


Eigenstate thermalization hypothesis and out of time order correlators

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The eigenstate thermalization hypothesis (ETH) implies a form for the matrix elements of local operators between eigenstates of the Hamiltonian, expected to be valid for chaotic systems. Another signal of chaos is a positive Lyapunov exponent, defined on the basis of Loschmidt echo or out of time order correlators. For this exponent to be positive, correlations between matrix elements unrelated by symmetry, usually neglected, have to exist. The same is true for the peak of the dynamic heterogeneity length χ_4 , relevant for systems with slow dynamics. These correlations, as well as those between elements of different operators, are encompassed in a generalized form of ETH.

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I. INTRODUCTION

In recent years there has been a renewal of interest in chaotic systems within a quantum mechanical description. A characterization of wave functions is the eigenstate thermalization hypothesis (ETH), an assumption that leads to a form for the matrix elements of local operators in the energy eigenbasis and can be viewed as an extension of ideas borrowed from random matrix theory. On the other hand, the Lyapunov exponent, a fingerprint of chaoticity in classical systems describing the sensitivity of the system's dynamics to a small perturbation [1], has been considered also for quantum systems based on the definition of suitable four-point correlation functions. Both the applicability of the ETH to chaotic systems and the emergence of a Lyapunov exponent in their dynamics are subjects of active current research. The simultaneous consideration of these two aspects for nonintegrable systems raises the question of how high-order correlation functions are described within the ETH ansatz, the subject of this paper.

In the following, we consider a Hamiltonian H which we assume is not integrable, and its eigenvalues e_i . We assume that there is a parameter (e.g., \hbar or the number of degrees of freedom N) that controls the level spacing δe . Because timescales and energy are inversely related, $t\delta e \sim \hbar$, this also defines the Heisenberg time $2\pi\hbar/\delta e$, the longest in the system's evolution ("a theoreticians's time" [2]). Between the smallest microscopic time and the Heisenberg time are physical times where some correlations are nonzero [3]. Because the level density is $\sim \hbar^N$ in the semiclassical limit, both small \hbar and large N may be cases where the separation of correlation and Heisenberg times is large, and we are thus allowed to consider an energy scale Δ that contains many levels, but is much smaller than the energy corresponding to the inverse of all interesting physical times.

The eigenstate thermalization hypothesis, as applied to expectation values [2,4–6], is usually stated as follows: For the matrix elements of a smooth observable A in the energy

eigenbasis, putting $e_+ \equiv (e_i + e_j)/2$,

$$A_{ij} = a(e)\delta_{ij} + R_{ij}F(e_+, e_i - e_j)e^{-S(e_+)/2}, \quad (1)$$

where both $a(e)$ and $F(e_+, \omega)$ are assumed to be smooth functions of their arguments. For macroscopic systems, as we shall mostly consider here, we may assume that $a(e)$ is a smooth function of the energy *density*. The origin of the factor $e^{-S(e_+)/2}$ comes from the spreading of weights over many levels, and is the only one compatible with $F(e_+, \omega)$ of order one, as we shall see. The $|R_{ij}|$ have expectations of order unity. Here, we may consider A 's that are real or complex Hermitian. We have

$$\begin{aligned} R_{ij} &= R_{ji} \quad \text{real, for } A \text{ real,} \\ R_{ij} &= R_{ji}^* \quad \text{complex with random phase, for } A \text{ complex.} \end{aligned} \quad (2)$$

Now, the off-diagonal elements R_{ij} are usually considered as independent random variables and therefore their product, apart from the obvious symmetries above, average to zero. Does this assumption, together with (1), contain everything that is necessary to calculate n -point functions? Clearly not, as this would ultimately mean that all higher correlation functions are obtainable as model-independent functionals of the one- and two-point functions, i.e., all models would be Gaussian. In particular, the correlations that determine the Lyapunov exponent [1,7–12], as well as the dynamic heterogeneities χ_4 , are not fully included—and the most important physical part is indeed absent—in (1).

Let us now be more clear on what ensemble we have in mind when we say "random." We may consider (i) energies averaged over windows of extension in energy Δ as above, (ii) a perturbation $H \rightarrow H + \Delta H_r$, where H_r is a local random Hamiltonian and the coupling is strong enough to couple nearby levels but sufficiently weak so as to not affect correlation functions at physical times (see Deutsch [4]), and (iii) a set of transformations U over physical observables described below ("typicality") (see Refs. [13–16]). Denoting

with an overbar averages over any of these ensembles, we may write

$$\overline{A_{i_1 i_1}} = \mathbf{F}_{e_+}^{(1)}, \quad \overline{A_{i_1 i_2} A_{i_2 i_1}} = e^{-S(e_+)} \mathbf{F}_{e_+}^{(2)}(e_{i_1} - e_{i_2}) \quad \text{for } i_1 \neq i_2, \quad (3)$$

where we have defined $\mathbf{F}_{e_+}^{(2)}(e_{i_1} - e_{i_2}) = |F(e_+, e_{i_1} - e_{i_2})|^2$.

In the following sections we introduce and justify an ansatz for higher correlations, and we show how it applies to the calculation of the out of time order correlators (OTOCs). We also give numerical support to the contention that the correlations between matrix elements are not negligible.

A. A convenient time axis

Here, we show how working on the shifted time axis (and correspondingly, a split Gibbs measure in the trace) is the natural strategy [8,17]. A way to introduce this construction is to consider two-point functions. The correlations and response functions are defined as (see, e.g., Ref. [18])

$$\begin{aligned} C(t, t') &= \frac{1}{2} \text{Tr}[[A(t), A(t')]_+ e^{-\beta H}] / Z, \\ R(t, t') &= \frac{i}{\hbar} \text{Tr}[[A(t), A(t')]_- e^{-\beta H}] \Theta(t - t') / Z, \end{aligned} \quad (4)$$

$$O_\beta(t, t') = \sum_i a^2(e_i) e^{-\beta e_i} / Z + \sum_{ij} e^{-\beta(e_i + e_j)/2 - S(e_+)} |R_{ij}|^2 |F(e_+, e_i - e_j)|^2 e^{it(e_i - e_j)} / Z. \quad (9)$$

We may hence change variables $e_+ = (e_i + e_j)/2$ and $\omega = e_i - e_j$. Replacing sums by integrals (see, e.g., Refs. [2,6]),

$$\sum_i \bullet \rightarrow \int d e e^{S(e)} \bullet, \quad (10)$$

and performing a saddle point integration for e_+ (we are assuming a thermodynamic system), we obtain

$$\begin{aligned} O_\beta(t, t') &= \int d e a^2(e) e^{-\beta e + S(e)} / Z + \int d e_+ d \omega e^{-\beta e_+ + S(e_+)} |F(e_+, \omega)|^2 e^{it\omega} / Z \\ &= [\mathbf{F}^{(1)}(e_\beta)]^2 + \int d \omega \mathbf{F}_{e_\beta}^{(2)}(\omega) e^{it\omega}, \end{aligned} \quad (11)$$

where e_β is the canonical energy at temperature β .

We first observe that the diagonal term must be of order one if $\lim_{t \rightarrow \infty} O(t) = a$ is finite, while the scale $e^{-S(e_+)}$ is necessary so that the result is of order one, as it should. In conclusion, we find that the function $|F|^2(\omega) = \mathbf{F}_{e_\beta}^{(2)}(\omega)$ is the Fourier transform of the correlation (7) which is defined along the shifted time axis. The usual correlation and response are obtained by shifting back the time axis, and in so doing collecting a factor $\cosh(\hbar\beta\omega/2)$ and $\sinh(\hbar\beta\omega/2)$, respectively [6]. Indeed, splitting the Gibbs measure gives a much simpler relation between time correlators and correlations between matrix elements, and this will be even more so in the sequel.

where we have made explicit \hbar . For $t_1 > t_2$,

$$\left\{ C - \frac{i\hbar}{2} R \right\} (t_1 - t_2) = \text{Tr}[A(t_1)A(t_2)e^{-\beta H}] / Z. \quad (5)$$

It is easy to derive the so-called Kubo-Martin-Schwinger (KMS) condition [19], the quantum version of fluctuation-dissipation relations, which in the time domain may be expressed as a symmetry property,

$$\left[\left\{ C - \frac{i\hbar}{2} R \right\} \left(\tau - \frac{i\beta\hbar}{2} \right) \right]^* = \left\{ C - \frac{i\hbar}{2} R \right\} \left(\tau - \frac{i\beta\hbar}{2} \right), \quad (6)$$

for all τ . This strongly suggests that the situation becomes simpler if one displaces the t axis by $\frac{i\beta\hbar}{2}$ and defines

$$O_\beta(t, t') = \text{Tr}[A(t)e^{-\frac{\beta}{2}H}A(t')e^{-\frac{\beta}{2}H}] / Z. \quad (7)$$

The KMS condition (6) is now the statement that $O_\beta(t, t')$ is real, which is immediate from Hermiticity and the cyclic property of the trace,

$$O_\beta(t, t') = [O_\beta(t, t')]^* = [O_\beta(t', t)]^*. \quad (8)$$

We shall see the generalization of this below.

