# Fractional Laplacians in bounded domains: Killed, reflected, censored, and taboo Lévy flights

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The fractional Laplacian $(-\Delta)^{\alpha/2}$, $\alpha \in (0, 2)$, has many equivalent (albeit formally different) realizations as a nonlocal generator of a family of $\alpha$-stable stochastic processes in $R^n$. On the other hand, if the process is to be restricted to a bounded domain, there are many inequivalent proposals for what a boundary-data-respecting fractional Laplacian should actually be. This ambiguity not only holds true for each specific choice of the process behavior at the boundary (e.g., absorbtion, reflection, conditioning, or boundary taboos), but extends as well to its particular technical implementation (Dirichlet, Neumann, etc., problems). The inferred jump-type processes are inequivalent as well, differing in their spectral and statistical characteristics, which may strongly influence the ability of the formalism (if uncritically adopted) to provide an unambiguous description of real geometrically confined physical systems with disorder. Specifically that refers to their relaxation properties and the near-equilibrium asymptotic behavior. In the present paper we focus on Lévy flight-induced jump-type processes which are constrained to stay forever inside a finite domain. This refers to a concept of taboo processes (imported from Brownian to Lévy-stable contexts), to so-called censored processes, and to reflected Lévy flights whose status still remains to be unequivocally settled. As a by-product of our fractional spectral analysis, with reference to Neumann boundary conditions, we discuss disordered semiconducting heterojunctions as the bounded domain problem.

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## I. MOTIVATION

Brownian motion in a bounded domain is a classic problem with ample coverage in the literature, specifically concerning the absorbing (Dirichlet) and reflecting (Neumann) boundary data (for the present purpose we disregard other boundary data choices). The coverage concerning their physical relevance is extensive as well [1,2].

Anticipating further discussion, we quite intentionally point out source papers dealing with reflected Brownian motion [3] and revealing at some length the method of eigenfunction expansions for the reflected and other boundary-data problems [4–8] (cf. also [9]). The latter method is as well an indispensable tool in the analysis of spectral properties of fractional Laplacians and related jump-type processes in bounded domains. Its direct link with the well-developed theory of heat semigroups for jump-type processes (mostly those with absorption or killing) allows one to address the statistics of exits from the domain, e.g., the first and mean first exit times, large time behavior, stationarity issues, and the probability of survival and its asymptotic decay (cf. [10–15]). Compare also [16,17] (Brownian case) and [18–20] (Lévy-stable case), where the role of the lowest eigenstates and eigenvalues (thence eigenvalue gaps) of the motion generator appears to be vital for the description of decay rates of killed stochastic processes. The spectral data of motion generators are relevant for quantifying long-living processes in a spatial trap (e.g., a bounded domain), eventually with an infinite lifetime.

Currently, the literature devoted to fractional Laplacians and related jump-type processes is extremely rich, albeit with no efficient interplay or communication between the physics- and mathematics-oriented communities. There is a definite prevalence of the very active purely mathematical research on this subject matter. On the other hand, even a concise listing of various real-world applications of the fractional calculus in science and engineering is beyond the scope of this introductory section (see, for example, [21–26]). Viewpoints of applied mathematicians can be consulted in [27–31]. The departure point of the present work is an apparent incompatibility of the implementation of reflecting boundary conditions for Lévy flights in the physics-motivated investigations in Refs. [32–36], set against varied (inequivalent) proposals available in the mathematical literature in Refs. [37–44].

We note that a general theory of censored Lévy processes has been developed [37] to handle jump-type processes which are not allowed to jump out of an open, sufficiently regular set $D \subset R^n$ (eventually closed, under suitable precautions [39]). Within this theory, reflected Lévy-stable processes have been introduced and so-called regional fractional Laplacians were identified as generators of these processes [38–40]. Nonlocal analogs of the Neumann boundary data have been associated with them in suitable ranges of the stability parameter [39]. We point out that other analogs of nonlocal Neumann data, imposed directly on the fractional Laplacian, were proposed as well [42].

The existence problem for jump-type processes with an infinite lifetime in a bounded domain seems to have been left aside in the physics literature (see, however, Refs. [16,17] in connection with diffusion processes and [15,45] for a preliminary discussion of the Cauchy process that is trapped in the interval). In contrast, permanently trapped Lévy-type processes (likewise diffusion processes) have their well-established place in the mathematical literature.

 

One category of such processes stems from the analysis of the long-time behavior of the survival probability in the case of absorbing enclosures. One may actually single out appropriately conditioned processes that never leave the domain once started within. Another category can be related to reflecting boundary data. In contrast to the reflected Brownian motion, this issue is conceptually more involved and as yet not free from ambiguities in the context of Lévy-stable processes. Thus, from both physical and applied mathematics points of view, constructing well-posed fractional (Lévy) transport models in bounded domains and keeping under control (the degree and physical relevance of) their possible inequivalence are of vital importance.

In the traditional Brownian lore, while defining the Laplacian in a bounded domain $D \subset R^n$, denoted tentatively by $\Delta_D$, we must account for various admissible boundary data that are local, i.e., set at the boundary $\partial D$ of an open set $D$. One may try to define a fractional power of the Laplacian by importing its locally defined boundary data on $\partial D$, through the so-called spectral definition $(-\Delta_D)^{\alpha/2}$ [27–30]. This operator is known to be different [31] from the outcome of the procedure in which one first executes the fractional power of the Laplacian and then imposes the boundary data, as embodied in the notation $(-\Delta)_D^{\alpha/2}$. In the case of absorbing boundaries, in contrast to $(-\Delta_D)^{\alpha/2}$, where Dirichlet conditions can be imposed locally at the boundary $\partial D$ of $D$, for $(-\Delta)_D^{\alpha/2}$ these data need to be imposed as exterior ones, i.e., in the whole complement $R^n\backslash D$ of $D$.

In passing we note that it is the nonlocality of fractional motion generators (fractional Laplacians) that is the main source of difficulties if the finite-size domain problems are to be considered. In the familiar to physicists lore of Riemann-Liouville fractional derivatives, defined in the Caputo sense, it is known that the divergence problems arise near the domain boundaries. In the study of transport properties of magnetically confined plasmas [23], a regularization of the otherwise singular fractional derivative of a general function has been accomplished by subtracting the boundary terms. A careful handling of such terms appears to be vital in the present research and allows one to make a clear distinction between e.g., absorbing, censored, and reflected processes and the corresponding fractional motion generators.

Reflected Brownian motions belong to the bounded domain paradigm [7,8] and likewise for the general family of censored Lévy flights (reflected case being included). If one resorts to the spectral definition of the fractional Laplacian on $D$, Neumann conditions can be imported directly from the Brownian framework and imposed locally.

This is not the case if censored Lévy processes and regional Laplacians become involved. In connection with reflected Lévy flights, a fairly nontrivial problem is to deduce a proper nonlocal analog of the Neumann boundary condition so that the existence status of the regional generator (and thence of the induced process) can be granted. An even more difficult issue is to provide a consistent (semi)phenomenological picture of the reflection mechanism, which should underlie (or directly follow from) the mathematical procedure. Nevertheless, Neumann type problems can be obtained in many ways, depending on the kind of reinjection we impose on the outside jumps [41].

The physically appealing reflection mechanism [a limiting infinite-well–trapping-interval case of the strongly anharmonic Langevin-Lévy evolution (cf. Refs. [32–36])] cannot be justified on the basis of the existing mathematical theory of reflected Lévy processes of Refs. [37,39,40] and needs a deeper discussion concerning its meaning and range of validity. On purely mathematical grounds, the resultant asymptotic probability density function can be readily recognized [15] as the so-called $\alpha$-harmonic function of the fractional Laplacian $(-\Delta)_D^{\alpha/2}$. Here we emphasize that the $\alpha$-harmonic function needs to be defined globally in $R$, although it may vanish beyond $D$, i.e., in $R\backslash D$. A strictly positive part (restricted exclusively to $D$) of the pertinent function, while normalized on $D$, can be interpreted as a probability density and, according to Ref. [36], sets a formal explanation of the "origin of the preferred concentration of flying objects near the boundaries in nonequilibrium systems."

We note that in the interior of $D$ this probability density function (PDF) rapidly diverges while approaching the boundary, in the whole parameter range $\alpha \in (0, 2)$. Such probability accumulation in the vicinity of the boundary barrier has been reported recently in the analysis of the fractional Brownian motion with a reflecting wall [46], albeit only in the superdiffusive regime $\alpha > 1$, while a probability depletion close to the barrier is a characteristic of the subdiffusive regime $\alpha < 1$.

Reflecting barriers were seldom seriously addressed by physicists in the case of Lévy-type processes, fractional diffusion, and continuous-time random-walk scenarios. On the other hand, their role in the so-called fractional Brownian motion and general anomalous diffusion problems has been analyzed [47–49], with observations that are different from the previously outlined ones. In addition, they remain incongruent with more mathematically oriented (including computer-assisted) research on reflected Lévy flights and fractional diffusion with reflection [50].

In the present paper, the main body of arguments has its roots in the theory of (nonlocally induced, Lévy-stable) Markov stochastic processes and spectral properties of their (nonlocal as well as fractional) generators. Hence many interesting research lines which refer to varied realizations of anomalous diffusions (processes with memory, those deriving from the continuous-time random walks, standard fractional Brownian motion, etc.) are disregarded. Nonetheless, we mention some source papers that investigate relaxation properties in the fractional transport that is governed by generalized Langevin equations, in particular in a finite domain [22,51,52] and in the context of the fractional Brownian motion [46,53], where depletion or accretion zones of particles near boundaries have been numerically predicted.

In the latter case [48], a clear specification is given of what pragmatically oriented researchers interpret as a reflection from the boundary (there are different prescriptions that may lead to inequivalent outcomes). The reflection recipe always refers to the trajectory behavior in the vicinity of the boundary. One needs to state clearly how to execute a reflection in the Monte Carlo pathwise simulations, i.e., not to cross the boundary once a jump of a given length would definitely take us away from the trapping enclosure. Compare, e.g., our discussion in Sec. VI and a related discussion of reflection conditions in Refs. [32–36]. None of these papers addresses the reflection boundary conditions for motion generators *per se*.

## II. VARIED (IN)EQUIVALENT FACES OF THE FRACTIONAL LAPLACIAN

### A. Fractional Laplacians in $R^n$

In the present paper, up to suitable adjustment of dimensional constants, the free stochastic evolution in $R^n$ refers to either the non-negative motion generator $-\Delta$ (Brownian motion) or $(-\Delta)^{\alpha/2}$ with $0 < \alpha < 2$ (Lévy-stable motion). One should keep in mind that it is $-(-\Delta)^{\alpha/2}$ which stands for a legitimate fractional relative of the ordinary Laplacian $\Delta$.

It is known that there are many formally different definitions of the fractional Laplacian which actually are equivalent [54]. For our purposes we will reproduce three equivalent in $R^n$ definitions of the symmetric Lévy-stable generator, which nowadays are predominantly employed in the literature [we do not directly refer to the popular notion of a fractional derivative, although formally one can write $(-\Delta)^{\alpha/2} \equiv -\partial^2/\partial|x|^{\alpha/2}$, whatever specific form of $(-\Delta)^{\alpha/2}$ is chosen].

The spatially nonlocal fractional Laplacian has an integral definition [involving a suitable function $f(x)$, with $x \in R^n$] in terms of the Cauchy principal value (P.V.), which is valid in space dimensions $n \geqslant 1$,

$$(-\Delta)^{\alpha/2} f(x) = \mathcal{A}_{\alpha,n} \lim_{\varepsilon\to 0^+} \int_{R^n \supset \{|y-x|>\varepsilon\}} \frac{f(x)-f(y)}{|x-y|^{\alpha+n}} dy, \quad (1)$$

where $dy \equiv d^n y$ and the (normalization) coefficient

$$\mathcal{A}_{\alpha,n} = \frac{2^\alpha \Gamma\left(\frac{\alpha+n}{2}\right)}{\pi^{n/2}|\Gamma\left(-\frac{\alpha}{2}\right)|} = \frac{2^\alpha \alpha \Gamma\left(\frac{\alpha+n}{2}\right)}{\pi^{n/2}\Gamma\left(1-\frac{\alpha}{2}\right)}. \quad (2)$$

Here one needs to employ $\Gamma(1-s) = -s\Gamma(-s)$ for any $s \in (0, 1)$. The normalization coefficient has been adjusted to secure that the integral definition stays in conformity with its Fourier transformed version. The latter actually gives rise to the widely (sometimes uncritically) used Fourier multiplier representation of the fractional Laplacian [27,29,30,54,55]

$$\mathcal{F}[(-\Delta)^{\alpha/2} f](k) = |k|^\alpha \mathcal{F}[f](k). \quad (3)$$

We recall again that it is $-(-\Delta)^{\alpha/2}$ which is a fractional analog of the Laplacian $\Delta$.

Another definition, which is quite popular in the literature in view of the more explicit dependence on the ordinary Laplacian, derives directly from the standard Brownian semigroup $\exp(t\Delta)$ (in passing we note that the Lévy semigroup reads $\exp[-t(-\Delta)^{\alpha/2}]$). The pertinent semigroup is explicitly built into the formula, originally related to Bochner's subordination concept [54–56]:

$$(-\Delta)^{\alpha/2} f = \frac{1}{|\Gamma\left(-\frac{\alpha}{2}\right)|} \int_0^\infty (e^{t\Delta} f - f) t^{-1-\alpha/2} dt. \quad (4)$$

Clearly, given an initial datum $f(x)$, we are dealing here directly with a solution of the standard (up to a dimensional coefficient) heat equation $f(x,t) = e^{t\Delta} f$ in the above integral formula.

We note that, based on tools from functional analysis (e.g., the spectral theorem), this definition of the fractional Laplacian extends to fractional powers of more general non-negative operators than $(-\Delta)$ proper. This point will receive more attention below.

*Remark 1.* While computing the singular integral (1), one needs to exert some care if a decomposition into a sum of integrals is involved. An alternative definition

$$(-\Delta)^{\alpha/2} f(x) = \frac{\mathcal{A}_{\alpha,n}}{2} \int_{R^n} \frac{2f(x) - f(x+y) - f(x-y)}{|y|^{n+\alpha}} dy, \quad (5)$$

if employed in suitable function spaces, is by construction free of singularities and does not involve the pertinent decompositions [27,29,31].

### B. Fractional Laplacians in a bounded domain

#### 1. Restricted (hypersingular) fractional Laplacian

As mentioned before, a domain restriction to a bounded subset $D \subset R^n$ is hard, if not impossible, to implement via the Fourier multiplier definition. The reason is an inherent spatial nonlocality of Lévy-stable generators.

We confine attention to the Dirichlet boundary data in a bounded domain. Here one begins, from the formal fractional operator definition, in $R^n$ and restricts its action to suitable functions with support in $D \subset R^n$. It is known that the standard Dirichlet restriction $f(x) = 0$ for all $x \in \partial D$ is insufficient. One needs to impose the so-called exterior Dirichlet condition $f(x) = 0$ for all $x \in R\backslash D$.

