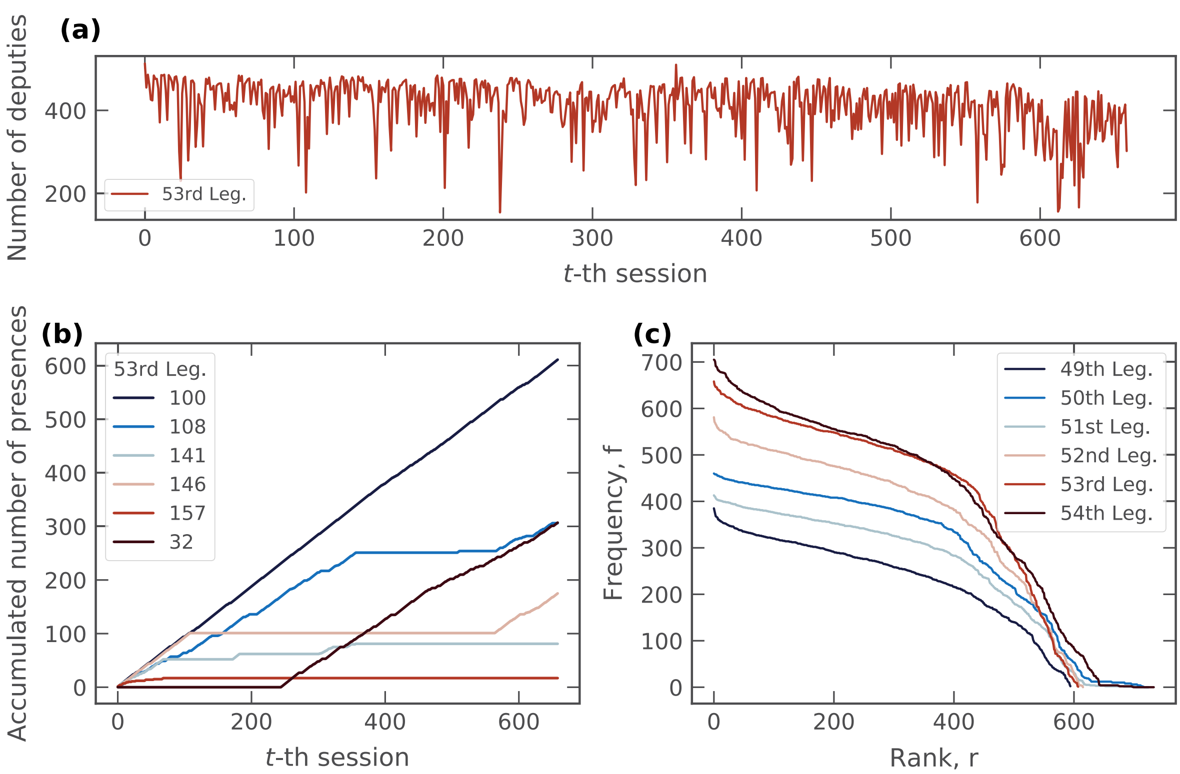

FIG. 1. Data aspects. Panel (a) shows the number of federal deputies of the 53rd legislature present in each session (t). Panel (b) illustrates the accumulated number of attendances of some federal deputies, which are represented by numeric labels. Panel (c) displays the rank distribution according to the frequency of the federal deputies present for the 49th–54th legislatures.

federal deputy is represented by a numeric label. We note from Fig. 1(b) that the federal deputy 100 was present in almost all sessions whereas deputies 32, 108, 141, 146, and 157 missed a significant number of consecutive sessions. Cases like these in which federal deputies do not participate in many sessions are common to the 49th–54th legislatures. Figure 1(c) shows the rank of the federal deputies of the 49th–54th legislatures according to their presence in the sessions. The fact that some federal deputies missed a significant number of consecutive sessions is manifested in Fig. 1(c) by the presence of two regimes: one approximately before 400 and another after this value.

In order to study the presence of federal deputies that regularly attended the sessions of a legislature, we will classify them according to their maximum fraction of missed sessions in a row. In this direction, the federal deputy 100 in Fig. 1(b) has a low maximum fraction of missed sessions in a row and, consequently, can be considered as a regular member of the 53rd legislature, after establishing a reasonable threshold. In contrast, for instance, the federal deputy 146 could not be considered as a regular member of the 53rd legislature because he (she) was absent more than 400 sessions in a row.

