Thermal power of heat flow through a qubit

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In this paper we consider the thermal power of a heat flow through a qubit between two baths. The baths are modeled as a set of harmonic oscillators initially at equilibrium, at two temperatures. Heat is defined as the change of energy of the cold bath, and thermal power is defined as expected heat per unit time, in the long-time limit. The qubit and the baths interact as in the spin-boson model, i.e., through qubit operator σ_z. We compute thermal power in an approximation analogous to a "noninteracting blip" (NIBA) and express it in the polaron picture as products of correlation functions of the two baths, and a time derivative of a correlation function of the cold bath. In the limit of weak interaction we recover known results in terms of a sum of correlation functions of the two baths, a correlation functions of the cold bath only, and the energy split.

I. INTRODUCTION

Heat and work in classical thermodynamics are properties of processes, and not states. Heat is further in classical thermodynamics energy transferred from the system to an uncontrolled environment such that it cannot later be retrieved to do useful work [1,2]. The translation of these concepts to the quantum domain is therefore not obvious, as discussed in an early review [3]. Quantum thermal power is average quantum heat per unit time, and is a centrally important topic for applications to quantum heat engines [4–6]. While heat and thermal power at weak coupling has been studied for a long time in the literature [7–9], the attention to systems interacting strongly with one or several baths is more recent.

The approach and the setting most closely related to the one in this paper was for a two-level quantum system interacting with one bath [10]. It hence gives access to heat exchanged with the bath when relaxing to the equilibrium state (a case not studied here). More recently in [11] a similar technique was applied to the case when the system is a chain of oscillators. Using other approaches, the paper [12] contains the general observation that it is inconsistent to include any fixed fraction of interaction energy in heat, and studies interactions with fermionic baths with the nonequilibrium Green's function technique. The paper [13] is on the other hand one of several using the collective coordinate technique while [14] uses (in strong coupling) a thermalization assumption for the parts of the system directly interacting with the bath. Numerical

simulations and a hierarchical equations of motion (HEOM) approach are used in [15,16]. Other recent contributions are [17–20].

The goal of this paper is to revisit these questions in perhaps the simplest nontrivial scenario with a steady heat flow: one qubit interacting strongly with two heat baths at different temperatures. We will start from the point of view that heat is energy change in a bath. Thermal power is thus expected energy change in a bath per unit time, in the long-time limit. For concreteness we will consider thermal power as heat per time to the cold bath, and thus a quantity that should be nonnegative in the long-time limit. We assume that the qubit interacts with the baths and with an external drive as in the spin-boson model which allows to reuse many results developed in that context [21]. At strong coupling, and in the approximation known as "noninteracting blip approximation" (NIBA), the stationary state of the qubit is then determined by equilibrium correlation functions of certain bath operators related to a polaron transform. Our main result is that in a similar approximation thermal power is determined by the same correlation functions and their derivative with respect to time.

The paper is organized such that the high-level discussions are given in the main body while some background and much of the calculations are presented in Appendexes. In the second section of the main body of the paper (Sec. II) we thus introduce our model, and in Sec. III we give dimensional arguments what the results should be, first in a version appropriate for weak coupling, and then in a version appropriate for strong coupling. Section IV contains an overview of the calculations, and states the results in path integral language while Sec. V states in the language of the correlation functions after the

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polaron transform. Section VI summarizes and discusses the results.

The background and details of the calculations are presented as follows. First, Appendix A contains a summary of the vast literature on the spin-boson problem, sufficient for our purposes, and Appendix B summarizes earlier work on quantum heat functionals [22–24] adapted to the spin-boson setting. Appendix C gives details of what these formulas mean for Ohmic baths. Appendix D, which is included for completeness, further translates this theory to when the interaction is through bath momentum. Appendix E discusses a type of contribution to quantum heat in the spin boson problem which is formally divergent, but which with a proper treatment of the boundaries is seen to be finite (independent of time). It hence does not contribute to thermal power. Appendix F gives an overall discussion of the other contributions that do contribute to thermal power, and how they can be computed by a transfer matrix technique. The final appendices contain the following special cases: Appendix G the quantum state; Appendix H the long-time limit of the quantum state; Appendix I the expected heat; and Appendix J the expected heat in the long term limit. This last quantity thus gives thermal power, as presented in Sec. IV.

II. MODEL

We consider one qubit interacting with two harmonic oscillator baths as in the spin-boson model [21]. Harmonic oscillator baths model, for instance, resistive elements in electrical circuits, and quantum harmonic oscillator baths hence model how such elements interact with other circuit elements at sufficiently low temperatures [25]. Circuits with superconducting elements that can be assimilated to qubits are widely investigated in scalable quantum information processing [26]. The state of one qubit interacting with two baths is hence a toy model of a quantum computer perturbed by a heat flow through the dynamical degrees of freedom of quantum computer itself. Quantum thermal power in this setting is conversely how well such a device can transport energy between two baths in the quantum regime.

The system, the baths and the interactions can thus be written down as a total Hamiltonian

$$H_{\text{TOT}} = H_S + H_C + H_H + H_{\text{CS}} + H_{\text{HS}}, \quad (1)$$

where "C" refers to the cold bath (temperature T_C) and "H" refers to the hot bath (temperature T_H).

The system Hamiltonian is

$$H_S = -\hbar\frac{\Delta}{2}\hat{\sigma}_x + \frac{\epsilon}{2}\hat{\sigma}_z, \quad (2)$$

where Δ is a rate [dimension (time)$^{-1}$], and ϵ is the level splitting. The bath Hamiltonian are

$$H_C = \sum_{b\in C} \frac{p_b^2}{2m_b} + \frac{1}{2}m_b\omega_b^2 q_b^2, \quad (3)$$

$$H_H = \sum_{b\in H} \frac{p_b^2}{2m_b} + \frac{1}{2}m_b\omega_b^2 q_b^2, \quad (4)$$

where the parameters m_b and ω_b are the mass and angular frequency of each oscillator and C and H also stand for the sets of oscillators in, respectively, the cold bath and the hot bath.

We will take the system-bath interactions to be described by

$$H_{\text{CS}} = -\sum_{b\in C} C_b q_b \hat{\sigma}_z, \quad (5)$$

$$H_{\text{HS}} = -\sum_{b\in H} C_b q_b \hat{\sigma}_z, \quad (6)$$

where C_b is the interaction coefficient between bath oscillator b and the qubit, q_b is the oscillator coordinate, and $\hat{\sigma}_z$ operates on the qubit. Pauli matrices are by convention dimensionless, and the coupling coefficients C_b hence have dimension (energy) × (length)$^{-1}$. In [21] the length scale (there called q_0) is taken to be the spatial distance between the minima of two potential wells. For a qubit formed out of a nonlinear oscillator the length scale could similarly be the typical spatial scale of the oscillator ground state $\sqrt{\frac{\hbar}{m\omega}}$.

We consider heat as related to two measurements on the cold bath, one at the beginning of the process and one at the end, which we assume to take values $\mathcal{E}_i$ and $\mathcal{E}_f$. In a quantum bath neither $\mathcal{E}_i$ nor $\mathcal{E}_f$ are known; all we can know is the probability of observing $\mathcal{E}_i$ at the beginning, and the probability of observing $\mathcal{E}_f$ at the end. Thermal power is then the expected change of bath energy per unit time $\frac{\langle\mathcal{E}_f-\mathcal{E}_i\rangle}{t_f-t_i}$.

Four remarks are in order. First, "measurement on the bath" is required in the theory we consider, as without measurement the bath energy does not have a definite value. However, expected heat per unit time can, as we will see, be expressed in terms of system properties alone. Thermal power hence does not make any direct references to measurement, the values of which can hence be taken to be unrecorded. We may thus imagine "measurement on the bath" to actually refer to interaction with a large superbath which forces the bath states to decohere, without assuming any direct control of the bath states by an experimenter. Second, we do not count any part of the interaction energy in the heat. While this issue is important and has been discussed at length on the classical side in the recent literature [23,27–31], it is reasonable to assume that the interaction energy between one qubit and a bath does not increase at a nonzero rate for long enough times. Third, in applications to superconducting circuits, the system-bath interaction may often more naturally be taken to be proportional to bath oscillator momentum variable p_b [25]. Since both q_b and p_b can be expressed in Fourier modes of the oscillator this can be expected to make no essential difference, as was indeed stated in [32] for the qubit state. For completeness we outline in Appendix D an argument that this is so also for heat (full distribution function of bath energy change). Lastly, in realistic mesoscopic devices effective temperatures of different parts may differ. Such situations fall outside what is considered here since the devices would then not be systems in thermal equilibrium that could be modeled as baths.

III. DIMENSIONAL ARGUMENTS

The long-time limit of the state of one qubit interacting with any number of baths is given by its density matrix, where the diagonal terms ("the populations") determine the

probability for the qubit to be, respectively, in the up state and in the down state. Suppose these probabilities are $P(\text{up})$ and $P(\text{down})$. Suppose further that the memory of the bath is short enough that when the system is in one state the bath does not remember in which states the system was before. We can then suppose that the expected energy given to the cold bath per unit time takes two values that depend on the system state, call them π_{up} and π_{down}. Thermal power can then be estimated as

$$\Pi = P(\text{up})\pi_{\text{up}} + P(\text{down})\pi_{\text{down}}. \qquad (7)$$

To turn this into a quantitative prediction we can suppose that qubit transitions happen with effective rates describing the interactions with the two baths, and call these rates $\Gamma^C_{\downarrow\uparrow}$, $\Gamma^H_{\downarrow\uparrow}$, $\Gamma^C_{\uparrow\downarrow}$, and $\Gamma^H_{\uparrow\downarrow}$. This approach is appropriate when the qubit is weakly coupled to the baths, and one considers sufficiently long time scales [7,8]. The up and down probabilities then depend on the rates as for a classical jump process, i.e., as

$$P(\text{up}) = \frac{\Gamma^C_{\downarrow\uparrow} + \Gamma^H_{\downarrow\uparrow}}{\Gamma^C_{\downarrow\uparrow} + \Gamma^H_{\downarrow\uparrow} + \Gamma^C_{\uparrow\downarrow} + \Gamma^H_{\uparrow\downarrow}},$$

$$P(\text{down}) = \frac{\Gamma^C_{\uparrow\downarrow} + \Gamma^H_{\uparrow\downarrow}}{\Gamma^C_{\downarrow\uparrow} + \Gamma^H_{\downarrow\uparrow} + \Gamma^C_{\uparrow\downarrow} + \Gamma^H_{\uparrow\downarrow}}.$$

Power is dimensionally energy per unit time. When interaction energy is negligible the characteristic scale of energy transferred to the cold bath must be ϵ in an up-to-down transition, and $-\epsilon$ in an down-to-up transition, and these happen with rates $\Gamma^C_{\uparrow\downarrow}$ and $\Gamma^C_{\downarrow\uparrow}$. This leads to the estimates of power in the two states as

$$\pi_{\text{up}} = \epsilon\Gamma^C_{\uparrow\downarrow}, \qquad (8)$$

$$\pi_{\text{down}} = -\epsilon\Gamma^C_{\downarrow\uparrow}, \qquad (9)$$

and overall expected power as

$$\Pi^{\text{weak}} = \epsilon[\Gamma^C_{\uparrow\downarrow}P(\text{up}) - \Gamma^C_{\downarrow\uparrow}P(\text{down})]. \qquad (10)$$

Expressions of this form are well known in the literature, e.g., in [9] [Eq. (5)], and essentially hold in weak coupling also without the assumption of a short bath memory time.