Let us tentatively assume that $f = g$ does not identically vanish in $R^n\backslash D$. The formal definition (1) (we use the principal value abbreviation P.V. for the integral) yields

$$\begin{aligned}(-\Delta)^{\alpha/2} f(x) &= \mathcal{A}_{\alpha,n} \text{P.V.} \int_{R^n} \frac{f(x)-f(y)}{|x-y|^{\alpha+n}} dy \\ &= \mathcal{A}_{\alpha,n} \left[ \text{P.V.} \int_D \frac{f(x)-f(y)}{|x-y|^{\alpha+n}} dy \right. \\ &\quad \left. + \int_{R^n\backslash D} \frac{f(x)-g(y)}{|x-y|^{\alpha+n}} dy \right]. \end{aligned} \quad (6)$$

Upon setting $g(x) = 0$ for all $x \in R\backslash D$, we arrive at the restricted fractional Laplacian $(-\Delta)_D^{\alpha/2}$, whose integral definition is shared with $(-\Delta)^{\alpha/2}$ [cf. Eq. (1)] but whose domain is restricted to functions vanishing in $R^n\backslash D$. Accordingly, for all $x \in D$ we have

$$(-\Delta)_D^{\alpha/2} f(x) = (-\Delta)^{\alpha/2} f(x) = h(x), \quad (7)$$

where the function $h(x)$ may not have the exterior property of $f(x)$. We point out that there is no restriction upon the integration volume, which is *a priori* $R^n$ and not solely $D \subset R^n$. Indeed,

$$\begin{aligned}(-\Delta)^{\alpha/2} f(x) &= \mathcal{A}_{\alpha,n} \left[ \text{P.V.} \int_D \frac{f(x)-f(y)}{|x-y|^{\alpha+n}} dy \right. \\ &\quad \left. + f(x) \int_{R^n\backslash D} \frac{dy}{|x-y|^{\alpha+n}} \right], \end{aligned} \quad (8)$$

so the exterior $R^n \backslash D$ contribution affects the outcome of (7) for all $x \in D$.

In the case of sufficiently regular domains, we encounter here a solvable spectral (eigenvalue) problem for the fractional Laplacian in a bounded domain $D$: $(-\Delta)_D^{\alpha/2}\phi_k(x) = \lambda_k\phi_k(x)$, $k \geqslant 1$. This spectral issue has ample coverage in the literature, and with regard to a detailed analysis of various eigenvalue problems for restricted fractional Laplacians, especially of the fully computable one-dimensional (1D) case in the interval $(-1, 1) = D \subset R$, we mention Refs. [57–63] (see also [55]).

We note that Eq. (8) can be converted to the form of the hypersingular Fredholm problem, discussed in detail in Refs. [60–63]. All involved singularities can be properly handled (are removable) and eventually one arrives at the formula [62]

$$(-\Delta)_D^{\alpha/2} f(x) \equiv -\mathcal{A}_{\alpha,n} \int_{\bar{D}} \frac{f(u)}{|u-x|^{n+\alpha}} du, \qquad (9)$$

where $x \in D$ and $D$ is the ultimate integration area. The $R \backslash D$ input, implicit in Eq. (8), has been completely eliminated.

#### 2. *Spectral fractional Laplacian*

We first impose the boundary conditions upon the Dirichlet Laplacian in a bounded domain $D$ i.e., at the boundary $\partial D$ of $D$. That is encoded in the notation $\Delta_{\mathcal{D}}$. Presuming to have in hand its $L^2(D)$ spectral solution [employed before in connection with (7)], we introduce a fractional power of the Dirichlet Laplacian $(-\Delta_{\mathcal{D}})^{\alpha/2}$ as

$$\begin{aligned}(-\Delta_{\mathcal{D}})^{\alpha/2} f(x) &= \sum_{j=1}^{\infty} \lambda_j^{\alpha/2} f_j \phi_j(x) \\ &= \frac{1}{\left|\Gamma\left(-\frac{\alpha}{2}\right)\right|} \int_0^{\infty} (e^{t\Delta_{\mathcal{D}}} f - f) t^{-1-\alpha/2} dt,\end{aligned} \qquad (10)$$

where $f_j = \int_D f(x)\phi_j(x)dx$ and $\phi_j$, $j = 1, 2, \ldots$, form an orthonormal basis system in $L^2(D)$: $\int_D \phi_j(x)\phi_k(x)dx = \delta_{jk}$.

We note that the spectral fractional Laplacian $(-\Delta_{\mathcal{D}})^{\alpha/2}$ and the ordinary Dirichlet Laplacian $\Delta_{\mathcal{D}}$ share eigenfunctions and their eigenvalues are related as well: $\lambda_j \leftrightarrow \lambda_j^{\alpha/2}$. The boundary data for $(-\Delta_{\mathcal{D}})^{\alpha/2}$ are imported from these for $\Delta_{\mathcal{D}}$. From the computational (computer-assisted) point of view, this spectral simplicity has been considered as an advantage, compared to other proposals (cf. [28,64,65]).

In contrast to the situation in $R^n$, the restricted $(-\Delta)_D^{\alpha/2}$ and spectral $(-\Delta_{\mathcal{D}})^{\alpha/2}$ fractional Laplacians are inequivalent and have entirely different sets of eigenvalues and eigenfunctions. Basic differences between them have been studied in [31] (see also [27,29,58]). An extended study of the intimately related subordinate killed Brownian motion in a domain can be found in Refs. [56,66]. We note one most obvious difference encoded in the very definitions: The boundary data for the restricted fractional Laplacian need to be exterior and set on $R^n \backslash D$, while those for the spectral one are set locally at the boundary $\partial D$ of $D$.

*Remark 2.* In the physics-oriented research, the existence of inequivalent fractional Laplacians in the case of bounded domains seems to have been overlooked. That led to vigorous discussions [67,68] about the validity of spectral (eigenvalue problems) solutions presented in Refs. [24,26]. See also [69] for simple, standard quantum-theory-motivated problems, e.g., the fractional infinite well or harmonic oscillator with mass $m \geqslant 0$. In particular, Laskin's spectral solution for the infinite well, as an outcome of his tacit assumptions, coincides precisely with the spectral Laplacian eigenvalue solution $\lambda_k^{\alpha/2}$ and $\phi_k(x)$, $k \geqslant 1$, given the standard solution $\lambda_k$ and $\phi_k(x)$, $k \geqslant 1$, of the quantum-mechanical infinite well. We point out an explicit usage of the spectral fractional Laplacian in Ref. [11], where the fractional process on the interval with absorbing ends has been resolved by means of the spectral affinity with an infinite-well problem [7,15,16].

*Remark 3.* Let us mention that solutions of the fractional infinite-well problem, together with that of an approximating sequence of deepening fractional finite wells, based on the restricted fractional Laplacian, have been addressed in Refs. [45,59–61,63,70–72]. Moreover, the fractional harmonic oscillator (including the so-called massless version, with the Cauchy generator involved) has been addressed in a number of papers (see, e.g., [13,60,73]; see also [74] for the quartic case).

*Remark 4.* In Ref. [12] the spectral definition of the fractional Laplacian has been comparatively mentioned, prior to the mathematically more refined analysis of Ref. [31]. In fact, a quantitative comparison has been made of the spectral and restricted Dirichlet problems on the interval. Numerical results for various average quantities have been found not to differ substantially. It has been noticed that the restricted Laplacian eigenfunctions are close to the spectral Laplacian eigenfunctions (trigonometric functions) except for the vicinity of the boundaries. An important observation was that the probability decay time rate formulas show detectable differences. In this connection it is also instructive to record an analogous situation while comparing the pure Brownian case with its spectral fractional relative.

*Remark 5.* It is useful to exemplify the differences between the restricted Laplacian and spectral Laplacian outcomes for the interval $(-1, 1)$. Namely, according to [72], one arrives at an approximate eigenvalue formula for $n \geqslant 1$ and $0 < \alpha < 2$: $\lambda_n = [\frac{n\pi}{2} - \frac{(2-\alpha)\pi}{8}]^\alpha - O(\frac{2-\alpha}{n\sqrt{\alpha}})$. We note that the eigenvalues of the ordinary (minus) Laplacian in the interval read $\lambda_n = [\frac{n\pi}{2}]^2$, $n \geqslant 1$. Up to dimensional coefficients, we have here the familiar quantum-mechanical spectrum of the infinite well, set on the interval in question. On the other hand, the spectral fractional well outcome is $\lambda_n^{\alpha/2} = [\frac{n\pi}{2}]^\alpha$, $n \geqslant 1$ (see [24,26,69]).

*Remark 6.* The ground-state function in the restricted case has been proposed in a quite complicated analytic form (actually, it is an approximate expression for a true eigenfunction) [71,72]. Prospective ground-state properties were also analyzed in minute detail, with numerical assistance [59–61]. The available data justify another approximation in terms of a function $\psi(x) = C_{\alpha,\gamma}[(1-x^2)\cos(\gamma x)]^{\alpha/2}$, where $C_{\alpha,\gamma}$ stands for the $L^2(D)$ normalization factor, while $\gamma$ is considered to be the best-fit parameter, allowing one to approach quite closely the computer-assisted eigenfunction outcomes [60]. This may be directly compared with the outcome for the spectral case, whose the ground state $\cos(\pi x/2)$ is shared with the ordinary Laplacian in the interval.

#### *3. Regional fractional Laplacian*

The regional fractional Laplacian has been introduced in conjunction with the notion of censored symmetric stable processes [37–41]. A censored stable process in an open set $D \subset R^n$ is obtained from the symmetric stable process by suppressing its jumps from $D$ to the complement $R^n \backslash D$ of $D$, i.e., by restricting its Lévy measure to $D$. Stated otherwise, a censored stable process in an open domain $D$ is a stable process forced to stay inside $D$. This is a clear difference from a number of proposals that define Neumann-type conditions (e.g., [42,43]), where outside jumps are admitted, albeit with an immediate return (resurrection; cf. [37,75]) to the interior of $D$.

The censorship idea resembles random processes conditioned to stay in a bounded domain forever [16,17]. However, we point out that the censoring concept is not the same [37] as that of the (Doob-type) conditioning. Instead, it is intimately related to reflected stable processes in a bounded domain with killing within the domain, or at least at its boundary, encompassing a class of processes (loosely interpreted as reflective) that do not approach the boundary at all [37,38].

In Ref. [38] the reflected stable processes in a bounded domain have been investigated, stringent criteria for their admissibility set, and their generators identified with regional fractional Laplacians on the closed region $\bar{D} = D \cup \partial D$. According to [38], censored stable processes of Ref. [37], in $D$ and for $0 < \alpha \leqslant 1$, are essentially the same as the reflected stable process.

In general [37], if $\alpha \geqslant 1$, the censored stable process never approaches $\partial D$. If $\alpha > 1$, the censored process may have a finite lifetime and may take values at $\partial D$.

Conditions for the existence of the regional Laplacian for all $x \in \bar{D}$, in spatial dimensions higher than one, have been set in Theorem 5.3 of [38]. For $1 \leqslant \alpha < 2$, the existence of the regional Laplacian for all $x \in \partial D$ is granted if and only if a derivative (this notion is not conventional and is adapted to the nonlocal setting) of a each function in the domain in the inward normal direction vanishes [38,40].

For our present purposes we assume $0 < \alpha < 2$ and $D \subset R^n$ being an open set. The regional Laplacian is assumed (a technical assumption employed in the mathematical literature) to act upon functions $f$ on an open set $D$ such that

$$\int_D \frac{|f(x)|}{(1+|x|)^{n+\alpha}} dx < \infty. \tag{11}$$

For such functions $f$, $x \in D$, and $\epsilon > 0$, we write [compare, e.g., Eqs. (6)–(9)]

$$(-\Delta)^{\alpha/2}_{D,\mathrm{reg}} f(x) = \mathcal{A}_{\alpha,n} \lim_{\varepsilon\to 0^+} \int_{y\in D\{|y-x|>\varepsilon\}} \frac{f(x)-f(y)}{|x-y|^{\alpha+n}} dy, \tag{12}$$

provided the limit (actually the Cauchy principal value) exists.

Note a subtle difference between the restricted and regional fractional Laplacians. The former is restricted exclusively by the domain property $f(x) = 0$, $x \in R^n \backslash D$. The latter is restricted exclusively by demanding the integration variable $y$ of the Lévy measure to be in $D$.

If we impose the Dirichlet domain restriction upon the regional fractional operator [$f(x) = 0$ for $x \in R^n \backslash D$, $D$ being an open set $D$], we can rewrite Eq. (8) as an identity relating the restricted and regional fractional Laplacians for all $x \in D$ [15,37,58],

$$(-\Delta)^{\alpha/2} f(x) = \left[(-\Delta)^{\alpha/2}_{D,\mathrm{reg}} + \kappa_D(x)\right] f(x), \tag{13}$$

where $\kappa_D(x) = \mathcal{A}_{\alpha,n} \int_{R^n\backslash D} |x-y|^{-(n+\alpha)} dy$ is non-negative and plays the role of the density of the killing measure [37].

Equations (12) and (13) actually indicate that the restricted fractional Laplacian can be obtained as a $\kappa_D$ perturbation of the regional one (see, e.g., [37]) and in principle we can move priority status from the restricted to the regional one, provided the exterior Dirichlet condition is respected by both operators. Indeed, while quantifying the random process associated with the generator $(-\Delta)^{\alpha/2}_{D,\mathrm{reg}} + \kappa_D(x)$ by invoking the concept of the Feynman-Kac kernel [37], one is tempted to view $\kappa_D(x)dt$ as the probability of extinction (killing) in the time interval $[t, t+\Delta t]$, and accordingly $\kappa_D(x)$ stands for the killing rate.

Proceeding otherwise, we can define the regional Laplacian as a perturbation of the restricted fractional one by the negative (not positive-definite) potential $(-\Delta)^{\alpha/2} - \kappa_D(x) = (-\Delta)^{\alpha/2}_{D,\mathrm{reg}}$. Then, however, $\kappa_D(x)$ would refer to the resurrection (birth or creation [37]) rate. One should be aware that to define the regional fractional Laplacian, various precautions concerning its domain must be observed to handle potentially divergent terms [$\kappa_D(x)$ being included].

If one were to replace $D$ in Eq. (12) by $\bar{D} = D \cup \partial D$ then, according to [37,38], one would arrive at the generator of a reflected stable process in $\bar{D}$ (cf. [38]), $(-\Delta)^{\alpha/2}_{\bar{D},\mathrm{reg}} f(x)$, provided suitable conditions (various forms of the Hölder continuity) upon functions in the domain of the nonlocal operator are respected. In particular, in the case of $1 \leqslant \alpha < 2$ it has been shown that $(-\Delta)^{\alpha/2}_{\bar{D},\mathrm{reg}} f(x)$ exists at a boundary point $x \in D$ if and only if the normal inward derivative vanishes: $(\partial f/\partial n)(x) = 0$. We note that the existence of the spectral solution (eigenvalue problem) for the regional Laplacian with (appropriately defined, generally not in an appealing derivative form) Neumann boundary data has been demonstrated in Ref. [40] (see also [37,39] for the one-dimensional discussion).