We will essentially consider a threshold for the maximum fraction of missed sessions in a row at 1/16, which corresponds approximately to three consecutive months of absence. Thus, only the regular federal deputies that were absent for continuous periods during at most three months are considered in our study. For instance, from the 608 federal deputies of the 53rd legislature, only 426 will be considered as regular members of this legislature, which also corresponds approximately to the position of the knee in Fig. 1(c). We remark that the choice of the threshold at values close to 1/16 produces similar results.

III. RANDOM WALK

In our empirical investigation, we employ $x_t^{(j)}$ as the number of parliamentary sessions the jth federal deputy attended in the first t sessions of a legislature. Thus, $x_t^{(j)} = \eta_0^{(j)} + \eta_1^{(j)} + \cdots + \eta_{t-1}^{(j)}$, with $\eta_{i-1}^{(j)}$ being equal to 1 (0) if the jth federal deputy was present (absent) in the ith session. In this way, one can also write

$$x_{t+1}^{(j)} = x_t^{(j)} + \eta_t^{(j)}, \quad (1)$$

with

$$\eta_t^{(j)} = \begin{cases} 1 & \text{if present} \\ 0 & \text{if absent} \end{cases} \quad (2)$$

and the initial condition $x_0^{(j)} = 0$. Note that Eqs. (1) and (2) can be formally viewed as a random walk of a particle, where $x_t^{(j)}$ and $\eta_t^{(j)}$ represent, respectively, its position and random step at time t [36].

An advantage of using a random walk approach is that it has proven to be very helpful to reveal several dynamic aspects in the most diverse contexts, for instance, in sports [37–39], in macroeconomics [40], to distinguish between

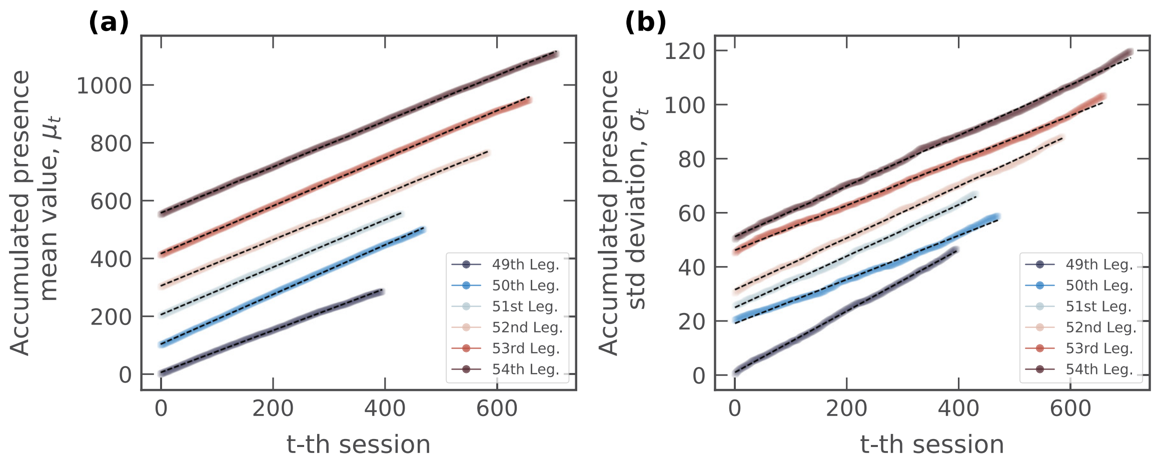

FIG. 2. Mean value and standard deviation. The dynamics of the accumulated mean presence of the federal deputies (μ_t) for the 49th–54th legislatures is presented in panel (a). Panel (b) exhibits the dynamics of the corresponding standard deviations (σ_t). The curves in both panels were vertically shifted for better visualization. The black dashed lines represent the fits. Both panels reveal a linear dependency of μ_t and σ_t on t for all the legislatures. The maximum fraction of missed sessions in a row considered is equal to 1/16.

autism disorders [41], and in applications to overstretched DNA [42].

IV. RESULTS

A. Average parliamentary presence

The average number of presences of the federal deputies until the tth session of a legislature is given by

$$\mu_t = \frac{1}{N}\sum_{j=1}^{N} x_t^{(j)}, \tag{3}$$

where N is the number of federal deputies considered in the legislature, which are essentially the ones that have a maximum fraction of missed sessions in a row not greater than 1/16 (see the end of Sec. II).