B. From matrix elements to correlations

Let us calculate for times small compared to the Heisenberg time the correlator Eq. (7), using the ansatz (1). We shall do this quickly here, and will justify the averaging involved in much more detail in the sections below:

II. MATRIX ELEMENTS AND MANY-TIME CORRELATION FUNCTIONS

In Sec. III, we shall show, using typicality considerations, that the expectations of the products of matrix elements in which the indices are not repeated vanish. In this section, we shall also argue that if an index is repeated more than twice, then the expectation breaks into a product of expectations. All in all, for the purposes of computing physical correlation functions, we shall only need to discuss the following generalization of (3) for different i_1, \dots, i_n ,

$$\overline{A_{i_1 i_2} A_{i_2 i_3} \cdots A_{i_n i_1}} = e^{-(n-1)S(e_+)} \mathbf{F}_{e_+}^{(n)}(e_{i_1} - e_{i_2}, \dots, e_{i_{n-1}} - e_{i_n}), \quad (12)$$

where $e_+ = (e_{i_1} + e_{i_2} + \dots + e_{i_n})/n$ and $\mathbf{F}_{e_+}^{(n)}(e_{i_1} - e_{i_2}, \dots, e_{i_{n-1}} - e_{i_n})$ is a smooth function of the energy differences and $a(e_+)$ is a smooth function of the average energy density e_+/N . The factor

$$\mathcal{N}^{(n)} \equiv e^{-(n-1)S(e_+)} \quad (13)$$

guarantees that $\mathbf{F}^{(n)}$ is of order one. The averages are as above. Expression (12) contains (3) as a particular case for $n = 1, 2$. A similar expression may be expected to hold if the matrices are different (e.g., $A_{ij}B_{ji}$), with an appropriate function $\mathbf{F}^{(n)}$ dependent on the operators involved.

Note that the usual ETH assumes an absolute value of the A 's of the order $e^{-S(e_+)/2}$, while the part contributing to higher correlations is much smaller, $\sim e^{-\frac{n-1}{n}S(e_+)}$ *per factor*. These correlations between $n > 2$ matrix elements are indeed smaller, but enter all expressions for n -point time correlations through sums with many more terms, so that their contribution to physical correlation functions is of the same order—just as what happens between diagonal and off-diagonal terms in (3). Indeed, as we shall see explicitly below, a sum over n indices of the average (12) picks up a factor $\sim e^{nS}$ from the density of levels, while we may estimate the sum of the (larger) fluctuating parts as being only a factor $\sim e^{nS/2}$, due to random signs.

We have restricted ourselves to different indices. In fact, when there is a repetition, one can see that what dominates are products of lower-order expectations under some assumptions on the fluctuating parts. Consider, for example, the case in which $i_r = i_1$, and all other indices are different,

$$\underbrace{A_{i_1, i_2} A_{i_2, i_3} \dots A_{i_{r-1}, i_1}}_{\mathcal{C}} \underbrace{A_{i_1, i_{r+1}} \dots A_{i_n, i_1}}_{\mathcal{D}} \equiv \mathcal{CD} \\ = \overline{A_{i_1, i_2} A_{i_2, i_3} \dots A_{i_{r-1}, i_1}} \overline{A_{i_1, i_{r+1}} \dots A_{i_n, i_1}} + \tilde{\mathcal{C}}\tilde{\mathcal{D}}, \quad (14)$$

where we have split into average and fluctuating parts $\mathcal{C} = \overline{\mathcal{C}} + \tilde{\mathcal{C}}$ and $\mathcal{D} = \overline{\mathcal{D}} + \tilde{\mathcal{D}}$. We assume that the average of fluctuating parts $\tilde{\mathcal{C}}\tilde{\mathcal{D}}$ scales at most as (12) (i.e., as the result without index repetitions). We wish to compare this with the product of averages, which contain $(r - 1)$ and $(n - r + 1)$ terms, respectively. Put $e_a = \frac{e_{i_1} + \dots + e_{i_{r-1}}}{(r-1)}$ and $e_b = \frac{e_{i_r} + \dots + e_{i_n}}{(n-r+1)}$, so that $ne_+ = (r - 1)e_a + (n - r + 1)e_b$. The size of the estimate (12) is the exponential of

$$-(n - 1)S(e_+) \\ = -(n - 1)S\left[\frac{(r - 1)e_a + (n - r + 1)e_b}{n}\right] \\ \leq -(n - 1)\left[\frac{(r - 1)}{n}S(e_a) + \frac{(n - r + 1)}{n}S(e_b)\right], \quad (15)$$

where in the inequality we have used the convexity of S (itself a consequence of the positivity of specific heat). This is to be compared with the estimate for the logarithm of the product,

$$\ln(\mathcal{N}^{(r-1)}\mathcal{N}^{(n-r+1)}) = -(r - 2)S(e_a) - (n - r)S(e_b). \quad (16)$$

Using the fact that $n - r + 1$ and $r - 1$ are non-negative, we check that the product form always dominates,

$$\overline{A_{i_1, i_2} A_{i_2, i_3} \dots A_{i_{r-1}, i_1} A_{i_1, i_{r+1}} \dots A_{i_n, i_1}} \\ \sim \overline{A_{i_1, i_2} A_{i_2, i_3} \dots A_{i_{r-1}, i_1}} \overline{A_{i_1, i_{r+1}} \dots A_{i_n, i_1}}. \quad (17)$$

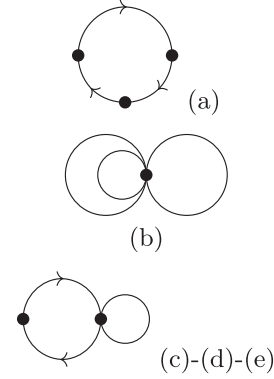


FIG. 1. Products corresponding to three-point functions (an alternative representation to Fig. 5): a loop and two cacti.

In short, $A_{i_1, i_2} A_{i_2, i_3} \dots A_{i_{r-1}, i_1}$ and $A_{i_1, i_{r+1}} \dots A_{i_n, i_1}$ are stochastic variables whose expectations factorize up to exponential accuracy if their covariances scale as (12). Their variances are always large, but contribute subdominantly to the sums over indices involved in the calculation of correlation functions.

Proceeding this way, one may use this argument to generalize (17). In particular, the fact that the ansatz (12) is all we need to calculate n -point correlations may be argued as follows. Consider a product $A_{i_1, j_2} \dots A_{i_n, j_n}$. We may represent it as a diagram by using as vertices all the *different* indices among the set $\{i_1, \dots, i_n, j_1, \dots, j_n\}$. Each factor A_{ij} is then represented by an arrow (for complex A_{ij} , real A_{ij} are represented by a simple edge) going from i to j . We consider

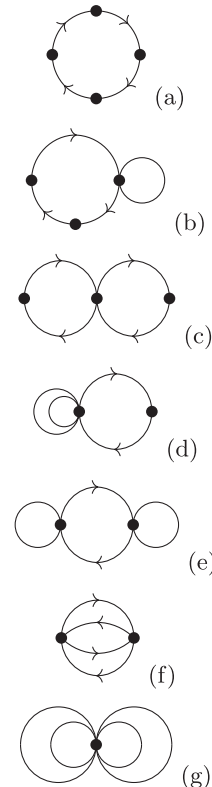


FIG. 2. Diagrams corresponding to four-point functions. All the diagrams are cacti apart from (f).

the following: (i) The simplest nonzero expectation, the one involved in ansatz (12), corresponds to a diagram with all distinct vertices lying on a single loop. (ii) Next, we consider a tree of loops, joined to one another at single vertices: a “cactus” diagram [a two-leaf cactus is (17)]. The entropic argument (15) may be generalized (see Appendix B) to argue that the expectation of such a product is dominated by a product of its constituent loops, each obtained from (12), under the assumption that the covariances scale as (12). This is the generalization of (17). (iii) Finally, we have products that are “non cactus,” for example, $A_{ij}A_{jk}A_{ki}A_{ik}A_{kj}A_{ji}$. These products may be decomposed in loops in more than one way [for example, $(A_{ij}A_{jk}A_{ki})(A_{ik}A_{kj}A_{ji}) = (A_{ij}A_{ji})(A_{ki}A_{ki})(A_{jk}A_{kj})$]. For these we make the assumption that their order in e^{-S} is the

same as the one of the decomposition in loops that dominates. These expectations may be large in order, but they have less free indices than the corresponding cactus, and hence pick up less factors e^S and their contribution to the n -point correlation functions is subdominant. In Figs. 1 and 2 we show examples of loops, cactus, and noncactus products. We shall not discuss the averages of noncactus products here, but let us note that we need to calculate them if we wish to determine the whole probability distribution $P(|A_{ij}|)$, through expectations of the moments $\langle |A_{ij}|^n \rangle$.

