Gyrokinetic Landau collision operator in conservative form

Qingjiang Pan* and Darin R. Ernst†
Plasma Science and Fusion Center, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA

(Received 3 November 2018; published 8 February 2019)

A gyrokinetic linearized exact (not model) Landau collision operator is derived by transforming the symmetric and conservative Landau form. The formulation obtains the velocity-space flux density and preserves the operator's conservative form as the divergence of this flux density. The operator contains both test-particle and field-particle contributions, and finite Larmor radius effects are evaluated in either Bessel function series or gyrophase integrals. While equivalent to the gyrokinetic Fokker–Planck form with Rosenbluth potentials [B. Li and D. R. Ernst, Phys. Rev. Lett. **106**, 195002 (2011)], the gyrokinetic conservative Landau form explicitly preserves the symmetry between test-particle and field-particle contributions, which underlies the conservation laws and the H theorem, and enables discretization with a finite-volume or spectral method to preserve the conservation properties numerically, independent of resolution. The form of the exact linearized field-particle terms differs from those of widely used model operators. We show the finite Larmor radius corrections to the field-particle terms in the exact linearized operator involve Bessel functions of all orders, while present model field-particle terms involve only the first two Bessel functions. This new symmetric and conservative formulation enables the gyrokinetic exact linearized Landau operator to be implemented in gyrokinetic turbulence codes for comparison with present model operators using similar numerical methods.

DOI: 10.1103/PhysRevE.99.023201

I. INTRODUCTION

Collisions are common and fundamentally important in laboratory and natural plasmas. They play an essential role in the classical and neoclassical transport in magnetic fusion confinement experiments, and irreversibly dissipate kinetic energy into thermal energy in kinetic turbulence. Considering the statistics of small-angle Coulomb collisions within a Debye sphere, which typically account for 90% of scattering events, using the Boltzmann collision operator results in the well-known Landau operator (or Fokker–Planck operator) accurate to $\mathcal{O}(1/\ln\Lambda)$, with $\ln\Lambda$ the Coulomb logarithm [1,2]. Both Landau and Fokker–Planck operators describe the collision effect as a divergence of a velocity-space flux density,

$$C_{ab}(f_a, f_b) = -\nabla\cdot\boldsymbol{J}_{ab}, \tag{1}$$

where $f_s = f_s(\boldsymbol{v})$ ($s = a, b$) are distribution functions in the a-b type of collisions, and $\nabla = \partial/\partial\boldsymbol{v}$. The flux density for the Landau operator is written in an integral form as

$$\boldsymbol{J}_{ab} = \Gamma_{ab}\int \boldsymbol{U}\cdot\left(\frac{f_a}{m_b}\nabla' f_b' - \frac{f_b'}{m_a}\nabla f_a\right)d^3v', \tag{2}$$

where m_s is the particle mass, $\Gamma_{ab} = 2\pi e_a^2 e_b^2 \ln\Lambda/m_a$, with e_s the particle charge, $f' = f(\boldsymbol{v}')$, $\nabla' = \partial/\partial\boldsymbol{v}'$, and the Landau tensor is $\boldsymbol{U} = \boldsymbol{S}/u$, with $\boldsymbol{u} = \boldsymbol{v} - \boldsymbol{v}'$ the relative velocity, $u = |\boldsymbol{u}|$, and $\boldsymbol{S} = \boldsymbol{I} - \boldsymbol{u}\boldsymbol{u}/u^2$ the orthogonal projection onto the plane perpendicular to $\boldsymbol{u}$. Physically, $\boldsymbol{v}$ and $\boldsymbol{v}'$ can be understood as the particle velocities prior to the elastic collision in the coarse-grained model of long-range Coulomb interactions. The flux density for the Fokker–Planck operator can be expressed as differentiations of Rosenbluth potentials,

$$\boldsymbol{J}_{ab} = \Gamma_{ab}\left(\frac{2}{m_b} f_a \nabla H - \frac{1}{m_a}\nabla\nabla G\cdot\nabla f_a\right), \tag{3}$$

with the Rosenbluth potentials defined as $G(\boldsymbol{v}) = \int f_b' u\, d^3v'$ and $H(\boldsymbol{v}) = \int f_b'/u\, d^3v'$. These two operators are equivalent [3]. For clarity, hereafter they are referred to as the Landau form and the Fokker–Planck form. While the Rosenbluth potentials satisfy Poisson-like equations and many analytical and numerical methods can be used to solve them in $\mathcal{O}(N\ln N)$ operations [4–7], previous studies suggest that the symmetric Landau form is useful in numerical simulations to preserve the conservation laws and the H theorem inherent in the operator [8–12], which are important for the physical fidelity of numerical solutions, especially for simulations with low to moderate velocity-space resolutions.

The Landau operator $C_{ab}(f_a, f_b)$ for unlike-species collisions is bilinear in f_a and f_b, and $C_{aa}(f_a, f_a)$ for like-species collisions is nonlinear in f_a. In situations such as the tokamak core, the turbulence fluctuation amplitude is much smaller than the equilibrium background ($|f - f_0| \ll f_0$), the fluctuation frequency is much less than the gyrofrequency ($\omega \ll \Omega$), and the gyroradius is much smaller than the scales of the background magnetic field, density, and temperature variations ($\rho \ll L$). In this ordering, referred to as the standard gyrokinetic ordering, the fast-scale gyration about the magnetic field can be averaged over, and the Fokker–Planck equation for the six-dimensional particle distribution function can be transformed to the

*panqj@psfc.mit.edu
†dernst@psfc.mit.edu

gyrokinetic equation describing evolution of the perturbed five-dimensional guiding-center distribution function [13–16]. For cases where departures from a Maxwellian distribution are relatively small, such as microturbulence in the core of magnetically confined fusion plasmas, a linearized collision operator is appropriate and consistent with the standard gyrokinetic ordering described in the foregoing references. To include finite Larmor radius (FLR) effects in the collision operator, the linearized collision operator is gyrokinetically transformed as follows. For convenience, the spatial coordinates are transformed to a Fourier representation in wave number $\mathbf{k}$. The non-Maxwellian, nonadiabatic part of the fluctuating guiding center distribution h is transformed to particle coordinates via the phase factor $e^{-i\boldsymbol{k}\cdot\boldsymbol{\rho}}$, where $\boldsymbol{\rho}$ is the gyroradius defined below. The collision operator then acts on the distributions in particle coordinates, and the result is transformed back to guiding center coordinates via the phase factor $e^{+i\boldsymbol{k}\cdot\boldsymbol{\rho}}$. Finally the result is averaged over the gyrophase [13,17,18]:

$$C_{ab}^{\mathrm{gk}}(h_a, h_b) = \left\langle e^{i\boldsymbol{k}\cdot\boldsymbol{\rho}_a} C_{ab}^{L}(h_a e^{-i\boldsymbol{k}\cdot\boldsymbol{\rho}_a}, h_b e^{-i\boldsymbol{k}\cdot\boldsymbol{\rho}_b})\right\rangle. \quad (4)$$

The distribution function is expanded as $f_s = f_{s0} + f_{s1} + \mathcal{O}(\epsilon^2)$, where the zeroth-order distribution is assumed to be Maxwellian, $f_{s0} = f_{sM} = n_s(2\pi T_s/m_s)^{-3/2}\exp[-m_s v^2/(2T_s)]$, and the first-order distribution $f_{s1} = -e_s\varphi f_{sM}/T_s + h_s$ consists of an adiabatic part associated with the electrostatic potential φ and a nonadiabatic part written in guiding-center coordinates $h_s = h_s(t, \boldsymbol{R}, v_\perp, v_\parallel)$. Here $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{\rho}_s$ is the guiding-center position, $\boldsymbol{\rho}_s = \boldsymbol{b}\times\boldsymbol{v}/\Omega_s$ is the gyroradius vector, $\boldsymbol{b} = \boldsymbol{B}/B$ is a unit vector in the direction of the magnetic field, $\Omega_s = e_s B/m_s c$ is the gyro-frequency, and $\epsilon \sim f_1/f_M \sim e\varphi/T \sim \omega/\Omega \sim \rho/L$ is the expansion parameter in the gyrokinetic ordering. Note that C_{ab}^L is a linear collision operator that acts on the particle distribution function, and $\langle\cdots\rangle = \oint d\phi/2\pi$ represents averaging over the gyrophase while holding $\mathbf{R}$ fixed. It is understood that Eq. (4) is valid for each Fourier component in wave number $\mathbf{k}$ and $h_s = h_{s\boldsymbol{k}}$ is implied.

