Mutual-friction-driven turbulent statistics in the hydrodynamic regime of superfluid ^{3}He-B

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It is well known that the turbulence that evolves from the tangles of vortices in quantum fluids at scales larger than the typical quantized vortex spacing ℓ has a close resemblance with classical turbulence. The temperature-dependent mutual friction parameter $\alpha(T)$ drives the turbulent statistics in the hydrodynamic regime of quantum fluids that involves a self-similar cascade of energy. From a simple theoretical analysis, here we show that superfluid ^{3}He-B in the presence of mutual damping exhibits a $k^{-5/3}$ Kolmogorov energy spectrum in the entire inertial range $\ell < r < L$ at temperature $T \lesssim 0.2T_c$, while at $T \gtrsim 0.2T_c$ dissipation begins to dominate larger eddies exhibiting a k^{-3} spectrum toward the energy pumping scale L. At $T \approx 0.35T_c$, eddies of all size, being highly affected by damping, exhibit a k^{-3} spectrum in the entire inertial range. The consistency of this result with the predictions of recent direct numerical simulations indicates that the present theoretical framework is applicable in quantifying the hydrodynamic regime of quantum turbulence.

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I. INTRODUCTION

As first envisaged by Feynman [1], superfluids obeying quantum hydrodynamics exhibit turbulent characteristics due to a three-dimensional dynamic tangle of reconnecting discrete thin vortex filaments, each carrying a fixed circulation [2]. The rapid advances in experimental techniques, particularly Andreev scattering [3] and NMR spectroscopy [4], have enabled us to visualize quantum vortices, and they have triggered many current theoretical and experimental developments [5,6] in the statistical characterization of turbulence in quantum fluids [2,4,7,8]. The current theoretical understanding of turbulent motions in quantum fluids is mostly based on Landau's phenomenological picture of the two-fluid model [9] where below the critical temperature T_c ($T_c = 2.17$ K for ^{4}He and $\approx$1 mK for ^{3}He-B) the hydrodynamics is described as a combination of two interpenetrating fluids, namely a viscous normal fluid (with viscosity $\nu_n > 0$, velocity u_n, and density ρ_n) and an inviscid superfluid (with viscosity $\nu_s = 0$, velocity u_s, and density ρ_s) both having a temperature-dependent relative density and moving with two different velocities. At finite temperature ($T \neq 0$), the normal and superfluid components get coupled to each other via thermal excitations, and the resulting dissipative force is modeled by a mutual friction term. In ^{3}He-B, the highly viscous normal fluid is effectively clamped to the walls, and the mutual friction, acting over all length scales, affects the dynamics of quantized vortices by damping the energy of Kelvin waves into the normal fluid. In the zero-temperature limit, the normal fluid is absent and quantum turbulence becomes entirely a tangle of quantized vortex lines that move in a fluid without viscosity. From the viewpoint of this two-fluid model, turbulence in Fermi superfluid ^{3}He-B is a relatively simpler problem than that of ^{4}He because of the very high viscosity of the normal component. The study of quantum turbulence in superfluid ^{3}He-B has gained prominence theoretically [10,11], numerically [12–15], and experimentally [3,8].

Similar to ^{4}He, there are three important length scales involved in superfluid turbulence of ^{3}He-B. The smallest length scale is the vortex core radius ξ ($\approx 10^{-8}$ m at low pressure). The quantization of circulation $\kappa = \hbar/m$ (m being the mass of the Cooper pairs) gives rise to an intermediate quantum length scale ℓ that is the mean separation between vortex lines ($\approx 10^{-5}$ m). The largest scale involved is the size of the largest eddies L, which is $\approx 10^{-2}$ m. Over the widely separated length scales $\ell \ll r \ll L$, nonlinear interaction between vortex filaments leads to the polarization and partial alignment of the vortex that mimics continuous hydrodynamic eddies, and thus a Kolmogorov energy cascade with larger eddies breaking into smaller ones is expected, as observed in numerical simulations [12] and experimental measurements [8]. At a scale $r \sim \ell$, the discrete nature of quantized vortex lines becomes important, and energy is believed to be transferred downscale by the interacting Kelvin waves (helical perturbations of the individual vortex lines), where it is radiated away by thermal quasiparticles (phonons).

In the hydrodynamic regime ($\ell \ll r \ll L$), the coarse-grained average superfluid velocity is obtained [11] by averaging over a small volume that is significantly larger than ℓ^3. As the normal fluid is extremely viscous, the description of the fluid motion in ^{3}He-B can be reduced to the coarse-grained hydrodynamic Euler equation for the superfluid velocity $u(\mathbf{r}, t) \equiv u_s(\mathbf{r}, t)$ as [10,11,14]

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla \mu(\rho) = -\mathcal{D} + \mathbf{f}, \tag{1}$$

where $\mathbf{f}(\mathbf{r}, t)$ is the force applied at the outer scale L to sustain the turbulence, $\mu(\rho)$ is the chemical potential, and $\mathcal{D}$ is the dissipative term representing the effective mutual friction acting at large scales. As suggested by Vinen [16], the simplest form of mutual friction in the purely hydrodynamic regime