*Remark 7.* To avoid possible misunderstandings and misuse of the concept of killing (and subsequently that of resurrection or birth [75–78]), let us recall basic Brownian motion intuitions that underlie the implicit path-integral formalism. Namely, operators of the form $\hat{H} = -\frac{1}{2}\Delta + V \geqslant 0$ with $V \geqslant 0$ give rise to transition kernels of diffusion-type Markovian processes with killing (absorption), whose rate is determined by the value of $V(x)$ at $x \in R$. This interpretation stems from the celebrated Feynman-Kac (path-integration) formula, which assigns to $\exp(-\hat{H}t)$ the positive integral kernel

$$\exp\left[-(t-s)\left(-\tfrac{1}{2}\Delta + V\right)\right](y,x) = \int \exp\left[-\int_s^t V(\omega(\tau))d\tau\right] d\mu_{s,y,x,t}(\omega).$$

In terms of Wiener paths, that kernel is constructed as a path integral over paths that get killed at a point $X_t = x$, with an extinction probability $V(x)dt$ in the time interval

$(t, t + dt)$. The killed path is henceforth removed from the ensemble of ongoing Wiener paths. The exponential factor $\exp[-\int_s^t V(\omega(\tau))d\tau]$ is here responsible for a proper redistribution of Wiener paths, so the evolution rule

$$\exp(tL)f(x) = \int_{R^n} k(x, 0; y, t)f(y)dy = E^x[f(X_t)],$$

with $-L = -\frac{1}{2}\Delta + V$, is well defined as an expectation value of the killed process $X(t)$, but given in terms of Brownian paths $W(t)$ with the Feynman-Kac weight,

$$E^x[f(X_t)] = E^x\left[f(W_t)\exp\left(-\int_0^t V(W_\tau)d\tau\right)\right].$$

The resurrection executed with the same $V(x)$ rate would actually refer to another Feynman-Kac weight (note the sign change) $\int \exp[\int_s^t V(\omega(\tau))d\tau]d\mu_{s,y,x,t}(\omega)$, which amounts to the replacement of $V$ by $-V$ in the expression for the motion generator $L$ (as an instructive example one may conceive of a replacement of the harmonic-oscillator potential by its inverted version; cf. [79]).

## III. RELEVANCE VERSUS IRRELEVANCE OF EIGENVALUE PROBLEMS FOR FRACTIONAL LAPLACIANS IN A BOUNDED DOMAIN: STOCHASTIC VIEWPOINT

### A. Transition densities

Let us restate our motivations in a more formal context (our notation is consistent with that in Ref. [10]). Namely, given the (negative-definite) motion generator $L$, we will consider the (contractive) semigroup evolution of the form

$$f(x, t) = T_t f(x) = \exp(tL)f(x) = \int_{R^n} k(x, 0; y, t)f(y)dy = E^x[f(X_t)], \tag{14}$$

where $t \geqslant 0$. In passing, we have here defined a local expectation value $E^x[\cdots]$, interpreted as an average taken at time $t > 0$, with respect to the process $X_t$ started in $x$ at $t = 0$, with values $X_t = y \in R^n$ that are distributed according to the positive transition (probability) density function $k(x, 0; y, t)$.

We in fact deal with a bit more general transition function $k(x, s; y, t)$, $0 \leqslant s < t$, that is symmetric with respect to $x$ and $y$ and time homogeneous. This justifies the notation $k(x, s; y, t) = k(t - s, x, y) = k(t - s, y, x)$ and subsequently $k(x, 0; y, t) = k(t, x, y) = k(t, y, x)$. The heat equation $\partial_t f(x, t) = Lf(x, t)$ for $t \geqslant 0$ is here presumed to follow. We recall that, given a suitable transition function, we recover the semigroup generator via $[Lf](x) = \lim_{t\to 0} \frac{1}{t} \int_{R^n}[k(t, x, y)f(y)dy - f(x)]$, in accordance with the (implicit strong continuity) assumption that actually $T_t = \exp(Lt)$.

For completeness, let us mention that the semigroup property $T_t T_s = T_{t+s}$ implies the validity of the composition rule $\int_{R^n} k(t, x, y)k(s, y, z)dy = k(t + s, x, z)$. Let $B \subset R^n$, a probability that a subset $B$ has been reached by the process $X_t$ started in $x \in R^n$, after the time lapse $t$, be inferred from $P[X_t \in B|X_s = x] = \int_B k(t - s, x, y)dy$, $0 \leqslant s < t$; it reads

$$P^x(X_t \in B) = \int_B k(t, x, y)dy = k(t, x, B). \tag{15}$$

Clearly, $P^x(X_t \in R^n) = 1$. In general, for time-homogeneous processes, we have $k(x, s; B, t) = \int_B k(t - s, x, y)dy$, $s < t$, hence we can rephrase the Chapman-Kolmogorov relation as

$$\begin{aligned}\int_{R^n} k(x, s; z, u)k(z, u, B, t)dz &= k(x, s; B, t) = k(t - s, x, B) \\ &= P[X_{t-s} \in B|X_s = x],\end{aligned} \tag{16}$$

where $s < u < t$.

### B. Absorbing boundaries and survival probability

Now we move on to killed Brownian and Lévy-stable motions in a bounded domain. Let us denote by $D$ a bounded open set in $R^n$. By $T_t^D$ we denote the semigroup given by the process $X_t$ that is killed on exiting $D$. Let $k_D(t, x, y)$ be the transition density for $T_t^D$. Then [57]

$$T_t^D f(x) = E^x[f(X_t); t < \tau_D] = \int_D k_D(t, x, y)f(y)dy, \tag{17}$$

provided $x \in D$, $t > 0$, and the first exit time $\tau_D = \inf\{t \geqslant 0, X_t \notin D\}$ actually stands for the killing time for $X_t$.

From the general theory of killed semigroups in a bounded domain it follows that in $L^2(D)$ there exists an orthonormal basis of eigenfunctions $\{\phi_n\}$, $n = 1, 2, \ldots$, of $T_t^D$ and corresponding eigenvalues $\{\lambda_n, n = 1, 2, \ldots\}$ satisfying $0 < \lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \cdots$. Accordingly, $T_t^D\phi_n(x) = e^{-\lambda_n t}\phi_n(x)$ holds, where $x \in D$, $t > 0$, and we also have

$$k_D(t, x, y) = \sum_{n=1}^{\infty} e^{-\lambda_n t}\phi_n(x)\phi_n(y). \tag{18}$$

The eigenvalue $\lambda_1$ is nondegenerate (e.g., simple) and the corresponding strictly positive eigenfunction $\phi_1$ is often called the ground-state function.

For the infinitesimal generator $L_D$ of the semigroup we have $L_D\phi_n(x) = -\lambda_n\phi_n(x)$. The corresponding heat equation $\partial_t f(x, t) = L_D f(x, t)$ holds true as well.

It is useful to introduce the notion of the survival probability for the killed random process in a bounded domain $D$. Namely, given $T > 0$, the probability that the random motion has not yet been absorbed (killed) and thus survives up to time $T$ is given by

$$P^x[\tau > T] = P^x[X_T \in D] = \int_D k_D(T, x, y)dy \tag{19}$$

and is named the survival probability up to time $T$.

Proceeding formally with Eqs. (4) and (5), under suitable integrability and convergence assumptions for the infinite series, one arrives at an asymptotic survival probability decay rule

$$P^x[\tau > T] = \sum_{n=1}^{\infty} e^{-\lambda_n T}a_n\phi_n(x) \Rightarrow a_1 e^{-\lambda_1 T}\phi_1(x), \tag{20}$$

where $a_n = \int_D \phi_n(y)dy$, $n = 1, 2, \ldots$. This familiar exponential decay law is characteristic of, e.g., the Brownian motion with absorbing boundary data. Its time rate is controlled by the largest eigenvalue $-\lambda_1$ of $\Delta_D$.

Thus, as far as the asymptotic properties are concerned, we do not need the full-fledged spectral solution (e.g., that of the eigenvalue problem for the fractional Laplacian). The lowest eigenvalue and its eigenfunction are what really matters. The degree of accuracy with which we approximate the asymptotic behavior may be improved by not ignoring the second eigenvalue (i.e., the fundamental gap [18]). That is also specific to the reflected motion with the lowest eigenvalue zero.

### C. Conditioned random motions in a bounded domain: Taboo processes

For the absorbing stochastic process with the transition density (17) (thus surviving up to time $T$), we introduce survival probabilities $P^y[\tau > T - t]$ and $P^x[\tau > T]$, respectively, at times $T - t$ and $T$, $0 < t < T$. We infer a conditioned stochastic process with the transition density

$$q_D(t, x, y) = k_D(t, x, y)\frac{P^y[\tau > T - t]}{P^x[\tau > T]}, \qquad (21)$$

which by construction survives up to time $T$ and is additionally conditioned to start in $x \in D$ at time $t = 0$ and reach the target point $y \in D$ at time $t < T$. An alternative construction of such processes, in the diffusive case, has been described in [16].

Given $t < T$, in the large time asymptotic of $T$, we can invoke (19), and once $T \to \infty$ limit is executed, Eq. (20) takes the form

$$q_D(t, x, y) \longrightarrow p_D(t, x, y) = k_D(t, x, y)\frac{\phi_1(y)}{\phi_1(x)}\exp(\lambda_1 t). \qquad (22)$$

We have arrived at the transition probability density $p_D(t, xy)$ of the probability conserving process, which never leaves the bounded domain $D$. The latter eternal lifetime property is shared with the censored processes of Ref. [37], but the pertinent taboo processes (cf. [16,17,80] for the origin of the term "taboo") appear to form an independent family of random motions. Its asymptotic (invariant) probability density is $\rho(y) = [\phi_1(y)]^2$, $\int_D \rho(y)dy = 1$ [that in view of the implicit $L^2(D)$ normalization of eigenfunctions $\phi_n$].

By employing (19) and the definition $\rho(y) = [\phi_1(y)]^2$, we readily check the stationarity property. We take $\rho(x)$ as the initial distribution (probability density) of points in which the process is started at time $t = 0$. The propagation towards target points, to be reached at time $t > 0$, induces a distribution $\rho(y, t)$. Stationarity follows from

$$\rho(y, t) = \int_D \rho(x)p_D(t, x, y)dx = \rho(y). \qquad (23)$$

Note that, in contrast to $k_D(t, x, y)$, the transition probability function $p_D(t, x, y)$ is no longer a symmetric function of $x$ and $y$ (cf. in this connection Ref. [19]).

*Remark 8.* The semigroup $T_t^D(\alpha) = \exp[-t(-\Delta)_D^{\alpha/2}]$, $t \geqslant 0$, of the stable process killed upon exiting from a bounded set $D$ has an eigenfunction expansion of the form (19). Basically, we never have in hand a complete set of eigenvalues and eigenfunctions and likewise we generically do not know a closed analytic form for the semigroup kernel $k_D(t, x, y)$ [Eq. (19)]. A genuine mathematical achievement has been established that when $\alpha \in (0, 2)$ and a bounded domain $D$ is a subset of $R^n$, then the stable semigroup $T_t^D(\alpha)$ is intrinsically ultracontractive. This technical (intrinsically ultracontractive) property actually means that for any $t > 0$ there exists $c_t$ such that for any $x, y \in D$ we have [57,81–83]

$$k_D(t, x, y) \leqslant c_t\phi_1(x)\phi_1(y).$$

Actually we have $k_D(t, x, y) = \sum_1^\infty e^{-\lambda_n t}\phi_n(x)\phi_n(y)$. Accordingly,

$$\frac{k_D(t, x, y)}{e^{-\lambda_1 t}\phi_1(x)\phi_1(y)} = 1 + \sum_2^\infty e^{-(\lambda_n - \lambda_1)t}\frac{\phi_n(x)\phi_n(y)}{\phi_1(x)\phi_1(y)}.$$

It follows that we have complete information about the (large time asymptotic) decay of relevant quantities

$$\lim_{t\to\infty}\frac{k_D(t, x, y)}{e^{-\lambda_1 t}\phi_1(x)\phi_1(y)} = 1$$

and (for $t > 1$)

$$e^{-(\lambda_2 - \lambda_1)t} \leqslant \sup_{x,y\in D}\left|\frac{k_D(t, x, y)}{e^{-\lambda_1 t}\phi_1(x)\phi_1(y)}\right| \leqslant C_{\alpha,D}e^{-(\lambda_2 - \lambda_1)t}.$$

Thus, what we actually need to investigate the large time regime of Lévy processes in the bounded domain $D$ is to know the two lowest eigenvalues $\lambda_1$ and $\lambda_2$ and the ground-state eigenfunction $\phi_1(x)$ (eventually its shape [84]) of the motion generator (see, e.g., [18]). The existence of conditioned Lévy flights, with a transition density (21) and an invariant probability density $\rho(y) = [\phi_1(y)]^2$, $\int_D \rho(y)\,dy = 1$, is here granted.

### D. Reflected motions in a bounded domain

Reflected random motions in the bounded domain are typically expected to live indefinitely, never leaving the domain, basically with a complete reflection from the boundary. (We cannot *a priori* exclude a partial reflection, which is accompanied by killing or transmission.) In the case of previously considered motions, a boundary may be regarded either as a transfer terminal to the so-called cemetery (killing or absorption) or as being completely inaccessible from the interior (conditioned processes). In both scenarios, the major technical tool was the eigenfunction expansion (11), where the spectral solution for the Laplacian with the Dirichlet boundary data has been employed. Thus, in principle, we should here use the notation $\Delta_{\mathcal{D}}$, where $\mathcal{D}$ indicates that the Dirichlet boundary data have been imposed at the boundary $\partial D$ of $D \subset R^n$.

Reflecting boundaries are related to Neumann boundary data and therefore we should rather use the notation $\Delta_{\mathcal{N}}$. In a bounded domain we deal with a spectral (eigenvalue) problem for $\Delta_{\mathcal{N}}$ with the Neumann data-respecting eigenfunctions and eigenvalues.

A major difference from the absorbing case is that the eigenvalue zero is admissible and the corresponding eigenfunction $\psi_0(x)$ determines an asymptotic (stationary, uniform in $D$) distribution $\rho_0(x) = [\psi_0(x)]^2$ [7,8]. In the Brownian context, the rough form of the related transition density looks like

$$k_{\mathcal{N}}(t,x,y) = \frac{1}{\mathcal{V}(D)} + \sum_{n=1}^{\infty} e^{-\kappa_n t}\psi_n(x)\psi_n(y), \quad (24)$$

where $\kappa_n$ are positive eigenvalues, $\psi_n(x)$ respect the Neumann boundary data, and $\mathcal{V}(D)$ denotes the volume of $D$ (interval length, surface area, etc.). We have $\psi_0(x) = 1/\sqrt{\mathcal{V}(D)}$.