The Fokker–Planck form linearized about a Maxwellian background is a natural choice for the collision operator. It inherits the conservation properties and the H theorem of the original nonlinear collision operator and is comprised of test-particle contribution and field-particle contribution, $C_{ab}^L(f_{a1}, f_{b1}) = C_{ab}^T(f_{a1}, f_{b0}) + C_{ab}^F(f_{a0}, f_{b1})$ [3,19]. The test-particle part consists of pitch-angle scattering and energy diffusion and can be written as [20]

$$\begin{aligned} C_{ab}^T(f_{a1}, f_{bM}) = {} & \frac{\nu_D^{ab}(v)}{2}\frac{\partial}{\partial \boldsymbol{v}}\cdot(v^2\boldsymbol{I} - \boldsymbol{v}\boldsymbol{v})\cdot\frac{\partial f_{a1}}{\partial \boldsymbol{v}} \\ & + \frac{1}{v^2}\frac{\partial}{\partial v}\left[\frac{\nu_\parallel^{ab}(v)}{2}v^4 f_{aM}\frac{\partial}{\partial v}\left(\frac{f_{a1}}{f_{aM}}\right)\right] \\ & + \frac{m_a}{T_b}\left(1 - \frac{T_b}{T_a}\right)\frac{1}{v^2}\frac{\partial}{\partial v}\left[\frac{\nu_\parallel^{ab}(v)}{2}v^5 f_{a1}\right], \end{aligned} \quad (5)$$

with the collision frequencies for pitch-angle scattering and energy diffusion given by

$$\nu_D^{ab}(v) \equiv \frac{4\pi e_a^2 e_b^2 \ln\Lambda}{m_a^2}\frac{dG_0(v)}{v^3 dv} = \hat{\nu}^{ab}\frac{\Phi(x_b) - \Psi(x_b)}{x_a^3} \quad (6)$$

and

$$\nu_\parallel^{ab}(v) \equiv \frac{4\pi e_a^2 e_b^2 \ln\Lambda}{m_a^2}\frac{d^2G_0(v)}{v^2 dv^2} = 2\hat{\nu}^{ab}\frac{\Psi(x_b)}{x_a^3}, \quad (7)$$

respectively. Here $G_0(v) = \int f_{b0}' u\, d^3v'$ is the Rosenbluth potential for the background, $\hat{\nu}^{ab} \equiv 4\pi n_b e_a^2 e_b^2 \ln\Lambda/(m_a^2 v_{Ta}^3)$ defines a basic collision frequency, $v_{Ts} = \sqrt{2T_s/m_s}$, $\Phi(x) \equiv 2\pi^{-1/2}\int_0^x \exp(-y^2)\,dy$ is the error function, $\Psi(x) \equiv [\Phi(x) - x\Phi'(x)]/(2x^2)$ is the so-called Chandrasekhar function, and $x_s \equiv v/v_{Ts}$. The field-particle part involves the Rosenbluth potentials of the perturbed field-particle distribution function and was generally considered intractable. Significant efforts have been made to construct various model operators to simplify the linearized operator [13,17,19,21]. Recently, Abel *et al.* [18] and Catto and Ernst [22] proposed a model operator for like-species collisions, which consists of $C_{aa}^T(f_{a1}, f_{aM})$ and two additional terms restoring the momentum and energy conservation. The model operator includes both pitch-angle scattering and energy diffusion, preferentially damps small structures, and satisfies the conservation laws and the H theorem. As described in Ref. [18], the two correcting terms are the standard momentum and energy restoring expressions for the pitch-angle scattering and energy diffusion that had appeared in Eqs. (21) and (22) of the seminal work by Hirshman and Sigmar [19], and, incidentally, Abel's operator in drift-kinetic limit (i.e., $k\rho = 0$) had been used for ion–ion collisions in earlier numerical studies of neoclassical transport [23,24]. Later this model operator was extended to treat collisions of multiple ion species with unequal temperatures and comparable masses while preserving the conservation laws and the H theorem [20].

Using Eq. (4), Li and Ernst [25] obtained the gyrokinetic version of the exact linearized field-particle operator for the first time. It involves a single two-dimensional velocity integral over the guiding-center distribution, and the gyrophase integral accounting for the FLR effects can be precomputed for a given velocity grid; thus the gyrokinetic linearized exact operator may be computationally affordable in large scale gyrokinetic simulations of plasma turbulence. A notable feature of the gyrokinetic exact field-particle operator is that the gyrophase integral is logarithmically singular at $u = 0$, namely when the colliding particles have the same velocity. In contrast, the collision frequencies $\nu_D^{ab}(v)$ and $\nu_\parallel^{ab}(v)$ in the test-particle operator diverge as $v \to 0$. In fact, both types of singularities originate from the Landau tensor, as will be shown in Appendix B. In numerical implementations, if the singularities are not treated similarly, so that errors due to the singular behavior do not cancel, the conservation properties could be affected since the integral kernel near the singularity of the field-particle operator makes the dominant contribution. In this paper, we reformulate the operator to overcome this obstacle. The key idea is linearizing and gyroaveraging the Landau form instead of the Fokker–Planck form while preserving the symmetry and the conservative structure, so that potential numerical errors associated with the singularity in the field-particle contribution can be canceled by the test-particle contribution, and the conservation laws are preserved regardless of velocity-space resolution.

The remainder of the paper is organized as follows. In Sec. II, properties of the linearized Landau operator such as the conservation laws and the H theorem are demonstrated based on the symmetry and the conservative structure inherited from the nonlinear operator. The gyrokinetic version of the linearized Landau operator as a divergence of a velocity-space flux density is derived in Sec. III. The flux density can be expressed as Bessel function series or equivalently in integral form. Section IV describes numerical methods, either finite-volume or spectral, to preserve the conservation properties. Conclusions and a discussion of future work are presented in Sec. V.

II. PROPERTIES OF THE LINEARIZED LANDAU OPERATOR

To derive the gyrokinetic version of linearized Landau collision operator, we begin by substituting $f_s = f_{s0} + f_{s1} + \mathcal{O}(\epsilon^2)$ into the nonlinear Landau form, Eqs. (1) and (2), to obtain

$$C_{ab} = C_{ab}(f_{a0}, f_{b0}) + C^L_{ab}(f_{a1}, f_{b1}) + \mathcal{O}(\epsilon^2), \tag{8}$$

where

$$C_{ab}(f_{a0}, f_{b0}) = -\Gamma_{ab}\nabla\cdot\int \boldsymbol{U}\cdot\left(\frac{f_{a0}}{m_b}\nabla' f'_{b0} - \frac{f'_{b0}}{m_a}\nabla f_{a0}\right)d^3v' \tag{9}$$

is the equilibrium operator formally at order $\mathcal{O}(\epsilon^0)$ and

$$C^L_{ab}(f_{a1}, f_{b1}) = C_{ab}(f_{a1}, f_{b0}) + C_{ab}(f_{a0}, f_{b1}) \tag{10}$$

is the linearized operator at order $\mathcal{O}(\epsilon)$, with the test-particle part and the field-particle part given by

$$C_{ab}(f_{a1}, f_{b0}) = -\Gamma_{ab}\nabla\cdot\int \boldsymbol{U}\cdot\left(\frac{f_{a1}}{m_b}\nabla' f'_{b0} - \frac{f'_{b0}}{m_a}\nabla f_{a1}\right)d^3v' \tag{11}$$

and

$$C_{ab}(f_{a0}, f_{b1}) = -\Gamma_{ab}\nabla\cdot\int \boldsymbol{U}\cdot\left(\frac{f_{a0}}{m_b}\nabla' f'_{b1} - \frac{f'_{b1}}{m_a}\nabla f_{a0}\right)d^3v', \tag{12}$$

respectively. Note that $C_{ab}(f_{a0}, f_{b0}) = 0$ when $f_{a0} = f_{aM}$, $f_{b0} = f_{bM}$, with $T_a = T_b = T$. This can be seen by using the relation

$$\boldsymbol{U}\cdot\left(\frac{f_{a0}}{m_b}\nabla' f'_{b0} - \frac{f'_{b0}}{m_a}\nabla f_{a0}\right) = \frac{f_{aM} f'_{bM}}{T}\boldsymbol{U}\cdot\boldsymbol{u} = 0. \tag{13}$$

When $T_a \neq T_b$, the $C_{ab}(f_{a0}, f_{b0})$ term pushes the plasma towards an equilibrium state between different species. For the same relation shown in Eq. (13), $C_{ab}(e_a\varphi f_{aM}/T_a, f_{bM}) = C_{ab}(f_{aM}, e_b\varphi f_{bM}/T_b) = 0$ when $T_a = T_b$; namely, the contribution from the adiabatic part of the first-order distribution function vanishes for the linearized operator.

The conservative structure of the Landau operator is preserved in the linearization. The (anti)symmetry of the Landau operator carries through to the symmetry between the test-particle part and field-particle part. The linearized operator thus inherits essential physical properties of the Landau operator. First, it conserves particles, momentum, and energy. To show this, consider

$$\begin{aligned}\int d^3v\,\phi_a C_{ab}(f_{a1}, f_{b0}) + \int d^3v\,\phi_b C_{ba}(f_{b0}, f_{a1}) &= 2\pi e_a^2 e_b^2 \ln\Lambda \int d^3v \int d^3v'\left(\frac{\nabla\phi_a}{m_a} - \frac{\nabla'\phi'_b}{m_b}\right)\\ &\quad\cdot\boldsymbol{U}\cdot\left(\frac{f_{a1}}{m_b}\nabla' f'_{b0} - \frac{f'_{b0}}{m_a}\nabla f_{a1}\right),\end{aligned} \tag{14}$$

which vanishes because $(\nabla\phi_a/m_a - \nabla'\phi'_b/m_b)\cdot\boldsymbol{U} = \boldsymbol{0}$ for $\phi_s = 1$ (particle conservation), $\phi_s = m_s\boldsymbol{v}$ (momentum conservation), and $\phi_s = m_s\boldsymbol{v}^2/2$ (energy conservation). While the particle conservation is independently satisfied by the test-particle part and the field-particle part, the momentum (and energy) loss via the test-particle operator of a-b collisions is exactly canceled by the momentum (and energy) gain via the field-particle operator of b-a collisions, and vice versa.