Correlations

The functions $\mathbf{F}^{(n)}$ are closely related to the time correlations [8],

$$O_\beta(t_1, \dots, t_n) = \text{Tr}[A(t_1) \dots e^{-\frac{\beta}{n}H} A(t_n) e^{-\frac{\beta}{n}H}] / Z = \left\{ \sum_{i_1, \dots, i_n} e^{-\frac{\beta}{n}(e_{i_1} + \dots + e_{i_n})} A_{i_1 i_2} A_{i_2 i_3} \dots A_{i_n, i_1} e^{it_1(e_{i_1} - e_{i_2}) + it_2(e_{i_2} - e_{i_3}) + \dots + it_n(e_{i_n} - e_{i_1})} \right\} / Z, \tag{18}$$

which clearly depend only on time differences. (From here onwards we set $\hbar = 1$.) Time correlations are invariant with respect to cyclic permutations of indices, a translation of all times $O_\beta(t_1, \dots, t_n) = O_\beta(t_1 + \tau, \dots, t_n + \tau)$, and satisfy the KMS condition [the generalization of (8)],

$$O_\beta(t_1, \dots, t_n) = [O_\beta(t_n, \dots, t_1)]^*. \tag{19}$$

The ordinary n -point correlations and responses can be obtained by analytic continuation. Shifting the various times by a suitable imaginary amount,

$$O_\beta(t_1 - i\beta/n, t_2 - 2i\beta/n, t_3 - 3i\beta/n, \dots) = \left\{ \sum_{i_1, \dots, i_n} e^{-\beta e_{i_1}} A_{i_1 i_2} A_{i_2 i_3} \dots A_{i_n, i_1} e^{ie_{i_1}(t_1 - t_n) + \dots + ie_{i_n}(t_n - t_{n-1})} \right\} / Z. \tag{20}$$

Let us consider, for the moment, the partial sum of (18),

$$D_\beta(t_1, \dots, t_n) = \left\{ \sum_{\substack{i_1, \dots, i_n \\ \text{different}}} e^{-\frac{\beta}{n}(e_{i_1} + \dots + e_{i_n})} A_{i_1 i_2} A_{i_2 i_3} \dots A_{i_n, i_1} e^{it_1(e_{i_1} - e_{i_2}) + it_2(e_{i_2} - e_{i_3}) + \dots + it_n(e_{i_n} - e_{i_1})} \right\} / Z. \tag{21}$$

Equation (19) clearly implies also that $D_\beta(t_1, \dots, t_n) = [D_\beta(t_n, \dots, t_1)]^*$ and

$$D_\beta(t_1 - i\beta/n, t_2 - 2i\beta/n, t_3 - 3i\beta/n, \dots) = \left\{ \sum_{\substack{i_1, \dots, i_n \\ \text{different}}} e^{-\beta e_{i_1}} A_{i_1 i_2} A_{i_2 i_3} \dots A_{i_n, i_1} e^{ie_{i_1}(t_1 - t_n) + \dots + ie_{i_n}(t_n - t_{n-1})} \right\} / Z. \tag{22}$$

We also consider the Fourier transform,

$$\hat{D}_\beta(\omega_1, \dots, \omega_{n-1}) = \frac{1}{(2\pi)^{(n-1)}} \int dt_1, \dots, dt_{n-1} D_\beta(t_1, \dots, t_{n-1}, 0) e^{-i(\omega_1 t_1 + \dots + \omega_{n-1} t_{n-1})}, \tag{23}$$

which may be written as

$$Z \hat{D}_\beta(\omega_1, \dots, \omega_{n-1}) = \sum_{\substack{i_1, \dots, i_n \\ \text{different}}} e^{-\frac{\beta}{n}(e_{i_1} + \dots + e_{i_n})} A_{i_1 i_2} A_{i_2 i_3} \dots A_{i_n, i_1} \delta(e_{i_1} - e_{i_2} - \omega_1) \delta(e_{i_2} - e_{i_3} - \omega_2) \dots \delta(e_{i_{n-1}} - e_{i_n} - \omega_{n-1}). \tag{24}$$

Equation (24) describes a function \hat{D}_β made of many deltas of random amplitudes, a “comb.” The definition implies no assumption. If we now perform an average of the operators in the spirit of a typicality argument (see the next section) we

obtain, assuming (12),

$$\overline{\hat{D}}_\beta(\omega_1, \dots, \omega_{n-1}) = \sum_{\substack{i_1, \dots, i_n \\ \text{different}}} e^{-\frac{\beta}{n}(e_{i_1} + \dots + e_{i_n})} \overline{A_{i_1 i_2} A_{i_2 i_3} \dots A_{i_{n-1} i_n}} \delta(e_{i_1} - e_{i_2} - \omega_1) \delta(e_{i_2} - e_{i_3} - \omega_2) \dots \delta(e_{i_{n-1}} - e_{i_n} - \omega_{n-1}) / Z, \quad (25)$$

$$\sum_{\substack{i_1, \dots, i_n \\ \text{different}}} \mathcal{N}^{(n)}(e_+) e^{-\beta e_+} \mathbf{F}_{e_+}^{(n)}(\omega_1, \dots, \omega_{n-1}) \delta(e_{i_1} - e_{i_2} - \omega_1) \delta(e_{i_2} - e_{i_3} - \omega_2) \dots \delta(e_{i_{n-1}} - e_{i_n} - \omega_{n-1}) / Z. \quad (26)$$

Smoothing over a scale of many eigenvalues, but small energy difference Δ , replacing sums by integrals (see, e.g., Refs. [2,6]) $\sum_i \bullet \rightarrow \int d e e^{S(e)} \bullet$, and performing saddle point integration for e_+ , we obtain

$$\overline{\hat{D}}_\beta(\omega_1, \dots, \omega_{n-1})|_\Delta = \mathbf{F}_{e_\beta}^{(n)}(\omega_1, \dots, \omega_{n-1}), \quad (27)$$

where e_β is the canonical energy at temperature $1/\beta$, associated with the partition function Z . This justifies the choice made for $\mathcal{N}^{(n)}$, that precisely cancels the level density. In going from (10) to (27) we have used that

$$\begin{aligned} S(e_{i_1}) + \dots + S(e_{i_n}) &= S(e_+ + e_{i_1} - e_+) + \dots + S(e_+ + e_{i_n} - e_+) \\ &= nS(e_+) + \beta[(e_{i_1} - e_+) + \dots + (e_{i_n} - e_+)] + S''(e_+)[(e_{i_1} - e_+)^2 + \dots + (e_{i_n} - e_+)^2]/2 + \dots \sim nS(e_+), \end{aligned} \quad (28)$$

where we have used that the quadratic and higher terms in $(e_i - e_+)$ become negligible—see the argument leading to (210) in Ref. [6]. Note the simplifying role of using a split Gibbs-Boltzmann factor, guaranteeing the cancellation of terms linear in $(e_i - e_+)$.

We can also write

$$D_\beta(t_1, \dots, t_{n-1}, 0) \sim \int d\omega_1, \dots, d\omega_{n-1} e^{i(\omega_1 t_1 + \dots + \omega_{n-1} t_{n-1})} \mathbf{F}_{e_\beta}^{(n)}(\omega_1, \dots, \omega_{n-1}), \quad (29)$$

where the \sim sign means that the approximate equality holds for times $t \ll \Delta^{-1}$, so that the Fourier transform implicitly smooths the amplitudes of the delta peaks in Eq. (24) and we may replace \hat{D} by \mathbf{F} : This allows us to write the equation for D_β without any other averaging.

In order to reconstruct the complete time correlation (18), we need to consider all the different possible ways in which indices may coincide. We shall do this in detail for three- and four-point functions below; let us state here the basic principle. Suppose we are calculating

$$\begin{aligned} D_\beta^{(\bullet)}(t_1, \dots, t_n) &= \left\{ \sum_{\substack{i_1, \dots, i_n \\ \text{different}}} e^{-\frac{\beta}{n}(e_{i_1} + \dots + e_{i_n})} \underbrace{A_{i_1 i_2} A_{i_2 i_3} \dots A_{i_{r-1} i_r}}_{\text{different}} \underbrace{A_{i_r i_{r+1}} \dots A_{i_n i_1}}_{\text{different}} e^{it_1(e_{i_1} - e_{i_2}) + it_2(e_{i_2} - e_{i_3}) + \dots + it_n(e_{i_n} - e_{i_1})} \right\} / Z \\ &= \left\{ \sum_{\substack{i_1, \dots, i_n \\ \text{different}}} e^{-\frac{\beta}{n}(e_{i_1} + \dots + e_{i_n})} \underbrace{A_{i_1 i_2} A_{i_2 i_3} \dots A_{i_{r-1} i_r}}_{\text{different}} \underbrace{A_{i_r i_{r+1}} \dots A_{i_n i_1}}_{\text{different}} e^{ie_{i_1}(t_1 - t_n) + \dots + ie_{i_n}(t_n - t_{n-1})} \right\} / Z, \end{aligned} \quad (30)$$