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of superfluid turbulence can be chosen as a linear function of the superfluid velocity $\mathbf{u}$ as $\mathcal{D} = \Gamma\mathbf{u}$, where $\Gamma \equiv \alpha(T)\,\omega_{\mathrm{T}}$ represents a scale-independent damping frequency. Here α is the temperature-dependent dimensionless mutual friction parameter and $\omega_{\mathrm{T}} = \nabla \times \mathbf{u}$ is the superfluid vorticity. This mutual friction causes the superfluid velocity $\mathbf{u}$ to decay with a time constant $\tau_d = 1/\Gamma$ irrespective of the size r of the eddies. The inertial-range cascade exists as long as $\tau_d \gg \tau_r$, where $\tau_r = r/u_r$ is the characteristic turnover time associated with the nonlinear transfer of energy from eddies of size r and characteristic velocity u_r. These two timescales define an effective scale-dependent superfluid Reynolds number [4,6] $\mathcal{R}'_e = \tau_d/\tau_r = u_r/\Gamma r \geqslant 1$. It can be expressed using Kolmogorov scaling for characteristic velocity, $u_r = (\varepsilon r)^{1/3}$, as $\mathcal{R}'_e = \varepsilon^{1/3}/\Gamma r^{2/3}$, indicating that dissipation will be important in eddies larger than the size given by $r_d = (\varepsilon/\Gamma^3)^{1/2}$. Thus an inertial-range Kolmogorov cascade characterized by the $k^{-5/3}$ spectrum is expected to exist on a length scale less than r_d [10,11,14]. For length scales $r_d \ll r \ll L$, theoretical predictions [10,11] and numerical simulations [14,15] suggest that the energy flux decays with increasing scales and the energy density follows a k^{-3} spectrum. This large-scale regime exists when the damping due to mutual friction exceeds some critical value while decreasing dissipation causes this regime to shrink. An interesting question, which has received much theoretical attention, is at what temperature range does the above classical scaling exist?

While the analytical results of L'vov *et al.* [10] for the possible energy spectra in superfluid ^{3}He-B are interesting and suggestive, they are based on an uncontrolled algebraic approximation for the energy flux. Their predictions, namely the crossover wave number that separates the small scale $k^{-5/3}$ and the large scale k^{-3} scaling regimes, have been verified through numerical simulations of the shell model [14]. The shell model is also an uncontrolled simplification of the basic equations of motion for the superfluid velocity field. Recently, direct numerical simulations (DNS) have been performed [15] on a gradually damped version of the coarse-grained Hall-Vinen-Bekarevich-Khalatnikov (HVBK) model, which is obtained from the standard HVBK model by adding a dissipation term proportional to the superfluid viscosity, as suggested in Ref. [17]. They used an integral closure [18] and obtained different turbulent energy spectra for different temperatures lying in the range $0 < T < 0.7\,T_c$. A k^{-3} spectrum at large scales and a $k^{-5/3}$ spectrum at small scales was obtained at $T < 0.37\,T_c$. Existing theoretical analyses due to L'vov *et al.* [10] and Vinen [11] are unable to quantify exactly this temperature-dependent distribution of the turbulent energy density.

Here we obtain a theoretical insight into the problem by employing the direct interaction picture of Kraichnan. The (Eulerian) direct interaction approximation (DIA), originally introduced and applied to incompressible homogeneous isotropic turbulence by Kraichnan [19], was the first "microscopic" theory of (three-dimensional) turbulence [20] that resembles the Dyson-Schwinger formulation of quantum field theory. In fact, it was the first detailed field-theoretic approach to the theory of turbulence based on the underlying Navier-Stokes dynamics of fluid motion. However, the Eulerian framework is unable to capture the correct timescales associated with the cascade from the timescales of sweeping of smaller eddies by the larger ones, and thereby it fails to extract the small effect of Kolmogorov cascade among the small eddies from the large effect of their being swept around. As a result, it incorrectly yields a $k^{-3/2}$ spectrum instead of the well-known Kolmogorov $k^{-5/3}$ spectrum. This failure was identified by Edwards [21] as a consequence of the divergence in the response integral due to the contributions coming from the large scales of motion (low wave numbers). To systematically eliminate this spurious effect of sweeping, Kraichnan reformulated the theory in a Lagrangian framework [22]. The Lagrangian version of the DIA gives excellent predictions for various statistical quantities such as Kolmogorov's universal form of the energy spectrum function and the skewness of the velocity gradient.