Second, the entropy production due to a-b collisions for the linearized case is given by

$$\begin{aligned}\frac{dS_a}{dt} &\equiv -\frac{d}{dt}\int f_a \ln f_a d^3v\\ &\simeq -\int C^L_{ab}(f_{a1}, f_{b1})(\ln f_{a0} + \hat{f}_{a1})d^3v,\end{aligned} \tag{15}$$

where the normalized perturbed distribution function $\hat{f}_{s1} = f_{s1}/f_{s0}$ is introduced. When f_{a0} and f_{b0} are Maxwellian distribution functions with equal temperatures, the first term from the mutual collisions between species a and b does not contribute to the overall entropy production due to the conservation laws, thus we have the H theorem based on the contribution from the second term,

$$\begin{aligned}\sum_{s=a,b}\frac{dS_s}{dt} &= -\left[\int d^3v\hat{f}_{a1}C^L_{ab}(f_{a1}, f_{b1}) + \int d^3v\hat{f}_{b1}C^L_{ba}(f_{b1}, f_{a1})\right]\\ &= 2\pi e_a^2 e_b^2 \ln\Lambda \int d^3v \int d^3v' f_{a0} f'_{b0}\\ &\quad\times\left(\frac{\nabla'\hat{f}'_{b1}}{m_b} - \frac{\nabla\hat{f}_{a1}}{m_a}\right)\cdot\boldsymbol{U}\cdot\left(\frac{\nabla'\hat{f}'_{b1}}{m_b} - \frac{\nabla\hat{f}_{a1}}{m_a}\right) \geqslant 0,\end{aligned} \tag{16}$$

due to the Cauchy-Schwarz inequality. Note that with unequal temperatures $T_a \neq T_b$ the dominant entropy production is from the equilibrium operator and is closely related to the collisional energy exchange between the Maxwellian distribution functions,

$$\begin{aligned}\sum_{s=a,b}\frac{dS_s}{dt} &= -\left[\int C_{ab}(f_{a0}, f_{b0})\ln f_{a0}\, d^3v + \int C_{ba}(f_{b0}, f_{a0})\ln f_{b0}\, d^3v\right]\\ &= \int C_{ab}(f_{a0}, f_{b0})\frac{m_a v^2}{2T_a}\, d^3v + \int C_{ba}(f_{b0}, f_{a0})\frac{m_b v^2}{2T_b}\, d^3v.\end{aligned} \tag{17}$$

The expression for energy exchange is well known (Ref. [26], p. 34):

$$\int C_{ab}(f_{a0}, f_{b0})\frac{m_a v^2}{2}d^3v = \frac{4\sqrt{2\pi}n_a n_b e_a^2 e_b^2 \ln\Lambda}{m_a m_b}(T_b - T_a)\left(\frac{T_a}{m_a} + \frac{T_b}{m_b}\right)^{-3/2}. \quad (18)$$

In the present context it can be calculated with the equilibrium operator obtained from replacing f_{a1} with f_{aM} in Eq. (5):

$$C_{ab}(f_{a0}, f_{b0}) = \frac{m_a}{T_b}\left(1 - \frac{T_b}{T_a}\right)\frac{1}{v^2}\frac{\partial}{\partial v}\left[\frac{v_\parallel^{ab}(v)}{2}v^5 f_{aM}\right]. \quad (19)$$

Therefore the H theorem for the equilibrium operator is established as

$$\sum_{s=a,b}\frac{dS_s}{dt} = \frac{4\sqrt{2\pi}n_a n_b e_a^2 e_b^2 \ln\Lambda}{m_a m_b} \times \left(\frac{T_a}{m_a} + \frac{T_b}{m_b}\right)^{-3/2}\frac{(T_b - T_a)^2}{T_a T_b} \geqslant 0. \quad (20)$$

Accordingly, the H theorem is satisfied and entropy production is positive for collisions between two species with equal or unequal temperatures.

III. GYROKINETIC LINEARIZED LANDAU OPERATOR IN CONSERVATIVE LANDAU FORM

A. Bessel function series for field-particle terms

The gyrokinetic version of the linearized operator describing collision effects on the nonadiabatic guiding-center distribution can be obtained via a guiding-center transformation and gyrophase averaging. Mathematically, this is achieved by substituting Eqs. (10)–(12) into Eq. (4), $C_{ab}^L(h_a, h_b) \to C_{ab}^{\rm gk}(h_a, h_b) = \langle e^{i\boldsymbol{k}\cdot\boldsymbol{\rho}_a} C_{ab}^L(h_a e^{-i\boldsymbol{k}\cdot\boldsymbol{\rho}_a}, h_b e^{-i\boldsymbol{k}\cdot\boldsymbol{\rho}_b})\rangle$. In order to carry out the derivation, a coordinate system in velocity space needs to be specified. In this paper, we adopt a cylindrical coordinate system as defined in Appendix A. For future reference, the gyrokinetic operator in spherical representation (speed and pitch angle) is given in Appendix C. The spherical representation is sometimes preferred because it diagonalizes the test-particle operator in the Fokker–Planck form, motivating some codes to use speed (or energy) and pitch-angle coordinates.

By using the vector relation $a\nabla\cdot\boldsymbol{A} = \nabla\cdot(a\boldsymbol{A}) - \boldsymbol{A}\cdot\nabla a$, the gyrokinetic test-particle operator can be split into two parts,

$$C_{ab}^{\rm gk}(h_a, f_{b0})/\Gamma_{ab} = -\left\langle\nabla\cdot\left[e^{i\boldsymbol{k}\cdot\boldsymbol{\rho}_a}\int \boldsymbol{U}\cdot\left(\frac{h_a}{m_b}e^{-i\boldsymbol{k}\cdot\boldsymbol{\rho}_a}\nabla' f_{b0}' - \frac{f_{b0}'}{m_a}\nabla(h_a e^{-i\boldsymbol{k}\cdot\boldsymbol{\rho}_a})\right)d^3v'\right]\right\rangle + \left\langle\nabla\cdot e^{i\boldsymbol{k}\cdot\boldsymbol{\rho}_a}\int \boldsymbol{U}\cdot\left(\frac{h_a}{m_b}e^{-i\boldsymbol{k}\cdot\boldsymbol{\rho}_a}\nabla' f_{b0}' - \frac{f_{b0}'}{m_a}\nabla(h_a e^{-i\boldsymbol{k}\cdot\boldsymbol{\rho}_a})\right)d^3v'\right\rangle. \quad (21)$$

The first part of Eq. (21) can be written as

$$\begin{aligned}
&-\left\langle\nabla\cdot\left[e^{i\boldsymbol{k}\cdot\boldsymbol{\rho}_a}\int d^3v'\boldsymbol{U}\cdot\left(\frac{h_a}{m_b}e^{-i\boldsymbol{k}\cdot\boldsymbol{\rho}_a}\nabla' f_{b0}' - \frac{f_{b0}'}{m_a}\nabla(h_a e^{-i\boldsymbol{k}\cdot\boldsymbol{\rho}_a})\right)\right]\right\rangle\\
&= -\frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}\left\langle v_\perp\int d^3v'\frac{f_{b0}'}{m_a}\left[U_{\perp\perp}\left(-\frac{\partial h_a}{\partial v_\perp} + \frac{i\boldsymbol{k}\cdot\boldsymbol{\rho}_a h_a}{v_\perp}\right) + U_{\perp\perp'}\frac{m_a}{m_b}\frac{h_a\partial f_{b0}'}{f_{b0}'\partial v_\perp'} + U_{\perp\parallel}\left(\frac{m_a}{m_b}\frac{h_a\partial f_{b0}'}{f_{b0}'\partial v_\parallel'} - \frac{\partial h_a}{\partial v_\parallel}\right) + U_{\perp\phi}\frac{-i\boldsymbol{k}\cdot\boldsymbol{v}_\perp h_a}{\Omega_a v_\perp}\right]\right\rangle\\
&\quad -\frac{\partial}{\partial v_\parallel}\left\langle\int d^3v'\frac{f_{b0}'}{m_a}\left[U_{\parallel\perp}\left(-\frac{\partial h_a}{\partial v_\perp} + \frac{i\boldsymbol{k}\cdot\boldsymbol{\rho}_a h_a}{v_\perp}\right) + U_{\parallel\perp'}\frac{m_a}{m_b}\frac{h_a\partial f_{b0}'}{f_{b0}'\partial v_\perp'} + U_{\parallel\parallel}\left(\frac{m_a}{m_b}\frac{h_a\partial f_{b0}'}{f_{b0}'\partial v_\parallel'} - \frac{\partial h_a}{\partial v_\parallel}\right) + U_{\parallel\phi}\frac{-i\boldsymbol{k}\cdot\boldsymbol{v}_\perp h_a}{\Omega_a v_\perp}\right]\right\rangle\\
&= -\frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}\left\langle v_\perp\int d^3v'\frac{f_{b0}'}{m_a}\left[-U_{\perp\perp}\frac{\partial h_a}{\partial v_\perp} + U_{\perp\parallel}\left(\frac{m_a}{m_b}\frac{h_a\partial f_{b0}'}{f_{b0}'\partial v_\parallel'} - \frac{\partial h_a}{\partial v_\parallel}\right) + U_{\perp\perp'}\frac{m_a}{m_b}\frac{h_a\partial f_{b0}'}{f_{b0}'\partial v_\perp'}\right]\right\rangle\\
&\quad -\frac{\partial}{\partial v_\parallel}\left\langle\int d^3v'\frac{f_{b0}'}{m_a}\left[-U_{\parallel\perp}\frac{\partial h_a}{\partial v_\perp} + U_{\parallel\parallel}\left(\frac{m_a}{m_b}\frac{h_a\partial f_{b0}'}{f_{b0}'\partial v_\parallel'} - \frac{\partial h_a}{\partial v_\parallel}\right) + U_{\parallel\perp'}\frac{m_a}{m_b}\frac{h_a\partial f_{b0}'}{f_{b0}'\partial v_\perp'}\right]\right\rangle. \quad (22)
\end{aligned}$$

Here the first identity results from projecting the ∇ operator and the Landau tensor onto the cylindrical coordinate basis. The relations in Eqs. (A3)–(A5) associated with the cylindrical coordinates are used. $U_{\mu\nu} \equiv \boldsymbol{e}_\mu\cdot\boldsymbol{U}\cdot\boldsymbol{e}_\nu$ is the projection of the Landau tensor, and is given explicitly in Eqs. (A7)–(A18). The $\frac{1}{v_\perp}\frac{\partial}{\partial\phi}$ term of the divergence does not survive the averaging over gyrophase. The apparent imaginary terms vanish in the second identity of Eq. (22) because f_{b0}' is assumed to be independent of ϕ', and

$$\begin{aligned}
\int_0^{2\pi}d\phi\int_0^{2\pi}d\phi' U_{\mu\nu}(\phi - \phi')g(\phi) &= -\int_0^{2\pi}d\phi\, g(\phi)\int_{\phi-0}^{\phi-2\pi}d\delta\, U_{\mu\nu}(\delta)\\
&= \int_0^{2\pi}d\phi\, g(\phi)\int_0^{2\pi}d\delta\, U_{\mu\nu}(\delta) = 0, \quad (23)
\end{aligned}$$

for $g(\phi) \in \{\sin\phi, \cos\phi\}$, where the first identity is from the change of integration variable ϕ' to $\delta = \phi - \phi'$, and the second identity results from the periodicity of $U_{\mu\nu}(\delta)$ in δ.