Figure 2(a) displays the evolution of μ_t for the 49th–54th legislatures. This figure reveals that μ_t linearly depends on t, i.e.,

$$\mu_t = \bar{\mu}_1 t. \tag{4}$$

This result directs to a rate $\bar{\mu}_1$ of parliamentary presence that does not depend on t. In general, the goodness of the fits, measured by R^2, is substantial for the 49th–54th legislatures ($R^2 > 0.999$). We stress that these results are robust, since small variations of the considered number N of federal deputies of a legislature yield the same results. However, $\bar{\mu}_1$ decreases as we consider more federal deputies as regular ones. This occurs because the number of deputies with less attendence increases with the value of the threshold for the maximum fraction of missed sessions in a row.

B. Standard deviation of the parliamentary presence

The following step in our investigation of the parliamentary presence is to analyze the fluctuation around the mean value μ_t. For this, we will employ the standard deviation

$$\sigma_t = \left[\frac{1}{N-1}\sum_{j=1}^{N}\left(x_t^{(j)} - \mu_t\right)^2\right]^{1/2}. \tag{5}$$

A particular but important case occurs when the steps $\eta_t^{(j)}$ are uncorrelated, leading to

$$\sigma_t = \bar{\sigma}_1 t^{\alpha}, \tag{6}$$

with $\alpha = 1/2$ and $\bar{\sigma}_1$ constant. A common practice in the study of random walks is to classify them according to the dependence of σ_t on t [43]. If σ_t obeys Eq. (6) with $\alpha = 1/2$, there is a normal behavior as in the usual Brownian motion. In contrast, if σ_t is not proportional to $t^{1/2}$, we are dealing with an anomalous behavior and the correlations among the $\eta_t^{(j)}$ play a significant role. Also, in a collection of objects (in our case an ensemble of federal deputies), it is common to use superdiffusive (subdiffusive) behavior to refer to instances with $\alpha > 1/2$ ($\alpha < 1/2$). In particular, if $\alpha = 1$, the superdiffusive regime is referred as ballistic.

Figure 2(b) focuses on the behavior of the standard deviation σ_t of the accumulated presence of the federal deputies of the 49th–54th legislatures. This figure indicates a superdiffusive regime with $\alpha \approx 1$ for all the legislatures, i.e., the parliamentary presence dynamics can be viewed as an anomalous diffusive process consistent with a ballistic behavior ($R^2 > 0.99$). As in the case of the mean presence μ_t, our results involving σ_t are robust, in the sense that small variations of the considered number of federal deputies of a legislature yield the same results.

However, in contrast to the mean presence, the slope coefficient, $\bar{\sigma}_1$, increases with the threshold for the maximum fraction of missed sessions in a row. Independent of this fact, the linear forms for μ_t and σ_t lead to coefficients of variation (σ_t/μ_t, the relative standard deviation) that are constant and equal to $\bar{\sigma}_1/\bar{\mu}_1$. This feature points toward a scaling relationship.

C. Probability density function of the parliamentary presence

The previous findings can be put in a more embracing framework by considering a probability density function (PDF) $p(x,t)$ to model the parliamentary presence of all federal deputies. In this direction, when we restrict our analysis to federal deputies that participated in a full legislature, we verify that practically none of them had presence equal to zero, $x_t^{(j)} = 0$. Thus, considering t is not so small, it seems feasible to consider that $p(x,t) \propto x^{a-1}$ (with $a > 1$) for small x. In addition, the maximum presence is t and it is very improbable federal deputies attended all sessions; therefore, it is also suitable to employ $p(x,t) \propto (t-x)^{b-1}$ (with $b > 1$) for x close to t. Note that we are considering x and t large enough to be adopted as continuous variables.