In superfluid turbulence, the two independent phenomena of vortex reconnection and mutual friction are the sources of dissipation. Vortex reconnection is particularly important at extremely low temperatures that generate Kelvin waves. It was confirmed experimentally that the turbulent dissipation at extremely low temperature is excited by vortex reconnection [23]. The dissipation rate at which the energy is fed into the Kelvin waves is given by $\varepsilon = G\kappa^3 l^{-4} = G\kappa^3\mathcal{L}^2$, where κ is the quantum of circulation and $\mathcal{L}$ is the vortex line density. The product $G\kappa$ can be interpreted as effective kinematic viscosity ν_0 so that $\varepsilon = \nu_0\kappa^2\mathcal{L}^2$ [24]. To apply Kraichnan's DIA, we introduce in the governing dynamical equation [Eq. (1)] a dissipative term $\nu_0\nabla^2\mathbf{u}$. Such a term has been incorporated in recent numerical simulations [15,17,25]. Thus, we write Eq. (1) as

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = \nu_0\nabla^2\mathbf{u} - \nabla\mu - \Gamma\mathbf{u} + \mathbf{f}, \tag{2}$$

and we apply Kraichnan's DIA to obtain analytic expressions for the perturbative corrections to the velocity-velocity correlation. A crossover wave number k_{cr} appears that, depending on the mutual damping frequency, dictates two different scaling regimes: an isotropic inertial-range Kolmogorov $k^{-5/3}$ regime at small scales ($k \gg k_{\mathrm{cr}}$), and a k^{-3} regime at large scales ($k \ll k_{\mathrm{cr}}$) with wave-number-dependent energy flux. Using available experimental data for the temperature-dependent mutual friction parameter [30], we show that the $k^{-5/3}$ spectrum occupies the entire inertial range $\ell < r < L$ at temperature $T \lesssim 0.2T_c$, while at $T \gtrsim 0.2T_c$ dissipation begins to dominate larger eddies exhibiting a k^{-3} spectrum. At $T \approx 0.35T_c$, eddies of all sizes are highly affected by damping, giving the k^{-3} spectrum in the entire inertial range.

II. DIA TREATMENT

To bring into play the physics associated with the response of the turbulent system to small perturbations, the DIA makes essential and intimate use of the response function $G_{ij}(\mathbf{x},\mathbf{x}',t,t') = \langle \delta u_i(\mathbf{x},t)/\delta f_j(\mathbf{x}',t')\rangle$, where the deviation $\delta u_i(\mathbf{x},t)$ in the velocity field is due to the infinitesimal disturbance $\delta f_j(\mathbf{x}',t')$ in the force field. Thus it captures the way in which the dynamical equation responds to perturbation. We write Eq. (2) in Fourier space as

$$\left(\frac{\partial}{\partial t} + \nu_0 k^2\right)u_i(\mathbf{k},t) + \frac{i}{2}P_{ijl}(\mathbf{k})\sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} u_j(\mathbf{p},t)u_l(\mathbf{q},t) = F_i(\mathbf{k},t), \tag{3}$$

with $F_i(\mathbf{k},t)=f_i(\mathbf{k},t)-\Gamma u_i(\mathbf{k},t)$, $P_{ijl}(\mathbf{k})=k_jP_{il}(\mathbf{k})+k_lP_{ij}(\mathbf{k})$ with $P_{ij}(\mathbf{k})=\delta_{ij}-k_ik_j/\mathbf{k}^2$, and the summation sign represents integrations on $\mathbf{p}$ and $\mathbf{q}$ with triad relation $\mathbf{p}+\mathbf{q}=\mathbf{k}$. Following Kraichnan's method, we introduce the full response function $G_{ij}(\mathbf{k};t,s)$ corresponding to the velocity field as $u_i(\mathbf{k},t)=\int_{-\infty}^{t}ds\ G_{ij}(\mathbf{k};t,s)F_j(\mathbf{k},s)$. We define the isotropic background field as $u_i^{(0)}(\mathbf{k},t)=\int_{-\infty}^{t}ds\ G_{ij}^{(0)}(\mathbf{k};t,s)f_j(\mathbf{k},s)$, and we follow Leslie's [20] suggestion of replacing the full response by the isotropic response, so that the full velocity field is

$$u_i(\mathbf{k},t)=u_i^{(0)}(\mathbf{k},t)-\Gamma\int_{-\infty}^{t}ds\,G_{ij}^{(0)}(\mathbf{k};t,s)u_i(\mathbf{k},s). \quad (4)$$

In the full (exact) velocity correlation tensor $Q_{ij}(\mathbf{k},t,t')$, defined as $\langle u_i(\mathbf{k},t)u_j(\mathbf{k}',t')\rangle=Q_{ij}(\mathbf{k};t,t')(2\pi)^3\delta^3(\mathbf{k}+\mathbf{k}')$, we substitute Eq. (4) iteratively and obtain the expansion $Q_{ij}=Q_{ij}^{(0)}+Q_{ij}^{(1)}+Q_{ij}^{(2)}+\cdots$, where the superscripts denote the order in powers of Γ. The zeroth-order isotropic correlation function is given by $(2\pi)^3\delta^3(\mathbf{k}+\mathbf{k}')Q_{ij}^{(0)}(\mathbf{k};t,t')=\langle u_i^{(0)}(\mathbf{k},t)u_j^{(0)}(\mathbf{k}',t')\rangle$. Using the properties of the isotropic correlation as well as response tensors, namely $Q_{ij}^{(0)}(\mathbf{k};t;t')=Q^{(0)}(k;t,t')P_{ij}(\mathbf{k})$ and $G_{ij}^{(0)}(\mathbf{k};t;t')=G^{(0)}(k;t,t')P_{ij}(\mathbf{k})$, we find the corrections to the correlation function up to $O(\Gamma^2)$ as