At strong coupling the above is, however, not correct because when the qubit flips there is also a change of interaction energy between qubit and the bath. When this is larger than the level splitting the characteristic scale of energy transferred to the bath can be very different from ϵ. Furthermore, in strong coupling one may assume combined effective mean switching rates $\Gamma_{\uparrow\downarrow}$ and $\Gamma_{\downarrow\uparrow}$, but it is not possible to disentangle the actions of the two baths into separate terms Γ^C and Γ^H.

A different argument can nevertheless be made using the assumption of short enough bath decorrelation time, or equivalently that Δ is small enough that the residence time of the qubit in one state is long enough. From one qubit jump to the next qubit jump the baths hence on the average behave as follows. Right after the jump into state s there will be some average interaction energy and some average bath energy, $\langle H^i_{\text{CS}}(s)\rangle$ and $\langle H^i_C(s)\rangle$. Between the jumps, when the qubit does not change its state, the sum of these energies is conserved, but in the same time interval the baths will equilibrate with the qubit. At the end of the interval the average interaction energy should hence vanish. This means

that during one residence time in state s the expected energy change of the bath should be the expected initial interaction energy, i.e., $\langle H^i_{\text{CS}}(s)\rangle$. By this reasoning one gets

$$\Pi^{\text{strong}} = P(\text{up})\Gamma_{\uparrow\downarrow}\langle H^i_{\text{CS}}(\text{up})\rangle + P(\text{down})\Gamma_{\downarrow\uparrow}\langle H^i_{\text{CS}}(\text{down})\rangle. \qquad (11)$$

What this derivation leaves out are precise statements of what is meant by $\langle H^i_{\text{CS}}(\text{up})\rangle$ and $\langle H^i_{\text{CS}}(\text{down})\rangle$. The main contribution of this paper is to derive such estimates systematically, and explain how the terms follow from the microscopic parameters of the model.

IV. THERMAL POWER AT STRONG COUPLING

We now describe an approach to thermal power at strong coupling based on the Feynman-Vernon formalism [33]. To calculate heat (energy change in a bath) we follow [22,23,31], related general results can also been found in [10,34] and [35]. Adapting the Feynman-Vernon formalism to describe the development of one spin interacting with one bath (the spin-boson problem) is already not trivial [21]. Here we have the complications that we are interested in heat in a spin interacting with two (or more) baths at different temperatures. Technical background and details have therefore been moved to the Appendixes as pointed to below and at the end of the Introduction; here we only outline the main idea of the calculation.

We focus on the energy changes of one bath, for concreteness we assume that is the cold bath. The starting point is to assume that initially the baths are independently at thermal equilibrium (at different temperatures), and the system as well as the energy of the cold bath are measured. After that measurement the state of the system and the baths is $\rho^{\text{eq}}_H \oplus |\mathcal{E}^{(C)}_i, i\rangle\langle\mathcal{E}^{(C)}_i, i|$ where ρ^{eq}_H is the equilibrium state of the hot bath (or baths), i indicates the state of the system after measurement, and $\mathcal{E}^{(C)}_i$ the state of the cold bath. We take $p_C(\Delta E, f|\mathcal{E}^{(C)}_i, i)$ to be the conditional probability of observing a final state $|f\rangle$ of the system and energy change of the cold bath ΔE, conditioned on total initial state.

Next we assume that the measured energy of the cold bath is not recorded. This means that we could also say that the cold bath decoheres by interacting with an unobserved cold superbath at the same temperature. The initial state of the cold bath is then a statistical mixture where $|\mathcal{E}^{(C)}_i\rangle$ appears with the Gibbs weight $Z^{-1}_C(\beta)\exp[-\beta E(\mathcal{E}^{(C)}_i)]$. Here β is the inverse temperature of the cold bath, and Z_C is the partition function. From here we consider the average distribution

$$\bar{p}_C(\Delta E, f|i) = \sum_{\mathcal{E}^{(C)}_i} p_C(\Delta E, f|\mathcal{E}^{(C)}_i, i)\frac{e^{-\beta E(\mathcal{E}^{(C)}_i)}}{Z_C(\beta)}, \qquad (12)$$

which can be rewritten

$$\bar{p}_C(\Delta E, f|i) = \sum_{\mathcal{E}_f, \mathcal{E}_i} Z^{-1}_C(\beta)e^{-\beta E(\mathcal{E}_i)}\mathbf{1}_{E(\mathcal{E}_f)-E(\mathcal{E}_i),\Delta E}$$

$$\times \langle\mathcal{E}_f, f|\rho_{\text{TOT}}(\mathcal{E}_i, i)|\mathcal{E}_f, f\rangle, \qquad (13)$$

where $\rho_{\text{TOT}}(\mathcal{E}_i, i)$ is the total density operator of the system and the bath at the end of the process, when the system and

the cold bath started in the pure state $|\mathcal{E}_i, i\rangle$ and the hot bath started in ρ_H^{eq}. Resolving the delta function one can write

$$\bar{p}_L(\Delta E, f|i) = \frac{1}{2\pi} \int e^{-i\nu\Delta E} G_{\text{if}}(\nu) d\nu, \qquad (14)$$

where

$$G_{\text{if}}(\nu) = \sum_{\mathcal{E}_f, \mathcal{E}_i} Z_B^{-1}(\beta) e^{-\beta E(\mathcal{E}_i)} e^{i\nu[E(\mathcal{E}_f) - E(\mathcal{E}_i)]}$$
$$\times \langle \mathcal{E}_f, f|\rho_{\text{TOT}}(\mathcal{E}_i, i)|\mathcal{E}_f, f\rangle. \qquad (15)$$

By linearity the Gibbs weight and the factor $e^{-i\nu E(\mathcal{E}_i)}$ can be taken inside the the big unitary transformation defining $\rho_{\text{TOT}}(\mathcal{E}_i, i)$. The above is therefore the same as

$$G_{\text{if}}(\nu) = \text{Tr}_{\text{CH}} \langle f|e^{i\nu H_C} \left(U e^{-i\nu H_C} \rho_i^{\text{TOT}}\right) U^\dagger|f\rangle, \qquad (16)$$

where $\rho_i^{\text{TOT}} = \rho_H^{\text{eq}} \oplus |i\rangle\langle i| \oplus \rho_C^{\text{eq}}$, and the trace is over the cold and the hot bath(s).

$G_{\text{if}}(\nu)$ codifies all the information on the distribution of energy change in a bath (here the cold bath), averaged over an initial equilibrium distribution of the baths at their respective temperatures and conditioned on the system starting in pure state $|i\rangle$ and finishing in pure state $|f\rangle$. Derivatives of $G_{\text{if}}(\nu)$ with respect to ν generate moments of the energy change. Here we are interested in the first derivative

$$\langle \Delta E_C \rangle = \frac{d}{d(i\nu)} G_{\text{if}}(\nu)|_{\nu=0}. \qquad (17)$$

Furthermore we are only interested in thermal power, the limit $\frac{1}{t}\langle \Delta E_C \rangle$ when t, the duration of the process, is long.

Stepping first back a bit, the calculation of $G_{\text{if}}(\nu)$ proceeds by representing U and $U^\dagger$ as path integrals. Path integrals for spins are known in general [36], and have recently been used by one of us to estimate the errors in quantum computing [37]. For the problem at hand a much simpler representation is however sufficient, where the spin paths X and Y representing U and $U^\dagger$ are piece-wise constant, taking values $\pm\frac{1}{2}$ [21]. The baths are composed of sets of harmonic oscillators interacting linearly with the spin, and their terms in U and $U^\dagger$ as well as ρ_H^{eq}, ρ_C^{eq}, and $e^{\pm i\nu H_C}$ can be represented as standard path integrals, which can be integrated out as many Gaussians [33]. The functional $G_{\text{if}}(\nu)$ can hence be represented as a double path integral of the spin paths X and Y weighted by an action, i.e., as $e^{\frac{i}{\hbar}\mathcal{A}[X,Y]}$. At $\nu = 0$ this is the same spin-boson path integral derived in [21], which represents the quantum operation of moving the density matrix of the spin at time zero to the density matrix of the spin at time t. For nonzero values of ν additional terms appear in $\mathcal{A}$, details are summarized in Appendix B.

In practice the spin-boson path integrals are quite cumbersome to do without replying on the "noninteracting blip approximation" (NIBA). The terms in $\mathcal{A}$ that arise from integrating out the bath(s) are double integrals with kernels, and NIBA means that those kernels should have short enough memory. More precisely, memory should be shorter than the duration of the periods when X and Y take the same value, $(\frac{1}{2}, \frac{1}{2})$ or $(-\frac{1}{2}, -\frac{1}{2})$, so that the bath can only remember the preceding such period. Since the switching rate of paths in the double path integral is given by the tunneling rate in the system Hamiltonian, NIBA is hence expected to hold when

that tunneling rate is small enough. The same reasoning essentially holds for nonzero values of ν. The setup is summarized in Appendixes A and B.

With caveats discussed in Appendix H the stationary state (for the spin) in the spin-boson problem can then (within NIBA) be determined by almost classical arguments. A transition from the up state $(\frac{1}{2}, \frac{1}{2})$ to the down state $(-\frac{1}{2}, -\frac{1}{2})$ proceeds through two channels labeled by which spin path goes first (X or Y), and the time (Δt) spent in the intermediate "blip" state $[(\frac{1}{2}, -\frac{1}{2})$ or $(-\frac{1}{2}, \frac{1}{2})]$. The two jumps occurs with intrinsic rates $i\frac{\Delta}{2}$ and $-i\frac{\Delta}{2}$. Altogether, for both kinds of channels, this gives $\frac{\Delta^2}{4}$.

The two baths are in equilibrium with respect to the spin before the jump, and integrating them out thus leads to characteristic functions first determined in [21] and given in general form, for the jth blip interval of length Δt_{2j-1}, as Eqs. (A10) and (A11) in Appendix A. For the important special case of an Ohmic bath these quantities can be computed in closed form, as discussed at the end of Appendix C. To lighten the notation we will, in the rest of this section, refer to these quantities as S_C and X_C for the cold bath, and S_H and X_H for the hot bath, the dependence on Δt_{2j-1} (and parameters) implicit.

Summing contributions from all channels thus gives an overall transition rate from up to down:

$$A = \frac{\Delta^2}{2} \int e^{-\frac{1}{\hbar}(S_C + S_H)} \cos\frac{1}{\hbar}(X_C + X_H - \epsilon\Delta t) d\Delta t \qquad (18)$$

and a similar overall transition rate from down to up

$$D = \frac{\Delta^2}{2} \int e^{-\frac{1}{\hbar}(S_C + S_H)} \cos\frac{1}{\hbar}(X_C + X_H + \epsilon\Delta t) d\Delta t. \qquad (19)$$

The stationary probability to be up is $\frac{D}{A+D}$. This expression is formally identical with the dimensional arguments in Sec. III: A may be identified with $\Gamma_{\uparrow\downarrow}$; and D with $\Gamma_{\downarrow\uparrow}$ [38].

The calculations of thermal power detailed in Appendixes E to J rely crucially on exact relations between the derivative of the action $\mathcal{A}$ with respect to the parameter ν at $\nu = 0$, and the derivatives of the two functions S and X with respect to the time argument. It is then convenient to introduce additional characteristic functions of the hot and the cold baths [39]

$$C_+^C(t) = e^{-\frac{1}{\hbar}S_C + \frac{i}{\hbar}X_C}, \qquad (20)$$

$$C_+^H(t) = e^{-\frac{1}{\hbar}S_H + \frac{i}{\hbar}X_H}, \qquad (21)$$

$$C_-^C(t) = e^{-\frac{1}{\hbar}S_C - \frac{i}{\hbar}X_C}, \qquad (22)$$

$$C_-^H(t) = e^{-\frac{1}{\hbar}S_H - \frac{i}{\hbar}X_H}. \qquad (23)$$

The quantity A introduced above is then

$$A = \frac{\Delta^2}{4} \int \left[C_+^C(t) C_+^H(t) e^{-i\frac{\epsilon t}{\hbar}} + C_-^C(t) C_-^H(t) e^{i\frac{\epsilon t}{\hbar}} \right] dt \qquad (24)$$

and similarly for D.