as in (14). We may replace this by the product average (17) and compute everything in terms of lower correlation functions. Another way of doing this is to observe that if we add in (18) to the times $(t_1 + s, t_2 + s, \dots, t_{r-1} + s, t_r, \dots, t_{n-1}, t_n)$, this produces a modification only in a factor $e^{is(e_{i_r} - e_{i_1})}$. Letting $s \rightarrow \infty$ will produce a rapidly oscillating factor whose contribution will cancel through random phases—the usual argument for the diagonal approximation. Hence, we may make the association $\lim_{s \rightarrow \infty} e^{is(e_{i_1} - e_{i_r})} \leftrightarrow \delta_{i_r i_1}$, and

$$\lim_{s \rightarrow \infty} O_\beta(t_1 + s, t_2 + s, \dots, t_{r-1} + s, t_r, \dots, t_n) = \left\{ \sum_{\substack{i_1, \dots, i_n \\ \text{different}}} \delta_{i_r i_1} e^{-\frac{\beta}{n}(e_{i_1} + \dots + e_{i_n})} A_{i_1 i_2} A_{i_2 i_3} \dots A_{i_n i_1} e^{it_1(e_{i_1} - e_{i_2}) + it_2(e_{i_2} - e_{i_3}) + \dots + it_n(e_{i_n} - e_{i_1})} \right\} / Z. \quad (31)$$

As a final remark, let us mention that if A is an observable satisfying ETH, consistently, A^n should also be. Let us here only argue that the n -element correlations of A contribute to $\langle A^n \rangle$,

$$\begin{aligned} \sum_i (A^n)_{ii} e^{-\beta e_i} / Z &= O_\beta(-i\beta/n, -2i\beta/n, -3i\beta/n, \dots) = D_\beta(-i\beta/n, -2i\beta/n, -3i\beta/n, \dots) + \text{other} \\ &\sim \int d\omega_1, \dots, d\omega_{n-1} e^{-i[i\beta\omega_1/n + 2i\beta\omega_2/n \dots + i(n-1)\beta\omega_{n-1}/n]} \mathbf{F}_{e_\beta}^{(n)}(\omega_1, \dots, \omega_{n-1}) + \text{other}, \end{aligned} \quad (32)$$

so we see that $\mathbf{F}_{e_\beta}^{(n)}$ definitely has a contribution to the diagonal part of a power of matrices. The full closure of the forms of expectations under matrix products is an interesting question to check in detail, but we shall do this in a future work.

III. “TYPICALITY” OF OPERATORS

In this section we use “typicality” arguments to justify the claim made above, that only expectations with every index repeated are nonzero.

A. Two-point functions

Consider a situation where A would be a full $M * M$ complex Hermitian matrix, in the basis of the Hamiltonian (i.e., the function F would be a constant). A notion of typicality that goes back to Von Neumann [13], and further discussed in Refs. [14,16] (here we follow Ref. [15]) is to assume that A may be replaced by $A^u = UAU^\dagger$, with U a Gaussian random matrix, without altering results such as correlations. The group associated with U depends on the symmetry of A : If it is a complex Hermitian, the U is a unitary matrix, while if A is real, it is orthogonal. We shall here restrict ourselves to the unitary case; the generalization to orthogonal or symplectic cases is straightforward.

This typicality cannot apply for rotations in the entire Hilbert space in reality, because they would destroy the band structure given by F that determines the time-correlation functions, but let us ignore this for the moment. We can then evaluate the correlation function as an average over U using

$$\overline{A_{ij}^u A_{ji}^u} = a\delta_{ij} + b, \quad (33)$$

where a, b are related to the invariants of the matrix, $M(a + b) = \text{Tr} A$ and $M(a + b)^2 + M(M - 1)b^2 = \text{Tr} A^2$.

Clearly, one may generalize this to n -point functions [15],

$$\overline{A_{i_1 j_1}^u \dots A_{i_n j_n}^u} = \overline{U_{i_1 \bar{i}_1} \dots U_{i_n \bar{i}_n} U_{j_1 \bar{j}_1}^* \dots U_{j_n \bar{j}_n}^* A_{\bar{i}_1 \bar{j}_1} \dots A_{\bar{i}_n \bar{j}_n}}, \quad (34)$$

where we assumed implicit summation over repeated indices. For the unitary group [15,20],

$$\begin{aligned} \overline{U_{i_1 \bar{i}_1} \dots U_{i_n \bar{i}_n} U_{j_1 \bar{j}_1}^* \dots U_{j_n \bar{j}_n}^*} \\ = \sum_{\sigma, \bar{\sigma}} \mathcal{C}(\sigma, \bar{\sigma}) \delta_{i_1 \sigma(j_1)} \dots \delta_{i_n \sigma(j_n)} \delta_{\bar{i}_1 \bar{\sigma}(\bar{j}_1)} \dots \delta_{\bar{i}_n \bar{\sigma}(\bar{j}_n)}, \end{aligned} \quad (35)$$

where $\sigma, \bar{\sigma}$ are all permutations and $\mathcal{C}(\sigma, \bar{\sigma})$ known combinatorial numbers. Therefore,

$$\begin{aligned} \overline{A_{i_1 j_1}^u \dots A_{i_n j_n}^u} \\ = \sum_{\sigma, \bar{\sigma}} \mathcal{C}(\sigma, \bar{\sigma}) A_{\bar{\sigma}(\bar{j}_1) \bar{j}_1} \dots A_{\bar{\sigma}(\bar{j}_n) \bar{j}_n} \delta_{i_1 \sigma(j_1)} \dots \delta_{i_n \sigma(j_n)} \\ = \sum_{\sigma} \mathcal{A}(\sigma) \delta_{i_1 \sigma(j_1)} \dots \delta_{i_n \sigma(j_n)}. \end{aligned} \quad (36)$$

The matrix A then enters in the average only through its invariants \mathcal{A} , implicitly defined above. Terms which are not invariant under rotation would be in fact modified by averaging. For $n = 2$, for instance, there are two terms proportional to $\delta_{i_1 j_1} \delta_{i_2 j_2}$ and $\delta_{i_1 j_2} \delta_{i_2 j_1}$ which for the particular case of Eq. (33) reduce to $\delta_{ij} * \delta_{ij} = \delta_{ij}$ and to $\delta_{ii} * \delta_{jj} = 1$.

In our case, we have to deal with averages of the particular kind,

$$\overline{A_{i_1 i_2}^u A_{i_2 i_3}^u \dots A_{i_n i_1}^u} = \sum_{\sigma} \mathcal{A}(\sigma) \delta_{i_1 \sigma(i_1)} \dots \delta_{i_n \sigma(i_n)}, \quad (37)$$

where the $\mathcal{A}(\sigma)$ are functions of the invariants of the matrix A . Clearly, the product $\delta_{i_1 \sigma(i_1)} \dots \delta_{i_n \sigma(i_n)}$ impose the equalities of indices in groups. For example, $\delta_{i_1 i_2} \delta_{i_2 i_3} \dots \delta_{i_n i_1}$ imposes the equality of all indices, while $\delta_{i_1 i_1} \delta_{i_2 i_2} \dots \delta_{i_n i_n} = 1$ imposes nothing.

Now, suppose we wish to compute using the typicality of operators a quantity such as (21), for the moment still for the case of a “full” $M * M$ random matrix. We should inject (37) into (18),

$$\begin{aligned} ZO_\beta(t_1, \dots, t_n) &\rightarrow \sum_{i_1, \dots, i_n} \sum_{\sigma} \mathcal{A}(\sigma) \delta_{i_1 \sigma(i_1)} \dots \\ &\delta_{i_n \sigma(i_n)} e^{it_1(e_{i_1} - e_{i_n}) + it_2(e_{i_2} - e_{i_1}) + \dots + it_n(e_{i_n} - e_{i_{n-1}}) - \frac{\beta}{n}(e_{i_1} + \dots + e_{i_n})}. \end{aligned} \quad (38)$$

For the case $n = 2$, Eq. (38) reads

$$O_\beta(t, t') = \mathcal{A}_1 \sum_{i \neq j} e^{i(t-t')(e_i - e_j)} e^{-\beta(e_i + e_j)/2} + \mathcal{A}_0, \quad (39)$$

where we have explicitly put all diagonal contributions in the second term.