$$Q_{ij}^{(1)}(k;t,t')=-2\Gamma P_{ij}(\mathbf{k})\int_{-\infty}^{t}ds\,G^{(0)}(k;t,s)Q^{(0)}(k;t',s) \quad (5)$$

and

$$Q_{ij}^{(2)}(\mathbf{k};t,t')=\Gamma^2P_{ij}(\mathbf{k})\Big[2\int_{-\infty}^{t}ds\int_{-\infty}^{s}ds'G^{(0)}(k;t,s)G^{(0)}(k;s,s')Q^{(0)}(k;t',s')$$
$$+\int_{-\infty}^{t}ds\int_{-\infty}^{t'}ds'G^{(0)}(k;t,s)G^{(0)}(k;t',s')Q^{(0)}(k;s,s')\Big] \quad (6)$$

We evaluate the above time integrals by assuming an exponential decay of the response and correlation functions [21], namely

$$G^{(0)}(k;t,t')=\theta(t-t')e^{-\eta(k)(t-t')} \text{ and } Q^{(0)}(k;t,t')=Q^{(0)}(k)e^{-\eta(k)|t-t'|},$$

where $\theta(t-t')$ is the Heaviside step function and $\eta(k)$ is the eddy damping factor. The equal-time correlations are obtained by setting $t=t'$ in Eqs. (5) and (6) and by choosing the proper region in s-s' space, as shown in Fig. 1. This yields

$$Q_{ij}^{(1)}(k;t,t)=-\Gamma P_{ij}(\mathbf{k})Q^{(0)}(k)/\eta(k) \quad (7)$$

and

$$Q_{ij}^{(2)}(k;t,t)=\Gamma^2P_{ij}(\mathbf{k})Q^{(0)}(k)/\eta(k)^2. \quad (8)$$

Now using the well-known isotropic Kolmogorov-Obukhov scaling [26] for the turbulent energy density, namely $E(k)=\frac{1}{(2\pi)^3}4\pi k^2Q^{(0)}(k)=C\varepsilon^{2/3}k^{-5/3}$, and the eddy damping rate $\eta(k)=a\,\varepsilon^{1/3}k^{2/3}$, we obtain

$$E(k)=C\varepsilon^{2/3}k^{-5/3}\Big[1-\frac{\Gamma}{a}\varepsilon^{-1/3}k^{-2/3}+\frac{\Gamma^2}{a^2}\varepsilon^{-2/3}k^{-4/3}\Big]. \quad (9)$$

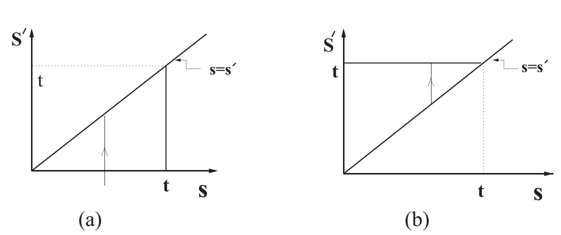

FIG. 1. The region of integration for the time integrals in s-s' space. (a) In the region $s>s'$, s varies from $-\infty$ to t while s' varies from $-\infty$ to s; (b) in the region $s<s'$, s varies from $-\infty$ to t while s' varies from s to t.

III. MUTUAL-FRICTION-DEPENDENT SCALING REGIME

To see how mutual friction leads to different energy spectra in different length scales, we express Eq. (9) in terms of a dimensionless mutual friction parameter $\gamma=\Gamma/(\varepsilon^{1/3}k_0^{2/3})$, where k_0 is the wave number corresponding to the outer scale L. Thus defining a crossover wave number

$$k_{\rm cr}=\gamma^{3/2}k_0/a^{3/2}=(\Gamma^3/a^3\varepsilon)^{1/2}, \quad (10)$$

we can write Eq. (9) as

$$\frac{E(k)}{\varepsilon^{2/3}k^{-5/3}}=C\Big[1-\Big(\frac{k_{\rm cr}}{k}\Big)^{2/3}+\Big(\frac{k_{\rm cr}}{k}\Big)^{4/3}\Big]. \quad (11)$$

Expressing Eq. (11) as $E(k)=C[\varepsilon(k)]^{2/3}k^{-5/3}$, we obtain the wave-number-dependent energy flux as

$$\frac{\varepsilon(k)}{\varepsilon}=\Big[1-\Big(\frac{k_{\rm cr}}{k}\Big)^{2/3}+\Big(\frac{k_{\rm cr}}{k}\Big)^{4/3}\Big]^{3/2}. \quad (12)$$

Thus we see that for wave numbers higher than $k_{\rm cr}$ ($k\gg k_{\rm cr}$), the spectrum returns to the classical Kolmogorov spectrum $E(k)=C\,\varepsilon^{2/3}\,k^{-5/3}$, while at small wave numbers ($k\ll k_{\rm cr}$) it gives

$$\frac{E(k)}{\varepsilon^{2/3}k^{-5/3}}=C\Big(\frac{k_{\rm cr}}{k}\Big)^{4/3}, \quad (13)$$

with $\varepsilon(k)=\varepsilon(k_{\rm cr}/k)^2$.