If one resorts to the spectral definition of the fractional Laplacian in a bounded domain, Neumann conditions are directly imported from these for the standard Laplacian and thence (see, e.g., [7,11,12]) the Laplacian eigenvalues $\lambda_k$, $k > 0$, need to be replaced by $\lambda_k^{\alpha/2}$. That is enough to map an eigenfunction expansion of the Laplacian-induced PDF and transition PDF to those appropriate for the fractional case (recalling the spectral definition of the fractional Laplacian; see also [31]).

*Remark 9.* Let us invoke the standard Laplacian in the interval (with an immediate passage to the spectral definition in mind). The case of reflecting boundaries in the interval is specified by Neumann boundary conditions for solutions of the diffusion equation $\partial_t f(x) = \Delta_{\mathcal{N}} f(x)$ in the interval $\bar{D} = [a,b]$. We need to have respected $(\partial_x f)(a) = 0 = (\partial_x f)(b)$ at the interval boundaries. The pertinent transition density reads [2,7]

$$k_{\mathcal{N}}(t,x,y) = \frac{1}{L} + \frac{2}{L}\sum_{n=1}^{\infty} \cos\left(\frac{n\pi}{L}(x-a)\right) \times \cos\left(\frac{n\pi}{L}(y-a)\right)\exp\left(-\frac{n^2\pi^2}{L^2}t\right). \quad (25)$$

The operator $\Delta_{\mathcal{N}}$ admits the eigenvalue 0 at the bottom of its spectrum, the corresponding eigenfunction being a constant whose square actually stands for a uniform probability distribution on the interval of length $L$ [$L = 2$ in the case of $(-1,1)$], to be approached in the asymptotic (large time) limit. Solutions of the diffusion equation with reflection at the boundaries of $D$ can be modeled by setting $p(x,t) = k_{\mathcal{N}}(t,x,x_0)$ while recalling that $p(x,0) = \delta(x-x_0)$. We can as well resort to $c(x,t) = \int_D k_{\mathcal{N}}(t,x,y)c(y)dy$ while recalling that $k(t,x,y) = k(t,y,x)$. Note that all $n \geqslant 1$ eigenvalues coincide with those of the absorbing case [11]. The eigenvalues of the spectral fractional operator for $n > 0$ would read $\lambda_n^{\alpha/2} = (n^2\pi^2/4)^{\alpha/2} = (n\pi/2)^{\alpha}$. For comparison we reproduce a rough analytic approximation of the restricted Laplacian eigenvalues $[\frac{n\pi}{2} - \frac{(2-\alpha)\pi}{8}]^{\alpha}$ [72] (see also [63] for an extended list of numerically generated sharp eigenvalues).

## IV. SPECTRAL PROPERTIES OF THE REGIONAL FRACTIONAL LAPLACIAN

All derivations and mathematically advanced demonstrations of the existence of regional fractional Laplacians and their link with (whatever that actually means) reflected Lévy flights and general censored processes involve subtleties concerning the behavior of functions in their domain, in the vicinity, and at the boundaries. Attempts to reconcile a semiphenomenologically prescribed boundary behavior of censored stochastic processes with the notion of (i) a regional fractional Laplacian and (ii) any conceivable forms of Neumann boundary data lead to inequivalent outcomes. This refers as well to the exploitation of Sobolev spaces instead of the more familiar for physicists $L^2(D)$ Hilbert spaces [40,85]. Other subtleties, like an issue of the Hölder continuity up to the boundary, need to be kept under control as well.

All that blurs or even hampers any pragmatic approach aiming at the deduction of approximate (if not sharp) eigenvalues, functional shapes of eigenfunctions, related (approximate) probability distributions, and estimates of probability killing rates or decay rates towards equilibrium if regional fractional Laplacians are interpreted as primary theoretical constructs, instead of the restricted ones. It is our purpose to follow the pragmatic routine, tested before in our investigation of fractional Laplacians with exterior Dirichlet boundary conditions [60,61,63]. We point out the existence of alternative computation routines (based on a direct discretization of fractional Laplacians) [59,86,87]. It is advantageous that we can directly compare our results with these of Ref. [58] for the regional Laplacian with the (exterior) Dirichlet boundary data. This test of a comparative validity of two different computation methods supports our subsequent analysis of the full-fledged spectral problem with no Dirichlet restriction, resulting in the implicit reflecting boundary conditions.

It is worthwhile to note that the main body of mathematical research refers to space dimensionalities $d \geqslant 2$. One should keep in mind, at least for comparative purposes, that the one-dimensional case (which is of interest for us below) differs from higher-dimensional ones in a number of technical points. Nevertheless, the one-dimensional case has an advantage of computability and allows for deeper insight into the properties of motion generators and processes, specifically into the issue of potentially dangerous divergences.

### A. Restricted versus regional fractional Laplacian

We depart from the formal definition (11)–(13) of the regional fractional Laplacian and following [15,58] concentrate on the one-dimensional spectral problem in the interval. The latter topic, for the Dirichlet fractional Laplacians, has been addressed before in a number of papers in the whole stability parameter range $0 < \alpha < 2$ and in more detail for the distinguished $\alpha = 1$ value (Cauchy process and motion generator) in Refs. [32–34,36,45,54,59–61,63,73].

We consider the regional fractional Laplacian (see, e.g., Ref. [38]) acting on the closed interval (including the end points) $-1 \leqslant x \leqslant 1$. To make clear the difference between the regional and ordinary fractional Laplacians, we begin with the fractional Laplacian on the whole real axis. We have [63]

$$(-\Delta)^{\alpha/2}\psi(x) = -A_\alpha \int_{-\infty}^{\infty} \frac{\psi(u)-\psi(x)}{|u-x|^{1+\alpha}}du,$$
$$A_\alpha = \frac{1}{\pi}\Gamma(1+\alpha)\sin\frac{\pi\alpha}{2}. \quad (26)$$

Since we assume the exterior Dirichlet condition $\psi(u) = 0$ for $R\backslash(-1,1)$ [cf. (8) and (9)], the division of the integral into separate contributions from within and outside the interval

$$(-\Delta)^{\alpha/2}\psi(x) = -A_\alpha \int_{-\infty}^{\infty} \frac{\psi(u)-\psi(x)}{|u-x|^{1+\alpha}}du = -A_\alpha\left[\int_{-\infty}^{-1} + \int_{-1}^{1} + \int_{1}^{\infty}\right]\frac{\psi(u)-\psi(x)}{|u-x|^{1+\alpha}}du \tag{27}$$

implies

$$\begin{aligned}(-\Delta)^{\alpha/2}\psi(x) &= -A_\alpha\left[-\psi(x)\int_{-\infty}^{-1}\frac{du}{|u-x|^{1+\alpha}} + \int_{-1}^{1}\frac{\psi(u)}{|u-x|^{1+\alpha}}du - \psi(x)\int_{-1}^{1}\frac{du}{|u-x|^{1+\alpha}} - \psi(x)\int_{1}^{\infty}\frac{du}{|u-x|^{1+\alpha}}\right]\\ &\equiv -A_\alpha\left[\int_{-1}^{1}\frac{\psi(u)}{|u-x|^{1+\alpha}}du - \psi(x)\int_{-\infty}^{\infty}\frac{du}{|u-x|^{1+\alpha}}\right] \equiv -A_\alpha\int_{-1}^{1}\frac{\psi(u)du}{|u-x|^{1+\alpha}} + \Delta I_{1\mathrm{D}}.\end{aligned} \tag{28}$$

Formally, the integral $\Delta I_{1\mathrm{D}}$ in (28) identically vanishes,

$$\Delta I_{1\mathrm{D}} = -\frac{A_\alpha\psi(x)}{\alpha}\left[\frac{1}{|u-x|^{\alpha}}\right]_{-\infty}^{\infty} = 0, \tag{29}$$

for all $0<\alpha<2$. This implies that the eigenvalue problem for the operator $|\Delta|^{\alpha/2}$ in the interval [i.e., actually the restricted fractional Laplacian $(-\Delta)_D^{\alpha/2}$ of Eq. (9)] is defined by the equation

$$(-\Delta)^{\alpha/2}\psi(x) = -A_\alpha\int_{-1}^{1}\frac{\psi(u)du}{|u-x|^{1+\alpha}} = E\psi(x). \tag{30}$$

The emergent spectrum is discrete $E\equiv E_n$, $n=1,2,\ldots$, nondegenerate, and positive $E_n>0$ (see, e.g., [57,62,63] and references therein).

The regional fractional Laplacian is defined similarly to Eq. (26) but directly in the interval $D=[-1,1]$ (which refers to the admitted integration area as well), i.e.,

$$(-\Delta)_{D,\mathrm{reg}}^{\alpha/2}\psi(x) = -A_\alpha\int_{-1}^{1}\frac{\psi(u)-\psi(x)}{|u-x|^{1+\alpha}}du = -A_\alpha\left[\int_{-1}^{1}\frac{\psi(u)du}{|u-x|^{1+\alpha}} - \psi(x)\int_{-1}^{1}\frac{du}{|u-x|^{1+\alpha}}\right]. \tag{31}$$

We can readily evaluate the integral

$$\int_{-1}^{1}\frac{du}{|u-x|^{1+\alpha}} = -\frac{1}{\alpha}\frac{1}{(1-x^2)^{\alpha}}[(1-x)^{\alpha}+(1+x)^{\alpha}] = -\frac{1}{A_\alpha}\kappa_D^{\alpha}(x), \tag{32}$$

where $D=[-1,1]$ and we obtain essentially the same $\kappa_D(x)$ that has emerged in the formula (13) [see also Eq. (2.13) in Ref. [58]]. We emphasize that, apart from a difference in the integration volumes for the involved integrals (13) and (32), the outcome of integrations is the same (see, e.g., [63]). We note a familiar [63] outcome $2/(x^2-1)$ for $\alpha=1$.

We thus arrive at the following spectral problem for the regional fractional Laplacian in $D=[-1,1]$:

$$-A_\alpha\int_{-1}^{1}\frac{\psi(u)du}{|u-x|^{1+\alpha}} - \frac{A_\alpha\psi(x)}{\alpha}\frac{(1-x)^{\alpha}+(1+x)^{\alpha}}{(1-x^2)^{\alpha}} = E\psi(x). \tag{33}$$

In short that reads

$$(-\Delta)_{D,\mathrm{reg}}^{\alpha/2}\psi(x) = (-\Delta)^{\alpha/2}\psi(x) - \kappa_D^{\alpha}(x)\psi(x) = E\psi(x), \tag{34}$$

with the persuasive interpretation of $(-\Delta)_{D,\mathrm{reg}}^{\alpha/2}\psi(x)$ as the additive perturbation of $|\Delta|^{\alpha/2}$ by the negative-definite potential $-\kappa_D^{\alpha}(x)$. The latter is an inverted version (cf. [79]) of the attractive singular (cf. [88]) potential $\kappa_D^{\alpha}(x)$. Its functional shape (the coefficient $A_\alpha$ has been omitted) for different values of $\alpha$ is reported in Fig. 1.

In passing we note that the concept of resurrected (after killing) Markov processes has been associated with the thus interpreted random noise generator $(-\Delta)_{D,\mathrm{reg}}^{\alpha/2}\psi(x)$ (see, e.g., [37,75]).

### B. Regional fractional Laplacian in $L^2(D)$, $D=[-1,1]$: Trigonometric base with Dirichlet boundary conditions

We do not know of any methods towards an analytic solution of the pertinent eigenvalue problem and therefore we revert to numerically assisted arguments, where an explicit diagonalization of Eq. (33) in the $L^2(D)$ trigonometric basis can be performed. To this end we will use the same even-odd base (and the computation method) as that in Ref. [63]. For any function $\psi_\alpha(x)$, which may possibly be a solution to the eigenvalue problem, we have the following expansions in the trigonometric basis of $L^2(D)$, $D=[-1,1]$:

$$\begin{aligned}\psi_{\alpha e}(x) &= \sum_{k=0}^{\infty} a_{k\alpha}\cos\frac{(2k+1)\pi x}{2},\\ \psi_{\alpha o}(x) &= \sum_{k=1}^{\infty} b_{k\alpha}\sin k\pi x.\end{aligned} \tag{35}$$

So defined would-be basis functions (35) satisfy the Dirichlet boundary conditions $\psi_{\alpha e,o}(\pm 1)=0$. The behavior of the derivative $d\psi(x=\pm 1)/dx$ is immaterial.

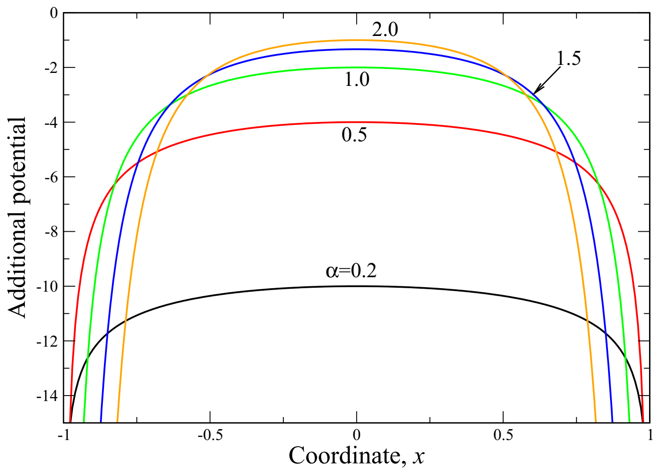


FIG. 1. Negative-definite singular potential (32) in the spectral problem for the regional fractional Laplacian for various values of $\alpha$ (figures near curves).

Continuing to the matrix representation of the eigenvalue problem, for the even subspace we have

$$\sum_{k=0}^{\infty} a_k(\gamma_{ki\alpha} + \beta_{ki\alpha}) = E a_{i\alpha}. \quad (36)$$

Explicitly, the matrix to be diagonalized has the form (we suppress an index $\alpha$ for the moment)

$$\hat{A}_{\text{even}} = \begin{pmatrix} \gamma_{00}+\beta_{00} & \gamma_{10}+\beta_{10} & \cdots & \gamma_{n0}+\beta_{n0} \\ \gamma_{10}+\beta_{10} & \gamma_{11}+\beta_{11} & \cdots & \gamma_{n1}+\beta_{n1} \\ \vdots & \cdots & \cdots & \vdots \\ \gamma_{n0}+\beta_{n0} & \gamma_{n1}+\beta_{n1} & \cdots & \gamma_{nn}+\beta_{nn} \end{pmatrix}, \quad (37)$$

where $\gamma_{ik} \equiv \gamma_{ki}$ are "old" [63] matrix elements coming from Eq. (30) [or the first term on left-hand side of Eq. (33)],

$$\gamma_{ik\alpha} = \int_{-1}^{1} f_{k\alpha}(x) \cos\frac{(2i+1)\pi x}{2} dx, \quad (38)$$

where $f_{k\alpha}(x)$ are defined by Eq. (16) of Ref. [63].