It is clear then, comparing with (1), what we are missing in this first typicality approach: We need to impose the fact that local operators may be typical with respect to rotations between eigenstates associated with nearby levels, but “in the large” they have a band structure given by $F(e_+, \omega)$, which so far is absent. One way to accomplish this [4] has already been mentioned above: It is to add to the Hamiltonian a small $[O(\Delta)]$ random perturbation that will effectively mix the level within a range of energy Δ . We follow here a slightly different route [15]. Let us then consider a realistic system, where there is a concentration of matrix elements near the diagonal. Imagine, for example, we wish to estimate which of the products $A_{ij} A_{kl}$ have a nonzero expectation value. We first introduce an energy scale Δ such that it contains many levels, but is small in the thermodynamic limit (or, in a semiclassical one). A modified form of typicality may be introduced as

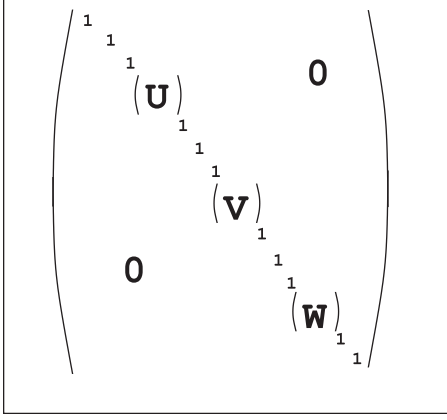


FIG. 3. A rotation group with three independent $\Delta * \Delta$ unitary transformation matrices.

follows: Consider the energy intervals

$$\left\{ e_i - \frac{\Delta}{2}, e_i + \frac{\Delta}{2} \right\}, \quad \left\{ e_j - \frac{\Delta}{2}, e_j + \frac{\Delta}{2} \right\},$$

$$\left\{ e_k - \frac{\Delta}{2}, e_k + \frac{\Delta}{2} \right\}, \quad \left\{ e_l - \frac{\Delta}{2}, e_l + \frac{\Delta}{2} \right\}, \quad (40)$$

and let us first assume that they are disjoint. We now define a group of unitary matrices of the form shown in Fig. 3, where $U^{(i)}, U^{(j)}, U^{(k)}, U^{(l)}$ are independent $\Delta * \Delta$ matrices. We now assume that there is typicality within this group, i.e.,

$$\overline{A_{ij}^u A_{kl}^u} = A_{i'j'}^u A_{k'l'}^u \overline{U_{i'i'}^{(\text{around } i)} U_{j'j}^{(\text{around } j)*} U_{kk'}^{(\text{around } k)} U_{l'l'}^{(\text{around } l)*}}. \quad (41)$$

The average is clearly zero, because each rotation is independent. In order to have the possibility of a nonzero average, we need that a matrix $U^{(a)}$ appears at least twice, once conjugated. For this to happen, we need that the indices (i, j, k, l) be at most in two intervals. Consider the case when (i, l) belong to an interval, and (j, k) to another, and the rotations associated with these two sets are U and V , respectively. We have then

$$\overline{A_{ij}^u A_{kl}^u} = A_{i'j'}^u A_{k'l'}^u \overline{U_{i'i'}^* V_{jj'}^* V_{kk'}^* U_{l'l'}^*} = \mathcal{A}_1(\text{intervals}) \delta_{jk} \delta_{il}. \quad (42)$$

Here and in the following, the argument “(intervals)” makes explicit the fact that the constant is dependent on where the intervals considered by the rotations are located, i.e., a function of the energies defining (up to a small uncertainty Δ) the two intervals. The other possibility is that (i, j) belong to an interval, and (k, l) to another. Then, we have

$$\overline{A_{ij}^u A_{kl}^u} = A_{i'j'}^u A_{k'l'}^u \overline{U_{i'i'}^* U_{jj'}^* V_{kk'}^* V_{l'l'}^*} = \mathcal{A}_0(\text{intervals}) \delta_{ij} \delta_{kl}, \quad (43)$$

a product of diagonal terms. This means that the nonzero expectation is

$$\overline{A_{ij}^u A_{ji}^u} = \mathcal{A}_0(\text{intervals}) \delta_{ij} + \mathcal{A}_1(e_i, e_j)$$

$$= \mathcal{A}_0(\epsilon_i) \delta_{ij} + \mathcal{N}^{(2)} \mathbf{F}_{e_+}^{(2)}(e_i - e_j). \quad (44)$$

The normalization $\mathcal{N}^{(2)}$ is for the moment arbitrary. We set it to $\mathcal{N}^{(2)} = e^{-S(e_+)}$ so that $\mathbf{F}^{(2)}$ remains of order one. For the first term we assume that $\mathcal{A}_0 = a^2$ may be at most of order

one, corresponding to an operator A having a finite limit for its large-time two-point function.

B. Three-point functions

Consider first a term such as $\overline{A_{i_1, j_1}^u A_{i_2, j_2}^u \cdots A_{i_n, j_n}^u}$ or, in particular, $\overline{A_{i_1, i_2}^u A_{i_2, i_3}^u \cdots A_{i_n, i_1}^u}$. The energies $e_{i_1}, e_{i_2}, \dots, e_{i_n}$ may be contained in at most n intervals of size Δ . If we proceed as before, with rotation along at most n intervals around the indices, we find as before that in order to have a nonzero average, a rotation must appear at least twice, one affecting a first and one affecting a second index. This means that we need to restrict ourselves to j indices such that the e_j are contained in the same intervals. In order to see how these averages work, let us develop in detail the case $n = 3$.

Our unitary matrix ensemble is as in Fig. 3. In the left-hand column of Fig. 4 we show the combinatorics of the energy intervals. Next, averaging over the independent transformations for every interval, we obtain a number of delta functions imposing some equalities of indices. The diagrams on the right have a solid line if the indices coincide, a dashed line if they do not coincide but are in the same energy block, and no line when they are in different blocks.

(1) If two indices are different $k \neq l$, but they belong to the same interval, this means that $|e_k - e_l| \rightarrow 0^+$: They are very close in terms of the parameters of the problem $\hbar, 1/N$, but do not coincide.

(2) If two indices are at different intervals, we assume that the average over the combined rotations depends smoothly on the energies e_k, e_l .

Typicality with respect to a random unitary transformation of this kind implies that matrix element products involving eigenstates associated with eigenvalues that are neighbors or differ by a few level separations may be completely different from matrix element products containing the diagonal elements. This is already so in the original ETH ansatz, where the diagonal elements A_{ii} scale differently (have a large expectation value) from the off-diagonal elements $A_{ij}, i \neq j$.

Going back to the diagrams in Fig. 4, the situation is summarized as follows. Putting $e_+ = (e_i + e_j + e_k)/3$:

(1) (i_1, i_2, i_3) in different blocks [Fig. 4(a)], then

$$\overline{A_{i_1, i_2}^u A_{i_2, i_3}^u A_{i_3, i_1}^u} = \mathcal{N}^{(3)} \mathbf{F}_{e_+}^{(3)}(e_{i_1} - e_{i_2}, e_{i_2} - e_{i_3}), \quad (45)$$

where $\mathcal{N}^{(3)}$ is a smooth function of e_+ fixed so that \mathbf{F} is of $O(1)$. We shall determine it below.

(2) Whenever two energies are different but are in the same block, we assume that the corresponding matrix element is simply given by a limit of Eq. (45). For example,

$$(b1): \overline{A_{i_1, i_2}^u A_{i_2, i_3}^u A_{i_3, i_1}^u}$$

$$= \mathcal{N}^{(3)} \lim_{(e_{i_2} - e_{i_3}) \rightarrow 0^+} \mathbf{F}_{e_+}^{(3)}(e_{i_1} - e_{i_2}, e_{i_2} - e_{i_3}),$$

$$(e1): \overline{A_{i_1, i_2}^u A_{i_2, i_3}^u A_{i_3, i_1}^u}$$

$$= \mathcal{N}^{(3)} \lim_{(e_{i_1} - e_{i_2}) \rightarrow 0^+} \lim_{(e_{i_2} - e_{i_3}) \rightarrow 0^+} \mathbf{F}_{e_+}^{(3)}(e_{i_1} - e_{i_2}, e_{i_2} - e_{i_3}). \quad (46)$$

This is natural, since the size Δ of the intervals is many level separations but otherwise arbitrary.