By using the same relations in Eqs. (A3)–(A5) and argument in Eq. (23), it is not difficult to show that the second part of the test-particle operator in Eq. (21) accounting for FLR effects can be written as

$$\begin{aligned}
&\left\langle (\nabla e^{i\boldsymbol{k}\cdot\boldsymbol{\rho}_a})\cdot\int d^3v' U\cdot\left(\frac{h_a}{m_b}e^{-i\boldsymbol{k}\cdot\boldsymbol{\rho}_a}\nabla' f'_{b0} - \frac{f'_{b0}}{m_a}\nabla(h_a e^{-i\boldsymbol{k}\cdot\boldsymbol{\rho}_a})\right)\right\rangle \\
&= -\frac{k^2\rho_a^2 h_a}{v_\perp^2}\int d^2v'\frac{f'_{b0}}{m_a}2\pi\oint\frac{d\delta}{2\pi}\left[\frac{1}{2}U_{\perp\perp}(\delta) + \frac{1}{2}U_{\phi\phi}(\delta)\right] \\
&= -\frac{k^2\rho_a^2 h_a}{v_\perp^2}\int d^2v'\frac{f'_{b0}}{m_a}2\pi\oint\frac{d\delta}{2\pi}\frac{1}{2u^3}[u^2 + (v_\parallel - v'_\parallel)^2],
\end{aligned} \tag{24}$$

where k is the wave number perpendicular to the magnetic field.

A notable feature of the gyrokinetic test-particle operator is that the FLR effects are completely separable from the drift-kinetic part. This feature is present in the Landau form, as well as the Fokker–Planck form [18,22]. In Appendix B, it is demonstrated that the test-particle operator in the Landau form is equivalent to the gyrokinetic version of Eq. (5). Specifically the drift-kinetic part of Eq. (22) corresponds to the drift-kinetic part of the gyrokinetic version of Eq. (5), and the FLR terms representing the gyrodiffusion in Eq. (24) can also be written as $[-\nu_D^{ab}(2v_\parallel^2 + v_\perp^2) - \nu_\parallel^{ab}v_\perp^2]k^2h_a/(4\Omega_a^2)$ in the Fokker–Planck form [13,18,21,22]. The gyrodiffusion increases secularly with the perpendicular wave number and thus preferentially damps the high-k modes of turbulent fluctuations.

The gyrokinetic field-particle operator also contains two parts. By projecting the ∇ operator and the Landau tensor onto the coordinate basis, the first part becomes

$$\begin{aligned}
&-\left\langle \nabla\cdot\left[e^{i\boldsymbol{k}\cdot\boldsymbol{\rho}_a}\int d^3v' U\cdot\left(\frac{f_{a0}}{m_b}\nabla'(h'_b e^{-i\boldsymbol{k}\cdot\boldsymbol{\rho}'_b}) - \frac{h'_b}{m_a}e^{-i\boldsymbol{k}\cdot\boldsymbol{\rho}'_b}\nabla f_{a0}\right)\right]\right\rangle \\
&= -\frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}\left\langle v_\perp\int d^3v' e^{i\boldsymbol{k}\cdot(\boldsymbol{\rho}_a - \boldsymbol{\rho}'_b)}\frac{f_{a0}}{m_b}\sum_n\left[c^n_{\perp\perp'}e^{in\delta}\left(\frac{\partial h'_b}{\partial v'_\perp} + \frac{e^{i\phi'} - e^{-i\phi'}}{2}\frac{k\rho'_b h'_b}{v'_\perp}\right)\right.\right. \\
&\quad\left.\left. - c^n_{\perp\perp}\frac{m_b}{m_a}\frac{h'_b\partial f_{a0}}{f_{a0}\partial v_\perp} + c^n_{\perp\parallel'}e^{in\delta}\left(\frac{\partial h'_b}{\partial v'_\parallel} - \frac{m_b}{m_a}\frac{h'_b\partial f_{a0}}{f_{a0}\partial v_\parallel}\right) - c^n_{\perp\phi'}e^{in\delta}\frac{e^{i\phi'} + e^{-i\phi'}}{2i}\frac{k\rho'_b h'_b}{v'_\perp}\right]\right\rangle \\
&\quad - \frac{\partial}{\partial v_\parallel}\left\langle\int d^3v' e^{i\boldsymbol{k}\cdot(\boldsymbol{\rho}_a - \boldsymbol{\rho}'_b)}\frac{f_{a0}}{m_b}\sum_n\left[c^n_{\parallel\perp'}e^{in\delta}\left(\frac{\partial h'_b}{\partial v'_\perp} + \frac{e^{i\phi'} - e^{-i\phi'}}{2}\frac{k\rho'_b h'_b}{v'_\perp}\right)\right.\right. \\
&\quad\left.\left. - c^n_{\parallel\perp}\frac{m_b}{m_a}\frac{h'_b\partial f_{a0}}{f_{a0}\partial v_\perp} + c^n_{\parallel\parallel'}e^{in\delta}\left(\frac{\partial h'_b}{\partial v'_\parallel} - \frac{m_b}{m_a}\frac{h'_b\partial f_{a0}}{f_{a0}\partial v_\parallel}\right) - c^n_{\parallel\phi'}e^{in\delta}\frac{e^{i\phi'} + e^{-i\phi'}}{2i}\frac{k\rho'_b h'_b}{v'_\perp}\right]\right\rangle,
\end{aligned} \tag{25}$$

where the Landau tensor projection $U_{\mu\nu}$ is periodic in δ and expanded in Fourier series, $U_{\mu\nu}(\delta) = \sum_n c^n_{\mu\nu}e^{in\delta}$, with $c^n_{\mu\nu}$ the nth expansion coefficient given by Eq. (A20). Notice that $U_{\mu\nu}$ is either even or odd in δ, thus $c^n_{\mu\nu}$ is either real or imaginary. To represent the FLR effects, the integral representations of Bessel functions $J^a_n \equiv \oint(d\phi/2\pi)\exp(ik\rho_a\sin\phi - in\phi)$ and $J^b_n \equiv \oint(d\phi'/2\pi)\exp(ik\rho'_b\sin\phi' - in\phi')$ can be applied and the first part of the gyrokinetic field-particle operator Eq. (25) can be further written as

$$\begin{aligned}
&-\frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}\left\{v_\perp\int 2\pi\, d^2v'\frac{f_{a0}}{m_b}\sum_n J^a_{-n}\left[c^n_{\perp\perp'}\left(J^b_n\frac{\partial h'_b}{\partial v'_\perp} + \frac{J^b_{n-1} - J^b_{n+1}}{2}\frac{k\rho'_b h'_b}{v'_\perp}\right)\right.\right. \\
&\quad\left.\left. - c^n_{\perp\perp}J^b_n\frac{m_b}{m_a}\frac{h'_b\partial f_{a0}}{f_{a0}\partial v_\perp} + c^n_{\perp\parallel'}J^b_n\left(\frac{\partial h'_b}{\partial v'_\parallel} - \frac{m_b}{m_a}\frac{h'_b\partial f_{a0}}{f_{a0}\partial v_\parallel}\right) - \frac{c^n_{\perp\phi'}}{i}\frac{J^b_{n-1} + J^b_{n+1}}{2}\frac{k\rho'_b h'_b}{v'_\perp}\right]\right\} \\
&\quad - \frac{\partial}{\partial v_\parallel}\left\{\int 2\pi\, d^2v'\frac{f_{a0}}{m_b}\sum_n J^a_{-n}\left[c^n_{\parallel\perp'}\left(J^b_n\frac{\partial h'_b}{\partial v'_\perp} + \frac{J^b_{n-1} - J^b_{n+1}}{2}\frac{k\rho'_b h'_b}{v'_\perp}\right)\right.\right. \\
&\quad\left.\left. - c^n_{\parallel\perp}J^b_n\frac{m_b}{m_a}\frac{h'_b\partial f_{a0}}{f_{a0}\partial v_\perp} + c^n_{\parallel\parallel'}J^b_n\left(\frac{\partial h'_b}{\partial v'_\parallel} - \frac{m_b}{m_a}\frac{h'_b\partial f_{a0}}{f_{a0}\partial v_\parallel}\right) - \frac{c^n_{\parallel\phi'}}{i}\frac{J^b_{n-1} + J^b_{n+1}}{2}\frac{k\rho'_b h'_b}{v'_\perp}\right]\right\}.
\end{aligned} \tag{26}$$

The Bessel function J_n^a is real because $\sin(k\rho_a \sin\phi - n\phi)$ is odd in ϕ. The same is true for J_n^b. It can be verified that all the terms in Eq. (26) are real, despite some terms appearing to be imaginary.