At the same time, $\beta_{ki}$ are "new" matrix elements coming from the second term on the left-hand side of (33), i.e., from the additive negative-definite potential (32),

$$\beta_{ik\alpha} = -\frac{A_\alpha}{\alpha}\int_{-1}^{1} \cos\frac{(2i+1)\pi x}{2} \cos\frac{(2k+1)\pi x}{2} \times \frac{(1-x)^\alpha + (1+x)^\alpha}{(1-x^2)^\alpha} dx. \quad (39)$$

The integrals for $\beta_{ik1}$ ($\alpha = 1$) can be calculated explicitly,

$$\beta_{ik1} = \frac{1}{\pi}\left\{(-1)^{k-i}\text{Ci}[2\pi(1+k+i)] + (-1)^{k+i+1}\text{Ci}[2\pi(k-i)] + (-1)^{k+i+1} \times \ln\frac{1+k+i}{k-i}\right\}. \quad (40)$$

Diagonal elements have the form

$$\beta_{ii1} = \frac{1}{\pi}\{-\gamma + \text{Ci}[2\pi(1+2i)] - \ln[2\pi(1+2i)]\}, \quad \gamma = 0.577\,216\ldots. \quad (41)$$

TABLE I. Six lowest eigenvalues of the regional fractional Laplacian for $\alpha = 0.5$, 1, and 1.5 calculated for the $2000 \times 2000$ matrix, composed with respect to the trigonometric base with Dirichlet boundary conditions. Energy levels are numbered according to standard convention: The lowest (ground) state is labeled $n = 1$ (even state with $k = 0$), $n = 2$ corresponds to the lowest odd state with $k = 1$, $n = 3$ corresponds to an even state with $k = 2$, etc. The corresponding eigenvalues from Ref. [58] are listed for comparison.

| $\alpha$ \ $i$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 0.5 | 0.0048 | 0.4495 | 0.8724 | 1.2189 | 1.5041 | 1.7809 |
| 0.5 [58] | 0.0038 | 0.4593 | 0.8626 | 1.2091 | 1.5149 | 1.7911 |
| 1.0 | 0.1177 | 1.1926 | 2.5888 | 4.0147 | 5.5328 | 7.0077 |
| 1.0 [58] | 0.1135 | 1.2026 | 2.5760 | 4.0292 | 5.5171 | 7.0245 |
| 1.5 | 0.8059 | 3.6475 | 7.7541 | 12.816 | 18.676 | 25.230 |
| 1.5 [58] | 0.8088 | 3.6509 | 7.7500 | 12.811 | 18.670 | 25.235 |

Here $\text{Ci}(x)$ is a cosine integral function [89]. Analogously, one proceeds with $\gamma_{ik1}$, which can be analytically computed as well [63].

For the odd subspace we have

$$\sum_{k=0}^{\infty} b_k(\eta_{ki\alpha} + \zeta_{ki\alpha}) = E b_{i\alpha}, \quad (42)$$

where $\eta_{ki\alpha}$ is given by Eq. (35) of Ref. [63] and

$$\zeta_{ik\alpha} = -\frac{A_\alpha}{\alpha}\int_{-1}^{1} \sin k\pi x \sin i\pi x \frac{(1-x)^\alpha + (1+x)^\alpha}{(1-x^2)^\alpha} dx. \quad (43)$$

For concreteness, we reproduce a computation outcome in the special (Cauchy) case $\alpha = 1$:

$$\zeta_{ik1} = \frac{1}{\pi}\left\{(-1)^{k-i}\text{Ci}[2\pi(k+i)] + (-1)^{k+i+1}\text{Ci}[2\pi(k-i)] + (-1)^{k+i+1}\ln\frac{k+i}{k-i}\right\}, \quad (44)$$

$$\zeta_{ii1} = \frac{1}{\pi}\{-\gamma + \text{Ci}[4i\pi] - \ln[4i\pi]\}. \quad (45)$$

For $\alpha \neq 1$ the same arguments are valid and can be safely reproduced step by step. The only difference is that matrix elements need to be calculated numerically. To obtain the results at a reasonable time cost, we use smaller-size matrices, around $50 \times 50$. This gives access to two decimal places for (lowest) eigenvalues and a fairly good approximation of the corresponding eigenfunctions.

The six lowest eigenvalues for Eq. (33) are reported in Table I for each of the stability indices $\alpha = 0.5$, 1.0, and 1.5. Quite good coincidence is seen with the spectral data reported in Table 1 of Ref. [58] (obtained by an alternative fractional Laplacian discretization method). Since our main purpose has been to test the computation method of [62,63] against an alternative proposal of Refs. [58,59], we refrain

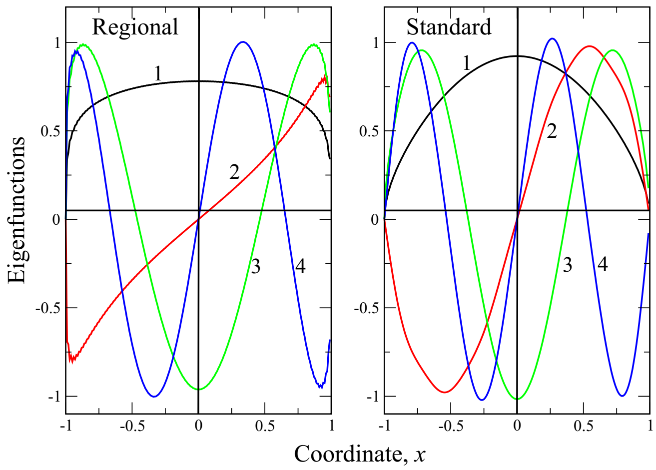


FIG. 2. Comparison between four lowest eigenfunctions of the (a) regional and (b) restricted fractional Laplacians for $\alpha = 1$. Eigenstate labels $n = 1, 2, 3, 4$ are indicated.

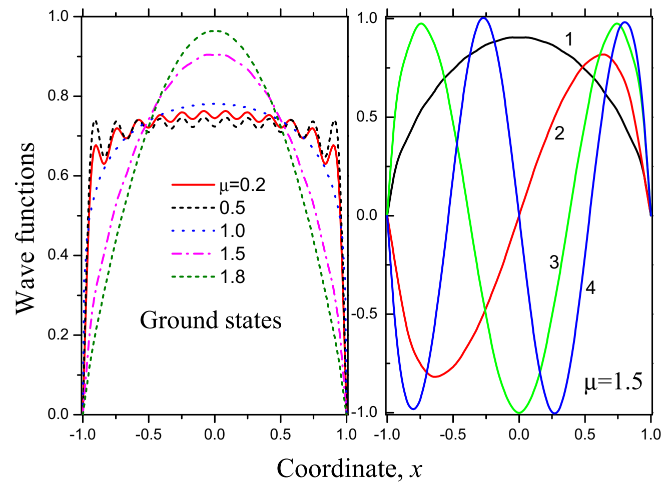


FIG. 4. (a) Comparison of ground states for the regional fractional operator (31) for different Lévy indices $\alpha$, shown in the panel. (b) Same as in Fig. 2 but for $\alpha = 1.5$. Figures near curves correspond to state labels (numbers 1,2,3,4).

from a comparative listing of other eigenvalues and other $\alpha$ choices; see, however, [58].

The four lowest eigenfunctions for $\alpha = 1$ are reported in Fig. 2. It can be seen [compare, e.g., Figs. 2(a) and 2(b)] that the eigenfunctions for regional and standard fractional Laplacians are qualitatively similar, with one exception. Namely, those for the regional operator show much sharper decay as $x \to \pm 1$ and their derivatives diverge to infinity as the boundary points are approached.

The more detailed comparative display of Fig. 3 gives further support to our statement about the sharp decay (steep descent down to zero) of the regional fractional Laplacian eigenfunctions set against those for the restricted one. Comparing Eqs. (30) and (33), we realize that the milder decay in the restricted case is a consequence of a strong repulsion (scattering) from the boundaries, which is encoded in the functional form of the (inverted) singular potential (32) in the eigenvalue problem of the form (13).

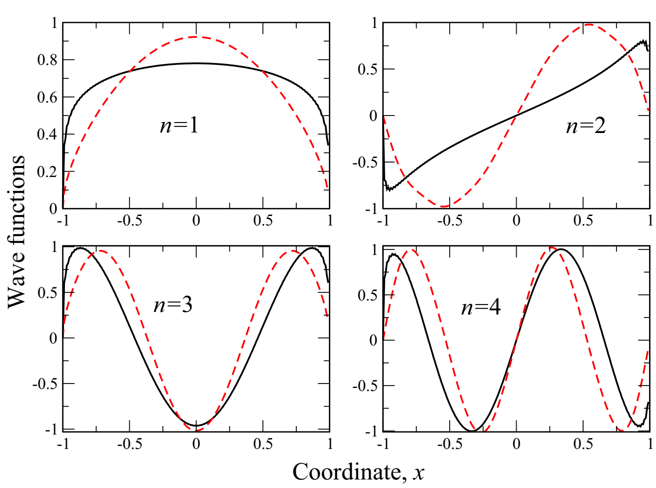


FIG. 3. Comparison of first four eigenfunctions of regional and restricted fractional Laplacians for $\alpha = 1$, while displayed in pairs corresponding to $n = 1, 2, 3, 4$. Black solid curves refer to the regional Laplacian and red dashed curves to the restricted one.

We note that our results are consistent with independent findings of Ref. [58] (which refers as well to a different computation method). Compare Fig. 7 therein, where the first and second eigenfunctions of the restricted and regional fractional Laplacians were compared. Additionally, those for the spectral fractional Laplacian have been depicted.

To give a glimpse of the $\alpha$ dependence of the spectral problems discussed, in Fig. 4 we display comparatively the $n = 1$ eigenfunctions (ground states) for several $\alpha \neq 1$ values [Fig. 4(a)]. We note that for $\alpha = 0.2$ and 0.5 the ground-state eigenfunctions show an oscillatory behavior, close to a maximum. The computation has been completed for $50 \times 50$ matrices (and checked for $30 \times 30$, which still can be viewed to provide a rough approximation). These oscillatory artifacts are expected to smoothen down with the growth in matrix size. For $\alpha = 1.5$ and 1.8, curves have been evaluated by employing $50 \times 50$ matrices and are found to be smooth.

Figure 4(b) depicts the four lowest eigenfunctions for $\alpha = 1.5$. The pertinent curves are qualitatively similar to those for $\alpha = 1$ (cf. Fig. 3).

### C. Regional fractional Laplacian: Trigonometric base with Neumann boundary conditions, an obstacle

On the basis of results reported in Figs. 2–4, it is possible to investigate the behavior of the derivative of the ground state in the vicinity of boundary points. Clearly, $\psi_1'(x \to \pm 1)$ becomes smaller as $\alpha$ increases. While for $0.2 < \alpha < 1$ the decay of $\psi_1(x)$ is steep, at $\alpha > 1$ the decay becomes progressively milder. Accordingly, as $\alpha$ increases, the derivative $\psi_n'(x \to \pm 1)$ of any state decreases. It is different from zero in the whole range $0 < \alpha < 2$.

It is thus natural to address the question of whether the traditional Neumann condition (vanishing of the derivative of the function at the boundaries) is at all feasible for the regional fractional Laplacian. The natural choice at this point

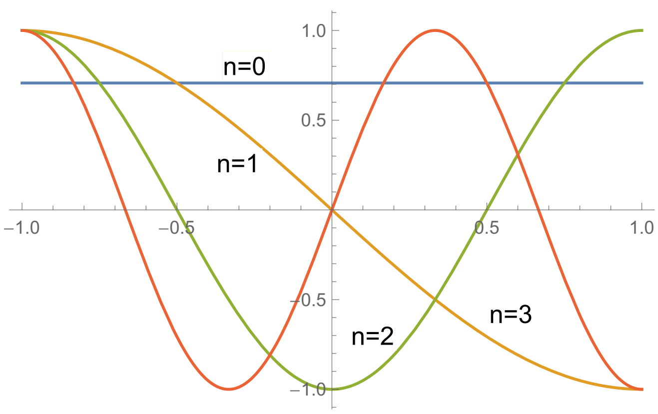


FIG. 5. Plot of four first functions of the Neumann base in the interval $[-1, 1]$. State labels $n = 0, 1, 2, 3$ are displayed near curves.

is to pass from the Dirichlet to the Neumann $L^2([-1, 1])$ basis [originally devised for the standard Laplacian in the interval; compare, e.g., Sec. III B, Eqs. (24) and (25)].

The pertinent Neumann base in the interval of length $L$ comprises $f_n = \cos(n\pi x/L)$, $0 < x < L$, $n = 0, 1, 2, \ldots$. In dimensionless units (measuring $x$ in units of $L$) the functions assume the form $f_n = \cos n\pi x$, $x \in [0, 1]$, $n = 0, 1, 2, 3, \ldots$. This basis system is orthogonal but not normalized: $\int_0^1 \cos n\pi x \cos k\pi x\, dx = \frac{1}{2}\delta_{nk}$, $n, k \neq 0$ or 1, $n = k = 0$. The passage to the interval $[-1, 1]$ is accomplished via a substitution $x \to (x+1)/2$, which gives rise to $f_n = \cos \frac{n\pi}{2}(x+1)$, $n = 0, 1, 2, \ldots$, $x \in [-1, 1]$. This basis system is orthonormal in $L^2([-1, 1])$ except for $f_0 = 1$. After incorporating the normalization coefficient, the orthonormal base with Neumann boundary conditions at end points of $[-1, 1]$ reads $\{f_0 = 1/\sqrt{2}, f_n = \cos \frac{n\pi}{2}(x+1), n = 1, 2, \ldots, x \in [-1, 1]\}$. The Neumann basis functions take nonzero values at the boundaries $x = \pm 1$ (see, e.g., Fig. 5) and this fact actually precludes the convergence of integrals that define matrix elements involved in the solution of the eigenvalue problem for the regional operator.

To demonstrate the divergence obstacle, we calculate explicitly the auxiliary function $g_{k\alpha}$ [see Eq. (15) of Ref. [63]] for $\alpha = 1$. We have

$$g_{k1} = -\frac{1}{\pi}\int_{-1}^{1} \frac{\cos \frac{k\pi}{2}(x+1)}{(z-x)^2} dz = \frac{1}{\pi}\left(\frac{\cos k\pi}{1-x} + \frac{1}{1+x}\right) + \frac{k}{2}\left\{\left[\mathrm{Ci}\frac{k\pi}{2}|x-1| - \mathrm{Ci}\frac{k\pi}{2}|x+1|\right]\sin\frac{k\pi}{2}(x+1) + \right.$$
$$\left. + \left[\mathrm{Si}\frac{k\pi}{2}(x+1) - \mathrm{Si}\frac{k\pi}{2}(x-1)\right]\cos\frac{k\pi}{2}(x+1)\right\}. \tag{46}$$

Now, if we pass to an explicit form of the matrix elements [cf. Eq. (18) of Ref. [63]]

$$\gamma_{kl1} = \int_{-1}^{1} \cos\frac{l\pi}{2}(x+1) g_{k1}(x) dx, \tag{47}$$

we see that the terms in Eq. (46) which contain $(1 \pm x)^{-1}$ generate the logarithmic divergence.