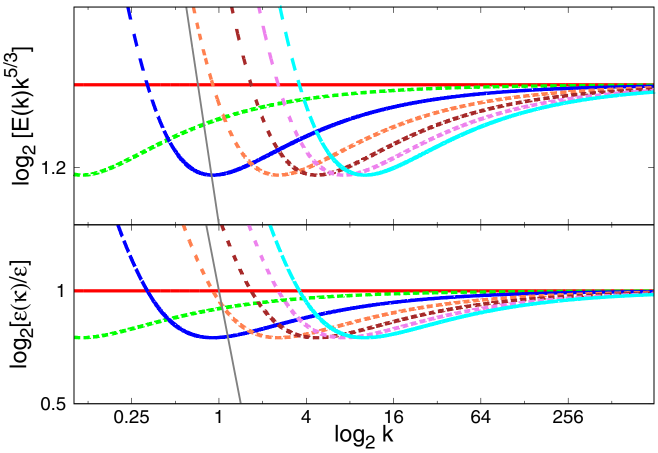

FIG. 2. Log-log plots of the energy density (top panel) and the energy flux (bottom panel) vs wave number for different values of γ. At small wave numbers and for increasing values of γ, the top panel shows the significant deviations of the energy density from $k^{-5/3}$ scaling (thick horizontal line) and approaches a k^{-3} scaling (thin line). At very large wave numbers, the bottom panel shows the almost constant energy flux, while with increasing values of γ, the energy flux decays and approaches a k^{-2} scaling (thin line). The values of γ for the deviated graphs (from green to cyan in color) are $\gamma = 0.05, 0.2, 0.4, 0.6, 0.8$, and 1.0 in both panels.

The above expressions involve two nondimensional constants C and a, which can be determined self-consistently from the dynamical equation using the Heisenberg approximation [27]. In the Heisenberg-type approximation, the self-consistent equation based on the DIA energy integral is obtained for the eddy-damping rate $\eta(k)$ so that it is dynamically determined without having to make a choice for it. This is in marked contrast to the approaches based on eddy-damped quasinormal Markovian closures (EDQNM) [28], where the eddy-damping rate is introduced by hand. By assuming a Kolmogorov-type scaling law for $E(k)$ and $\eta(k)$ so that the unphysical $k^{-3/2}$ DIA spectrum does not get accommodated in the calculations, the ratio a^2/C can be determined self-consistently in the Heisenberg approximation, as explicitly shown in Ref. [27]. Within the same scheme, the flux integral for energy can also be evaluated self-consistently, giving a ratio a/C^2. From these two amplitude ratios, the universal numbers C and a are dynamically determined, without having to set the numerical value of any constant by hand. Thus, taking the values of C and a from the theoretical estimates obtained via the Heisenberg approximation [27], namely $a^2/C = 7/60$, $a = 0.4269$, and $C = 1.5618$, we plot in Fig. 2 the left-hand sides of Eqs. (11) and (12) against a wide range of wave numbers k for different values of γ. Thus, as obtained by L'vov *et al.* [10], our calculations indicate that with increasing scales (decreasing k), the energy flux $\varepsilon(k)$ varies as k^{-2} and the energy density follows a k^{-3} spectrum, while at small scales (large k) the energy flux remains constant and the energy density follows the classical Kolmogorov $k^{-5/3}$ spectrum. Below we shall discuss a few important points that can be drawn from our above analytical results.

(i) The scale-dependent Reynolds number described at the outset can be expressed as [4] $\mathcal{R}'_e = \varepsilon^{1/3}k^{2/3}/\Gamma = 1/\alpha$, where we set $\Gamma \equiv \alpha\,\omega_{\rm T}$. Thus, the condition $\mathcal{R}'_e \gg 1$ corresponds to $\alpha \ll 1$. The role of parameter α as the inverse superfluid Reynolds number was demonstrated in the experiments described in Ref. [4], where it was shown that the turbulence develops only when $\alpha < 1$. From the expression for $k_{\rm cr}$, we see that $\gamma = a = 0.4269$ corresponds to $k_{\rm cr} = k_0$ $(r_d = L)$ so that the classical Kolmogorov cascade is realized in the entire inertial range $\ell < r < L$. As the value of γ increases from $\gamma = 0.4269$, $k_{\rm cr}$ is shifted to higher values and, eventually, the $k^{-5/3}$ energy range is restricted in the high wave number $k > k_{\rm cr}$ while a k^{-3} spectrum begins to develop at low wave numbers $k < k_{\rm cr}$. This is because the mutual damping begins to play a dominating role causing a decay of energy flux in the region of small k.

TABLE I. Values of the critical temperature T_c against the mutual friction parameter α, extracted from the graphical plots of Ref. [30].