Similarly, the second part of the field-particle operator can be expressed as

$$\left\langle (\nabla e^{i\boldsymbol{k}\cdot\boldsymbol{\rho}_a}) \cdot \int d^3v' U \cdot (f_{a0}\nabla'(h_b' e^{-i\boldsymbol{k}\cdot\boldsymbol{\rho}_b'}) - h_b' e^{-i\boldsymbol{k}\cdot\boldsymbol{\rho}_b'}\nabla f_{a0})\right\rangle$$
$$= -\frac{k\rho_a}{v_\perp}\int 2\pi\, d^2v' \frac{f_{a0}}{m_b}\sum_n \frac{J^a_{-(n+1)} - J^a_{-(n-1)}}{2}\left[c^n_{\perp\perp'}\left(J_n^b\frac{\partial h_b'}{\partial v_\perp'} + \frac{J^b_{n-1} - J^b_{n+1}}{2}\frac{k\rho_b' h_b'}{v_\perp'}\right)\right.$$
$$\left. - c^n_{\perp\perp}J_n^b\frac{m_b}{m_a}\frac{h_b'\partial f_{a0}}{f_{a0}\partial v_\perp} + c^n_{\perp\parallel'}J_n^b\left(\frac{\partial h_b'}{\partial v_\parallel'} - \frac{m_b}{m_a}\frac{h_b'\partial f_{a0}}{f_{a0}\partial v_\parallel}\right) - \frac{c^n_{\perp\phi'}}{i}\frac{J^b_{n-1} + J^b_{n+1}}{2}\frac{k\rho_b' h_b'}{v_\perp'}\right]$$
$$+ \frac{k\rho_a}{v_\perp}\int 2\pi d^2v' \frac{f_{a0}}{m_b}\sum_n \frac{J^a_{-(n+1)} + J^a_{-(n-1)}}{2i}\left[c^n_{\phi\perp'}\left(J_n^b\frac{\partial h_b'}{\partial v_\perp'} + \frac{J^b_{n-1} - J^b_{n+1}}{2}\frac{k\rho_b' h_b'}{v_\perp'}\right)\right.$$
$$\left. - c^n_{\phi\perp}J_n^b\frac{m_b}{m_a}\frac{h_b'\partial f_{a0}}{f_{a0}\partial v_\perp} + c^n_{\phi\parallel'}J_n^b\left(\frac{\partial h_b'}{\partial v_\parallel'} - \frac{m_b}{m_a}\frac{h_b'\partial f_{a0}}{f_{a0}\partial v_\parallel}\right) - \frac{c^n_{\phi\phi'}}{i}\frac{J^b_{n-1} + J^b_{n+1}}{2}\frac{k\rho_b' h_b'}{v_\perp'}\right], \quad (27)$$

where all the terms are real. For the field-particle operator, the second part accounts for FLR effects since it vanishes in the drift-kinetic limit $k\rho = k\rho' = 0$, and the first part also contains FLR effects via the Bessel functions. This is in contrast to the test-particle operator, for which the drift-kinetic part is completely separable from the FLR terms. We note that Eqs. (26) and (27) involve Bessel functions of all orders, revealing a different wave-number dependence than the model operators which involve only J_0 and J_1.

B. Symmetric integral form

The operator in Bessel function series requires a proper truncation of the infinite summation for analytical analyses and numerical implementations. Approximations can be performed based on the significance of the FLR effects in the limits of $k\rho \ll 1$ and $k\rho \gg 1$ (e.g., Catto and Tsang [13] and references therein). In order to treat arbitrary wave number and fully assess the FLR effects in numerical simulations, here we pursue the operator in integral form [25]. Since both the test-particle and field-particle parts of the gyrokinetic operator are proven real, we can factor out the gyrophase average $\oint d\phi/(2\pi)$, the gyrophase integration $\oint d\phi'/(2\pi)$, and the gyrophase-dependent part of the integrand, then combine them into precomputable and real-valued gyrophase integrals. The fact that the gyrophase integrals are independent of the distribution and can be precomputed should result in significant time savings in simulations. The resultant gyrokinetic operator involves two-dimensional velocity-space integrals and can be written in vector form similar to Eqs. (10)–(12):

$$C^{\mathrm{gk}}_{ab}(h_a, h_b) = -\nabla\cdot\boldsymbol{J}_{ab} + (\text{FLR terms}). \quad (28)$$

Here $\nabla = \boldsymbol{e}_\perp\partial/\partial v_\perp + \boldsymbol{e}_\parallel\partial/\partial v_\parallel$, and the flux density $\boldsymbol{J}_{ab}(\boldsymbol{v})$ in the two-dimensional velocity space is the sum of the test-particle flux density

$$\boldsymbol{J}^T_{ab} = \Gamma_{ab}\int 2\pi\, d^2v'\left(\frac{h_a}{m_b}\boldsymbol{I}^T_E\cdot\nabla' f_{b0}' - \frac{f_{b0}'}{m_a}\boldsymbol{I}^T_D\cdot\nabla h_a\right) \quad (29)$$

and the field-particle flux density

$$\boldsymbol{J}^F_{ab} = \Gamma_{ab}\int 2\pi\, d^2v'\left(\frac{f_{a0}}{m_b}\boldsymbol{I}^F_E\cdot\nabla' h_b' - \frac{h_b'}{m_a}\boldsymbol{I}^F_D\cdot\nabla f_{a0}\right), \quad (30)$$

with the 2×2 tensors $\boldsymbol{I}^T_E$ and $\boldsymbol{I}^T_D$ for the drag and diffusion coefficients of the test-particle part given by

$$\boldsymbol{I}^T_E(\boldsymbol{v},\boldsymbol{v}') = \begin{pmatrix} I^T_{\perp\perp'} & I^T_{\perp\parallel} \\ I^T_{\parallel\perp'} & I^T_{\parallel\parallel}\end{pmatrix} \equiv \oint\frac{d\phi}{2\pi}\oint\frac{d\phi'}{2\pi}\begin{pmatrix} U_{\perp\perp'} & U_{\perp\parallel} \\ U_{\parallel\perp'} & U_{\parallel\parallel}\end{pmatrix} \quad (31)$$

and

$$\boldsymbol{I}^T_D(\boldsymbol{v},\boldsymbol{v}') = \begin{pmatrix} I^T_{\perp\perp} & I^T_{\perp\parallel} \\ I^T_{\parallel\perp} & I^T_{\parallel\parallel}\end{pmatrix} \equiv \oint\frac{d\phi}{2\pi}\oint\frac{d\phi'}{2\pi}\begin{pmatrix} U_{\perp\perp} & U_{\perp\parallel} \\ U_{\parallel\perp} & U_{\parallel\parallel}\end{pmatrix}, \quad (32)$$

respectively, and $\boldsymbol{I}^F_E$ and $\boldsymbol{I}^F_D$ for field-particle drag and diffusion coefficients given by

$$\boldsymbol{I}^F_E(\boldsymbol{v},\boldsymbol{v}') = \begin{pmatrix} I^F_{\perp\perp'} & I^F_{\perp\parallel} \\ I^F_{\parallel\perp'} & I^F_{\parallel\parallel}\end{pmatrix} \equiv \oint\frac{d\phi}{2\pi}\oint\frac{d\phi'}{2\pi} \times\cos(k\rho'\sin\phi' - k\rho\sin\phi)\begin{pmatrix} U_{\perp\perp'} & U_{\perp\parallel} \\ U_{\parallel\perp'} & U_{\parallel\parallel}\end{pmatrix} \quad (33)$$

and

$$\boldsymbol{I}^F_D(\boldsymbol{v},\boldsymbol{v}') = \begin{pmatrix} I^F_{\perp\perp} & I^F_{\perp\parallel} \\ I^F_{\parallel\perp} & I^F_{\parallel\parallel}\end{pmatrix} \equiv \oint\frac{d\phi}{2\pi}\oint\frac{d\phi'}{2\pi} \times\cos\left(k\rho'\sin\phi' - k\rho\sin\phi\right)\begin{pmatrix} U_{\perp\perp} & U_{\perp\parallel} \\ U_{\parallel\perp} & U_{\parallel\parallel}\end{pmatrix}. \quad (34)$$

The FLR terms in the gyrokinetic collision operator are proportional to the perpendicular wave number and can be cast into

$$(\text{FLR terms})/\Gamma_{ab} = -\frac{k^2\rho_a^2 h_a}{v_\perp^2}\int 2\pi\, d^2v' \frac{f'_{b0}}{m_a} I^T_{\rm FLR} + \frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}\left(v_\perp \int 2\pi\, d^2v' \frac{f_{a0}}{m_b}\frac{k\rho'_b h'_b}{v'_\perp} I^{F,1}_{\rm FLR}\right) + \frac{\partial}{\partial v_\parallel}\left(\int 2\pi\, d^2v' \frac{f_{a0}}{m_b}\frac{k\rho'_b h'_b}{v'_\perp} I^{F,2}_{\rm FLR}\right)$$
$$+\frac{k\rho_a}{v_\perp}\int 2\pi\, d^2v' \frac{f_{a0}}{m_b}\left[\frac{\partial h'_b}{\partial v'_\perp} I^{F,3}_{\rm FLR} + \left(\frac{\partial h'_b}{\partial v'_\parallel} - \frac{m_b}{m_a}\frac{h'_b \partial f_{a0}}{f_{a0}\partial v_\parallel}\right) I^{F,4}_{\rm FLR} - \frac{m_b}{m_a}\frac{h'_b \partial f_{a0}}{f_{a0}\partial v_\perp} I^{F,5}_{\rm FLR} + \frac{k\rho'_b h'_b}{v'_\perp} I^{F,6}_{\rm FLR}\right], \tag{35}$$

with the additional gyrophase integrals given by

$$I^T_{\rm FLR} = \oint \frac{d\phi}{2\pi}\oint \frac{d\phi'}{2\pi}(\sin^2\phi U_{\perp\perp} + 2\sin\phi\cos\phi U_{\perp\phi} + \cos^2\phi U_{\phi\phi}), \tag{36}$$