The choice of $\alpha = 1$ may be considered special, but explicit computations allow one to identify jeopardies to be met if the Neumann base is used. Actually, the situation is more intricate. Below we will see that for $\alpha \geqslant 1$ the divergence problem persists, while for $\alpha < 1$ one can handle the integrals.

Interestingly, these observations appear to stay in conformity with the mathematically rigorous discussion of divergence jeopardies in the case of censored Lévy flights [37] and of reflected Lévy flights proper [40], where the stability parameter ranges $0 < \alpha < 1$ and $1 \leqslant \alpha < 2$ were found to refer to qualitatively different jump-type processes.

We now return to the above functions $f_n = \cos \frac{k\pi}{2}(x+1)$ $(k = 0, 1, 2, \ldots)$ comprising a complete orthonormal basis system in $L^2(D)$, $D = [-1, 1]$. While seeking a solution $\psi_\alpha(x)$ of the eigenvalue problem for the fractional Laplacian, we consider the expansion $\psi_\alpha(x) = \sum_{k=0}^{\infty} a_{k\alpha} f_k(x)$.

To quantify a possible outcome of the choice of Neumann boundary conditions, it is sufficient to consider the first integral term in Eq. (31), which is a remnant of the action of the restricted fractional Laplacian proper upon $\psi(x)$. Clearly, the second term (the inverted potential) does not depend on the choice of the (Dirichlet vs Neumann) boundary conditions.

Following the arguments of [63], we invoke Eq. (15) therein and note that for an auxiliary function $g_{k\alpha}(x)$ $(0 < \alpha < 2)$ we actually have

$$g_{k\alpha}(x) = -A_\alpha \int_{-1}^{1} \frac{f_k(u)}{|u-x|^{1+\alpha}} = \{x - u = t,\ u = x - t,\ du = dt\}$$
$$= -A_\alpha \int_{x-1}^{x+1} \frac{f_k(x-t)dt}{t^{1+\alpha}} = -A_\alpha \times \begin{cases} \frac{dt}{t^{1+\alpha}} = dv, & u = f_k(x-t) \\ v = -\frac{1}{\alpha t^\alpha}, & du = -f_k'(x-t)dt \end{cases}$$
$$= -A_\alpha \left[ -\frac{f_k(x-t)}{\alpha t^\alpha}\Big|_{x-1}^{x+1} - \frac{1}{\alpha}\int_{x-1}^{x+1} \frac{f_k'(x-t)dt}{t^\alpha} \right]$$
$$= \frac{A_\alpha}{\alpha}\left[ \frac{f_k(1)}{(x-1)^\alpha} - \frac{f_k(-1)}{(x+1)^\alpha} + \int_{x-1}^{x+1} \frac{f_k'(x-t)dt}{t^\alpha} \right]. \tag{48}$$

If we were to impose the Dirichlet boundary conditions $f_k(\pm 1) = 0$, the first two terms in Eq. (48) would vanish, leaving us with a convergent integral. In the non-Dirichlet regime the situation becomes more complicated, since we may encounter divergent matrix elements. We will analyze the latter obstacle in more detail.

To this end let us calculate the matrix elements $\gamma_{\alpha kl} = \int_{-1}^{1} g_{k\alpha}(x) f_l(x) dx$ of the (so far) *ordinary* fractional Laplacian

$$\gamma_{\alpha kl} = \frac{A_\alpha}{\alpha} \int_{-1}^{1} f_l(x) \left\{ \frac{f_k(1)}{(x-1)^\alpha} - \frac{f_k(-1)}{(x+1)^\alpha} + \int_{x-1}^{x+1} \frac{f_k'(x-t)dt}{t^\alpha} \right\} dx. \tag{49}$$

It can be shown that the double integral in Eq. (49) is convergent (of course in the sense of the Cauchy principal value) for all $0 < \alpha < 2$. At the same time if $f_k(\pm 1) \neq 0$, we should consider the first two integrals separately. We have

$$I_1 = \int_{-1}^{1} \frac{f_l(x)dx}{(x-1)^\alpha}, \quad I_{-1} = \int_{-1}^{1} \frac{f_l(x)dx}{(x+1)^\alpha}. \tag{50}$$

The dangerous points are $x = 1$ for $I_1$ and $x = -1$ for $I_{-1}$. Clearly, in the Dirichlet case $q_l(\pm 1) = 0$, no convergence problem arises.

However, if $f_l(\pm 1) = \text{const} \equiv C$ (with respect to the convergence issue, it does not matter whether these constants can or cannot be different at $x = 1$ and $x = -1$), we have

$$I_1(x \to 1) = C \int_{-1}^{1} \frac{dx}{(x-1)^\alpha} = C \frac{(x-1)^{1-\alpha}}{1-\alpha} \bigg|_{-1}^{1} = \{x - 1 = \delta,\ x \to 1,\ \delta \to 0\} \sim \frac{C}{1-\alpha} \lim_{\delta \to 0} \delta^{1-\alpha}$$
$$= \begin{cases} 0, & \alpha < 1 \\ \ln \delta \to \infty, & \alpha = 1 \\ \infty, & 1 < \alpha < 2. \end{cases} \tag{51}$$

The same behavior is shared by $I_{-1}$ near $x = -1$. This shows that, regardless of the value of the derivative $f_l'(\pm 1)$, the integrals (50) are divergent in the range $1 \leqslant \alpha < 2$.

We point out that a departure point for our discussion was the Neumann basis system and thus standard Neumann boundary data (vanishing of the derivative at the end points of $[-1, 1]$) were implicit. The outcome (51) tells us that the regional fractional Laplacian may react consistently with Neumann boundary data in the range $0 < \alpha < 1$.

This observation stays in conformity with the results of mathematical papers [39,40] where the range $0 < \alpha < 1$ has been singled out for the unquestionable identification of the regional fractional Laplacian on a closed bounded domain as the generator of a reflected $\alpha$-stable process. Interestingly, no traditional form of the Neumann condition has been used. On the other hand, a traditional looking (merely on the formal, notational level) Neumann-type boundary condition has been found to be necessary for the existence of the reflected process in the range $1 \leqslant \alpha < 1$.

## V. REGIONAL FRACTIONAL LAPLACIAN: SIGNATURES OF REFLECTING BOUNDARIES

The explicit spectral solution for the regional fractional Laplacian in the non-Dirichlet regime is not known in the literature, except for some general existence statements [37,39,40,44]. The low part of that spectrum (shapes of ground and first excited eigenstates, related eigenvalues) remains unknown as well. The spectral solution reported in [58] refers explicitly to the Dirichlet boundary data for the regional fractional Laplacian.

Our major purpose is to deduce the spectral solution that would have something in common with physicists' intuitions about reflected random motions in a bounded domain. To this end we will address the spectral problem for the regional fractional Laplacian more carefully, avoiding the decomposition of the (nonlocal operator action-defining) integral expression into a sum of integrals, of the form (27) or (31)–(34). We will not impose any explicit form of the Neumann condition (or any of its analogs that can be found in the literature [39–43]). The only Neumann input will be related to the choice of the basis system in $L^2(D)$, the latter Hilbert space not being the one favored by mathematicians [85]. The lowest eigenvalues and shapes of related eigenfunctions will be deduced with numerical assistance.

The structure of the expression for the fractional regional operator (31) shows that the term $\psi(x)$ balances $\psi(u)$ in the integrand numerator making the integral convergent. Indeed, by inserting $u = x + \delta$ in the integrand (31) and expanding at small $\delta = u - x$ in power series, we obtain that around the dangerous point $u = x$, the integral takes the form $\int \frac{u-x}{|u-x|^{1+\alpha}} du$, which is convergent for $0 < \alpha < 1$. Note that it has the form $\int dt/t^\alpha \sim t^{1-\alpha} = 0$ at $\alpha < 1$ and exists as the Cauchy principal value for $1 \leqslant \alpha \leqslant 2$. We have previously discussed this question in Ref. [14]; see also Eqs. (9) and (10) and the surrounding discussion. We will analyze that issue in more detail below.

Accordingly, to calculate safely (i.e., without divergences) the spectrum of the regional operator (31), we should not split the integral into a sum of terms containing, respectively, $\psi(u)$ and $\psi(x)$, but rather consider them together. [We point out that in the case of the restricted fractional Laplacian with Dirichlet boundary data, we were actually urged to split the integral into a sum because the term with $\psi(x)$ vanishes identically in this case; see Ref. [63] for details.]

Let us make use of the modulus property

$$|u - x| = \begin{cases} u - x, & u > x \\ x - u, & u < x \end{cases} \tag{52}$$

and rewrite the limits of integration in Eq. (31) accordingly, thus arriving at (here again $D = [-1, 1]$)

$$(-\Delta)^{\alpha/2}_{D,\mathrm{reg}}\psi(x) = -A_\alpha \int_{-1}^{1} \frac{\psi(u)-\psi(x)}{|u-x|^{1+\alpha}}du = -A_\alpha\left[\int_{-1}^{x} \frac{\psi(u)-\psi(x)}{(x-u)^{1+\alpha}}du + \int_{x}^{1} \frac{\psi(u)-\psi(x)}{(u-x)^{1+\alpha}}du\right]. \tag{53}$$

We perform the substitution $x - u = t$ $(u = x - t)$ in the first integral to obtain

$$P_1(x) = \int_0^{x+1} \frac{\psi(x-t)-\psi(x)}{t^{1+\alpha}}dt. \tag{54}$$

Next we substitute $u - x = t$ $(u = x + t)$ in the second integral

$$P_2(x) = \int_0^{1-x} \frac{\psi(x+t)-\psi(x)}{t^{1+\alpha}}dt. \tag{55}$$

The regional operator can be rewritten in the form

$$(-\Delta)^{\alpha/2}_{D,\mathrm{reg}}\psi(x) = -A_\alpha[P_1(x) + P_2(x)]. \tag{56}$$

We will demonstrate that for $0 < \alpha < 1$ the integrals $P_i(x)$ $(i = 1, 2)$ are convergent as $t \to 0$. We have, in the lowest order in $t$,

$$\psi(x-t) \approx \psi(x) - t\psi'(x), \quad \psi(x+t) \approx \psi(x) + t\psi'(x). \tag{57}$$

Substitution of (57) into (54) and (55) yields

$$P_1(x) = -\psi'(x)\int_0^{x+1} \frac{t\,dt}{t^{1+\alpha}} = -\psi'(x)\left.\frac{t^{1-\alpha}}{1-\alpha}\right|_0^{x+1} = -\psi'(x)\frac{(x+1)^{1-\alpha}}{1-\alpha}, \quad \alpha < 1 \tag{58}$$

$$P_2(x) = \psi'(x)\int_0^{1-x} \frac{t\,dt}{t^{1+\alpha}} = \psi'(x)\left.\frac{t^{1-\alpha}}{1-\alpha}\right|_0^{1-x} = \psi'(x)\frac{(1-x)^{1-\alpha}}{1-\alpha}, \quad \alpha < 1. \tag{59}$$

We note that at $1 \leqslant \alpha \leqslant 2$ we should interpret $P_1 + P_2$ as a whole, but in terms of the Cauchy principal value procedure. The limiting behavior near zero gives rise to a cancellation similar to that in Eq. (10) from Ref. [14].

In the lowest order in $t$, the final form of the regional fractional operator reads

$$(-\Delta)^{\alpha/2}_{D,\mathrm{reg}}\psi(x) \approx -A_\alpha\frac{\psi'(x)}{1-\alpha}[(1-x)^{1-\alpha} - (1+x)^{1-\alpha}]. \tag{60}$$

The expression (60) is already free from divergences.

We note that the procedure (57) can be extended to an arbitrary order in $t$ to yield the representation of the regional fractional Laplacian (nonlocal integral operator) through an infinite series of locally defined differential operators. For our present purposes we find that series idea is impractical, being too computing time consuming. Therefore, we will compute explicitly the integrals (53) replacing $\psi(x)$ by $f_k(x)$, which are the elements of the Neumann basis in $L^2(D)$, $n = 0, 1, 2, \ldots$. We note the structure of the integrands in Eqs. (53)–(60). Namely, the Lévy index $\alpha$ is contained only in the denominators, while the functions $\psi(x)$ do not have this index. The same situation occurs for the functions $f_k(x)$. Below, for convenience, we include the index $\alpha$ in the definition of $f_k(x)$, i.e., we set formally $f_k(x) \equiv f_{k\alpha}(x)$. We have

$$f_{k\alpha}(x) = -A_\alpha[P_{1,k\alpha}(x) + P_{2,k\alpha}(x)],$$

$$P_{1,k\alpha}(x) = \int_0^{x+1} \frac{\cos\frac{k\pi}{2}(x-t+1) - \cos\frac{k\pi}{2}(x+1)}{t^{1+\alpha}}dt = \cos\frac{k\pi}{2}(x+1)P_{11,k\alpha}(x) + \sin\frac{k\pi}{2}(x+1)P_{12,k\alpha}(x); \tag{61}$$

$$P_{2,k\alpha}(x) = \int_0^{1-x} \frac{\cos\frac{k\pi}{2}(x+t+1) - \cos\frac{k\pi}{2}(x+1)}{t^{1+\alpha}}dt = \cos\frac{k\pi}{2}(x+1)P_{21,k\alpha}(x) - \sin\frac{k\pi}{2}(x+1)P_{22,k\alpha}(x); \tag{62}$$

$$P_{11,k\alpha}(x) = \int_0^{x+1} \frac{\cos\frac{k\pi t}{2} - 1}{t^{1+\alpha}}dt, \quad P_{12,k\alpha}(x) = \int_0^{x+1} \frac{\sin\frac{k\pi t}{2}}{t^{1+\alpha}}dt; \tag{63}$$

$$P_{21,k\alpha}(x) = \int_0^{1-x} \frac{\cos\frac{k\pi t}{2} - 1}{t^{1+\alpha}}dt, \quad P_{22,k\alpha}(x) = \int_0^{1-x} \frac{\sin\frac{k\pi t}{2}}{t^{1+\alpha}}dt. \tag{64}$$

To derive the expressions for $P_{ij,k\alpha}$ $(i, j = 1, 2)$, we use the trigonometric identity $\cos(A - B) = \cos A\cos B + \sin A\sin B$ so that, for example, in (61),

$$\cos\frac{k\pi}{2}(x+1-t) - \cos\frac{k\pi}{2}(x+1) = \cos\frac{k\pi}{2}(x+1)\cos\frac{k\pi t}{2} + \sin\frac{k\pi}{2}(x+1)\sin\frac{k\pi t}{2} - \cos\frac{k\pi}{2}(x+1) = \cos\frac{k\pi}{2}(x+1)\left(\cos\frac{k\pi t}{2} - 1\right) + \sin\frac{k\pi}{2}(x+1)\sin\frac{k\pi t}{2}.$$

Dividing this expression by $t^{1+\alpha}$ and integrating yields Eq. (63) for $P_{11,k\alpha}(x)$ and $P_{12,k\alpha}(x)$. The derivation of Eqs. (64) is the same. In other words, the second subscript in the functions $P_{ij,k\alpha}$ $(i, j = 1, 2)$ appears simply because the initial functions $P_{i,k\alpha}(x)$ $(i = 1, 2)$ [Eqs. (61) and (62)] contain two terms, each of which adds one more index $j = 1, 2$.