As determined in the Appendix, the rate of energy change in the cold bath while the system is, respectively, in the up and

the down state can be written, compare Eq. (J10),

$$
\pi_{\mathrm{up}} = -i\hbar\frac{\Delta^2}{4}\int dt\, e^{-i\frac{\epsilon t}{\hbar}}\frac{dC_+^C(t)}{dt}C_+^H(t)
$$
$$
+ i\hbar\frac{\Delta^2}{4}\int dt\, e^{i\frac{\epsilon t}{\hbar}}\frac{dC_-^C(t)}{dt}C_-^H(t), \qquad (25)
$$

$$
\pi_{\mathrm{down}} = -i\hbar\frac{\Delta^2}{4}\int dt\, e^{i\frac{\epsilon t}{\hbar}}\frac{dC_+^C(t)}{dt}C_+^H(t)
$$
$$
+ i\hbar\frac{\Delta^2}{4}\int dt\, e^{-i\frac{\epsilon t}{\hbar}}\frac{dC_-^C(t)}{dt}C_-^H(t). \qquad (26)
$$

In the above results, C_+^C, C_+^H, C_-^C, and C_-^H are the influence functionals from integrating out the baths when the forward and backward paths of the spin are fixed and opposite. These influence functionals are formally $\mathrm{Tr}[U\rho^{\mathrm{eq},\uparrow}V^\dagger]$ with different unitary operators applied to the left and to the right. Differentiating U and $V^\dagger$ with respect to time brings down $-\frac{i}{\hbar}(H_B + H_I)$ and $\frac{i}{\hbar}(H_B + H_I')$ with different interaction Hamiltonians on the two sides because the spin coordinate is different on the two sides. The bath Hamiltonians are the same and their contributions therefore cancel. The remaining terms are expectation values of the interaction Hamiltonians, conditional on which state the spin started from, which path jumped first, and the blip duration. In this way Eqs. (25) and (26) give a precise meaning to the dimensional estimate (11), where $\langle\cdots\rangle$ means averaging over the quantum fluctuations in the bath during a blip of length Δt, over the kind of blip (which spin jumps first) and over the length of the blip.

V. POLARON TRANSFORM PICTURE

Another interpretation of the results in (25) and (26) is based on the polaron transform. Changing $\hat\sigma_z$ from up to down has the same effect on the bath energy as instantaneously shifting the position of every bath oscillator q_b by an amount $2\frac{C_b}{m_b\omega_b^2}$. Such a shift is generated by $\hat B_+ = \exp(i2\sum_b\frac{C_b}{\hbar m_b\omega_b^2}\hat p_b)$ where $\hat p_b$ is the momentum operator of oscillator b. Similarly $\hat B_- = \exp(-i2\sum_b\frac{C_b}{\hbar m_b\omega_b^2}\hat p_b)$ has the same effect on the bath energy as changing $\hat\sigma_z$ from down to up.

The function $C_-(t) = e^{-\frac{1}{\hbar}S - \frac{i}{\hbar}X}$ for the cold or hot bath (C or H) is therefore the same as $\langle\hat B_-(t)\hat B_+(0)\rangle_{\mathrm{eq}}$ where the operators are in the Heisenberg picture, and the average is over the bath in equilibrium. Similarly $C_+(t) = e^{-\frac{1}{\hbar}S + \frac{i}{\hbar}X}$ is the same as $\langle\hat B_-(0)\hat B_+(t)\rangle_{\mathrm{eq}}$. The effective jump rates are thus

$$
A = \frac{\Delta^2}{4}\int \langle\hat B_-(0)\hat B_+(t)\rangle_{C,\mathrm{eq}}\langle\hat B_-(0)\hat B_+(t)\rangle_{H,\mathrm{eq}}e^{-i\frac{\epsilon t}{\hbar}}
$$
$$
+ \langle\hat B_-(t)\hat B_+(0)\rangle_{C,\mathrm{eq}}\langle\hat B_-(t)\hat B_+(0)\rangle_{H,\mathrm{eq}}e^{i\frac{\epsilon t}{\hbar}}\,dt \qquad (27)
$$

and similarly for D. The above may be used to derive the weak-interaction limit since then $\hat B_+ \approx \mathbf{1} + i2\sum_b\frac{C_b}{\hbar m_b\omega_b^2}\hat p_b$ and $\hat B_- \approx \mathbf{1} - i2\sum_b\frac{C_b}{\hbar m_b\omega_b^2}\hat p_b$, and (linear terms cancel)

$$
\langle\hat B_-(t)\hat B_+(0)\rangle_{\mathrm{eq}} \approx 1 + \frac{4}{\hbar^2}\sum_b\frac{C_b^2}{m_b^2\omega_b^4}\langle\hat p_b(0)\hat p_b(t)\rangle_{\mathrm{eq}}. \qquad (28)
$$

Except for ϵ very small this gives the effective jump rate proportional to the sum of the spectral powers of the cold and hot bath at frequency $\epsilon/\hbar$, which can be compared, e.g., to [9] [Eq. (3)].

In a similar manner one may also consider Eqs. (25) and (26). The derivatives $\frac{dC_+^C(t)}{dt}$ and $\frac{dC_-^C(t)}{dt}$ translate (in weak coupling) to $\frac{4}{\hbar^2}\sum_b\frac{C_b^2}{m_b^2\omega_b^4}\langle\hat p_b(0)\frac{d\hat p_b(t)}{dt}\rangle_{C,\mathrm{eq}}$ and $\frac{4}{\hbar^2}\sum_b\frac{C_b^2}{m_b^2\omega_b^4}\langle\frac{d\hat p_b(t)}{dt}\hat p_b(0)\rangle_{C,\mathrm{eq}}$. The dependence on the hot bath is only to higher orders in the interaction coefficients, and therefore drops out. Given that $C_+^C(0)$ and $C_-^C(0)$ are both equal to 1, one may integrate by parts, which gives

$$
\pi_{\mathrm{up}} \approx \frac{\Delta^2}{4}\epsilon\int dt\, e^{-i\frac{\epsilon t}{\hbar}}C_+^C(t) + \frac{\Delta^2}{4}\epsilon\int dt\, e^{i\frac{\epsilon t}{\hbar}}C_-^C(t), \qquad (29)
$$

$$
\pi_{\mathrm{down}} \approx -\frac{\Delta^2}{4}\epsilon\int dt\, e^{i\frac{\epsilon t}{\hbar}}C_+^C(t) - \frac{\Delta^2}{4}\epsilon\int dt\, e^{-i\frac{\epsilon t}{\hbar}}C_-^C(t), \qquad (30)
$$

which is of the same form as Eqs. (8) and (9).

VI. DISCUSSION

In this paper we considered thermal power (heat per unit time) through a qubit interacting with two or several baths as in the spin-boson problem [21]. By an extension of the Feynman-Vernon influence functional method it has been shown how to compute the distribution of energy changes in a bath or baths of harmonic oscillators interacting with a general quantum system [10,22,24,35]. Here we adapted this approach to the situation where the system in one spin. This is not only a simplification since the effective dynamics of the spin is only Markovian when the bath temperature is sufficiently high. When that is so, and when the the spin degree of freedom actually only represents the two lowest energy eigenstates in a much larger system, higher states will also be excited, and the system is no longer well described as a spin. When the physical model is appropriate, the dynamics of the spin is therefore non-Markovian. The "noninteracting blip approximation" (NIBA) can be seen as the simplest form of non-Markovianity, where the memory of the bath(s) is (are) shorter than the times the system stays in one diagonal state (up or down). Although non-Markovian in continuous time the quantum dynamics is then Markovian in the discrete variables labeling the diagonal state. For the quantum state those variables can be summed out completely, while for the quantum heat they can be handled by a transfer matrix technique.

The advantage of the Feynman-Vernon method is that, while each oscillator in the bath is only perturbed slightly, and the system-bath interaction hence assumed linear in the harmonic oscillator coordinates, the accumulated effect on the system from all the bath oscillators can be large. A Feynman-Vernon theory of energy changes in a bath is thus a way to model quantum heat in a system interacting strongly with its environment. In this paper we only considered the expected value, but, in principle, higher moments can also be computed by the formulas given in [24]. Furthermore, we only considered the stationary case (constant drive) and the long-time limit which can be analyzed by Laplace transforms.

The general structure of quantum thermal power Π through one spin was discussed around Eq. (7) above by dimensional arguments. For convenience we restate this equation here:

$$\Pi = P(\text{up})\pi_{\text{up}} + P(\text{down})\pi_{\text{down}},$$

where $P(\text{up})$ and $P(\text{down})$ are the stationary probabilities of the spin to be, respectively, up or down, and π_{up} and π_{down} are the average thermal power in these two states. When coupling to the baths are weak, one can consider system transitions due to interactions with the baths as independent processes. The two terms π_{up} and π_{down} are then the transition rates by interacting with the cold bath, times the energy split ϵ. This well-known result can also be derived by perturbation theory.

When coupling between the system and the baths is strong the effective transition rates of the system do not separate. The formula for one of the rates (the rate A, up to down) is given in (24) and is the integral over time of the product of two factors that depend on time, one from each bath. Alternatively on can consider the polaron picture discussed in Sec. V: generally the transition rates are bath correlation functions of exponential quantities which can only be approximated as sums of standard correlation functions, one for each bath, in a weak-coupling limit.

The main technical result derived in this paper is then expressions for π_{up} and π_{down} in strong coupling. Like the effective transition rates these quantities do not separate, but can be written as time integrals of products of quantities from, respectively, the cold and the hot bath, see Eqs. (25) and (26). By nontrivial relations these quantities are for the hot bath *the same* as in the transition rate, and for the cold bath proportional to the *time derivative* of that in the transition rate. By an argument given towards the end of Appendix J the long-time limits of the expected energy changes in the hot and cold baths are equal in magnitude, and opposite in sign.

While the final result is simple, the intermediate calculations are not, as seems to be the case for most path integral treatments of the spin-boson problem, compare [21] as well as the later literature [7,40–42]. For the quantum state a much simpler approach is possible using the polaron transform directly [43,44]. Since our result for thermal power can also be expressed in terms of quantities after a polaron transform, it would be interesting to know if it can also be found in a simpler manner. We leave this question to future work, as well as the numerical determination of the terms (25) and (26) in thermal power.

We end by noting that for a qubit interacting with two baths the prediction of NIBA may be not only incorrect, but also physically inadmissible. The limits of the validity of NIBA may thus be qualitatively different in nonequilibrium compared to equilibrium. This question appears not have been raised before, and deserves further study. We further note that in NIBA the condition that thermal power to the cold bath be positive appears different than the admissibility condition on the state. Conceivably there may hence be situations where NIBA is appropriate for the quantum state, but not for quantum thermodynamics. This issue also deserves further study.

Note added. It was recently brought to our attention that a strong-coupling theory of thermal power of heat flow through a qubit using the polaron transform was presented in [45].