$$I^{F,1}_{\rm FLR} = \oint \frac{d\phi}{2\pi}\oint \frac{d\phi'}{2\pi}\sin(k\rho'\sin\phi' - k\rho\sin\phi)(\sin\phi' U_{\perp\perp'} + \cos\phi' U_{\perp\phi'}), \tag{37}$$

$$I^{F,2}_{\rm FLR} = \oint \frac{d\phi}{2\pi}\oint \frac{d\phi'}{2\pi}\sin(k\rho'\sin\phi' - k\rho\sin\phi)(\sin\phi' U_{\parallel\perp'} + \cos\phi' U_{\parallel\phi'}), \tag{38}$$

$$I^{F,3}_{\rm FLR} = \oint \frac{d\phi}{2\pi}\oint \frac{d\phi'}{2\pi}\sin(k\rho'\sin\phi' - k\rho\sin\phi)(\sin\phi U_{\perp\perp'} + \cos\phi U_{\phi\perp'}), \tag{39}$$

$$I^{F,4}_{\rm FLR} = \oint \frac{d\phi}{2\pi}\oint \frac{d\phi'}{2\pi}\sin(k\rho'\sin\phi' - k\rho\sin\phi)(\sin\phi U_{\perp\parallel} + \cos\phi U_{\phi\parallel}), \tag{40}$$

$$I^{F,5}_{\rm FLR} = \oint \frac{d\phi}{2\pi}\oint \frac{d\phi'}{2\pi}\sin(k\rho'\sin\phi' - k\rho\sin\phi)(\sin\phi U_{\perp\perp} + \cos\phi U_{\phi\perp}), \tag{41}$$

$$I^{F,6}_{\rm FLR} = \oint \frac{d\phi}{2\pi}\oint \frac{d\phi'}{2\pi}\cos(k\rho'\sin\phi' - k\rho\sin\phi)(\sin\phi\sin\phi' U_{\perp\perp'} + \sin\phi\cos\phi' U_{\perp\phi'} + \cos\phi\sin\phi' U_{\phi\perp'} + \cos\phi\cos\phi' U_{\phi\phi'}). \tag{42}$$

It is straightforward to verify that the gyrokinetic operator in integral form is equivalent to the operator in Bessel function series given in Eqs. (22), (24), (26), and (27). It simply involves expanding $U_{\mu\nu}$ of the gyrophase integrals in Fourier series and rewriting the integrals as Bessel function series. As in the Bessel function series, in integral form the FLR effects are completely separated from the drift-kinetic part for the test-particle operator and partially separated from the drift-kinetic part for the field-particle operator. The test-particle operator and the field-particle operator in integral form are treated on an equal footing. From the definitions of gyrophase integrals (31)–(34), we have $\boldsymbol{I}^T_E = \boldsymbol{I}^F_E$ and $\boldsymbol{I}^T_D = \boldsymbol{I}^F_D$ when $k\rho = k\rho' = 0$. Thus the symmetry of the linearized Landau operator in three-dimensional velocity space, Eqs. (10)–(12), is transformed to the symmetry of the gyrokinetic operator in two-dimensional velocity space in the drift-kinetic limit. It is well known that the gyrophase integrals in the drift-kinetic limit can be written in terms of complete elliptic integrals [10,25,27]. For example, one of the gyrophase integrals is

$$I^T_{\perp\perp} = I^F_{\perp\perp} = \frac{2}{\pi\lambda^3}\frac{E(\kappa)}{1-\kappa^2}(v_\parallel - v'_\parallel)^2 + \frac{\lambda}{\pi v_\perp^2}\left[(\lambda^2 - 2v_\perp v'_\perp)\frac{K(\kappa)}{\lambda^2} - E(\kappa)\right], \tag{43}$$

where $\lambda^2 = (v_\perp + v'_\perp)^2 + (v_\parallel - v'_\parallel)^2$, $\kappa^2 = 4v_\perp v'_\perp/\lambda^2$, and K and E are complete elliptic integrals of the first kind and the second kind, respectively. $I_{\perp\perp}$ diverges when $v_\perp = v'_\perp$ and $v_\parallel = v'_\parallel$ because $K(\kappa)$ logarithmically diverges as $\kappa \to 1$. Similar analysis can be applied to other gyrophase integrals.

Let us discuss the conservation properties of the gyrokinetic operator. Because the particle position and velocity coordinates are mixed in the gyrokinetic phase space via the guiding-center transformation, the conservation laws cannot be simply expressed as the invariance of the first three velocity moments at fixed guiding-center positions; rather, they apply at the particle positions. To address this, we write the FLR effects for the collision operator in the general form Eq. (4) as a divergence of flux density in particle position space, so that the particle conservation laws require the first three velocity moments to be conserved by gyrokinetic collisions in the drift-kinetic limit [18]. Recall that the symmetry between the test-particle operator and the field-particle operator allows for the proof of conservation laws in Eq. (14) for the linearized Landau operator. Since the symmetry is inherited by the gyrokinetic version of the linearized operator, the conservation laws in the drift-kinetic limit can be demonstrated in a similar fashion. Consider

$$-\left(\int d^2v\, \phi_a \nabla\cdot \boldsymbol{J}^T_{ab} + \int d^2v\, \phi_b \nabla\cdot \boldsymbol{J}^F_{ba}\right)$$
$$= 2\pi e_a^2 e_b^2 \ln\Lambda \int d^2v \int 2\pi\, d^2v'$$
$$\times\left[\frac{h_a}{m_b}\left(\frac{\nabla\phi_a}{m_a}\cdot \boldsymbol{I}^T_E - \frac{\nabla'\phi'_b}{m_b}\cdot \boldsymbol{I}'^F_D\right)\cdot\nabla' f'_{b0} - \frac{f'_{b0}}{m_a}\left(\frac{\nabla\phi_a}{m_a}\cdot \boldsymbol{I}^T_D - \frac{\nabla'\phi'_b}{m_b}\cdot \boldsymbol{I}'^F_E\right)\cdot\nabla h_a\right], \tag{44}$$

where the primed tensors are obtained from their corresponding unprimed ones by swapping $\boldsymbol{v}$ and $\boldsymbol{v}'$. We need to show Eq. (44) vanishes for $\phi_s = 1$ (particle conservation), $\phi_s = m_s v_\parallel$ (parallel momentum conservation), and $\phi_s = m_s(v_\perp^2 + v_\parallel^2)/2$ (energy conservation). Particle conservation is satisfied separately by the test-particle part and the field-particle part. The parallel momentum conservation is guaranteed since in the drift-kinetic limit we have $\boldsymbol{I}_E^T = \boldsymbol{I}_E^F$ and $\boldsymbol{I}_D^T = \boldsymbol{I}_D^F$, thus

$$(0 \quad 1) \cdot \left(\boldsymbol{I}_E^T - \boldsymbol{I}_D'^F\right) = \left(I_{\parallel\perp'}^T - I_{\parallel\perp}'^F \quad I_{\parallel\parallel}^T - I_{\parallel\parallel}'^F\right) = \boldsymbol{0}, \quad (45)$$

$$(0 \quad 1) \cdot \left(\boldsymbol{I}_D^T - \boldsymbol{I}_E'^F\right) = \left(I_{\parallel\perp}^T - I_{\parallel\perp'}'^F \quad I_{\parallel\parallel}^T - I_{\parallel\parallel}'^F\right) = \boldsymbol{0}. \quad (46)$$

For energy conservation, it can be verified that

$$(v_\perp \quad v_\parallel) \cdot \boldsymbol{I}_E^T - (v_\perp' \quad v_\parallel') \cdot \boldsymbol{I}_D'^F = \boldsymbol{0}, \quad (47)$$

$$(v_\perp \quad v_\parallel) \cdot \boldsymbol{I}_D^T - (v_\perp' \quad v_\parallel') \cdot \boldsymbol{I}_E'^F = \boldsymbol{0}, \quad (48)$$

by substituting the expressions for $U_{\mu\nu}$ given in Appendix A into the gyrophase integrals.

IV. CONSERVATIVE DISCRETIZATION OF THE LANDAU FORM

To discretize the gyrokinetic linear Landau operator such that it obeys corresponding discrete conservation laws in the drift-kinetic limit even for simulations with low to moderate resolutions, it is essential to observe that the conservative structure of the nonlinear operator is inherited by the gyrokinetic linearized operator, and the symmetry of the nonlinear operator leads to the symmetry between test-particle operator and field-particle operator. The proof of the conservation laws for the continuous case in Eq. (44) suggests a weak formulation of the conservative form in Eq. (28) as

$$\frac{\partial}{\partial t}\int d^2v\, \phi_a h_a = \int d^2v\, \nabla\phi_a \cdot \boldsymbol{J}_{ab}, \quad (49)$$

where ϕ_a are test functions depending on specific numerical schemes and the boundary terms are dropped by using appropriate zero-flux boundary conditions. The key observation on this weak formulation is that it directly measures the changes of ϕ_a moments due to collisions. If we manage to find a scheme such that (1) the set of monomials $\{1, v_\parallel, v^2\}$, which measure the conservation quantities, can be represented by test functions, and (2) the discretization of the right-hand side of Eq. (49) respects the relations required for the continuous conservation laws [shown in Eq. (44)] for each velocity pair $(\boldsymbol{v}, \boldsymbol{v}')$,

$$\frac{\nabla\phi_a}{m_a} \cdot \boldsymbol{I}_E^T - \frac{\nabla'\phi_b'}{m_b} \cdot \boldsymbol{I}_D'^F = \boldsymbol{0}, \quad (50)$$

$$\frac{\nabla\phi_a}{m_a} \cdot \boldsymbol{I}_D^T - \frac{\nabla'\phi_b'}{m_b} \cdot \boldsymbol{I}_E'^F = \boldsymbol{0}, \quad (51)$$

then this scheme will obey the discrete conservation laws.