The integrals $P_{ij,k\alpha}$ $(i, j = 1, 2)$ will be calculated numerically. For reference purposes we list the integrals $P_{ij,k\alpha}$ $(i, j = 1, 2)$ in terms of variable $z = k\pi t/2$, which we use for actual numerical calculations

$$P_{11,k\alpha}(x)=\left(\frac{k\pi}{2}\right)^{\alpha}\int_0^{(k\pi/2)(x+1)}\frac{\cos z-1}{z^{1+\alpha}}dz,$$
$$P_{12,k\alpha}(x)=\left(\frac{k\pi}{2}\right)^{\alpha}\int_0^{(k\pi/2)(x+1)}\frac{\sin z}{z^{1+\alpha}}dz,$$
$$P_{21,k\alpha}(x)=\left(\frac{k\pi}{2}\right)^{\alpha}\int_0^{(k\pi/2)(1-x)}\frac{\cos z-1}{z^{1+\alpha}}dz,$$
$$P_{22,k\alpha}(x)=\left(\frac{k\pi}{2}\right)^{\alpha}\int_0^{(k\pi/2)(1-x)}\frac{\sin z}{z^{1+\alpha}}dz. \quad (65)$$

Also, to remove the (spurious) divergences at $z=0$ elegantly, we render the integrals in the form

$$P_{ij,k\alpha}(x)=\int_0^{\delta}+\int_{\delta}^{1\pm x}\equiv P_{ij,k0\alpha}(x)+\Delta P_{ij,k\alpha},\qquad \delta<0.01. \quad (66)$$

The explicit expressions for $\Delta P_{ij,k\alpha}$ read

$$\Delta P_{11,k\alpha}=\Delta P_{21,k\alpha}=\left(\frac{k\pi}{2}\right)^{\alpha}\frac{1}{2}\frac{\delta^{2-\alpha}}{2-\alpha},$$
$$\Delta P_{12,k\alpha}=\Delta P_{22,k\alpha}=\left(\frac{k\pi}{2}\right)^{\alpha}\frac{\delta^{1-\alpha}}{1-\alpha}. \quad (67)$$

Finally, the elements of the matrix $(\gamma_{kl\alpha})$, to be diagonalized in order to find the desired spectrum, are

$$\gamma_{kl\alpha}=-A_{\alpha}\int_{-1}^{1}\cos\frac{l\pi}{2}(x-t+1)\times\left\{\cos\frac{k\pi}{2}(x-t+1)[P_{11,k\alpha}(x)+P_{21,k\alpha}(x)]+\sin\frac{k\pi}{2}(x-t+1)[P_{12,k\alpha}(x)-P_{22,k\alpha}(x)]\right\}dx. \quad (68)$$

The results of numerical calculations of the spectrum for $(\gamma_{kl\alpha})$ [Eq. (68)] are reported in Table II. As computational times are very long (around 6 h for a $20\times20$ matrix and around 24 h for $30\times30$ one), we limit ourselves to $20\times20$ matrices. However, the qualitative features of the spectrum remain the same as those for larger matrices. We have checked that by test calculations of some of the eigenfunctions, by means of $30\times30$ matrices.

It can be seen that the Neumann base, in view of our matrix size limitations (small matrices, to lower the computing time), gives a slightly inaccurate estimation for the lowest (ground-state) eigenvalue. This is probably due to the fact that the lowest function of the Neumann base $f_0=1/\sqrt{2}$ is simply constant.

To convince ourselves that the obtained eigenvalues are signatures of reflecting boundaries, we have computed several (approximate) eigenfunctions. The representative plot of the four lowest eigenfunctions for $\alpha=0.5$ is portrayed in Fig. 6.

We have checked that for other $\alpha\in(0,1)$ the eigenfunctions become fairly close to those found for $\alpha=0.5$. Also, they are not distant from the Neumann basis (trigonometric) functions. Strictly speaking, up to a sign of resulting eigenfunctions, $\psi_n$ may happen to be opposite to that of the Neumann trigonometric functions. Except for the ground state, the sign issue is immaterial, since it is $|\psi_n|^2$, which stands for the probability density. The main feature of the eigenfunctions is that they appear to satisfy the Neumann boundary conditions, interpreted in terms of standard vanishing derivatives.

TABLE II. Comparison of the six lowest eigenvalues $E_i$ of the regional fractional Laplacian, evaluated in the Neumann base, for different $\alpha$ in the range $0<\alpha<1$ (calculated by means of the $20\times20$ matrices), with those obtained by imposing the Dirichlet boundary condition. Data are taken from Table I in Sec. IV of the present paper and from Table I in Ref. [58] (for comparison purposes we have abstained from using the label zero; the displayed data should be read as the first, second, third, etc., eigenvalue of the regional Laplacian in question). The first column, modulo the computing inaccuracy, is a clear signature of reflecting boundary data that were imported by executing the diagonalization in the Neumann basis. The ground-state function in the Neumann base is constant and the respective eigenvalue should equal zero. Our data show that the numerically computed lowest eigenvalue should actually be identified with zero, up to inaccuracies appearing in the third decimal place.

| $\alpha$ \ $i$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 0.2 | 0.0000 | 0.1871 | 0.3075 | 0.3972 | 0.4696 | 0.5307 |
| 0.2 [58] | 0.0003 | 0.1878 | 0.3085 | 0.3981 | 0.4700 | 0.5306 |
| 0.5 | 0.001 | 0.4499 | 0.8505 | 1.1959 | 1.5018 | 1.7785 |
| 0.5 [58] | 0.0038 | 0.4593 | 0.8626 | 1.2091 | 1.5149 | 1.7911 |
| 0.7 | 0.004 | 0.6319 | 1.3138 | 1.9591 | 2.5665 | 3.1420 |
| 0.7 [58] | 0.0170 | 0.6729 | 1.3646 | 2.0140 | 2.6231 | 3.1993 |
| 0.9 | 0.006 | 0.8290 | 1.8987 | 3.0064 | 4.1148 | 5.2156 |
| 0.9 [58] | 0.0640 | 0.9799 | 2.0823 | 3.2054 | 4.3230 | 5.4300 |

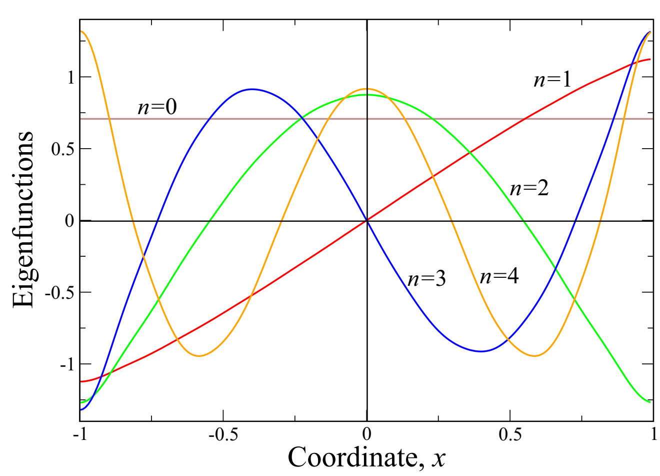


FIG. 6. Representative ($\alpha=0.5$) plot of the first five eigenfunctions of the regional fractional Laplacian, found via the diagonalization in the Neumann base; the $n=0$ function is constant, being a clear signature of the reflecting boundary conditions. The corresponding eigenvalue is for all practical purposes zero. It is instructive to compare the shapes of eigenfunctions with the elements of the Neumann base, depicted in Fig. 5.

We have paid special attention to the range $0 < \alpha < 1$, because in this parameter regime the integrals (53) are convergent. The case of $1 < \alpha < 2$ needs more attention, since these integrals exist only in the sense of the Cauchy principal value. That enforces slight modifications of the matrix diagonalization procedure, previously adopted to calculate the spectrum of the regional fractional Laplacian in the Neumann base.

On one hand, the calculation of the matrix elements (integrals) is now more time consuming, because we need to bypass the singularity at $u = x$ by means of the Cauchy principal value. On the other hand, the convergence of the matrix method is generally much worse for the base with Neumann boundary conditions than that for Dirichlet ones.

For instance, for the Dirichlet base, the acceptable accuracy of computation outcomes has been achieved already for $20 \times 20$ matrices. For the Neumann base we need matrices of size $1000 \times 1000$ to achieve an acceptable accuracy level (the smallest possible case is $600 \times 600$). That substantially increases the computational time. The integration procedure for $1 < \alpha < 2$ incurs a further increase of the computation time.

By these reasons, we have explicitly checked the signatures of reflection by an explicit computation of the ground-state eigenvalue for $\alpha = 1.2$, 1.5, and 1.8, followed by a computation of the shape of the corresponding eigenfunction. As expected, the eigenvalues come up (for all practical purposes) as zero, and eigenfunctions are indeed constant.

In passing we note that the integrals (53) are identically zero for the constant function $\psi(x)$. This is actually the signature of reflection, i.e., the zeroth ground-state eigenvalue.

## VI. IMPENETRABLE BARRIERS FOR LÉVY FLIGHTS: STOCHASTIC BEHAVIOR IN THE VICINITY OF THE BOUNDARY

The notion of censored stable processes, as introduced in Ref. [37], is verbally rather loose: A censored stable process in an open set $D \subset R^n$ is obtained by suppressing its jumps from $D$ to the complement $R^n \backslash D$. Alternatively, it is a process forced to stay inside $D$. Jumps are censored, i.e., those that would (according to the jump-size probability law) land beyond $D$ are simply canceled. Next the process is resurrected at the stopping point and started anew. Regarding the resurrection of the Markov process, see, e.g., [37,75].

*Remark 10.* The above concept of resurrection exploits the starting anew property for the stopped stochastic process. It shows some affinity for the family of stochastic processes with resetting, where the wandering particle can be reset to an initial location, at a certain rate, and next the process is started anew. It is known that in the diffusion with stochastic resetting one arrives at nonequilibrium stationary states with non-Gaussian fluctuations for the particle position. Apart from the above (reset) affinity, we have not found any probability accumulation outcomes that would resemble those reported in [32,33] and in the fractional Brownian motion on the interval [53].

Bogdan *et al.* [37] exclude from consideration so-called taboo processes which are related to the concept of the Doob $h$ transform [16,17,80] and are known not to leave the open set $D$. It has been demonstrated that what is referred to in Ref. [37] as a censored process (actually, a recurrent censored symmetric $\alpha$-stable process) is different (in law) from the symmetric stable one conditioned not to leave $D$ (cf. pp. 104–105 in Ref. [37]). This is by no mean a no-go statement for taboo processes; quite simply they do not involve any pointwise censoring mechanism, since it is the conditioning that does the job (of not reaching the boundaries).

It is quite clear that the above loose definition encompasses both taboo and censored processes plus (upon admitting that the boundary of $D$ can be reached by the process) $\alpha$-stable versions of reflected processes. The main issue addressed in the mathematical literature has been to strengthen the concept of reflection by inventing nonlocal analogs of Neumann boundary conditions [39–43].

Various scenarios of the behavior of the censored process in the vicinity of the boundary have been formulated as well. One of them is described as follows [42,43]. When the process exits $D$, it immediately returns to $D$. The way it returns is, if a process exits to a point $x \in R^n \backslash \overline{D}$, its return to $y \in D$ is realized with a probability density being proportional to $|x - y|^{-n-\alpha}$, hence not necessarily to the stopping point.

The concept of resurrection for a Markov process has been invoked here as well [37,75]. It amounts to an immediate resurrection of the process after eliminating (censorship) the inadmissible jump (from $D$ to $R^n \backslash D$), through a procedure of gluing together stable processes: At a stopping point (and time) of the tentatively terminated process, we glue its copy that actually gives birth to a process started anew at the stopping point. The process proceeds up to the next stopping time (i.e., censored jump) and the gluing procedure is repeated. Such a continually resurrected process is bound not to leave $D$, as required.

Leaving aside the many technicalities concerning the proper mathematical formulation of what a reflected stable process should actually be, we will move on to a brief discussion of the physicists' viewpoint of impenetrable barriers and eventually of the concept of reflection (e.g., that of reflecting boundaries) [11,12,32–36].

Motivated by the pathwise simulations, physicists coin their own recipes on how to implement the condition of reflection from the barrier in terms of the the sample path behavior (that on the computer simulation level). For example, in Ref. [32] the condition of reflection is ensured by wrapping the trajectory, destined to hit the barrier (or crossing the barrier), around the hitting point location while preserving the assigned length. On the other hand, it is mentioned that for jump-type processes the location boundary is not hit by the majority of discontinuous sample paths and returns (or recrossings of the boundary location) should be excluded from consideration, which excludes the wrapping scenario from further consideration. Another viewpoint mentioned in Ref. [32] refers to a simulation of the reflecting boundary by an infinitely high hard (e.g., impenetrable) wall, quantum mechanically interpreted as the infinite well [60,61], yielding an immediate reflection once its vicinity (and not necessarily the boundary itself) is reached. In contrast, in Ref. [33] another proposal for the introduction of the reflection scenario has been outlined (an idea inspired by [34–36]), with a focus on a numerical simulation of sample trajectories of the Langevin-type Lévy-stable evolution in the binding extremally anharmonic potential $V_n(x) \sim x^n$ with $n \gg 1$ (set, e.g., $n = 800$ for concreteness).

Here a departure point is the observation (strictly speaking, questionable in the quantum setting) that in the limit $n \to \infty$ the potential $V_n(x)$ mimics the infinite-well enclosure, with boundaries at end points of the interval $[-1, 1]$ (interpreted in [33] as reflecting). The random motion is described in terms of the Langevin-type equation $dX(t)/dt = -V'_n(x) + \zeta_\alpha(t)$, where $\zeta_\alpha(t)$ is a formal encoding of the symmetric white $\alpha$-stable noise (cf. [33]). The limiting stationary probability density of the associated fractional Fokker-Planck equation has been found [36] in the form of the $L^1([-1, 1])$ normalized function

$$\rho_\alpha(x) = (2)^{1-\alpha} \frac{\Gamma(\alpha)}{\Gamma^2(\alpha/2)} (1 - x^2)^{\alpha/2-1}, \tag{69}$$

a result valid for all $0 < \alpha \leqslant 2$. The special case of the Cauchy noise ($\alpha = 1$) has been addressed in Ref. [34] by an independent reasoning, with the outcome

$$\rho_1(x) = \frac{1}{\pi} \frac{1}{\sqrt{1 - x^2}}, \tag{70}$$

valid for all $|x| < 1$. This function blows up to infinity at the boundaries of the interval $[-1, 1]$.