APPENDIX A: SUMMARY OF SPIN-BOSON THEORY AND NIBA

The calculations in Sec. IV are for the quantum thermal power and two baths what Leggett and collaborators did in the 1980's for the development of the quantum state and one bath [21]. This Appendix summarizes relevant results from that earlier calculation. For ease of comparison (here and in later related Appendixes) we follow the notation of [21]. We restate the system (qubit) Hamiltonian:

$$H_S = -\hbar\frac{\Delta}{2}\hat{\sigma}_x + \frac{\epsilon}{2}\hat{\sigma}_z, \tag{A1}$$

where Δ is a rate [dimension (time)$^{-1}$], and ϵ is the level splitting. The bath Hamiltonians are, in classical notation,

$$H_C = \sum_{b\in C}\frac{p_b^2}{2m_b} + \frac{1}{2}m_b\omega_b^2 q_b^2, \tag{A2}$$

$$H_H = \sum_{b\in H}\frac{p_b^2}{2m_b} + \frac{1}{2}m_b\omega_b^2 q_b^2, \tag{A3}$$

where the parameters m_b and ω_b are the mass and angular frequency of each oscillator and C and H also stand for the sets of oscillators in, respectively, the cold bath and the hot bath. The system-bath interactions are similarly

$$H_{\text{CS}} = -\sum_{b\in L}C_b q_b\hat{\sigma}_z, \tag{A4}$$

$$H_{\text{HS}} = -\sum_{b\in R}C_b q_b\hat{\sigma}_z, \tag{A5}$$

where C_b is the interaction coefficient between bath oscillator b and the qubit, and $\hat{\sigma}_z$ operates on the qubit. The coupling coefficients C_b have dimension (energy) $\times$ (length)$^{-1}$.

The Feynman-Vernon transition probability of a general quantum system interacting with two baths is

$$P_{\text{if}} = \text{Tr}_{\text{CH}}\langle f|U\left(|i\rangle\langle i|\oplus\rho_{\text{CH}}^{\text{eq}}\right)U^{\dagger}|f\rangle, \tag{A6}$$

where the initial state of the baths $\rho_{\text{CH}}^{\text{eq}}$ is the product state of two thermal states ρ_C^{eq} and ρ_H^{eq}, at two temperatures. U is the big unitary expressing the forward time evolution due to the total Hamiltonian given by (A1)–(A5), and $U^{\dagger}$ (the adjoint) is the backward time evolution.

The bath coordinates in Eq. (A6) can be integrated out to yield

$$P_{\rm if} = \int_{\rm if} \mathcal{D}X\mathcal{D}Y e^{\frac{i}{\hbar}S_S[X] - \frac{i}{\hbar}S_S[Y] + \frac{i}{\hbar}S_i^C[X,Y] + \frac{i}{\hbar}S_i^H[X,Y] - \frac{1}{\hbar}S_r^C[X,Y] - \frac{1}{\hbar}S_r^H[X,Y]}, \tag{A7}$$

where X is the system coordinate in the forward system path (part of the representation of U), Y is the system coordinate in the backward system path (part of the representation of $U^\dagger$), and $\int_{\rm if}$ denotes the projection on initial and final states (integrals over initial and final positions of the system in the forward and backward path). The result of integrating out the cold bath is $\frac{i}{\hbar}S_i^C[X,Y] - \frac{1}{\hbar}S_r^C[X,Y]$, and the result of integrating out the hot bath is $\frac{i}{\hbar}S_i^H[X,Y] - \frac{1}{\hbar}S_r^H[X,Y]$. The real terms ($S_r$) depend on the difference $X - Y$ at two different times while the imaginary terms (S_i) depend on the difference $X - Y$ at a later time, and the sum $X + Y$ at an earlier time.

For the system and bath interaction described by Eqs. (A1)–(A5) the system paths X and Y can be represented as piece-wise constant, taking value $\frac{1}{2}$ when the spin is up, and $-\frac{1}{2}$ when the spin is down. This means that at any one time the forward-backward system path pair can take only four values $(\frac{1}{2}, \frac{1}{2})$, $(-\frac{1}{2}, -\frac{1}{2})$, $(\frac{1}{2}, -\frac{1}{2})$, and $(-\frac{1}{2}, \frac{1}{2})$. The two first are in the terminology of [21] called *sojourns* and correspond to populations, the diagonal elements of the density matrix, up and down. The last first are in the terminology of [21] called *blips* and correspond to coherences, the off-diagonal elements of the density matrix. The kind of sojourn and blip can be indicated by variables $\chi = X + Y$ and $\xi = X - Y$, both taking values ± 1. A given double path in X and Y, conventionally starting from the up sojourn, can therefore be represented as

$$\sigma = (\chi_0 = 1, \Delta t_0, \xi_1, \Delta t_1, \chi_1, \Delta t_2, \xi_2, \Delta t_3, \ldots), \tag{A8}$$

where $\Delta t_0, \Delta t_2, \ldots$ are the durations of the sojourns and $\Delta t_1, \Delta t_3, \ldots$ are the durations of the blips. The first sojourn starts at time t_0 and the nth sojourn at time $t_{2n} = t_0 + \sum_{j=0}^{2n-1} \Delta t_j$; the nth blip starts at time $t_{2n+1} = t_0 + \sum_{j=0}^{2n} \Delta t_j$.

The $\hat{\sigma}_x$ terms in Eq. (A1) translate to weights in the integrations $\mathcal{D}X$ and $\mathcal{D}Y$ in Eq. (A7) which are $+i(\frac{\Delta}{2})$ if the forward path (X) jumps, and $-i(\frac{\Delta}{2})$ if the backward path (Y) jumps. Everything else is included in the total exponent in Eq. (A7) which one can write as

$$\mathcal{S}(\sigma) = \sum_j \left(-i\frac{\epsilon}{\hbar}\right)\Delta t_{2j-1} - \frac{1}{\hbar}\left[S_j^C(\Delta t_{2j-1}) + S_j^H(\Delta t_{2j-1})\right] + \frac{i}{\hbar}\xi_j\chi_{j-1}\left[X_{j,j-1}^C(t_{2j-2}, t_{2j-1}, t_{2j}) + X_{j,j-1}^H(t_{2j-2}, t_{2j-1}, t_{2j})\right]$$

$$+ \sum_j \sum_{k<j} -\frac{1}{\hbar}\xi_j\xi_k\left[\Lambda_{jk}^C(t_{2j}, t_{2j+1}, t_{2k}, t_{2k+1}) + \Lambda_{jk}^H(t_{2j}, t_{2j+1}, t_{2k}, t_{2k+1})\right]$$

$$+ \sum_j \sum_{k<j-1} \frac{i}{\hbar}\xi_j\chi_k\left[X_{jk}^C(t_{2j}, t_{2j+1}, t_{2k-1}, t_{2k}) + X_{jk}^H(t_{2j}, t_{2j+1}, t_{2k-1}, t_{2k})\right], \tag{A9}$$

where all terms are integrals over time of the terms in the exponent in Eq. (A7). The first line in above hence represent the terms $\frac{i}{\hbar}S_S[X] - \frac{i}{\hbar}S_S[Y]$ which have only one time integral, and which are nonzero only for blips, the terms $-\frac{1}{\hbar}S_r^C[X,Y] - \frac{1}{\hbar}S_r^H[X,Y]$, with both terms in the same blip, and $\frac{i}{\hbar}S_i^C[X,Y] + \frac{i}{\hbar}S_i^H[X,Y]$ with the sojourn immediately before the blip. The second and third line in Eq. (A9) correspond to times separated by at least one sojourn.

The *noninteracting blip approximation* (NIBA) of [21] is to ignore the second and third line of (A9), and to assume that $X_{j,j-1}^C$ and $X_{j,j-1}^H$ only depend on the associated blip duration Δt_{2j-1}. The validity of this approximation was discussed in depth in [21] and in the later literature, see e.g., [7,40–42]. Here we only note that it is essentially an expansion in small tunneling rates Δ, as lucidly explained in Refs. [43,44], with long blip durations suppressed as a result of the interaction between the system and the baths.

The content of NIBA is thus expressed in the following two characteristic functions of the baths, which we write for the cold bath as

$$X_{j,j-1}^C(\Delta t_{2j-1}) = \sum_{b\in C} \frac{C_b^2}{2m_b\omega_b^3} \sin\omega_b\Delta t_{2j-1}, \tag{A10}$$

$$S_j^C(\Delta t_{2j-1}) = \sum_{b\in C} \frac{C_b^2}{2m_b\omega_b^3} \coth\left(\frac{\omega_b\hbar\beta_C}{2}\right)(1 - \cos\omega_b\Delta t_{2j-1}). \tag{A11}$$

In the above the sums are over oscillators in the cold bath and β_C is the inverse temperature of the cold bath. The formulas for the contributions from the hot bath are analogous.

It is customary to also write the above functions as Q_1 and Q_2 as these are equivalent in NIBA. If one does not assume NIBA, $X_{j,j-1}$ would, however, be the sum of three terms Q_1 with different arguments, where the one above is the shortest time.

APPENDIX B: HEAT AND NIBA

The starting point is the generating function of energy changes in the cold bath

$$G_{if}(\nu) = \text{Tr}_{CH}\langle f|e^{i\nu H_C}U\left(|i\rangle\langle i| \oplus \rho_{CH}^{eq}e^{-i\nu H_C}\right)U^\dagger|f\rangle. \quad (B1)$$

This equation is the same as Eq. (A6) above, except that exponentials of the Hamiltonian of the cold bath have been

inserted at the initial and final time. It is assumed in Eq (B1) that $e^{-i\nu H_C}$ commutes with the initial density matrix of the baths ρ_{CH}^{eq}; this issue, related to strong coupling, will be discussed below.

As for Eq. (A6) we can introduce path integral representations of U and $U^\dagger$ and integrate out the two baths. The result must analogously to Eq. (A7) look like

$$G_{if}(\nu) = \int_{if} \mathcal{D}X\mathcal{D}Y e^{\frac{i}{\hbar}S_S[X]-\frac{i}{\hbar}S_S[Y]+\frac{i}{\hbar}\left(S_i^C[X,Y]+S_i^H[X,Y]\right)-\frac{1}{\hbar}\left(S_r^C[X,Y]+S_r^H[X,Y]\right)+\frac{i}{\hbar}\mathcal{J}_\nu[X,Y]+\frac{i}{\hbar}\tilde{\mathcal{J}}_\nu[X,Y]}, \quad (B2)$$

where the two new functionals $\mathcal{J}_\nu$ and $\tilde{\mathcal{J}}_\nu$, which represent the distribution of energy changes in the cold bath, are quadratic in X and Y. The two terms are for later convenience separated as to and, respectively, depending antisymmetric and symmetric combinations in the exchange of times. In earlier contributions the same two functionals and their kernels were distinguished by superscripts (2) and (3) [22,23,31]. Here we choose to streamline the formalism, additionally because the similar functional with superscript (1) does not appear; for a discussion, see [23].

When ν is equal to zero $G_{if}(\nu)$ is equal to P_{if}, and the two functionals $\mathcal{J}_\nu$ and $\tilde{\mathcal{J}}_\nu$ must vanish. In this paper we are concerned with the terms linear in ν which are given by

$$\mathcal{J}_\nu[X,Y] \approx \nu \int_{t_i}^{t_f} dt \int_{t_i}^t ds (X_t Y_s - X_s Y_t) h(t-s), \quad (B3)$$

$$\tilde{\mathcal{J}}_\nu[X,Y] \approx \nu \int_{t_i}^{t_f} dt \int_{t_i}^t ds (X_t Y_s + X_s Y_t) \tilde{h}(t-s), \quad (B4)$$

with two kernels

$$h(t-s) = i\hbar \sum_{b\in C} \frac{C_b^2}{2m_b} \coth\left(\frac{\beta\hbar\omega_b}{2}\right)\sin\omega_b(t-s), \quad (B5)$$

$$\tilde{h}(t-s) = \hbar \sum_{b\in C} \frac{C_b^2}{2m_b}\cos\omega_b(t-s). \quad (B6)$$

These two kernels are the same as $h^{(2)}$ and $h^{(3)}$ in [22], except for a factor $\hbar$.