One type of scheme involves expanding the distribution functions and the test functions with a discrete orthogonal polynomial basis [12,28,29], namely

$$h_s(\boldsymbol{v}_j) = \sum_i h_{si}\lambda_i(\boldsymbol{v}_j), \quad (52)$$

$$\phi_s(\boldsymbol{v}_j) = \sum_i \phi_{si}\lambda_i(\boldsymbol{v}_j), \quad (53)$$

where the $\lambda_i(\boldsymbol{v}_j)$ represents the ith element of the two-dimensional polynomial basis evaluated at the grid point $\boldsymbol{v}_j$. The grid point locations are determined by the quadrature of the basis polynomials for given boundary conditions [28]. The operator, formulated in the spherical representation in Appendix C, is suitable for the spectral type discretization. The tensor product of the Legendre polynomials in pitch-angle coordinate $P_l(\xi)$ $(\xi = v_\parallel/v)$ and the Chebyshev polynomials in speed coordinate $T_n(v)$ $(v = \sqrt{v_\perp^2 + v_\parallel^2})$ can serve as a favorable two-dimensional basis. The Legendre polynomials are eigenfunctions of both test-particle operator and field-particle operator. The basis in speed should have the characteristic that, in the regions where the distribution function shows strong variation, the next basis functions in the series are different enough from preceding basis functions so that few additional basis functions are needed to effectively span the space. This will ensure rapid decay of the error with the number of basis functions included [29]. For perturbed distributions in turbulence, which display structure at low speeds, Chebyshev polynomials are appropriate. To verify the conservation laws, we note that $\{1, v_\parallel, v^2\}$ are represented by the basis since $1 = P_0T_0$, $v_\parallel = P_1T_1$, and $v^2 = (P_0T_0 + P_0T_2)/2$. In addition, the two-dimensional velocity-space integration in Eq. (49) is evaluated with a quadrature rule, and at the quadrature points the derivatives of $\{1, v_\parallel, v^2\}$ obtained by differentiating the basis polynomials [28] are exact. Thus Eqs. (50) and (51) are respected at each pair of quadrature points $(\boldsymbol{v}_j, \boldsymbol{v}_{j'}')$.

Alternatively, the gyrokinetic linearized Landau operator can be discretized with a finite-volume method as in Yoon *et al.* [10] and Hager *et al.* [11]. This method was originally designed for a two-dimensional nonlinear Landau operator in the drift-kinetic limit based on a weak formulation of the symmetric and conservative form. Because the symmetric and conservative form is preserved in present formulation of the gyrokinetic linearized operator, the finite-volume method is applicable to Eq. (28) after replacing the nonlinear flux density with the linear flux density. The scheme requires a velocity grid $(v_\perp, v_\parallel)$ uniformly spaced in both directions to ensure conservation numerically. The FLR terms in Eq. (35) containing purely FLR effects also need to be evaluated. The conservative terms associated with $I_{\rm FLR}^{F,1}$ and $I_{\rm FLR}^{F,2}$ can be incorporated into the drift-kinetic part simply by adding corresponding terms in the definition of the flux density $\boldsymbol{J}_{ab}$. The differentiations and integrations in the nonconservative FLR terms can be carried out by centered finite-differencing and numerical summations, respectively. The FLR terms are proportional to the perpendicular wave number and do not affect the conservation laws.

V. DISCUSSION AND FUTURE WORK

In this paper, the gyrokinetic exact linearized Landau collision operator is formulated. Two key properties of the nonlinear Landau operator—symmetry and conservative structure—are explicitly preserved in the linearization and subsequent guiding-center transformation and gyrophase averaging. These two mathematical properties underline the physical properties including the conservation laws and the H theorem. It is verified that the gyrokinetic operator in the Landau form is equivalent to the operator in the nonconservative and nonsymmetric Fokker–Planck form [25]. The present formulation addresses the potential numerical difficulty associated with the logarithmic singularity in the gyrophase integral of the field-particle operator in the Fokker–Planck form. By treating the test-particle contribution and the field-particle contribution symmetrically, potential numerical errors associated with the logarithmic singularity in the test-particle operator of *a*-*b* collisions are balanced by the field-particle operator of *b*-*a* collisions, and vice versa. Thus overall conservation is ensured despite numerical errors.

This work is motivated in part by previous results which show that the inherent differences between the exact linearized drift-kinetic field operator and present model operators produce different results for collisional fluxes and flows in magnetically confined fusion plasmas. In addition to the strong effects of electron collisions, ion collisions (subject to FLR corrections) affect turbulence through the neoclassical polarization and zonal flow damping [30,31], as well as neoclassical distortions of the background distribution. The latter can break symmetry, leading to momentum transport [32], and neoclassical distortions are particularly relevant to steep gradient regions such as the tokamak edge pedestal [33]. When Abel's model operator is applied to classical ion transport, the coefficient of heat flux transport perpendicular to the magnetic field is found to be about 50% greater than the result from the exact linearized Fokker–Planck operator (see the paragraph between Eqs. (21) and (22) of Ref. [22]). The exact linearized Fokker–Planck operator, without FLR corrections, has been used in several kinetic neoclassical codes [10,29,33–37]. The codes of Refs. [29,33,35,37] closely agree over a wide range of collisionality (e.g., Fig. 4 of Ref. [33], also a similar comparison with the NEO code was performed). Neoclassical results from the exact linearized Fokker–Planck operator differ significantly from results of model operators [37], showing errors as large as 10–15% for the neoclassical particle fluxes and 20–30% for the neoclassical ion energy fluxes. In the same work, the model referred to as *ad hoc* Fokker–Planck (actually a drift-kinetic generalization of the Abel operator for multiple species) resulted in 25% larger parallel neoclassical flows at high collisionality and 15% smaller bootstrap current than the exact linearized Fokker–Planck operator. Before this, it was reported [34] that the exact linearized Fokker–Planck operator resulted in 20% differences relative to the most accurate calculation of the bootstrap current at that time. Similar differences in neoclassical fluxes, using the exact linearized operator relative to widely accepted calculations using model operators, were observed [35].

The gyrokinetic operator in conservative form can be implemented with either spectral or finite-volume numerical schemes. In continuum codes, spectral methods [29] can achieve rapid convergence in velocity space. Particle codes can utilize methods similar to those in Refs. [10,11] to ensure numerical conservation. The initial implementation of the gyrokinetic exact linearized Landau operator is underway in the gyrokinetic code GENE [38,39], using the finite-volume scheme described in Sec. IV. The δf version of GENE can apply a velocity grid $(v_\perp, v_\parallel)$ that is equally distant in both directions and has the model operators by Abel *et al.* [18] and Sugama *et al.* [20] implemented [40]. We have utilized this framework so far to implement the drift-kinetic Landau operator and are continuing to include FLR corrections. Progress on the implementation, including numerical verification of the conservation laws and comparisons of the exact operator with model operators in physical applications, such as the turbulence driven by microinstabilities, will be reported in the future. Using this new formulation, it will be possible to assess the accuracy of present model operators in gyrokinetic turbulence simulations.

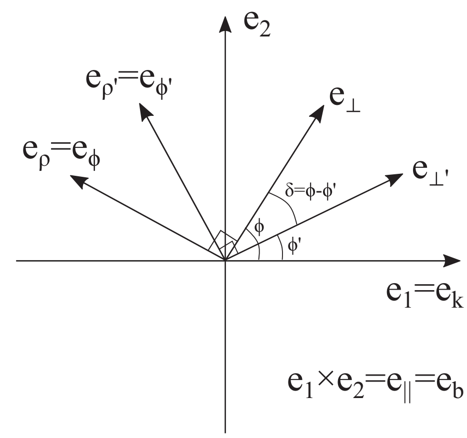

FIG. 1. The cylindrical coordinate system for velocity of species *a*, and for velocity of species *b* with primed subscripts, in *a*-*b* type collisions.

ACKNOWLEDGMENTS

We thank G. W. Hammett for suggesting that we formulate the gyrokinetic exact linearized operator in conservative form. We thank P. Crandall and F. Jenko for useful discussions on the implementations of model operators in the gyrokinetic code GENE. This work is supported by U. S. DOE Contract No. DE-FC02-08ER54966.

APPENDIX A: CYLINDRICAL COORDINATES AND ASSOCIATED USEFUL RELATIONS

Figure 1 shows the cylindrical coordinate system used in this work. In this coordinate system, the velocity can be decomposed as

$$\boldsymbol{v} = v_\perp(\boldsymbol{e}_1 \cos\phi + \boldsymbol{e}_2 \sin\phi) + v_\parallel \boldsymbol{e}_\parallel, \quad \text{(A1)}$$

and the gyroradius is

$$\boldsymbol{\rho} = \frac{v_\perp}{\Omega}\boldsymbol{e}_\rho = \frac{v_\perp}{\Omega}\boldsymbol{e}_\parallel \times \boldsymbol{e}_\perp = \frac{v_\perp}{\Omega}(-\boldsymbol{e}_1 \sin\phi + \boldsymbol{e}_2 \cos\phi). \quad \text{(A2)}$$

From Eq. (A2), we have $\partial\boldsymbol{\rho}/\partial v_\perp = \boldsymbol{\rho}/v_\perp$, $\partial\boldsymbol{\rho}/\partial v_\parallel = 0$, and $\partial\boldsymbol{\rho}/\partial\phi = -\boldsymbol{v}_\perp/\Omega$. Thus

$$\frac{\partial}{\partial v_\perp}\left(e^{-i\boldsymbol{k}\cdot\boldsymbol{\rho}} g\right) = e^{-i\boldsymbol{k}\cdot\boldsymbol{\rho}}\left(\frac{\partial g}{\partial v_\perp} - \frac{i\boldsymbol{k}\cdot\boldsymbol{\rho} g}{v_\perp}\right), \quad \text{(A3)}$$

$$\frac{\partial}{\partial v_\parallel}(e^{-i\boldsymbol{k}\cdot\boldsymbol{\rho}}g) = e^{-i\boldsymbol{k}\cdot\boldsymbol{\rho}}\frac{\partial g}{\partial v_\parallel}, \tag{A4}$$

$$\frac{\partial}{\partial \phi}(e^{-i\boldsymbol{k}\cdot\boldsymbol{\rho}}g) = e^{-i\boldsymbol{k}\cdot\boldsymbol{\rho}}\frac{i\boldsymbol{k}\cdot\boldsymbol{v}_\perp}{\Omega}g, \tag{A5}$$

for a distribution function $g = g(v_\perp, v_\parallel)$ (either f_{s0} or h_s) that is independent of gyrophase. Without loss of generality, $\boldsymbol{e}_1$ may be chosen in the direction of the perpendicular wave number so that $\boldsymbol{k} = k\boldsymbol{e}_1, \boldsymbol{k}\cdot\boldsymbol{\rho} = -k\rho\sin\phi$, and $\boldsymbol{k}\cdot\boldsymbol{v}_\perp = kv_\perp\cos\phi$.