A direct way to accomplish the above reasoning is to suppose that $p(x,t)$ is proportional to the product of the two limiting behaviors, i.e., $p(x,t) \propto x^{a-1}(t-x)^{b-1}$. This approach leads to the PDF

$$p(x,t) = \frac{1}{B(a,b)}\,\frac{x^{a-1}}{t^{a}}\left(1-\frac{x}{t}\right)^{b-1}, \tag{7}$$

where $a > 1$, $b > 1$, $B(a,b)$ is the beta function, and $0 \leqslant x \leqslant t$. A first consequence of this choice for $p(x,t)$ is that x scales with t, since $p(x,t) = h(x/t)/t$, where $h(y) = y^{a-1}(1-y)^{b-1}/B(a,b)$ (with $0 \leqslant y \leqslant 1$) is the standard beta PDF [44]. The PDF given in Eq. (7) leads to a mean value and standard deviation of x proportional to t and to a constant coefficient of variation for all t, hence recovering our previous results. In the same direction, note also that the mode, obtained from the condition $dp(x,t)/dx = 0$, leads to $x \propto t$. Other theoretical discussions related to the PDF (7) are given in the next subsection.

In order to verify that the PDFs given in Eq. (7) is robust in our study for all t and x, we need to compare it with the empirical PDF's, adjusting the parameters a and b from the data. A way to proceed in this direction is to calculate the mean value and standard deviation of x using $p(x,t)$ and to enforce that they are equal to the empirical μ_t and σ_t. This procedure leads to the following condition:

$$\frac{a(\mu_t,\sigma_t,t)}{\mu_t} = \frac{b(\mu_t,\sigma_t,t)}{t-\mu_t} = \frac{\mu_t\, t - \sigma_t^2 - \mu_t^2}{\sigma_t^2\, t}. \tag{8}$$

By using the calculated values for μ_t and σ_t [as the circles in Figs. 2(a) and 2(b)] to obtain the parameters a and b above, we verify a good agreement among the PDF's for different values of t. In addition, by applying the Kolmogorov-Smirnov test [45], a measure of this goodness is the fraction of sessions that have p values greater than 0.01. We verified that this fraction is greater than 0.7 for the 49th–54th legislatures. Similar values are obtained for other choices of the threshold for the maximum fraction of missed sessions in a row close to 1/16.

Since μ_t and σ_t are well adjusted by Eqs. (4) and (6) with $\alpha = 1$, we can utilize these relationships to fix the values of a and b independently of the t values in each legislature. Thus, by using these two equations in Eq. (8), the parameters a and b are obtained via

$$\frac{a}{\bar{\mu}_1} = \frac{b}{1-\bar{\mu}_1} = \frac{\bar{\mu}_1 - \bar{\sigma}_1^2 - \bar{\mu}_1^2}{\bar{\sigma}_1^2}. \tag{9}$$