It is a nontrivial fact [22] that h and $\tilde{h}$ are proportional to time derivatives of the Feynman-Vernon kernels

$$\tilde{h}(\tau) = \hbar\frac{d}{d\tau}k_i(\tau), \quad k_i = \sum_b \frac{C_b^2}{2m_b\omega_b}\sin\omega_b\tau, \quad (B7)$$

$$h(\tau) = -i\hbar\frac{d}{d\tau}k_r(\tau),$$

$$k_r = \sum_b \frac{C_b^2}{2m_b\omega_b}\coth\left(\frac{\omega\hbar\beta}{2}\right)\cos\omega_b\tau. \quad (B8)$$

Similar relations between second integrals of these kernels will be crucial in the following.

We can now represent $G_{if}(\nu)$ in a similar way to Eq. (A9) with new terms stemming from $\mathcal{J}$ and $\tilde{\mathcal{J}}$. We can write these as

$$J(\sigma) \approx \frac{1}{2}\nu \sum_j \sum_{k<j} \xi_j\chi_k X_{jk}^{(1)} - \nu\frac{1}{2}\sum_j\sum_{k\geqslant j}\xi_j\chi_k X_{jk}^{(1)}, \quad (B9)$$

$$\tilde{J}(\sigma) \approx -\frac{1}{2}\nu\sum_j S_j^{(1)} + \frac{1}{2}\nu\sum_j S_j^{(1')} - \frac{1}{2}\nu\sum_j\sum_{k<j}\xi_j\xi_k\Lambda_{jk}^{(1)}$$

$$+ \frac{1}{2}\nu\sum_j\sum_{k<j}\chi_j\chi_k\Lambda_{jk}^{(1')}. \quad (B10)$$

In the above $X_{jk}^{(1)}$ are the first-order terms in ν from the kernels antisymmetric in the time exchange. In contrast to the imaginary Feynman-Vernon kernel, both the blip-sojourn and sojourn-blip terms appear. Furthermore $S_j^{(1)}$ and $S_j^{(1')}$ are the first-order terms in ν from the kernels symmetric in the time exchange where both times fall in the same time interval. In contrast to the real Feynman-Vernon kernel, there are such terms from both blips and sojourns. Finally $\Lambda_{jk}^{(1)}$ and $\Lambda_{jk}^{(1')}$ are terms from two intervals of the same kind, either two blips or two sojourns.

A NIBA-like approximation to Eq. (B9) means to include only the terms from an adjacent blip and sojourn. These are on the one hand terms like $-\frac{1}{2}\nu\xi_j\chi_{j-1}X_{j,j-1}^{(1)}$, and on the other $\frac{1}{2}\nu\xi_j\chi_j X_{j,j}^{(1)}$ both of which depend on time increments as discussed for $X_{j,j-1}$ above. Only one of these time increments is for a blip interval (the same blip interval), and we are therefore led to

$$X_{j,j-1}^{(1)} \approx X_{j,j}^{(1)} \approx K(\Delta t_{2j-1})$$

$$\equiv i\hbar\sum_{b\in C}\frac{C_b^2}{2m_b\omega_b^2}\coth\left(\frac{\beta\hbar\omega_b}{2}\right)\sin\omega_b\Delta t_{2j-1}. \quad (B11)$$

From this we have the NIBA-like approximation

$$J(\sigma) \approx \frac{1}{2}\nu\sum_j \xi_j(\chi_{j-1}-\chi_j)K(\Delta t_{2j-1}). \quad (B12)$$

Comparing to Eqs. (B8) and (A11) we see that

$$K(\tau) = i\hbar\frac{d}{d\tau}S_j^C(\tau). \quad (B13)$$

A NIBA-like approximation to Eq. (B10) is a bit more involved, for two reasons. First the two terms $S_j^{(1)}$ and $S_j^{(1')}$

both need to be included, and they are both diverging in the bath cutoff frequency. This requires a separate discussion which we give below in Appendix E. Second, the terms on the second line of Eq. (B10) cannot be neglected entirely. This is so because the interaction of two neighboring sojourns $(\frac{1}{2}\nu\chi_j\chi_{j-1}\Lambda_{j,j-1}^{(1')})$ has one term which depends on the intervening blip time, and which hence gives

$$\Lambda_{j,j-1}^{(1')} \approx \tilde{K}(\tau) \equiv -\hbar \sum_{b \in C} \frac{C_b^2}{2m_b\omega_b^2}\cos\omega_b\tau. \quad (B14)$$

Comparing to Eq. (A10) we see that

$$\tilde{K}(\tau) = -\hbar\frac{d}{d\tau}X_{j,j-1}^C(\tau). \quad (B15)$$

APPENDIX C: OHMIC BATHS

Ohmic baths have spectra (density of states) that are continuous up to some very large upper cutoff Ω and increase quadratically with frequency. The number of oscillators with frequencies in the interval $[\omega, \omega + d\omega]$ is $f(\omega)d\omega$ can then be taken to be

$$\begin{aligned} f(\omega) &= \frac{2}{\pi}\omega_c^{-3}\omega^2 & \omega < \Omega, \\ f(\omega) &= 0 & \omega > \Omega, \end{aligned} \quad (C1)$$

where ω_c is some characteristic frequency less than Ω. The total number of oscillators is then $\frac{2}{3\pi}(\frac{\Omega}{\omega_c})^3$.

An alternative version is to take a smooth cutoff

$$f(\omega) = \frac{2}{\pi}\omega_c^{-3}\omega^2\exp\left(-\frac{\omega}{\Omega}\right). \quad (C2)$$

In this case the number of bath oscillators is $\frac{12}{\pi}(\frac{\Omega}{\omega_c})^3$.

The system-bath interactions are characterized by two parameters η_C and η_H such that for an oscillator in the cold bath

$$C_b = \sqrt{\omega_c^3 m_\omega\eta_L} \quad (C3)$$

and for an oscillator in the hot bath

$$C_b = \sqrt{\omega_c^3 m_\omega\eta_R}. \quad (C4)$$

For the spin-coupling problem the dimensions of η_C and η_H are (mass) × (length)^2 × (time)$^{-1}$, i.e., the action.

The terms $X_{j,j-1}(\tau)$ and $S_j(\tau)$ in Eqs. (A10) and (A11) were computed in [21] as $\eta\tan^{-1}(\Omega\tau)$, and $\frac{1}{2}\eta\log(1 + \Omega^2\tau^2) + \eta\log(\frac{\hbar\beta}{\pi\tau}\sinh\frac{\pi\tau}{\hbar\beta})$. The first is essentially a sign function. The second starts as $\frac{\eta}{2}\Omega^2\tau^2$ in the interval $\tau \ll \Omega^{-1}$, then grows as $\eta\log\Omega\tau + \frac{\eta}{2\Omega^2\tau^2}$ in the interval $\Omega^{-1} \ll \tau \ll \hbar\beta$ and finally behaves as $\eta\log\frac{\Omega\hbar\beta}{2\pi} + \frac{\eta\pi}{\hbar\beta}|\tau|$ when $\tau \gg \hbar\beta$. The derivative $\partial_\beta S_j(\tau)$ evaluates to $\eta/\beta(1 - \frac{\pi\tau}{\hbar\beta}\coth\frac{\pi\tau}{\hbar\beta})$. which is always negative. $S_j(\tau)$ is hence an increasing function of bath temperature. The second derivative $\partial_{\beta\tau}S_j(\tau)$ evaluates to $\frac{\pi\eta}{\hbar\beta^2}(-\coth\frac{\pi\tau}{\hbar\beta} + \frac{\pi\tau}{\hbar\beta}\sinh^{-2}\frac{\pi\tau}{\hbar\beta})$. which is also always negative. $\partial_t S_j(\tau)$ is hence also an increasing function of bath temperature.

K and $\tilde{K}$ can be computed from Eqs. (B13) and (B15): $\tilde{K}$ is essentially a delta function on the bath cutoff frequency scale Ω^{-1}, while K is basically a delta function on the time scale $\hbar\beta$, and for large τ a constant.

APPENDIX D: INTERACTION THROUGH BATH MOMENTUM

Theorem D1. Let a system described by coordinate X interact with by a bath of harmonic oscillators described by coordinate and momenta (q_b, p_b) through a combined bath and interaction Hamiltonian $\sum_b \frac{1}{2m_b}(p_b + m_bC_bX)^2 + \frac{1}{2}m_b\omega_bq_q^2$. The coupling coefficients C_b vanish at the beginning and the end of the process. Then the generating function of the change of bath energy is the same is if the combined bath and interaction Hamiltonian would have been $\sum_b \frac{1}{2m_b}p_b^2 + \frac{1}{2}m_b\omega_b^2(q_q - \frac{C_b}{\omega_b}X)^2$.

The proof proceeds by adapting the calculation in [24], in the following steps.

(1) The action corresponding to the Hamiltonian coupled through momentum is $\int\frac{1}{2}m_b\dot{q}_b^2 - m_bC_bX\dot{q}_b - \frac{1}{2}m_b\omega_bq_q^2$. By an integration by parts the term linear in $\dot{q}_b$ is changed to *boundary terms* $+ \int m_b\frac{d}{dt}(C_bX)q_b$.

(2) The path integral of the bath oscillator with fixed initial and final positions can then be considered to be that of a Lagrangian $\int\frac{1}{2}m_b\dot{q}_b^2 - \frac{1}{2}m_b\omega_b^2q_b^2 + m_b\frac{d}{dt}(C_bX)q_b$. This path integral can then be done as in Feynman-Vernon theory giving integrals of the external drive [here $m_b\frac{d}{dt}(C_bX)$] multiplying the initial and final positions of the oscillator, and a constant.

(3) The integrals are of the type (u in the notation of [24], Appendix A) $\frac{1}{\sin\omega_bt}\int_0^t\sin\omega_b(t - s)m_b\frac{d}{ds}(C_bX)(s)ds$. By a partial integration they can be combined with the boundary terms to give $\frac{m_b\omega_b}{\sin\omega_bt}\int_0^t\cos\omega_b(t - s)C_bXds$, multiplying the initial position of the bath oscillator in the forward path. There are four terms of this type with two sign changes compared to [24], Appendix A.

(4) The constant (B in the notation of [24], Appendix A) is two terms of the type $\frac{1}{m_b\omega_b\sin\omega_bt}\int_0^t\int_0^s\sin\omega_b(t - s)\sin\omega_bs'm_b\frac{d}{ds}(C_bX)(s)m_b\frac{d}{ds'}(C_bX)(s')ds'ds$. By two integrals by parts the sines are turned into cosines multiplying $(C_bX)(s)(C_bX)(s')$, and there is a change of sign. Additionally there is a boundary term $-\frac{m_b}{2}\int C_b^2X^2ds$, the same as appears in the complete square $-\frac{1}{2}m_b\omega_b^2(q_q - \frac{C_b}{\omega_b}X)^2$.

(5) The integration over the initial and final coordinates of the bath oscillator proceeds as in [24], Appendix A, and gives in fact the same result, with $m_bC_b\omega_b$ appearing instead of C_b. One of the authors (E.A.) points out that there is an error in Eqs. (25) and (A14) in [24]: the constant appearing in the kernel $\mathcal{J}^{(2)}$ should read $(yz' - y'z)/\Delta$ [instead of $(y'z' - yz)/\Delta$]. To linear order in the parameter ν these two quantities are, however, the same, hence there is no difference to the present paper.