$1/\alpha$	0.3	0.8	3	7	11	15	42	350	∞
T/T_c	0.72	0.59	0.39	0.37	0.35	0.32	0.27	0.2	0

(ii) Equation (13) yields $E(k) = (C/a^2)\Gamma^2 k^{-3} = 8.5714\,\Gamma^2 k^{-3}$. The critical value of α for which this k^{-3} energy spectrum is realized for any k can be obtained by substituting this $E(k)$ into the expression

$$\omega_{\rm T}^2 \equiv \langle|\omega|^2\rangle \approx 2\int_{k_0}^{k_\ell} k^2\, E(k)dk, \tag{14}$$

giving $\alpha_{\rm cr} \simeq a/\sqrt{2C\ln(k_\ell/k_0)}$. The value $k_\ell/k_0 \sim 10^3$ can be inferred from the experimental work of Walmsley *et al.* [29], where the mean intervortex distance $\ell \sim 0.03$ mm and the largest scale is $L = 4.5$ cm. In fact, the value $k_\ell/k_0 \sim 10^3$ was effectively used by Boué *et al.* [14] to obtain $\alpha_{\rm cr}$. This is also evident from the numerical simulation of a shell model carried out by Wacks and Barenghi [25] where a forcing wave number $k_0 = 2^{-4}$ was chosen and $k_l \approx 10^3$ was obtained. Thus for $k_\ell/k_0 \simeq 10^3$, we obtain $\alpha_{\rm cr} \simeq 0.0919$. Corresponding to $\alpha = \alpha_{\rm cr}$, there is a value of γ that can be obtained by setting $k_{\rm cr} = k_\ell$ in the definition of $k_{\rm cr}$. Thus, for $k_\ell/k_0 \approx 10^3$ [14] we find $\gamma \approx 42.69$. For $\alpha \ll \alpha_{\rm cr}$, the dissipation is negligible and the Kolmogorov $k^{-5/3}$ spectrum can be substituted for $E(k)$ in Eq. (14). This yields a relation between the coefficients γ and α as $\gamma \simeq 153.0588\,\alpha$ for small α, where we set $k_\ell/k_0 \approx 10^3$. This gives $\alpha \simeq 0.0028$ for $\gamma = 0.4269$ at which the Kolmogorov spectrum is realized in the entire inertial range.

(iii) For our estimated value of $\alpha_{\rm cr} \approx 0.0919$, the temperature data can be obtained from the graph for the temperature dependence of the mutual friction parameter, given in the experimental work of Bevan *et al.* [30]. From their experimental graph, we extracted some temperature data and displayed them in Table I. We see that $1/\alpha_{\rm cr} \approx 10.88$ corresponds to $T \approx 0.35T_c$. Further, $\alpha \simeq 0.0028$ corresponds to $1/\alpha \approx 358$, giving $T \approx 0.2T_c$. Thus, the $k^{-5/3}$ Kolmogorov energy spectrum is realized in the entire inertial range $\ell < r < L$ at temperature $T \lesssim 0.2T_c$, while at $T \gtrsim 0.2T_c$ dissipation begins to dominate larger eddies exhibiting a k^{-3} spectrum toward large scales. At $T \approx 0.35T_c$, eddies of all sizes are highly affected by damping, giving the k^{-3} spectrum in the entire inertial range. For $0.2T_c \lesssim T \lesssim 0.35T_c$, there appear two different scaling regimes at different length scales, given by $E(k) \sim k^{-x}$ with $x = 5/3$ at small scales and $x = 3$ at large scales. This is in remarkably good agreement with the prediction from the recent DNS performed by Biferale *et al.* [15] where such scaling was observed at $T < 0.37T_c$.

IV. CONCLUDING REMARKS

To summarize, we obtain a crossover wave number $k_{\rm cr}$ that shifted toward higher values for increasing strength of mutual damping frequency Γ, and it separates two regimes corresponding to the scaling of turbulent energy density: a classical Kolmogorov spectrum $E(k) \sim k^{-5/3}$ for $k \gg k_{\rm cr}$ and a k^{-3} spectrum for $k \ll k_{\rm cr}$. Our expression for the crossover wave number [Eq. (10)] differs from that of Ref. [14] where $k_{\rm cr} = k_0|\gamma - \gamma_{\rm cr}|^{-2/3}$ and, accordingly, $k_{\rm cr}$ diverges there at $\gamma = \gamma_{\rm cr}$. In fact, their expression for $k_{\rm cr}$ was obtained by employing Leith differential closure [31] for the energy flux in a suitable form proposed by Nazarenko [32], and thereby they arrived at the subcritical and supercritical energy spectra. In our case, $k_{\rm cr}$ corresponds to $r_d \equiv (\varepsilon/\Gamma^3)^{1/2}$ (obtained from dimensional analysis), as defined in Sec. I. The value of $k_{\rm cr}$ changes gradually with the mutual friction parameter γ: with increasing temperature (i.e., with increasing value of γ), $k_{\rm cr}$ shifted toward a higher value ($k_{\rm cr} = k_0$ at $\gamma = 0.4269$). Thus it distinguishes two different scaling regimes for turbulent energy density.