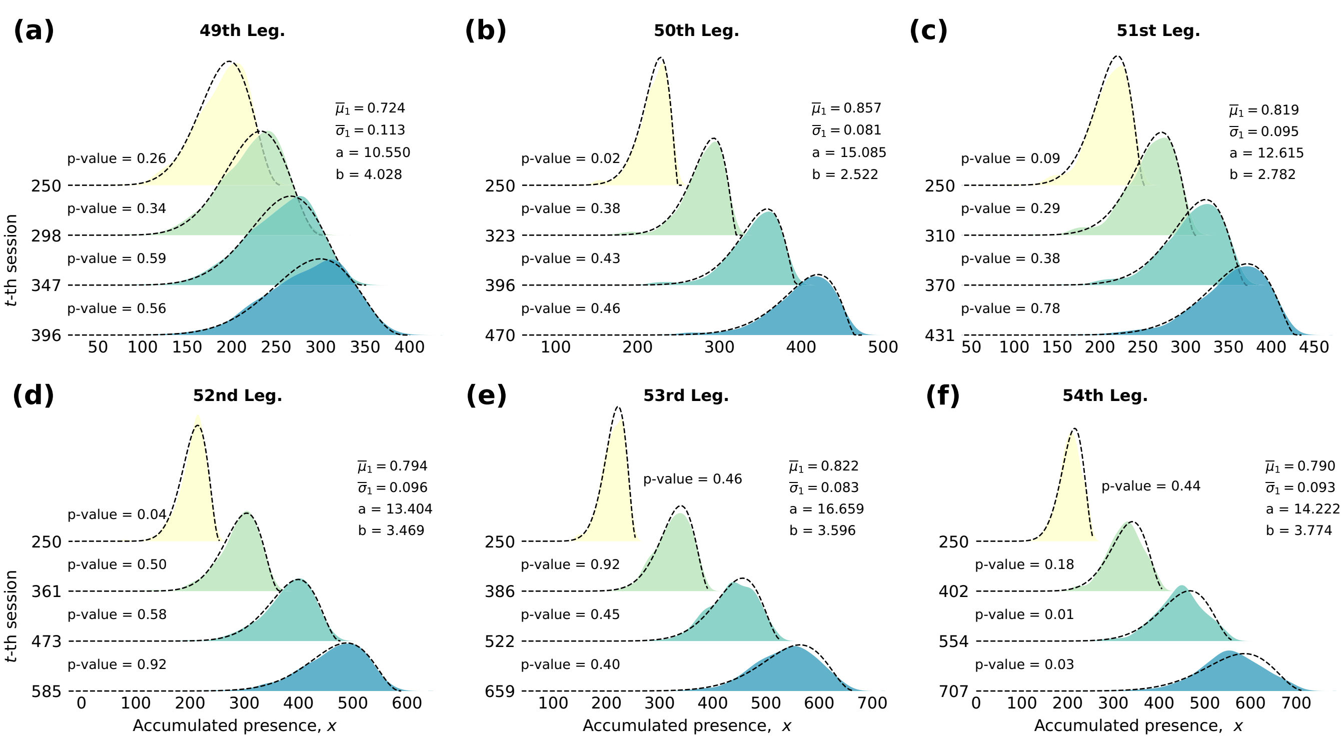

FIG. 3. Probability distribution. Panels (a)–(f) illustrate the agreement between the empirical PDF (colored ones) and the beta one (black dashed) given by Eq. (7), using a and b given by Eq. (9). Observe that the same parameters, for each legislature, adjust the distributions as the number of parliamentary sessions increase (p value $\geqslant 0.01$). Here, it was also used 1/16 as the maximum fraction of missed sessions in a row.

Considering the fraction of missed sessions in a row equal to $1/16$, the values of these parameters are displayed in Fig. 3, for some sessions within each legislature. This figure also illustrates how well the PDF (7) adjusts the data. The behavior of $\overline{\mu}_1$, $\overline{\sigma}_1$, a, and b as a function of the fraction of missed sessions in a row is displayed in Fig. 4. Note, in particular, that the results presented in Figs. 4(a) and 4(b) are consistent with Fig. 1(c) and the comments in the end of Subsecs. IV A and IV B. Observe also that if $1/16 \lesssim \xi \lesssim 1/4$ the parameters $\overline{\mu}_1$, $\overline{\sigma}_1$, a, and b can be seen in a first approximation as constants, where ξ is the fraction of missed sessions in a row. For $\xi \geqslant 1/4$ (one year or more of continuous absence), it is difficult to affirm that a deputy participated in a full legislature.

D. Modeling the parliamentary presence PDF

The distribution of presences plays a central role in our analysis. Here we are going to revisit it in connection with Bernoulli trials; we consider, in a first approximation, that the presence of each deputy is independent of the others in any session. In this context, let S_t^N be the number of accumulated presences of the N deputies in the first t sessions. Then, if the probability of any deputy being present at a session were q, the probability of having k presences until the tth session would be given by a binomial distribution [46]

$$w_t(q, N, k) = \binom{Nt}{k} q^k (1-q)^{Nt-k}. \tag{10}$$

The hypothesis that all deputies have the same probability of being present in a session is quite unlikely. Thus, we consider that q is a random variable distributed according to a probability density $h(q)$. In this case, the probability of having k presences until the tth session is

$$P(S_t^N = k) = \int_0^1 w_t(q, N, k)\, h(q)\, dq. \tag{11}$$

As a consequence,

$$P(S_t^N \leqslant Ntx) = \int_0^1 \left[\sum_{k=0}^{\lfloor Ntx \rfloor} w_t(q, N, k)\right] h(q)\, dq. \tag{12}$$

In Eq. (12), the term within brackets can be interpreted as $P(T_t^N \leqslant Ntx)$, where T_t^N is a sum of Nt independent variables that have a Bernoulli distribution with parameter q. Consequently, the weak law of large numbers [46] implies that

$$\lim_{N\to\infty} \sum_{k=0}^{\lfloor Ntx \rfloor} w_t(q, N, k) = \begin{cases} 1 & \text{if } q < x \\ 0 & \text{if } q > x. \end{cases} \tag{13}$$