In summary, the only difference to coupling through coordinate is hence that if the coupling coefficient to bath momentum is C, then the equivalent coupling coefficient to bath coordinate is $m\omega C$, as is also required dimensionally.

APPENDIX E: SINGULAR NIBA HEAT TERMS

In this Appendix we estimate the contributions $S_j^{(1)}$ and $S_j^{(1')}$ to Eq. (B10). Both these terms are second integrals of the kernel $\tilde{h}$ in Eq. (B6) over one blip or one sojourn interval,

hence proportional to

$$\text{Expr}(\Delta t) = \int_{t_i}^{t_{i+1}} ds \int_{t_i}^{s} ds' \sum_{b \in C} \frac{C_b^2}{2m_b} \cos \omega_b(s - s')$$

$$= \sum_{b \in C} \frac{C_b^2}{2m_b \omega_b^2} (1 - \cos \omega_b \Delta t). \quad \text{(E1)}$$

For an Ohmic bath with sharp cutoff this expression is $\frac{2\eta}{\pi}(\Omega - \delta_\Omega(\Delta t))$ where $\delta_\Omega(\Delta t)$ a delta-function smoothened at time scale Ω^{-1}. The contribution to $G_{\text{if}}(\nu)$ from $n+1$ sojourns and n blips is hence

$$\text{Expr} = \frac{2\eta}{\pi}\Omega - \frac{2\eta}{\pi}\Omega + \frac{2\eta}{\pi}\Omega \ldots$$
$$- \frac{2\eta}{\pi}\delta_\Omega(t_1 - t_0) + \frac{2\eta}{\pi}\delta_\Omega(t_2 - t_1) \ldots. \quad \text{(E2)}$$

While the first line sums to a large number it does not scale with the time, and there will hence not be any contribution to thermal power from these terms.

The large terms are, in fact, an artifact from assuming that the baths are in equilibrium at the start and the end of the process while still interacting strongly with the system. It has been known for quite some time that this leads to problems already for the open quantum system state [46–49]. One way to resolve the problem for heat is to assume that the interaction coefficients C_b depend on time, and vanish in the beginning of the process [23]. Assuming as in [23] and in analogy to Eq. (C3) above that $C_b(s) = \sqrt{\omega_c^3 m_\omega \eta(s)}$ we have instead of the above

$$\text{Expr} = \sum_i (-1)^i \left(\frac{1}{4}[\dot{\eta}(t_{i+1}) - \dot{\eta}(t_i)] + \frac{1}{4}\int_{t_i}^{t_{i+1}} \frac{(\dot{\eta})^2}{\eta} ds \right). \quad \text{(E3)}$$

In the above the bath cutoff frequency has been taken to infinity. Clearly if the function $\eta(s)$ is constant except at the boundaries this does not give anything proportional to the duration of the process.

APPENDIX F: NONSINGULAR NIBA HEAT TERMS: GENERAL FORMALISM

The main idea is to write the sum $G_i(\nu) = \sum_f G_{\text{if}}(\nu)$ as a matrix product (transfer matrix formalism). The formulation is as follows.

(1) Starting state i is by convention "up." The starting vector is therefore $\chi_0 = \binom{1}{0} = (\uparrow, \uparrow)$.

(2) End vector, when we sum over the final state of the system, is $\chi_n = \binom{1}{1} = (\uparrow, \uparrow) + (\downarrow, \downarrow)$.

(3) The phase terms at the jumps are determined by the translation tables

Sojourn $\rightarrow$ blip					
Sart state	χ	_end state_	ξ	_forward/backward_	_factor_
$\uparrow, \uparrow$	$+1$	$\uparrow, \downarrow$	$+1$	B	$-i\frac{\Delta}{2}$
$\uparrow, \uparrow$	$+1$	$\downarrow, \uparrow$	-1	F	$i\frac{\Delta}{2}$
$\downarrow, \downarrow$	-1	$\uparrow, \downarrow$	$+1$	F	$i\frac{\Delta}{2}$
$\downarrow, \downarrow$	-1	$\downarrow, \uparrow$	-1	B	$-i\frac{\Delta}{2}$

and

Blip $\rightarrow$ sojourn					
Start state	ξ	_end state_	χ	_forward/backward_	_factor_
$\uparrow, \downarrow$	$+1$	$\uparrow, \uparrow$	$+1$	B	$-i\frac{\Delta}{2}$
$\uparrow, \downarrow$	$+1$	$\downarrow, \downarrow$	-1	F	$i\frac{\Delta}{2}$
$\downarrow, \uparrow$	-1	$\uparrow, \uparrow$	$+1$	F	$i\frac{\Delta}{2}$
$\downarrow, \uparrow$	-1	$\downarrow, \downarrow$	-1	B	$-i\frac{\Delta}{2}$

(4) To every transition _sojourn $\rightarrow$ blip_ are associated terms $e^{\frac{i}{\hbar}\chi_{j-1}\xi_j(X_{j,j-1} + \frac{1}{2}\nu K)}$. Combine this and the phase factor to a matrix $\frac{i\Delta}{2}\mathbf{T}$.

(5) To every _blip interval_ is associated the terms $e^{-\frac{1}{\hbar}S_j - \frac{i}{\hbar}\epsilon(t_{2j} - t_{2j-1})}$. Call this the diagonal matrix $\mathbf{\Lambda}$.

(6) To every transition _blip $\rightarrow$ sojourn_ is associated a term $e^{-\frac{i}{\hbar}\chi_j\xi_j\nu\frac{1}{2}K}$ Combine this and the phase factors to a matrix $\frac{i\Delta}{2}\mathbf{S}$.

(7) To every transition _sojourn $\rightarrow$ sojourn_ is additionally associated as term $e^{\frac{i}{\hbar}\chi_j\chi_{j-1}\nu\frac{1}{2}\tilde{K}}$. This is the same for both signs of the blip in between.

(8) The transition _sojourn $\rightarrow$ sojourn_ is then given by a matrix $\mathbf{M}$ formed by $\mathbf{S\Lambda T}$ and the modifications due to $\tilde{K}$. By matrix multiplication one finds

$$\mathbf{M} = e^{-\frac{1}{\hbar}S} \begin{pmatrix} 2\cos\frac{1}{\hbar}(X - \epsilon t)e^{\frac{i}{\hbar}\frac{1}{2}\nu\tilde{K}} & -2\cos\frac{1}{\hbar}(X + \nu K + \epsilon t)e^{-\frac{i}{\hbar}\frac{1}{2}\nu\tilde{K}} \\ -2\cos\frac{1}{\hbar}(X + \nu K - \epsilon t)e^{-\frac{i}{\hbar}\frac{1}{2}\nu\tilde{K}} & 2\cos\frac{1}{\hbar}(X + \epsilon t)e^{\frac{i}{\hbar}\frac{1}{2}\nu\tilde{K}} \end{pmatrix}. \quad \text{(F1)}$$

For simplicity the blip interval is written t.

(9) The whole generating function can hence, within NIBA, be written as

$$G_i(\nu) = (1 \ \ 1)\left(\sum_n (-1)^n \left(\frac{\Delta}{2}\right)^{2n} \mathbf{M}^n \right) \binom{1}{0}, \quad \text{(F2)}$$

where all the blip times are implicit in the matrices $\mathbf{M}$ on the right-hand side.

To analyze Eq. (F2) in a stationary setting (the bias ϵ and all other parameters are constant in time) one takes a Laplace transform. Every sojourn interval then yields a factor λ^{-1}, and the n'th term in Eq. (F2) hence a factor λ^{-1-n}. For the Laplace

transform of the matrix it is convenient to write

$$\tilde{\mathbf{M}}(\lambda) = 2 \begin{pmatrix} A(\lambda) & -B(\lambda, \nu) \\ -C(\lambda, \nu) & D(\lambda) \end{pmatrix}, \qquad (\text{F3})$$

where

$$A = \int dt e^{-\lambda t} e^{-\frac{1}{\hbar}S} \cos \frac{1}{\hbar}(X - \epsilon t) e^{\frac{i}{\hbar}\frac{1}{2}\nu\tilde{K}}, \qquad (\text{F4})$$

$$B = \int dt e^{-\lambda t} e^{-\frac{1}{\hbar}S} \cos \frac{1}{\hbar}(X + \nu K + \epsilon t) e^{-\frac{i}{\hbar}\frac{1}{2}\nu\tilde{K}}, \quad (\text{F5})$$

$$C = \int dt e^{-\lambda t} e^{-\frac{1}{\hbar}S} \cos \frac{1}{\hbar}(X + \nu K - \epsilon t) e^{-\frac{i}{\hbar}\frac{1}{2}\nu\tilde{K}}, \quad (\text{F6})$$

$$D = \int dt e^{-\lambda t} e^{-\frac{1}{\hbar}S} \cos \frac{1}{\hbar}(X + \epsilon t) e^{\frac{i}{\hbar}\frac{1}{2}\nu\tilde{K}}. \qquad (\text{F7})$$

All S, X, K, and $\tilde{K}$ depend on the blip time t (at least in principle).

The Laplace transform of the generating function is

$$\hat{G}_i(\nu, \lambda) = \int dt e^{-\lambda t} G_i(\nu, t)$$

$$= \lambda^{-1}(1 \quad 1)\left[\sum_n (-1)^n \lambda^{-n}\left(\frac{\Delta}{2}\right)^{2n} \tilde{\mathbf{M}}^n\right]\begin{pmatrix}1\\0\end{pmatrix}. \quad (\text{F8})$$

APPENDIX G: GENERATING FUNCTION AT $\nu = 0$

The special case of $\nu = 0$ is an important check because that should give the quantity computed by Leggett in [21]: $\tilde{P}(\lambda) = \int dt e^{-\lambda t}\langle\sigma_z\rangle(t)$. The relation is $\langle\sigma_z\rangle(t) = 2 \cdot \text{Prob}(\text{"up"}, t) - 1$ and hence $\tilde{P}(\lambda) = \tilde{G}_{\text{if}}(\nu = 0, \lambda) - \lambda^{-1}$ where i and f are both "up." The formula found by Leggett is

$$\tilde{P}(\lambda) = \frac{1 - \tilde{h}/\lambda}{\lambda + \tilde{g}} \quad ([21], \text{Eq. (7.6))}, \qquad (\text{G1})$$

where

$$\tilde{g} = \int dt e^{-\lambda t} \Delta^2 e^{-\frac{1}{\hbar}S} \cos \frac{1}{\hbar}X \cos \frac{\epsilon t}{\hbar} \quad ([21], \text{Eq. (7.5a))}, \qquad (\text{G2})$$

$$\tilde{h} = \int dt e^{-\lambda t} \Delta^2 e^{-\frac{1}{\hbar}S} \sin \frac{1}{\hbar}X \sin \frac{\epsilon t}{\hbar} \quad ([21], \text{Eq. (7.5b))}. \qquad (\text{G3})$$

We hence consider Eq. (F8) at $\nu = 0$. We have the simplification that $C = A$ and $B = D$, and the Laplace transform matrix is hence

$$\tilde{\mathbf{M}}(\lambda) = 2\begin{pmatrix} A & -D \\ -A & D \end{pmatrix}. \qquad (\text{G4})$$

The eigenvalues of this matrix are 0 and $2(A + D)$. Positive powers of this matrix ($n \geqslant 1$) are thus simply

$$(\tilde{\mathbf{M}}(\lambda))^n = [2(A + D)]^{n-1}\tilde{\mathbf{M}}(\lambda), \qquad (\text{G5})$$

which means that

$$G_{\text{if}}(\nu = 0, \lambda) = \lambda^{-1} - \lambda^{-2}\left(\frac{\Delta}{2}\right)^2 \frac{2A}{1 + \lambda^{-1}\frac{\Delta^2}{2}(A + D)} \qquad (\text{G6})$$

We may identify $\frac{\Delta^2}{2}(A + D) = \tilde{g}$ and $\frac{\Delta^2}{2}A = \frac{1}{2}(\tilde{g} + \tilde{h})$ and so

$$G_{\text{if}}(\nu = 0, \lambda) = \lambda^{-1} - \lambda^{-2}\frac{1}{2}\frac{\tilde{g} + \tilde{h}}{1 + \lambda^{-1}\tilde{g}}. \qquad (\text{G7})$$

This means that

$$\tilde{P} = \lambda^{-1} - \lambda^{-2}\frac{\tilde{g} + \tilde{h}}{1 + \lambda^{-1}\tilde{g}} = \lambda^{-1}\frac{\lambda + \tilde{g} - \tilde{g} - \tilde{h}}{\lambda + \tilde{g}}, \quad (\text{G8})$$

which is Eq. (G1), as required. The result $G_i(\nu = 0, t) = 1$ (normalization of the system state) follows from $(1 \quad 1)\tilde{\mathbf{M}} = 0$, which means that $G_i(\nu = 0, \lambda) = \lambda^{-1}$ (only the $n = 0$ term survives).