We have shown that the Kolmogorov $k^{-5/3}$ regime occupies the entire inertial-range $l \ll r \ll L$ when $\gamma = 0.4269$, which corresponds to $T \approx 0.2T_c$. For $\gamma > 0.4269$ ($T \gtrsim 0.2T_c$), the energy flux $\varepsilon(k)$ decays with decreasing k, giving a k^{-3} regime at low wave numbers. As pointed out in Ref. [10], at some critical value of the mutual friction ($\alpha = \alpha_{\rm cr}$) there exists the self-similar balance between the energy flux and the mutual-friction energy dissipation, leading to the scale-invariant energy spectrum $E(k) \sim k^{-3}$. As such, we have obtained an expression for $\alpha_{\rm cr}$ from Eq. (14), giving $\alpha_{\rm cr} = 0.0919$, which corresponds to $T \approx 0.35\,T_c$. Our most remarkable finding that the two different scaling regimes exist for $T \lesssim 0.35\,T_c$ is in excellent agreement with the recent DNS results ($T \lesssim 0.37\,T_c$) [15]. The numerical value $\alpha_{\rm cr} \simeq 0.34$ predicted via the simulation of the shell model [14] corresponds to $T \simeq 0.42T_c$ and deviates from the DNS result [15].

To contrast with the previous parametrization, it may be noted that in Refs. [10,14], the formulations were based on the differential closure for the energy flux [31], which, although it gives the scaling exponent for energy spectra, does not accurately control the numerical prefactors. Without using such an uncontrolled algebraic expression for differential closure, we have derived the expression for turbulent energy density from the governing dynamical equation. As such, we have assumed the universal Kolmogorov scaling for the energy spectra, namely $E(k) = C\varepsilon^{2/3}k^{-5/3}$, corresponding to the isotropic background velocity field together with the universal scaling law for the eddy damping rate $\eta(k) = a\varepsilon^{1/3}k^{2/3}$. As a result, the numerical prefactors in the corresponding expression for energy spectra involve the universal numbers C and a. Further, the values of γ and α are also determined by these two universal numbers. Thus the value of the transition temperature would vary for different sets of C and a values. We have taken self-consistently calculated values of C and a based on the Heisenberg approximation. If one takes the EDQNM results [33], namely $C = 1.6$ and $a = 0.28$, the value of $\alpha_{\rm cr}$ becomes $\alpha_{\rm cr} = 0.0595$, which corresponds to $T \simeq 0.31T_c$.

While comparing with the DNS results of Biferale *et al.* [15], we find that our estimate $T \approx 0.35T_c$ deviates from their numerical prediction $T \approx 0.37T_c$ by 5.4%. In Ref. [14], the value $1/\alpha_{\rm cr} = 2.9411$ gives $T \approx 0.42T_c$ and thus the errors in the estimation of the temperature is 13.5% in comparison to the results of Biferale *et al.* [15]. Further, if one uses the EDQNM results (quoted above), $1/\alpha_{\rm cr} = 16.8067$, giving $T \simeq 0.31T_c$; the corresponding error is 16.2%.

In Kraichnan's DIA that we have adopted here, both the velocity field and the response functions have to be evaluated perturbatively and recursively at each order of the perturbative scheme. Such perturbative evaluation of the velocity field and the response function G_{ij} at higher order is a difficult task. To simplify the calculations, we made use of Leslie's suggestion [20] of replacing the full response by the isotropic response. Although it is a crude approximation, this makes the problem analytically tractable. From the theoretical point of view, it would be really interesting to see how the present results will get modified while relaxing such a crude approximation. An idea of tackling the full response function was proposed by Leslie himself in his book [20]. The idea is to construct the DIA equations for the response and energy transport rate starting from the governing dynamical equation (2) and then to treat them via a suitable approximation for eddy viscosity such as the Heisenberg approximation. In spite of this drawback, the present approach offers a theoretical framework to characterize the statistical behavior of superfluid turbulence in the hydrodynamic regime directly from the governing equation of motion that may potentially be used to attack much wider problems of superfluid turbulence.

[1] R. P. Feynman, *Progress in Low Temperature Physics* (North-Holland, Amsterdam, 1955), Vol. I.

[2] R. J. Donnelly, *Quantized Vortices in Helium II* (Cambridge University Press, Cambridge, 1991); S. R. Stalp, L. Skrbek, and R. J. Donnelly, Phys. Rev. Lett. **82**, 4831 (1999).

[3] S. N. Fisher, A. J. Hale, A. M. Guénault, and G. R. Pickett, Phys. Rev. Lett. **86**, 244 (2001); S. N. Fisher, M. J. Jackson, Y. A. Sergeev, and V. Tsepelin, Proc. Natl. Acad. Sci. (USA) **111**, 4659 (2014); M. A. Silaev, Phys. Rev. Lett. **108**, 045303 (2012); A. W. Baggaley, V. Tsepelin, C. F. Barenghi, S. N. Fisher, G. R. Pickett, Y. A. Sergeev, and N. Suramlishvili, *ibid.* **115**, 015302 (2015); M. J. Jackson, D. I. Bradley, A. M. Guénault, R. P. Haley, G. R. Pickett, and V. Tsepelin, Phys. Rev. B **95**, 094518 (2017).

[4] A. P. Finne *et al.*, Nature (London) **424**, 1022 (2003); J. Low Temp. Phys. **136**, 249 (2004).