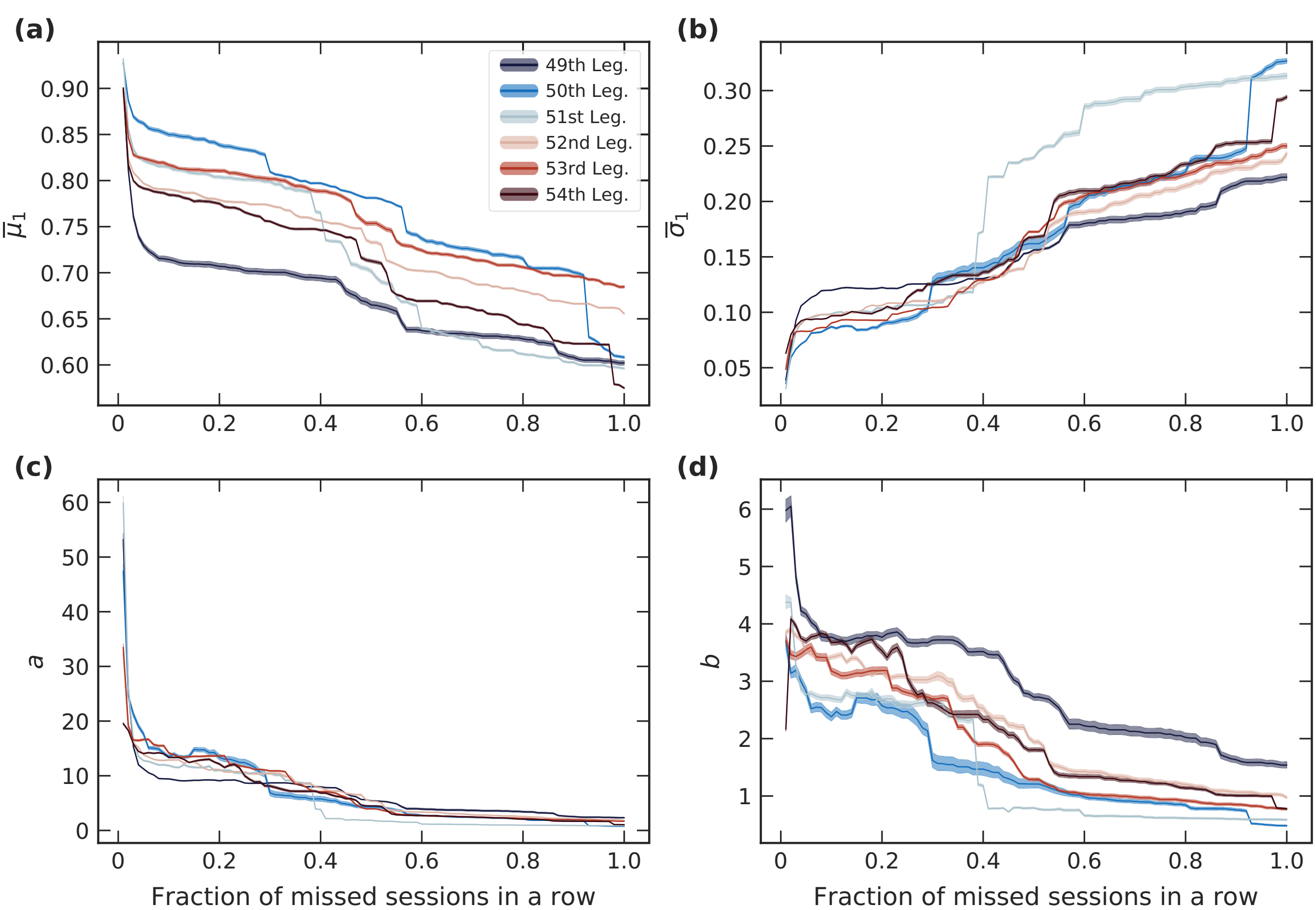

FIG. 4. Parameters values. Considering all sessions of each legislature, panels (a)–(d) illustrate all the rates $\overline{\mu}_1$ and $\overline{\sigma}_1$ and parameters a and b, obtained via Eq. (9), for all fractions of missed sessions in a row. The solid lines are the values while the bands in the curves are the intervals of confidence of 95%.

Because the number N of deputies under analysis is finite, Eq. (13) holds as a first approximation.