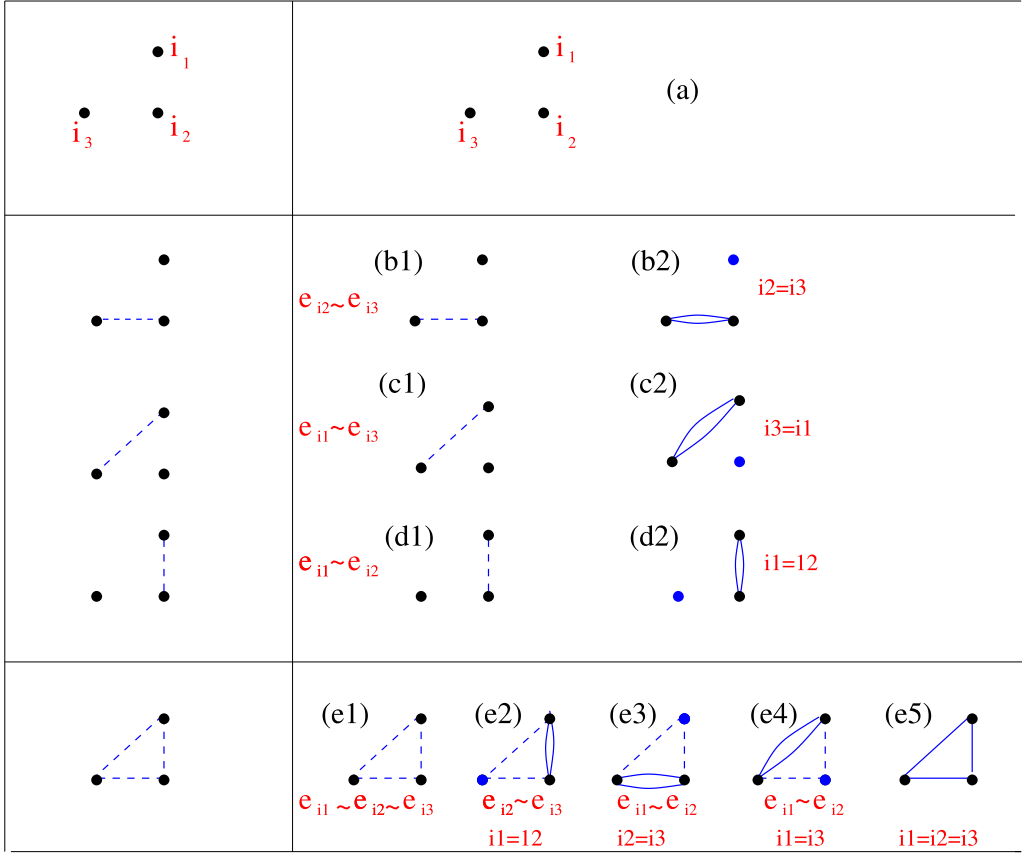


FIG. 4. A dotted line in the diagram on the left column means that the indices are in the same energy interval, and in a diagram on the right column that they are in the same interval but are not equal. In the diagrams of the right columns a solid line stands for a Kronecker delta, imposing the equality of indices. A double line imposes the same equality twice (e.g., $\delta_{ij}\delta_{ji}$), while a blue dot a delta with equal indices (e.g., δ_{ii}): They are indicated only for counting purposes, to show that there are three Kronecker symbols in each diagram.

(3) If two indices coincide—there is a solid line in the diagram—then the elements may scale in a different way, and their form is not obtained as a limit of $\mathbf{F}_{e_+}^{(3)}(e_{i_1} - e_{i_2}, e_{i_2} - e_{i_3})$.

All in all, the situation is reduced to Fig. 5, with a scaling function for each set of indices that differ. We can now write, for times that are smaller than the Heisenberg time, so that we may replace the matrix products by their averages,

$$\begin{aligned}
 O_\beta(t_1, t_2, t_3) &= \text{Tr}[A(t_1)e^{-\frac{\beta}{3}H}A(t_2)e^{-\frac{\beta}{3}H}A(t_3)e^{-\frac{\beta}{3}H}]/Z \\
 &= \left\{ \overline{\sum_{i_1, i_2, i_3} e^{-\frac{\beta}{3}(e_{i_1} + e_{i_2} + e_{i_3})} A_{i_1 i_2} A_{i_2 i_3} A_{i_3, i_1} e^{it_1(e_{i_1} - e_{i_3}) + it_2(e_{i_2} - e_{i_1}) + it_3(e_{i_3} - e_{i_2})}} \right\} / Z \\
 &= \left\{ \sum_{i_1, i_2, i_3} + \sum_{i_1, i_1, i_1} + \sum_{i_1, i_2, i_2} + \sum_{i_1, i_2, i_1} + \sum_{i_1, i_1, i_3} \right\} / Z \\
 &\sim D_\beta(t_1, t_2, t_3) + D_\beta^{(b)}(t_1, t_2, t_3) + D_\beta^{(c)}(t_1, t_2, t_3) + D_\beta^{(d)}(t_1, t_2, t_3) + D_\beta^{(e)}(t_1, t_2, t_3), \tag{47}
 \end{aligned}$$

in direct correspondence with the diagrams of Fig. 5 [we have not made explicit the supraindex (*a*) for the function corresponding to all indices different: It will be understood by default]. Here and in what follows, a bar over $\overline{\sum}$ means that the sum is unrestricted, while a \sum without a bar assumes all indices are different, unless they have explicitly the same name.

As mentioned above, to clarify the role of the sum over repeated indices in the time domain, consider the limit of times larger than the Heisenberg time, $\lim_{s \rightarrow \infty} O_\beta(t_1, t_2 + s, t_3)$. From Eq. (18) we see that the rapidly oscillating factors $e^{is(e_{i_2} - e_{i_3})}$ will cancel

everything by random phases, except for the diagonal terms with $i_2 = i_3$. Hence

$$\lim_{s \rightarrow \infty} O_\beta(t_1, t_2 + s, t_3) = \left\{ \overline{\sum_{i_1, i_2} e^{-\frac{\beta}{3}(e_{i_1} + 2e_{i_2})} A_{i_1 i_2} A_{i_2 i_2} A_{i_2, i_1} e^{i(t_1 - t_3)(e_{i_1} - e_{i_2})}} \right\} / Z = D_\beta^{(c)}(t_1, t_3) + D_\beta^{(b)}. \quad (48)$$

C. Four-point functions

We shall consider also the four-point function, as it is the one that is used to define the Lyapunov exponent and the dynamic heterogeneity length χ_4 [21],

$$\begin{aligned} ZO_\beta(t_1, t_2, t_3, t_4) &= \text{tr} \left[e^{-\frac{\beta}{4}H} e^{it_1 H} A e^{-it_1 H} e^{-\frac{\beta}{4}H} e^{it_2 H} A e^{-it_2 H} e^{-\frac{\beta}{4}H} e^{it_3 H} A e^{-it_3 H} e^{-\frac{\beta}{4}H} e^{it_4 H} A e^{-it_4 H} \right] \\ &= \overline{\sum_{ijkl} e^{-\frac{\beta}{4}e_i} e^{i(t_1 - t_4)e_i} A_{ij} e^{i(t_2 - t_1)e_j} e^{-\frac{\beta}{4}e_j} A_{jk} e^{i(t_3 - t_2)e_k} e^{-\frac{\beta}{4}e_k} A_{kl} e^{i(t_4 - t_3)e_l} e^{-\frac{\beta}{4}e_l} A_{li}}. \end{aligned} \quad (49)$$

We now wish to express (49) as a sum of terms with all possible index contractions, i.e., corresponding to the diagrams in Fig. 6. Let us first consider the sum with all indices different, corresponding to Fig. 6(a),

$$ZD_\beta(t_1, t_2, t_3, t_4) = \sum_{ijkl} e^{-\frac{\beta}{4}e_i} e^{i(t_1 - t_4)e_i} A_{ij} e^{i(t_2 - t_1)e_j} e^{-\frac{\beta}{4}e_j} A_{jk} e^{i(t_3 - t_2)e_k} e^{-\frac{\beta}{4}e_k} A_{kl} e^{i(t_4 - t_3)e_l} e^{-\frac{\beta}{4}e_l} A_{li}, \quad (50)$$

or its Fourier transform $\hat{D}_\beta(e_{12}, e_{23}, e_{34})$,

$$\hat{D}_\beta(e_{12}, e_{23}, e_{34}) \equiv \sum_{ijkl} \delta(e_i - e_j - e_{12}) \delta(e_j - e_k - e_{23}) \delta(e_k - e_l - e_{34}) e^{-\frac{\beta}{4}e_i} A_{ij} e^{-\frac{\beta}{4}e_j} A_{jk} e^{-\frac{\beta}{4}e_k} A_{kl} e^{-\frac{\beta}{4}e_l} A_{li} / Z. \quad (51)$$

Again, the difference between $\hat{D}_\beta(e_{12}, e_{23}, e_{34})$ and $\mathbf{F}_\beta^{(4)}(e_{12}, e_{23}, e_{34})$ is that $\hat{D}_\beta(e_{12}, e_{23}, e_{34})$ is a ‘‘comb’’ of delta functions, while $\mathbf{F}_\beta^{(4)}(e_{12}, e_{23}, e_{34})$ is a version that has

been smoothed either by averaging over an energy window or by averaging over unitary transformations as described above. Equation (12) is, for the case of four different indices,

$$\overline{A_{ij} A_{jk} A_{kl} A_{li}} = \mathbf{F}_{e_+}^{(4)}(e_i - e_j, e_j - e_k, e_k - e_l) e^{-3s(e_+)}. \quad (52)$$

Other types of contributions to the four-point correlation function, corresponding to the diagrams in Fig. 6, are listed in Appendix A.