APPENDIX H: LONG TERM LIMIT OF THE GENERATING FUNCTION AT $\nu = 0$

On physical grounds it is reasonable to assume that for long times the generating function is

$$G_{\text{if}}(\nu, t)_{\nu=0} = p + \sum_k q_k e^{-t r_k}, \qquad (\text{H1})$$

where p is the long term limit of the probability to be up, and q_k and r_k are some constants. The Laplace transform is then

$$\hat{G}_{\text{if}}(\nu, \lambda)_{\nu=0} = p\lambda^{-1} + \sum_k \frac{q_k}{\lambda + r_k}, \qquad (\text{H2})$$

from which follows

$$p = \lim_{\lambda \to 0} \lambda \tilde{G}_i(\nu, \lambda)_{\nu=0}. \qquad (\text{H3})$$

Inserting Eq. (G6) we have

$$p = \frac{D}{A + D}, \qquad (\text{H4})$$

where in the integrals defining A and D the Laplace transform parameter λ is zero.

A physical density matrix of the qubit must lie inside the Bloch sphere. A necessary condition for $\frac{D}{A+D}$ and $\frac{A}{A+D}$ to be the diagonal elements of a stationary density matrix in the long-time limit is hence that they fall between 0 and 1. For a qubit interacting with one bath at one temperature this was shown to be always the case in [21], even when the density matrix computed under these assumption of NIBA is not correct.

For our case of one qubit interacting with two baths the situation is more involved, and we state it as the following theorem.

Theorem H.1. Consider $S = S_C + S_H$ and $X = X_C + X_H$ as an even and an odd function on the whole line. Let $\hat{F}(\omega)$ be the Fourier transform of $e^{-\frac{1}{\hbar}S + \frac{i}{\hbar}X}$ and $\hat{F}^*(\omega) = \hat{F}(-\omega)$ the Fourier transform of $e^{-\frac{1}{\hbar}S - \frac{i}{\hbar}X}$. Then $\frac{D}{A+D}$ and $\frac{A}{A+D}$ are possible diagonal elements of a density matrix if $|\mathcal{I}\hat{F}(\frac{\epsilon}{\hbar})| < |\mathcal{R}\hat{F}(\frac{\epsilon}{\hbar})|$.

The proof is by simple translation. We may write

$$\frac{A}{A + D} = \frac{1}{2} + \frac{1}{2}\frac{\int dt e^{-\frac{1}{\hbar}S} \sin \frac{1}{\hbar}X \sin \frac{1}{\hbar}\epsilon t}{\int dt e^{-\frac{1}{\hbar}S} \cos \frac{1}{\hbar}X \cos \frac{1}{\hbar}\epsilon t} \qquad (\text{H5})$$

and the condition

$$0 \leqslant \frac{A}{A + D} \leqslant 1 \qquad (\text{H6})$$

is hence the same as

$$\left| \int dt e^{-\frac{1}{\hbar}S} \sin \frac{1}{\hbar} X \sin \frac{1}{\hbar} \epsilon t \right| \leqslant \left| \int dt e^{-\frac{1}{\hbar}S} \cos \frac{1}{\hbar} X \cos \frac{1}{\hbar} \epsilon t \right|.$$
(H7)

Multiplying out and identifying terms says that the imaginary part of the Fourier transform should be smaller in absolute value than the real part, at the frequency of the level splitting. Note that the theorem does not give a condition for NIBA with two baths to be correct, only a condition for it to give physically admissible populations.

With two caveats one may interpret Eq. (H4) in an almost classical manner. First we can (trivially) rewrite it as

$$p = \frac{\frac{\Delta^2}{2}D}{\frac{\Delta^2}{2}A + \frac{\Delta^2}{2}D},$$
(H8)

where (at $\lambda = 0$)

$$\frac{\Delta^2}{2}A = \left(i\frac{\Delta}{2} \right)\left(-i\frac{\Delta}{2} \right) \int dt e^{-\frac{1}{\hbar}S}(e^{\frac{i}{\hbar}(X-\epsilon t)} + e^{\frac{i}{\hbar}(-X+\epsilon t)}),$$
(H9)

$$\frac{\Delta^2}{2}D = \left(i\frac{\Delta}{2} \right)\left(-i\frac{\Delta}{2} \right) \int dt e^{-\frac{1}{\hbar}S}(e^{\frac{i}{\hbar}(X+\epsilon t)} + e^{\frac{i}{\hbar}(-X-\epsilon t)}).$$
(H10)

The two terms in $\frac{\Delta^2}{2}A$ are the integrals over time t of the influence functionals of two particular spin histories, where the state is $(\uparrow, \uparrow)$ before time zero, then at time zero either the forward or the backward path jumps to down, and then at time t the other path follows. The two terms $(i\frac{\Delta}{2})$ and $(-i\frac{\Delta}{2})$ are the jump rate amplitudes [dimension (time)$^{-1}$] for the two paths. These combined with the integral over time t hence gives a quantity analogous to the probability that the state transits from $(\uparrow, \uparrow)$ to $(\downarrow, \downarrow)$ per unit time. The two terms in $\frac{\Delta^2}{2}D$ may similarly be taken to represent the total rate of the state transiting from $(\downarrow, \downarrow)$ to $(\uparrow, \uparrow)$.

The first of the two caveats is that by the above A and D may have different signs so that one of $\frac{A}{A+D}$ and $\frac{D}{A+D}$ is negative, and the other is larger than one. If so, NIBA would not give a physically admissible state. The second is that even when $\frac{A}{A+D}$ and $\frac{D}{A+D}$ are both between 0 and 1, both A and D could be negative. NIBA would in that case give a physically admissible state, but not one that can be described as from a classical jump process.

APPENDIX I: DERIVATIVES OF GENERATING FUNCTION FORMULA AT $\nu = 0$

The expected energy change of the bath is given by the derivative of the generating function (F8) with respect to $i\nu$ taken at $\nu = 0$. At any ν this quantity is

$$\frac{d}{d(i\nu)}\hat{G}_i(\nu, \lambda) = -\lambda^{-2}\left(\frac{\Delta}{2}\right)^2 (1 \ \ 1)\left[\sum_l (-1)^l \lambda^{-l}\left(\frac{\Delta}{2}\right)^{2l}\tilde{\mathbf{M}}^l \right]\frac{d\tilde{\mathbf{M}}}{d(i\nu)}\left[\sum_k (-1)^k \lambda^{-k}\left(\frac{\Delta}{2}\right)^{2k}\tilde{\mathbf{M}}^k \right]\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$
(I1)

At $\nu = 0$ the sums on the left and the right simplify as above. *On the left* only the zeroth order term ($l = 0$) survives, while *on the right* we have

$$\left[\sum_k (-1)^k \lambda^{-k}\left(\frac{\Delta}{2}\right)^{2k}\tilde{\mathbf{M}}^k \right]\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \lambda^{-1}\left(\frac{\Delta}{2}\right)^2 \frac{2A}{1 + \lambda^{-1}\frac{\Delta^2}{2}(A+D)}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$
(I2)

The dependence on ν comes either through the function K, or the function $\tilde{K}$. In the first case only the off-diagonal elements (B and C) depend on ν, and the total expression is

$$\frac{d}{d(i\nu)}\hat{G}_i(\nu, \lambda)|_{\nu=0, \text{through } K} = \lambda^{-2}\left(\frac{\Delta}{2}\right)^2 2\dot{C} - \lambda^{-3}\left(\frac{\Delta}{2}\right)^4 \frac{2A}{1 + \lambda^{-1}\frac{\Delta^2}{2}(A+D)}2(\dot{C} - \dot{B}),$$
(I3)

where $\dot{C} = \frac{dC}{d(i\nu)}|_{\nu=0}$ through K and $\dot{B} = \frac{dB}{d(i\nu)}|_{\nu=0}$ through K. These derivatives follow from Eqs. (F5) and (F6) and are

$$\dot{B} = \int dt e^{-\lambda t} e^{-\frac{1}{\hbar}S} \sin \frac{1}{\hbar}(X + \epsilon t)\left(\frac{i}{\hbar}K \right),$$
(I4)

$$\dot{C} = \int dt e^{-\lambda t} e^{-\frac{1}{\hbar}S} \sin \frac{1}{\hbar}(X - \epsilon t)\left(\frac{i}{\hbar}K \right).$$
(I5)

Following Eq. (B13) we can rewrite this as

$$\dot{B} = \int dt e^{-\lambda t} e^{-\frac{1}{\hbar}S} \sin \frac{1}{\hbar}(X + \epsilon t)\left(-\frac{dS}{dt} \right),$$
(I6)

$$\dot{C} = \int dt e^{-\lambda t} e^{-\frac{1}{\hbar}S} \sin \frac{1}{\hbar}(X - \epsilon t)\left(-\frac{dS}{dt} \right).$$
(I7)

In the second case of dependence through $\tilde{K}$ the derivative matrix is

$$\frac{d\mathbf{M}}{d(i\nu)}\bigg|_{\nu=0,\,\text{through }\tilde{K}} = e^{-\frac{1}{\hbar}S}\begin{pmatrix} \cos\frac{1}{\hbar}(X-\epsilon t)\frac{dX}{dt} & \cos\frac{1}{\hbar}(X+\epsilon t)\frac{dX}{dt} \\ \cos\frac{1}{\hbar}(X-\epsilon t)\frac{dX}{dt} & \cos\frac{1}{\hbar}(X+\epsilon t)\frac{dX}{dt} \end{pmatrix},$$ (I8)

where we used Eq. (B15). Together with Eq. (I3) we have hence also

$$\frac{d}{d(i\nu)}\hat{G}_i(\nu,\lambda)|_{\nu=0,\,\text{through }\tilde{K}} = -\lambda^{-2}\left(\frac{\Delta}{2}\right)^2 2A' + \lambda^{-3}\left(\frac{\Delta}{2}\right)^4\frac{2A}{1+\lambda^{-1}\frac{\Delta^2}{2}(A+D)}2(A'-D'),$$ (I9)

where

$$A' = \int dt\, e^{-\lambda t}e^{-\frac{1}{\hbar}S}\cos\frac{1}{\hbar}(X-\epsilon t)\left(\frac{dX}{dt}\right),$$ (I10)

$$D' = \int dt\, e^{-\lambda t}e^{-\frac{1}{\hbar}S}\cos\frac{1}{\hbar}(X+\epsilon t)\left(\frac{dX}{dt}\right).$$ (I11)