In passing we note that for $\alpha = 2$ a uniform Brownian distribution $1/L$ arises. That would suggest a link to reflected processes, but this observation is misleading.

The probability density functions (69) and (70) do not belong to the $L^2([-1, 1])$ inventory associated with the regional fractional Laplacian. Furthermore, they refer to random motion scenarios that cannot reach the boundary and definitely comply with the stopping scenarios used in the numerical simulations in Ref. [33]. That view is supported by the fact that (70) coincides with the familiar probability distribution function for the classical harmonic oscillator. Its high-probability areas correspond to the long residence time for a classical particle in harmonic motion.

From the physical point of view, the most interesting observation of Ref. [33] in this context is that the extremely anharmonic and stopping motion scenarios in the presence of the symmetric $\alpha$-stable noise actually yield the same statistics of simulation outcomes (cf. Sec. II A 2 in [33]). Moreover, a detectable deviation from these outcomes has been reported if the wrapping scenario is employed. Actually, irrespective of the stability index $\alpha$, the wrapping assumption, in the pathwise simulation procedure of Refs. [32,33], has been found to lead to a uniform asymptotic distribution in the interval, like in the reflecting Brownian motion. The pertinent figures have not been reproduced in Ref. [33], but are available from the authors [90].

At this point we will discuss the concept of the stopping scenario [90]. In the course of the pathwise simulation of the jump-type process, any jump longer than the distance of the point of origin from the boundary is canceled (the process is stopped). In the $\varepsilon$ vicinity of the boundary, $\epsilon \ll 1$, all jumps in the direction of the barrier are canceled (it is an explicit censorship at work). The process remains stopped until the probability law produces a jump in the direction opposite to the boundary, and then it continues.

From a mathematical point of view, it is clear that probability functions (69) and (70) are not eigenfunctions of the regional fractional Laplacian. Actually, after completing them in $R\backslash(-1, 1)$ by assigning them the value zero beyond $(-1, 1)$, we realize that $\rho_\alpha(x)$ is the so-called $\alpha$-harmonic eigenfunction of the original fractional Laplacian $(-\Delta)^{\alpha/2}$ [Eq. (1)], defined on the whole of $R^1$ and associated with the eigenvalue zero

$$(-\Delta)^{\alpha/2} \rho_\alpha(x) = 0 \tag{71}$$

(cf. [30,57]). One should keep in mind that the above identity needs somewhat involved calculations in the case of arbitrary $1 < \alpha < 2$, but can be straightforwardly checked in the case of $\alpha = 1$. The calculation exploits in full a nonlocality of the fractional Laplacian, and terms that account for $R(-1, 1)$ cannot be disregarded [91,92].

Thus a stochastic process that is consistent with the PDF (70) is not the reflected one, but rather the censored one (see, e.g., [37]). The pertinent censored process never crosses or reaches the boundary, which is a property shared with taboo processes in the impenetrable enclosure [cf. the fractional infinite-square-well spectral problem or the related taboo process in the interval (so-called ground-state process) [15,58–63]]. Nonetheless, the associated stationary probability distributions appear to be very different, behaving reciprocally at the boundary: quick decay with a decreasing distance $\mathcal{D}$ from the boundary $\sim\mathcal{D}^{\alpha/2}$ (taboo process) versus a blowup to infinity $\sim 1/\mathcal{D}^{\alpha/2-1}$ in the same regime (censored process). Clearly, the taboo case refers to a probability depletion in the vicinity of the boundary, while its accumulation close to the boundary characterizes the censored process considered.

## VII. OUTLOOK AND PROSPECTS: SOME ADVANTAGES OF THE SPECTRAL LORE FOR FRACTIONAL LAPLACIANS IN BOUNDED DOMAINS

### A. General considerations

We have paid special attention to spectral problems involved with nonlocal motion generators (fractional Laplacians of arbitrary Lévy index $0 < \alpha < 2$) whose adjustment to account for finite (bounded) spatial geometries is a source of ambiguities in the physics literature. Like in the case of standard diffusion processes, the lowest eigenfunctions and eigenvalues of the pertinent operators are decisive for deducing measurable properties of relaxation processes towards equilibrium or their near-equilibrium behavior.

Our motivations stem basically from a simple looking inquiry into the problem of all admissible stochastic processes in a bounded domain, which may be consistently associated (derived or inferred from) with the primordial Lévy noise. That sets the broad conceptual context of Lévy flights in bounded domains and directly involves the delicate issue of properly defined boundary data for fractional Laplacians in finite geometries.

To this end, we have adopted a numerically assisted approach to the eigenvalue problem of fractional Laplacians in the interval, developed by us earlier [62] and based on the expansion of the sought eigenfunctions in the properly tailored orthonormal base of $L^2(D)$. In the present paper we have considered two such bases, namely, trigonometric ones associated with $D = [-1, 1]$.

The first one is familiar from the standard quantum-mechanical infinite-well problem [93] and comprises functions with Dirichlet boundary conditions, i.e., vanishing at the boundaries. The second one is that obeying the Neumann boundary conditions, i.e., basic functions are allowed to take arbitrary nonzero values at the boundaries, while their derivatives are required to vanish. In both cases the spectral fractional Laplacian (see Sec. II) simply imports all basic properties of the standard Laplacian. Things become complicated if other definitions of the fractional Laplacian in $D$ are considered.

In the course of the study, we have demonstrated that the Dirichlet base secures much better convergence properties from the numerical procedure than the Neumann one. The latter base implies particularly bad convergence properties in the stability parameter range $1 < \alpha < 2$, where integrals of importance [see, e.g., Eq. (53)] exist only in the sense of Cauchy principal values. Not incidentally, in all studies devoted to censored stochastic processes [37,39], the range $0 < \alpha \leqslant 1$ has been considered separately from $1 < \alpha < 2$, in which major difficulties were encountered, while attempting to define the notion of reflected jump-type processes and to devise a consistent form of Neumann boundary data.

We note in passing that in so-called fractional quantum mechanics [26], the range $1 \leqslant \alpha < 2$ is introduced *a priori* as the one in which this theory is supposed to make sense (it is not quite a necessary assumption [94] that essentially narrows the framework). We point out that a good reason for that may be the nonexistence of probability currents out of the pertinent range [94]. Nevertheless, consistent spectral solutions for Laplacians in bounded Dirichlet domains are known to exist in the whole range $0 < \alpha < 2$. On the other hand, spectral problems beyond the range $0 < \alpha < 1$ make the Dirichlet boundary data the only (for all practical purposes) reliable choice (compared to the Neumann or Robin ones), for purely pragmatic reasons. This conforms with the standard quantum-mechanical approach, where Neumann or Robin boundary data are definitely out of favor, while the Dirichlet data are prevalent.

In contrast, all these data are encountered often and are amply discussed in the theory of Brownian motion and diffusion-type processes. In connection with the eigenvalue problems addressed, we note that one may in principle invoke the variational approach, which is customary in ordinary quantum mechanics. Such an approach works for self-adjoint operators (see, e.g., [95]) and hence should be valid for trial functions with both Dirichlet and Neumann boundary conditions in settings more general than the standard Laplacian settings, i.e., for fractional Laplacians as well. It would be interesting to study the accuracy of the variational method for our two classes of (trigonometric) trial functions.

### B. Fractional spectral problems: Exemplary association with realistic physical systems

Our discussion of Sec. III on relevance versus irrelevance of eigenvalue problems for fractional Laplacians pertains to an interplay (intertwine) between the theory of stochastic processes (stochastic modeling) in a bounded domain and the fractionally generalized quantum model systems (which refers to so-called fractional quantum mechanics [24–26]; see also [94]) which are spatially confined. Leaving aside an explicit probabilistic (stochastic processes) viewpoint, the formalism developed can be applied to physical systems of confined geometry (like quantum wells and/or surfaces or interfaces), where the disorder and other kinds of intrinsic randomness (hence the fractional Lévy noise) can be interpreted on the quantum level.

One class of such systems is an electronic ensemble, which tunnels through potential barriers in so-called spintronic devices (see [96–98] and references therein). One of the realistic examples here is heterostructures like the $LaAlO_3/SrTiO_3$ interface [99–101]. To describe the experimental data related to the above tunneling statistics, fractional derivatives should be introduced in the conventional quantum-mechanical problem of a tunneling particle.

To be specific, the fractional generalization of the Schrödinger equation, describing the electronic properties of a heterojunction between materials 1 and 2 reads (see Ref. [100] for details)

$$\mathcal{H}\Psi(z) \equiv \begin{pmatrix} s_c(z)+U(z) & iv(-\Delta)^{1/2} \\ iv(-\Delta)^{1/2} & -s_v(z)+U(z) \end{pmatrix} \begin{pmatrix} \varphi(z) \\ \chi(z) \end{pmatrix} = \varepsilon \begin{pmatrix} \varphi(z) \\ \chi(z) \end{pmatrix}, \tag{72}$$

where $\varepsilon$ is the eigenenergy and $\Psi(z)$ is the spinor wave function of the electron at the interface. Here $(-\Delta)^{1/2}$ is the fractional Laplacian for Lévy index $\alpha = 1$, $z$ is the dimensionless coordinate along the interface (normalized by the interface width $a$) [100], and we use atomic units $\hbar = c = 1$. The spatial confinement of the problem lies in the interfacial potential $U(z)$,

$$U(z) = U_0\delta(z) + U_1 P(z), \qquad P(z) = [\theta(z) + \theta(d-z)], \tag{73}$$

where the first term, containing the $\delta$ function, signifies the potential barrier exactly at the interface, while the second term denotes the barrier of height $U_1$ and width $d$, through which the electrons can tunnel. Here $\theta(z)$ is the Heaviside (unit step) function. The influence of technologically unavoidable disorder is modeled by the introduction of the fractional derivative $(-\Delta)^{1/2}$ instead of ordinary gradient $d/dz$. We use the additional notation [100]

$$\begin{aligned} s_c(z) &= s_{c1}[1-\theta(z)] + s_{c2}\theta(z), \\ s_v(z) &= s_{v1}[1-\theta(z)] + s_{v2}\theta(z), \\ v(z) &= v_1[1-\theta(z)] + v_2\theta(z), \end{aligned} \tag{74}$$

with

$$s_{c1,2} = \Delta_{1,2} + \frac{k_\perp^2}{2m_{c1,2}}, \quad s_{v1,2} = \Delta_{1,2} + \frac{k_\perp^2}{2m_{v1,2}}, \tag{75}$$

where $\Delta_{1,2}$ and $m_{c,v,1,2}$ are, respectively, the energy gaps and effective masses of interface-forming semiconductors 1 and 2. Indices $c$ and $v$ mean, respectively, the conduction and valence bands, so $m_{1c}$ denotes the electron effective mass in the conduction band of semiconductor 1 (see Refs. [100,101] for details). Also, $k_\perp$ is (unimportant for the interfacial problem) the electron momentum component, parallel to the interface [100].

The explicit form of the Schrödinger equation (72) reads

$$(s_c + U - \varepsilon)\varphi + iv(-\Delta)^{1/2}\chi = 0,$$
$$iv(-\Delta)^{1/2}\varphi + (-s_v + U - \varepsilon)\chi = 0. \quad (76)$$

Then we can express the component $\chi$ through $\varphi$ from the second of Eqs. (76) to obtain the following eigenproblem for $\varphi$:

$$[s_c(z) + U(z)]\varphi(z) + v(z)(-\Delta)^{1/2} \times \left[\frac{v(z)}{-s_v(z) + U(z) - \varepsilon}(-\Delta)^{1/2}\varphi(z)\right] = \varepsilon\varphi(z). \quad (77)$$

We note that the eigenproblem (77) is indeed nonlinear as the eigenenergy $\varepsilon$ enters also the denominator. Our analysis shows that for small $\varepsilon$ the problem (77) is reducible to that for 1D regional fractional derivative. The best way to solve it is to expand a solution in the orthogonal set of functions with Neumann boundary conditions. The complete solution of the problem (77) can be obtained only numerically. Our preliminary studies of that solution show that it describes many experimentally observable salient features of the interfaces like high electronic concentration at the interface (leading, e.g., to metallic conductivity; see [99] and references therein), which appears due to disorder, described by fractional derivatives.

One more interesting physical problem is the onset of chaos in excitons, induced by the Rashba spin-orbit interaction [102]. This situation is described by one more confined (although in two dimensions, making the solution much more complicated than in the 1D case) problem. Then one is dealing with the quantum version of the Kepler problem, i.e., the hydrogen atom (see, e.g., [103]). The fractional Hamiltonian of the 2D problem has the form

$$\frac{(-\Delta)^{\alpha/2}}{2m}\psi(x, y) - \frac{\psi(x, y)}{r} = E\psi(x, y), \quad (78)$$

where $r = (x^2 + y^2)^{1/2}$ and we once more use atomic units. Here the spatial confinement is due to the Coulomb potential $1/r$. Similar to the case of ordinary quantum mechanics [93,95], the discrete spectrum exists only at $E < 0$. It can be shown that the problem (78) admits the separation of angular and radial variables, leaving us with an ordinary differential equation for the function $\psi(r)$. The solution of the latter problem is still much more complicated than those in one dimension. Our preliminary analysis shows that the function $\psi(r)$ can be obtained as the expansion of the complete set of eigenfunctions, corresponding to the ordinary [i.e., with the conventional Laplacian in Eq. (78)] 2D hydrogen atom. The method is similar to that of Ref. [62]. Note that the problem (78) can be formulated in three dimensions with additional confinement in the potential well. The latter case corresponds to excitons confined to quantum wells in disordered semiconductors. One more way of generalizing the problem (78) is to take into account the fractional analog of the Rashba spin-orbit interaction [102]. Although the latter problem becomes very complex (for example, it contains now, similar to Eq. (72), a spinor wave function), its solution can be obtained along the lines of Ref. [62], i.e., by the expansion over a properly tailored orthonormal basis with Dirichlet boundary conditions. The problems discussed, related to the fractionally quantized Kepler one, are extremely important for photovoltaic applications (see Ref. [104] for appropriate references), where possible chaotic behavior [105,106] can disrupt the solar cell functionality. Namely, the chaotic behavior has been obtained explicitly in the form of nonuniform electronic trajectories [105]. At the same time, in the quantized version only weak effects like energy levels repulsion have been revealed [106]. It is tempting to reveal the quantum chaotic features (along with direct quantum trajectories simulations) of this problem by replacing the ordinary 2D Laplacian by the fractional one in the Schrödinger equation (78). In this case, the level repulsion and other features like non-Poissonian energy level statistics [107] would heavily depend on the Lévy index $\alpha$.

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