IV. THE LYAPUNOV EXPONENT AND THE OUT OF TIME ORDER CORRELATOR

The Lyapunov exponent is derived from the OTOC $O_\beta(t, 0, t, 0)$ by assuming [8] that there is a time regime, larger than the correlation time, such that

$$O_\beta(t, 0, t, 0) \sim C_o - \frac{\hbar^2 C_1}{N^a} e^{\lambda t} + \dots \sim C_o - e^{\lambda(t - t_e)}, \quad (53)$$

for times $t_e \gg t \gg 1/\lambda$, where $t_e = \frac{a \ln N / \hbar^2 C_1}{\lambda}$ is (an estimate of) the Ehrenfest time, the time taken for a minimal packet to spread throughout phase space [8,22,23] (for a detailed discussion of the connections between the fidelity decay and the Lyapunov exponent, see Refs. [1,12]). The quantum exponent extracted from the OTOC coincides with the classical one in some instances of the semiclassical limit [22], but this is not necessarily the case [24]. It should be also noted in passing, as implicitly stated in Ref. [24], that the Lyapunov exponent so defined corresponds to an average taken *before* the logarithm, and not *after* as is the normal definition: In the language of disordered systems it is an ‘‘annealed’’ rather than a ‘‘quenched’’ average.

The terms contributing to the OTOC can be separated in those resulting from different sums over indices,

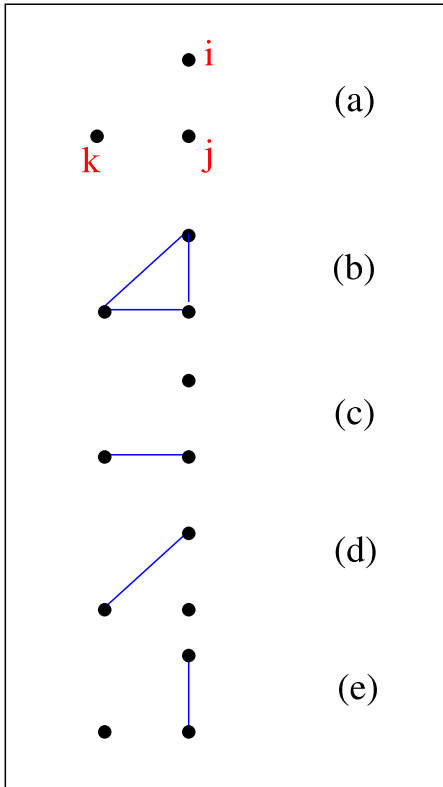


FIG. 5. The final situation: Products as in diagram (a), diagrams (c)–(e), and diagram (b) have different scaling functions.

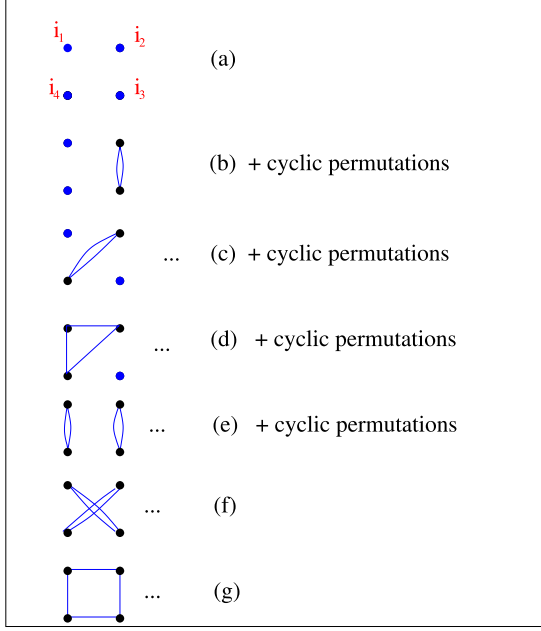


FIG. 6. Products corresponding to four-point functions.

corresponding to the diagrams of Fig. 6. As we have seen above, the only term that is not directly deducible, with a model-independent [25] formula (see below) from two- or three-point correlation functions [26], is the sum with all indices different, so we expect that the exponential growth is given by this term,

$$D_\beta(t, 0, t, 0) = \frac{1}{Z} \sum_{ijkl} e^{i(e_i - e_j + e_k - e_l)t} \times A_{ij} A_{jk} A_{kl} A_{li} e^{-\frac{\beta}{4}(e_i + e_j + e_k + e_l)t}, \quad (54)$$

a real, even function of time. In Fourier space it reads

$$\begin{aligned} \tilde{G}_\beta(\omega) &= \int dt e^{-i\omega t} D_\beta(t, 0, t, 0) \\ &= \frac{1}{Z} \sum_{ijkl} \delta(e_i - e_j + e_k - e_l - \omega) e^{-\frac{\beta}{4}e_i} A_{ij} e^{-\frac{\beta}{4}e_j} \\ &\quad \times A_{jk} e^{-\frac{\beta}{4}e_k} A_{kl} e^{-\frac{\beta}{4}e_l} A_{li}. \end{aligned} \quad (55)$$

The smoothed version of this reads

$$G_\beta(\omega) = \overline{\tilde{G}_\beta(\omega)}|_\Delta = \int \mathbf{F}_\beta^{(4)}(\omega - \omega', \omega'', \omega') d\omega' d\omega''. \quad (56)$$

What can we expect of the general features of (56)? First of all, because $D_\beta(t, 0, t, 0)$ is expected to be nonzero for times smaller than the Ehrenfest time, this means that $G_\beta(\omega)$ should be smooth on a scale $\delta\omega \ll 1/t_e$ (by which we mean that convoluted with a Gaussian having a width smaller than that, it remains unaltered). We expect, however, oscillations of $\delta\omega \sim 1/t_e$, if D extends to t_e . The information we seek is in the envelope of these oscillations.

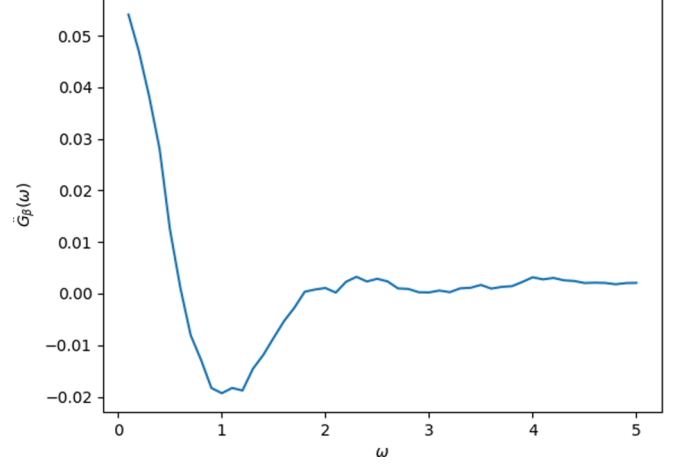


FIG. 7. The function $\hat{G}_\beta(\omega)$ for a system of size $N = 12$ and where we took $\Delta = 0.3$ and β corresponding to ϵ_+ in the middle of the spectrum.

A rapidly oscillating function suggests that one has to study it for complex variables, which leads us to compute the Laplace transform,

$$\begin{aligned} \mathcal{G}(s) &= \int_0^\infty dt e^{-ts} D_\beta(t, 0, t, 0) = \int d\omega \int_0^\infty dt e^{-ts+i\omega t} \tilde{G}_\beta(\omega) \\ &= \int d\omega \frac{s \tilde{G}_\beta(\omega)}{s^2 + \omega^2} \\ &= \frac{s}{Z} \sum_{ijkl} \frac{e^{-\frac{\beta}{4}e_i} A_{ij} e^{-\frac{\beta}{4}e_j} A_{jk} e^{-\frac{\beta}{4}e_k} A_{kl} e^{-\frac{\beta}{4}e_l} A_{li}}{(e_i - e_j + e_k - e_l)^2 + s^2} \\ &= \int d\omega \frac{s G_\beta(\omega)}{s^2 + \omega^2}. \end{aligned} \quad (57)$$

In the last equality we have substituted the function for the smoothed version, a safe thing to do for s much larger than the level spacing.

The function $\mathcal{G}(s)$ is analytic outside the imaginary s axis [cf. Eq. (57)] for t_e finite. However, in the limit of diverging Ehrenfest time, a form (53) implies that $\mathcal{G}(s)$ has a pole in zero (rounded off at a scale $s \sim 1/t_e$) and a pole in $s = \lambda$ of residue $\sim e^{-t_e \lambda}$, rounded off at a scale $(s - \lambda) \sim 1/t_e$. The rounding off of poles reestablishes analyticity for t_e finite.

The bound to Lyapunov [8] means that $\lim_{t_e \rightarrow \infty} e^{st_e} [\mathcal{G}(s) - C_0/s]$ is analytical for $s > \frac{2\pi}{\beta}$.