APPENDIX J: LONG-TIME LIMIT OF THE DERIVATIVE

On physical grounds it is reasonable to assume that the derivative of the generating function with respect to its argument is for long times

$$\frac{d}{d(i\nu)}G_i(\nu,t)_{\nu=0} = \Pi \cdot t + b + \sum_k c_k e^{-t\lambda_k},$$ (J1)

where Π is the long-time limit of the power (heat per unit time), and b, c_k, and λ_k are some constants. The Laplace transform is then

$$\frac{d}{d(i\nu)}\tilde{G}_i(\nu,\lambda)_{\nu=0} = \Pi\lambda^{-2} + b\lambda^{-1} + \sum_k\frac{c_k}{\lambda+\lambda_k}$$ (J2)

from which follows

$$\Pi = \lim_{\lambda\to 0}\lambda^2\frac{d}{d(i\nu)}\tilde{G}_i(\nu,\lambda)_{\nu=0}.$$ (J3)

Inserting the various formulas above we have

$$\Pi = \frac{D}{A+D}\frac{\Delta^2}{2}\left(\int dt\, e^{-\frac{1}{\hbar}S}\sin\frac{1}{\hbar}(X-\epsilon t)(-\partial_t S_C) + \int dt\, e^{-\frac{1}{\hbar}S}\cos\frac{1}{\hbar}(X-\epsilon t)(\partial_t X_C)\right) + \frac{A}{A+D}\frac{\Delta^2}{2}$$

$$\times\left(\int dt\, e^{-\frac{1}{\hbar}S}\sin\frac{1}{\hbar}(X+\epsilon t)(-\partial_t S_C) + \int dt\, e^{-\frac{1}{\hbar}S}\cos\frac{1}{\hbar}(X+\epsilon t)(\partial_t X_C)\right),$$ (J4)

where in the integrals defining A and D the Laplace transform parameter λ is zero, and where the subscript C indicates that only the quantities for the cold bath are considered. Clearly we now have an expression for power similar to the dimensional formula (7). For the case of only one bath we can integrate by parts in (J4) to get

$$\text{One bath: } \Pi = \frac{D}{A+D}\frac{\Delta^2}{2}\left((\epsilon)\int dt\, e^{-\frac{1}{\hbar}S}\cos\frac{1}{\hbar}(X-\epsilon t)\right) + \frac{A}{A+D}\frac{\Delta^2}{2}\left((-\epsilon)\int dt\, e^{-\frac{1}{\hbar}S}\cos\frac{1}{\hbar}(X+\epsilon t)\right)$$

$$= \frac{D}{A+D}\frac{\Delta^2}{2}(\epsilon A) + \frac{A}{A+D}\frac{\Delta^2}{2}(-\epsilon D) = 0,$$ (J5)

which is the expected result. In the long term limit the thermal power from one qubit equilibrating with one bath must vanish. If we were to consider heat to the hot bath, all that would change (J4) is that the time derivatives would be $\partial_t S_H$ and $\partial_t X_H$. By adding the same argument as in Eq. (J5) shows that the the sum of thermal power to the cold bath and the hot bath cancel.

In the case of two baths and heat to one bath it is on the other hand more convenient to write $S = S_C + S_H$ and $X = X_C + X_H$ and to introduce the kernels [50]

$$C_+^C(t) = e^{-\frac{1}{\hbar}S_C + \frac{i}{\hbar}X_C},$$ (J6)

$$C_+^H(t) = e^{-\frac{1}{\hbar}S_H + \frac{i}{\hbar}X_H},$$ (J7)

$$C_-^C(t) = e^{-\frac{1}{\hbar}S_C - \frac{i}{\hbar}X_C},$$ (J8)

$$C_-^H(t) = e^{-\frac{1}{\hbar}S_H - \frac{i}{\hbar}X_H},$$ (J9)

in terms of which (J4) can be written

$$\text{Two baths: } \Pi = \frac{D}{A+D}\frac{\Delta^2}{4}\left(-i\hbar\int dt\, e^{-\frac{i\epsilon t}{\hbar}}\frac{dC_+^C(t)}{dt}C_+^H(t) + i\hbar\int dt\, e^{\frac{i\epsilon t}{\hbar}}\frac{dC_-^C(t)}{dt}C_-^H(t)\right) + \frac{A}{A+D}\frac{\Delta^2}{4}$$

$$\times\left(-i\hbar\int dt\, e^{\frac{i\epsilon t}{\hbar}}\frac{dC_+^C(t)}{dt}C_+^H(t) + i\hbar\int dt\, e^{\frac{-i\epsilon t}{\hbar}}\frac{dC_-^C(t)}{dt}C_-^H(t)\right). \tag{J10}$$

This is the formulation used in Secs. IV and V in the main text.

Physically, thermal power to the cold bath must be positive. Referring to the discussion at the end of Appendix H we may identify A as $\frac{1}{2}\hat{F}^*(\frac{\epsilon}{\hbar})$ and D as $\frac{1}{2}\hat{F}(\frac{\epsilon}{\hbar})$ and the terms in parentheses in Eq. (J10) as Fourier components of the function $H(t) = i\hbar\frac{dC_+^C(t)}{dt}C_+^H(t)$. Thermal power would then be $-\frac{\Delta^2}{2\mathcal{R}[\hat{F}(\frac{\epsilon}{\hbar})]}\mathcal{R}[\hat{F}(\frac{\epsilon}{\hbar})\hat{H}^*(\frac{\epsilon}{\hbar})]$.

[1] K. Sekimoto, *Stochastic Energetics*, Lecture Notes in Physics, Vol. 799 (Springer, New York, 2010).
[2] U. Seifert, Rep. Prog. Phys. **75**, 126001 (2012).
[3] M. Esposito, U. Harbola, and S. Mukamel, Rev. Mod. Phys. **81**, 1665 (2009).
[4] R. Kosloff and A. Levy, Annu. Rev. Phys. Chem. **65**, 365 (2014).
[5] J. P. Pekola, Nat. Phys. **11**, 118 (2015).
[6] S. Vinjanampathy and J. Anders, Contemp. Phys. **57**, 545 (2016).
[7] U. Weiss, *Quantum Dissipative Systems, 4th Ed.* (World Scientific, Singapore, 2012).
[8] R. Alicki and K. Lendi, *Quantum Dynamical Semigroups and Applications*, Lecture Notes in Physics, Vol. 286 (Springer, New York, 1987).
[9] B. Karimi and J. P. Pekola, Phys. Rev. B **94**, 184503 (2016).
[10] M. Carrega, P. Solinas, M. Sassetti, and U. Weiss, Phys. Rev. Lett. **116**, 240403 (2016).
[11] T. Motz, M. Wiedmann, J. T. Stockburger, and J. Ankerhold, New J. Phys. **20**, 113020 (2018).
[12] M. Esposito, M. A. Ochoa, and M. Galperin, Phys. Rev. B **92**, 235440 (2015).
[13] D. Newman, F. Mintert, and A. Nazir, Phys. Rev. E **95**, 032139 (2017).
[14] A. Ronzani, B. Karimi, J. Senior, Y.-C. Chang, J. T. Peltonen, C. Chen, and J. P. Pekola, Nature Phys. **14**, 991 (2018).
[15] A. Kato and Y. Tanimura, J. Chem. Phys. **143**, 064107 (2015).
[16] A. Kato and Y. Tanimura, J. Chem. Phys. **145**, 224105 (2016).
[17] K. Goyal and R. Kawai, arXiv:1712.09129.
[18] W. Dou, M. A. Ochoa, A. Nitzan, and J. E. Subotnik, Phys. Rev. B **98**, 134306 (2018).
[19] C. Kwon, J. Um, J. Yeo, and H. Park, arXiv:1810.13134.
[20] M. Perarnau-Llobet, H. Wilming, A. Riera, R. Gallego, and J. Eisert, Phys. Rev. Lett. **120**, 120602 (2018).
[21] A. J. Leggett, S. Chakravarty, A. T. Dorsey, M. P. A. Fisher, A. Garg, and W. Zwerger, Rev. Mod. Phys. **59**, 1 (1987).
[22] E. Aurell and R. Eichhorn, New J. Phys. **17**, 065007 (2015).
[23] E. Aurell, Entropy **19**, 595 (2017).
[24] E. Aurell, Phys. Rev. E **97**, 062117 (2018).

[25] M. H. Devoret, in *Les Houches Session LXIII*, edited by S. Reynaud, E. Giacobino, and J. Zinn-Justin, (Elsevier Science, Amsterdam, 1995).
[26] G. Wendin, Rep. Prog. Phys. **80**, 106001 (2017).
[27] U. Seifert, Phys. Rev. Lett. **116**, 020601 (2016).
[28] P. Talkner and P. Hänggi, Phys. Rev. E **94**, 022143 (2016).
[29] C. Jarzynski, Phys. Rev. X **7**, 011008 (2017).
[30] H. J. D. Miller and J. Anders, Phys. Rev. E **95**, 062123 (2017).
[31] E. Aurell, Phys. Rev. E **97**, 042112 (2018).
[32] A. Caldeira and A. Leggett, Physica A **121**, 587 (1983).
[33] R. P. Feynman and F. Vernon, Ann. Phys. (NY) **24**, 118 (1963).
[34] M. Carrega, P. Solinas, A. Braggio, M. Sassetti, and U. Weiss, New J. Phys. **17**, 045030 (2015).
[35] K. Funo and H. T. Quan, Phys. Rev. E **98**, 012113 (2018).
[36] A. Atland and B. Simons, *Condensed Matter Field Theory* (Cambridge University Press, Cambridge, England, 2006).
[37] E. Aurell, J. Stat. Phys. **171**, 745 (2018).
[38] The sum $A + D$ is proportional to the quantity called $\tilde{g}$ in [21] (at zero Laplace transform parameter), and the difference $D - A$ is proportional to $-\tilde{h}$. The magnetization is $\frac{D-A}{A+D}$, which equals $-\tilde{h}/\tilde{g}$ in the notation of [21].
[39] Equivalent functions were introduced in the previous literature, but not exactly for these quantities.
[40] M. Grifoni and P. Hänggi, Phys. Rep. **304**, 229 (1998).
[41] M. Grifoni, M. Winterstetter, and U. Weiss, Phys. Rev. E **56**, 334 (1997).
[42] L. Hartmann, I. Goychuk, M. Grifoni, and P. Hänggi, Phys. Rev. E **61**, R4687 (2000).
[43] C. Aslangul, N. Pottier, and D. Saint-James, J. Phys. (France) **47**, 1657 (1986).
[44] H. Dekker, Phys. Rev. A **35**, 1436 (1987).
[45] D. Segal and A. Nitzan, Phys. Rev. Lett. **94**, 034301 (2005).
[46] H. Grabert, U. Weiss, and P. Talkner, Z. Phys. B **55**, 87 (1984).
[47] G. W. Ford, J. T. Lewis, and R. F. O'Connell, Phys. Rev. Lett. **55**, 2273 (1985).
[48] M. Rosenau da Costa, A. O. Caldeira, S. M. Dutra, and H. Westfahl, Phys. Rev. A **61**, 022107 (2000).
[49] G.-L. Ingold, P. Hänggi, and P. Talkner, Phys. Rev. E **79**, 061105 (2009).
[50] Similar kernels were introduced in the literature before, but not exactly these ones, hence the new notation.