[5] C. F. Barenghi, V. S. L'vov, and P. E. Roche, Proc. Natl. Acad. Sci. (USA) **111**, 4683 (2014); C. F. Barenghi, L. Skrbek, and K. R. Sreenivasan, *ibid.* **111**, 4647 (2014).
[6] V. Eltsov, R. Hänninen, and M. Krusius, Proc. Natl. Acad. Sci. (USA) **111**, 4711 (2014); R. Hänninen and A. W. Baggaley, *ibid.* **111**, 4667 (2014); P. Walmsley, D. Zmeev, F. Pakpour, and A. Golov, *ibid.* **111**, 4691 (2014); A. C. White, B. P. Anderson, and V. S. Bagnato, *ibid.* **111**, 4719 (2014).
[7] G. P. Bewley, D. P. Lathrop, and K. R. Sreenivasan, Nature (London) **441**, 588 (2006).
[8] D. I. Bradley, S. N. Fisher, A. M. Guénault, M. R. Lowe, G. R. Pickett, A. Rahm, and R. C. V. Whitehead, Phys. Rev. Lett. **93**, 235302 (2004); D. I. Bradley, S. N. Fisher, A. M. Guenault, M. R. Lowe, G. R. Pickett, A. Rahm, and R. C. V. Whitehead, *ibid.* **96**, 035301 (2006); **101**, 065302 (2008).
[9] L. D. Landau, J. Phys. USSR **5**, 71 (1941).
[10] V. S. L'vov, S. V. Nazarenko, and G. E. Volovik, JETP Lett. **80**, 535 (2004).
[11] W. F. Vinen, Phys. Rev. B **71**, 024513 (2005).
[12] T. Araki, M. Tsubota, and S. K. Nemirovskii, Phys. Rev. Lett. **89**, 145301 (2002); C. F. Barenghi, S. Hulton, and D. C. Samuels, *ibid.* **89**, 275301 (2002).
[13] S. Fujiyama and M. Tsubota, J. Low Temp. Phys. **158**, 428 (2010).
[14] L. Boué, V. L'vov, A. Pomyalov, and I. Procaccia, Phys. Rev. B **85**, 104502 (2012).
[15] L. Biferale, D. Khomenko, V. L'vov, A. Pomyalov, I. Procaccia, and G. Sahoo, Phys. Rev. B **95**, 184510 (2017).
[16] W. F. Vinen, Phys. Rev. B **61**, 1410 (2000).
[17] L. Boue, V. S. L'vov, Y. Nagar, S. V. Nazarenko, A. Pomyalov, and I. Procaccia, Phys. Rev. B **91**, 144501 (2015).
[18] V. S. L'vov, S. V. Nazarenko, and O. Rudenko, Phys. Rev. B **76**, 024520 (2007).
[19] R. H. Kraichnan, J. Fluid Mech. **5**, 497 (1959).
[20] D. C. Leslie, *Developments in the Theory of Turbulence* (Clarendon, Oxford, 1973).
[21] S. F. Edwards, J. Fluid. Mech. **18**, 239 (1964).
[22] R. H. Kraichnan, Phys. Fluids **8**, 575 (1965); **8**, 995 (1965).
[23] V. B. Eltsov, R. de Graaf, P. J. Heikkinen, J. J. Hosio, R. Hänninen, M. Krusius, and V. S. L'vov, Phys. Rev. Lett. **105**, 125301 (2010).
[24] W. F. Vinen and J. J. Niemela, J. Low Temp. Phys. **128**, 167 (2002).
[25] D. H. Wacks and C. F. Barenghi, Phys. Rev. B **84**, 184505 (2011); L. Boué, V. L'vov, A. Pomyalov, and I. Procaccia, Phys. Rev. Lett. **110**, 014502 (2013).
[26] A. N. Kolmogorov, C. R. (Dokl.) Acad. Sci. SSSR **30**, 299 (1941); A. M. Obukhov, Izv. Akad. Nauk. SSSR, Ser. Geogr. Geophys. **13**, 58 (1949).
[27] K. Dutta and M. K. Nandy, Phys. Rev. E **84**, 036315 (2011).
[28] S. A. Orszag, J. Fluid Mech. **41**, 363 (1970); A. Pouquet *et al.*, *ibid.* **72**, 305 (1975); O. Métais and M. Lesieur, J. Atmos. Sci. **43**, 857 (1986).
[29] P. M. Walmsley, A. I. Golov, H. E. Hall, A. A. Levchenko, and W. F. Vinen, Phys. Rev. Lett. **99**, 265302 (2007).
[30] T. D. C. Bevan, A. J. Manninen, J. B. Cook, A. J. Armstrong, J. R. Hook, and H. E. Hall, Phys. Rev. Lett. **74**, 750 (1995); T. D. C. Bevan, A. J. Manninen, J. B. Cook, H. Alles, J. R. Hook, and H. E. Hall, J. Low Temp. Phys. **109**, 423 (1997).
[31] C. Leith, Phys. Fluids **10**, 1409 (1967).
[32] C. Connaughton and S. Nazarenko, Phys. Rev. Lett. **92**, 044501 (2004).
[33] L. M. Smith and W. C. Reynolds, Phys. Fluids A **4**, 364 (1992).