A. Numerical results

Following Ref. [6], we compute the correlation functions of a one-dimensional model of hard-core bosons with the Hamiltonian

$$\hat{H} = -J \sum_{j=1}^{L-1} (\hat{b}_j^\dagger \hat{b}_{j+1} + \text{H.c.}) + V \sum_{j < l} \frac{\hat{n}_j \hat{n}_l}{|j - l|^3} + g \sum_j x_j^2 \hat{n}_j. \quad (59)$$

The number of bosons was set to be $L/2$. The three terms in this Hamiltonian describe, from left to right, hopping, dipolar interactions, and a harmonic potential. Here, x_j is the distance

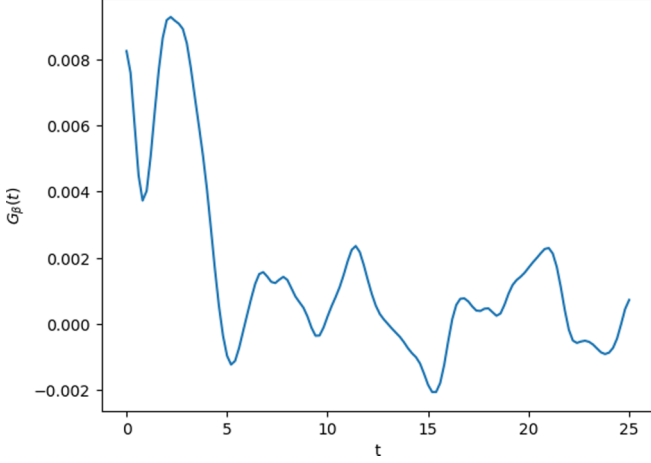


FIG. 8. The function $G_\beta(t)$ for a system of size $N = 10$ and where we took $\Delta = 0.3$ and $\beta = 0$.

of site j from the center of a trap. For $V \neq 0$, this model is nonintegrable.

Figure 7 shows the function $\hat{G}_\beta(\omega)$ computed for a system of size $N = 12$ where we considered as an observable the site occupation at the center of the chain, and Fig. 8 shows its Fourier transform. The sizes are small, but we clearly see correlations that are absent in the usual assumption of independence of matrix elements unrelated by symmetry. Figure 9 shows the contribution of the all terms except the ones in Fig. 7: We see a behavior very close to that of the two-point function computed in Ref. [6].

V. CONCLUSIONS

We have argued that the matrix elements of local operators in the basis of a chaotic Hamiltonian retain correlations that are small but contribute to higher correlation functions. The

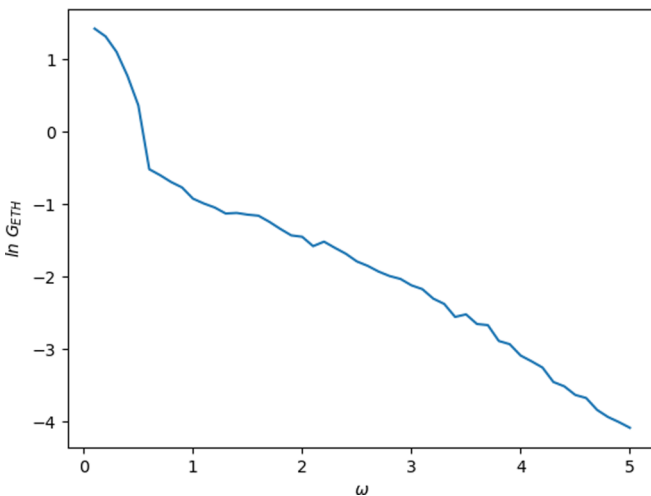


FIG. 9. The contribution to the OTOC of all terms except the one with all the indices different, the Fourier transform of $O_\beta(t, 0, t, 0) - D_\beta(t, 0, t, 0)$. The curve is very similar to the two-point curve, and is hence unimportant for the Lyapunov times. The system of size $N = 12$ and we took $\Delta = 0.3$ and ϵ_+ in the middle of the spectrum.

set of nonzero expectations of the products of matrix elements is in a one-to-one relation with the set of connected correlators. Even if the ETH ansatz with independent elements may be shown to close under products [2,6], so that, for example, the square of an operator of the ETH form is also of the ETH form, the neglected correlations between elements contribute and disregarding them may lead to incorrect results. A simple argument to convince oneself of this is that, if expectations of diagonal and variances of off-diagonal elements sufficed to determine all n -time correlations, then all could be written as model-independent functionals of one- and two-time correlations, which is for general models obviously not the case. An interesting generalization of the present work is to derive ETH forms for the joint distribution of various operators, under the same assumptions of this paper.

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APPENDIX A: CONTRIBUTIONS FROM OTHER DIAGRAMS

The other contributions to $O_\beta(t, 0, t, 0)$ associated with the diagrams in Fig. 6 derive from the two-point function within the standard ETH or its extension for the three-point function,

$$\begin{aligned} D^{(b)}(t) &= \frac{1}{Z} \sum_{i,j,k} e^{-\beta(2\epsilon_j + \epsilon_k + \epsilon_i)/4} e^{i(\epsilon_i - \epsilon_j)t} e^{i(\epsilon_j - \epsilon_k)t} A_{ij} A_{jj} A_{jk} A_{ki} \\ &\sim \int d\epsilon_1 d\epsilon_2 e^{\beta(\epsilon_1 + \epsilon_2)/12} e^{i(\epsilon_1 - \epsilon_2)t} \mathbf{F}_{\epsilon_\beta}^{(3)}(\epsilon_1, \epsilon_2) \mathbf{F}^{(1)} \\ &\quad \times (\epsilon_\beta + 3\epsilon_1/4 - \epsilon_2/4), \end{aligned} \quad (\text{A1})$$

where we made the change of variables $\epsilon_1 = \epsilon_i - \epsilon_j$, $\epsilon_2 = \epsilon_k - \epsilon_j$,

$$\begin{aligned} D^{(c)}(t) &= \frac{1}{Z} \sum_{i,j,k} e^{-\beta(2\epsilon_i + \epsilon_k + \epsilon_j)/4} e^{i(\epsilon_i - \epsilon_j)t} e^{i(\epsilon_i - \epsilon_k)t} A_{ij} A_{ji} A_{ik} A_{ki} \\ &\sim \left[\int d\epsilon_1 e^{-\beta\epsilon_1/4} e^{i\epsilon_1 t} \mathbf{F}_{\epsilon_\beta}^{(2)}(\epsilon_1) \right]^2, \end{aligned} \quad (\text{A2})$$

with $\epsilon_1 = \epsilon_i - \epsilon_j$, $\epsilon_i - \epsilon_k$,

$$\begin{aligned} D^{(d)}(t) &= \frac{1}{Z} \sum_{i,j} e^{-\beta(3\epsilon_i + \epsilon_j)/4} e^{i(\epsilon_i - \epsilon_j)t} A_{ii} A_{ii} A_{ij} A_{ji} \\ &\sim \int d\epsilon_1 e^{-\beta\epsilon_1/2} e^{i\epsilon_1 t} \mathbf{F}_{\epsilon_\beta}^{(2)}(\epsilon_1) [\mathbf{F}^{(1)}(\epsilon_\beta + \epsilon_1/4)]^2, \end{aligned} \quad (\text{A3})$$

with $\epsilon_1 = \epsilon_i - \epsilon_j$,

$$\begin{aligned} D^{(e)}(t) &= \frac{1}{Z} \sum_{i,j} e^{-\beta(\epsilon_i + \epsilon_j)/2} A_{ij} A_{jj} A_{ji} A_{ii} \\ &\sim \int d\epsilon_1 \mathbf{F}_{\epsilon_\beta}^{(2)}(\epsilon_1) \mathbf{F}^{(1)}(\epsilon_\beta + \epsilon_1/2) \mathbf{F}^{(1)}(\epsilon_\beta - \epsilon_1/2), \end{aligned} \quad (\text{A4})$$

with $\epsilon_1 = \epsilon_i - \epsilon_j$.

Figure 6(f) corresponds to a diagram that is not a cactus and therefore does not contribute (see the comment in Sec. II). Finally, one has

$$D^{(g)}(t) = \frac{1}{Z} \sum_i e^{-\beta \epsilon_i} A_{ii}^4 \sim [\mathbf{F}^{(1)}(\epsilon_\beta)]^4. \quad (\text{A5})$$

APPENDIX B: COUNTING ARGUMENT

Let us consider a loop with repetitions characterized by $E = \epsilon_1 + \epsilon_2 + \dots + \epsilon_n$. This is partitioned in k subloops of sizes n_a and energies $E_a = \epsilon_{i_1} + \dots + \epsilon_{i_{n_a}}$ with $a = 1, \dots, k$. We define $\lambda_a = n_a/n$, with $0 \leq \lambda_a \leq 1$, $\sum_{a=1}^k \lambda_a = 1$, and

$x_a = E_a/n_a$. It holds,

$$\begin{aligned} (n-1)S(\epsilon_+) &= (n-1)S\left(\sum_{a=1}^k \lambda_a x_a\right) \\ &\geq (n-1) \sum_{a=1}^k \lambda_a S(x_a) \geq \sum_{a=1}^k (n_a-1)S(x_a), \end{aligned} \quad (\text{B1})$$

where in the first inequality we have used the convexity of the entropy and in the second the fact that $\lambda_a \leq 1$. Therefore, if we assume that the covariances of such loops (and their generalization to multipoint functions) scale as $e^{-(n-1)S(\epsilon_+)}$, we see that the expectation of the maximal contraction in loops dominates the expectation of the diagram.

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