It follows from Eqs. (12) and (13), after using the dominated convergence theorem [46], that

$$\lim_{N\to\infty} P\left(S_t^N \leqslant Ntx\right) = \int_0^x h(q)\,dq. \tag{14}$$

Thus,

$$\lim_{N\to\infty} P\left(\frac{S_t^N}{N} \leqslant y\right) = \int_0^y \frac{h(x/t)}{t}\,dx, \tag{15}$$

i.e., the accumulated number of presences in t sessions per number of deputies converges weakly to a random variable with PDF $p(x,t) = h(x/t)/t$. Note that if $h(q)$ is not heavily concentrated in some value [i.e., $h(q) = \delta(q - q_0)$], this result recovers the empirical linear dependence of mean value and standard deviation with t, previously discussed in this section.

A well-known distribution with support in [0,1] which can assume a huge set of different shapes is the standard beta PDF [44]. This justifies the use of $h(q) = q^{a-1}(1-q)^{b-1}/B(a,b)$. By considering this choice for $h(q)$, it follows from Eq. (15) that Eq. (7) is the PDF of the limiting distribution of S_t^N/N.

As a further remark, we would like to point out that although we obtained the PDF given in Eq. (7), it can also be connected with a phenomenological nonlinear anomalous diffusion equation. As a guide in this direction, we start considering the function

$$f(x,t) = \frac{M}{t^{1/(1+\theta+\nu)}}\left(1 - \frac{A\,|x|^{\theta+2}}{t^{(\theta+2)/(1+\theta+\nu)}}\right)^{1/(\nu-1)}, \tag{16}$$

whose form resembles Eq. (7) except for the factor x^{a-1}. This $f(x,t)$ is known [47] to be a solution of equation

$$\frac{\partial f}{\partial t} = \mathcal{D}\,\frac{\partial}{\partial x}\left(x^{-\theta}\,\frac{\partial f^{\nu}}{\partial x}\right). \tag{17}$$

The constants A and the normalization one M depend on the parameters θ, ν, and $\mathcal{D}$.

Equations (16) and (17) describe a unified scenario of diffusive processes. In fact, if $\theta = 0$ and $\nu = 1$, we have the usual diffusion equation and it can be verified that Eq. (16) reduces to its well-known Gaussian solution in the limit $\nu \to 1$. Another particular case occurs when $\theta \neq 0$ and $\nu = 1$, leading to the oldest anomalous diffusion equation, the Richardson one [48]. In this case, the expression (16) yields a stretched Gaussian solution in the limit $\nu \to 1$. For $\theta = 0$ and $\nu \neq 1$, the porous media equation and its fundamental solution are recovered [49,50]. For general values of θ and ν, we obtain an interpolation of the phenomena described by these limiting cases. Thus, Eqs. (17) and (16) cover a rich class of normal and anomalous of diffusive phenomena.

A further generalization of a diffusion equation can be quickly obtained by incorporating a time dependence in the diffusion coefficient, e.g., proportional to t^{β}. For instance, we may use this proportionality for $\mathcal{D}$ in Eq. (17). In this case, the new equation can be reduced to the one with $\mathcal{D}$ constant if we employ $\tau \propto t^{\beta+1}$ as a new time variable. Hence, the solution of the new equation will be given by Eq. (16), replacing t by τ.

Concerning the distribution we dealt with in this paper, i.e., Eq. (7), we observe that its structure resembles the one in Eq. (16), except for the factor x^{a-1}. The presence of this extra term demands a further modification of Eq. (17). Since this term is a power of x, we expect that the required modification should also involve a power of x, e.g., x^{γ}. Aiming at an effective modification of the solution (16) that might incorporate the factor x^{a-1}, we ought to put x^{γ} under the influence of the inner spatial derivative. Thus, we consider

$$\frac{\partial f}{\partial t} = D\,\frac{\partial}{\partial x}\left[x^{-\theta}\,t^{\beta}\,\frac{\partial}{\partial x}(x^{\gamma} f^{\nu})\right]. \tag{18}$$

Finally, we can verify that the PDF given in Eq. (7) satisfies Eq. (18) if $\theta = -a$, $D = [B(a,b)]^{1/(b-1)}/b$, $\beta = a/(b-1)$, $\gamma = b(1-a)/(b-1)$, and $\nu = b/(b-1)$. The reader who is interested in certifying that Eq. (7) is a solution of Eq. (18) is referred to the Appendix.

V. CONCLUSIONS

We have investigated aspects of the parliamentary presence for the 49th–54th Brazilian Chambers of Deputies, concerning a period of 24 years, from 1991 to 2015. In general, by analyzing the presence of each federal deputy, we observed two distinct groups: the ones that were present in most of the sessions and the ones that barely attended the sessions [Figs. 1(b) and 1(c)]. Our analysis was focused on the former group, comprising about 400 federal deputies [see Fig. 1(c)]. We observed universal behaviors for statistical quantities such as mean value and standard deviation, where both displayed linear trends (see Fig. 2). In particular, the dependence of the standard deviation with the number of sessions implied that an anomalous diffusive process, with the ballistic regime, governs the parliamentary presences.

Among the 3243 parliamentary sessions, some sessions had fewer than ten federal deputies, three of which had only one deputy present, while only one session had all the 513 seats occupied. Motivated by the fact that the probability of a federal deputy being present in none or all the sessions of a legislature is essentially null, we proposed that the distribution of presence of all deputies is actually a beta distribution [Eq. (7)]. Employing this PDF and the fact that its coefficient of variance is constant, we calculated its parameters utilizing the empirical data and compared each other, for all sessions, via the Kolmogorov-Smirnov test. We observed that numerous sessions accept the null hypothesis (p-value $\geqslant 0.01$) that they can be drawn from a beta distribution with the same parameters. This suggests a consistency of how federal deputies who were present in most of the sessions attend the sessions: A few are always present, a few go sometimes, and most of them go regularly (see Fig. 3). In addition to the fact that the beta distribution appears naturally with our arguments, we also verified that it satisfies an anomalous diffusion equation [Eq. (18)]. As a last concern, we remark that future studies considering other parliaments and other models would be important in order to verify the degree of universality of these findings.

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APPENDIX: DERIVATION OF EQ. (18) FROM EQ. (7)

From Eqs. (16) and (17), we can conclude that a function

$$f(x,t) = \frac{C}{t}\left(1-\frac{x}{t}\right)^{b-1}, \tag{A1}$$

where $C > 0$ and $b > 0$ are constants, is a solution of the equation

$$\frac{\partial f}{\partial t} = \frac{1}{bC^{1/(b-1)}}\frac{\partial}{\partial x}\left[xt^{1/(b-1)}\frac{\partial}{\partial x}f^{b/(b-1)}\right], \tag{A2}$$

which is essentially Eq. (17) with a diffusion coefficient that depends on time. Now, let us consider the problem of finding a diffusion equation for the function

$$g(x,t) = C\frac{x^{a-1}}{t^a}\left(1-\frac{x}{t}\right)^{b-1}. \tag{A3}$$

We note immediately that $g(x,t) = (x/t)^{a-1}f(x,t)$. Hence, we have

$$\frac{\partial}{\partial t}\left[\left(\frac{x}{t}\right)^{1-a}g\right] = \frac{t^{\nu-1}}{C^{\nu-1}b}\frac{\partial}{\partial x}\left\{x\frac{\partial}{\partial x}\left[\left(\frac{x}{t}\right)^{1-a}g\right]^{\nu}\right\}, \tag{A4}$$

where $\nu = b/(b-1)$. Employing the product rule of differentiation in the first spatial derivative of the right-hand side of this equation and after simplifying and rearranging the resulting terms, we can obtain

$$\frac{a-1}{t}g + \frac{\partial g}{\partial t} = \frac{C^{1-\nu}}{b}(1-a)t^{a(\nu-1)}x^{a-1}\frac{\partial}{\partial x}(x^{1-a}g)^{\nu} + \frac{C^{1-\nu}}{b}\frac{\partial}{\partial x}\left[x^a t^{a(\nu-1)}\frac{\partial}{\partial x}(x^{1-a}g)^{\nu}\right]. \tag{A5}$$

By using Eq. (A3), we can verify that the first term on the left-hand side of Eq. (A5) is equal to the corresponding one on the right-hand side. Therefore, the function g satisfies the equation

$$\frac{\partial g}{\partial t} = \frac{C^{1-\nu}}{b}\frac{\partial}{\partial x}\left[x^a t^{a(\nu-1)}\frac{\partial}{\partial x}(x^{1-a}g)^{\nu}\right], \tag{A6}$$

which is Eq. (18) with an appropriate choice of the parameters.

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Correction: The surname of the second author contained an error and has been fixed.