The projection of the Landau tensor is defined as

$$U_{\mu\nu} \equiv \boldsymbol{e}_\mu \cdot \frac{\boldsymbol{I}u^2 - \boldsymbol{uu}}{u^3}\cdot\boldsymbol{e}_\nu \tag{A6}$$

with $\boldsymbol{u} = \boldsymbol{v} - \boldsymbol{v}'$, $\mu \in \{\perp, \parallel, \phi\}$, and $\nu \in \{\perp, \parallel, \phi, \perp', \parallel', \phi'\}$. Here the unprimed quantities are for species a and primed quantities are for species b in a-b collisions. By using $\boldsymbol{u} = v_\perp\boldsymbol{e}_\perp - v'_\perp\boldsymbol{e}'_\perp + (v_\parallel - v'_\parallel)\boldsymbol{e}_\parallel$ and Fig. 1, $U_{\mu\nu}$ can be explicitly written out and they are listed below for a reference:

$$U_{\perp\perp} = \frac{1}{u^3}[u^2 - (v_\perp - v'_\perp\cos\delta)^2], \tag{A7}$$

$$U_{\perp\parallel} = U_{\perp\parallel'} = U_{\parallel\perp} = -\frac{1}{u^3}(v_\perp - v'_\perp\cos\delta)(v_\parallel - v'_\parallel), \tag{A8}$$

$$U_{\perp\phi} = U_{\phi\perp} = -\frac{1}{u^3}(v_\perp - v'_\perp\cos\delta)v'_\perp\sin\delta, \tag{A9}$$

$$U_{\perp\perp'} = \frac{1}{u^3}[u^2\cos\delta + (v_\perp - v'_\perp\cos\delta)(v'_\perp - v_\perp\cos\delta)], \tag{A10}$$

$$U_{\perp\phi'} = \frac{1}{u^3}[u^2\sin\delta - (v_\perp - v'_\perp\cos\delta)v_\perp\sin\delta], \tag{A11}$$

$$U_{\parallel\parallel} = U_{\parallel\parallel'} = \frac{1}{u^3}[u^2 - (v_\parallel - v'_\parallel)^2], \tag{A12}$$

$$U_{\parallel\phi} = U_{\phi\parallel} = U_{\phi\parallel'} = -\frac{1}{u^3}(v_\parallel - v'_\parallel)v'_\perp\sin\delta, \tag{A13}$$

$$U_{\parallel\perp'} = \frac{1}{u^3}(v_\parallel - v'_\parallel)(v'_\perp - v_\perp\cos\delta), \tag{A14}$$

$$U_{\parallel\phi'} = -\frac{1}{u^3}(v_\parallel - v'_\parallel)v_\perp\sin\delta, \tag{A15}$$

$$U_{\phi\phi} = \frac{1}{u^3}[u^2 - (v'_\perp\sin\delta)^2], \tag{A16}$$

$$U_{\phi\perp'} = \frac{1}{u^3}[-u^2\sin\delta + v'_\perp\sin\delta(v'_\perp - v_\perp\cos\delta)], \tag{A17}$$

$$U_{\phi\phi'} = \frac{1}{u^3}(u^2\cos\delta - v_\perp v'_\perp\sin^2\delta), \tag{A18}$$

with $\delta = \phi - \phi'$. $U_{\mu\nu}$ is periodic in δ and can be expanded in Fourier series as

$$U_{\mu\nu}(\delta) = \sum_{n\in\mathbb{Z}} c^n_{\mu\nu}e^{in\delta}, \tag{A19}$$

where the expansion coefficients

$$c^n_{\mu\nu} = -\frac{1}{2\pi}\int_\phi^{\phi-2\pi} d\delta U_{\mu\nu}(\delta)e^{-in\delta} = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\delta U_{\mu\nu}(\delta)e^{-in\delta} \tag{A20}$$

can be obtained by first expanding $U_{\mu\nu}$ as function of ϕ' and then changing the variable from ϕ' to δ. Note that $U_{\mu\nu}$ is either even or odd in δ, thus $c^n_{\mu\nu}$ is either real or imaginary.

APPENDIX B: PROOF OF THE EQUIVALENCE BETWEEN GYROKINETIC LANDAU FORM AND GYROKINETIC FOKKER–PLANCK FORM

The approach to proving the equivalence of gyrokinetic test-particle operator in the Landau form with the Fokker–Planck form is first writing Eq. (5) in conservative form in cylindrical coordinates, then obtaining the gyrokinetic version via substitution to Eq. (4) and comparing it with the Landau form given in Eqs. (22) and (24).

Equation (5) can be written in conservative form as

$$C^T_{ab}(f_{a1}, f_{bM}) = \nabla\cdot\boldsymbol{J}_{ab}, \tag{B1}$$

where $\boldsymbol{J}_{ab}$ is the sum of the pitch-angle-scattering flux density and energy-diffusion flux density given by

$$\boldsymbol{J}^{PA}_{ab} = \frac{1}{2}\big(v^2\boldsymbol{I} - \boldsymbol{vv}\big)\cdot\frac{\partial(\nu_D f_{a1})}{\partial\boldsymbol{v}} \tag{B2}$$

and

$$\boldsymbol{J}^E_{ab} = \boldsymbol{e}_v\left[\frac{\nu^{ab}_\parallel}{2}v^4 f_{aM}\frac{\partial}{\partial v}\left(\frac{f_{a1}}{f_{aM}}\right) + \frac{m_a}{T_b}\left(1 - \frac{T_b}{T_a}\right)\frac{\nu^{ab}_\parallel}{2}v^5 f_{a1}\right], \tag{B3}$$

respectively. Notice that since $\nu^{ab}_D(v)$ depends only on v, it can be absorbed into the pitch-angle-scattering operator. Projecting the flux density $\boldsymbol{J}_{ab}$ onto the basis of cylindrical coordinates, we obtain

$$C^T_{ab}(f_{a1}, f_{bM}) = \frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}(v_\perp J^\perp_{ab}) + \frac{\partial J^\parallel_{ab}}{\partial v_\parallel} + \frac{1}{v_\perp}\frac{\partial J^\phi_{ab}}{\partial\phi}, \tag{B4}$$

with

$$J^\perp_{ab} = \frac{1}{2}\big(\nu^{ab}_\parallel(v) - \nu^{ab}_D(v)\big)v_\parallel v_\perp\frac{\partial f_{a1}}{\partial v_\parallel} + \frac{1}{2}\big(\nu^{ab}_\parallel v^2_\perp + \nu^{ab}_D v^2_\parallel\big)\frac{\partial f_{a1}}{\partial v_\perp} + \frac{1}{2}\nu^{ab}_\parallel v_\perp v^2\frac{m_a f_{a1}}{T_b}, \tag{B5}$$

$$J^\parallel_{ab} = \frac{1}{2}\big(\nu^{ab}_\parallel(v) - \nu^{ab}_D(v)\big)v_\parallel v_\perp\frac{\partial f_{a1}}{\partial v_\perp} + \frac{1}{2}\big(\nu^{ab}_\parallel v^2_\parallel + \nu^{ab}_D v^2_\perp\big)\frac{\partial f_{a1}}{\partial v_\parallel} + \frac{1}{2}\nu^{ab}_\parallel v_\parallel v^2\frac{m_a f_{a1}}{T_b}, \tag{B6}$$

$$J^\phi_{ab} = \frac{\nu^{ab}_D}{2}\frac{v^2}{v_\perp}\frac{\partial f_{a1}}{\partial\phi}. \tag{B7}$$

The gyrokinetic form for the nonadiabatic part of the guiding-center distribution function obtained by substituting Eqs. (B4)–(B7) into Eq. (4) eliminates the gyrophase dependence and contains FLR effects:

$$C^{\rm gk}_{ab}(h_a, f_{bM}) = \frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}\big(v_\perp J^{\perp,\rm gk}_{ab}\big) + \frac{\partial J^{\parallel,\rm gk}_{ab}}{\partial v_\parallel} + \frac{k^2 h_a}{4\Omega^2_a}\big[-\nu^{ab}_D(2v^2_\parallel + v^2_\perp) - \nu^{ab}_\parallel v^2_\perp\big], \tag{B8}$$

with the gyrokinetic flux densities given by

$$J_{ab}^{\perp,\mathrm{gk}} = \frac{1}{2}\big(\nu_{\parallel}^{ab}(v) - \nu_D^{ab}(v)\big)v_{\parallel}v_{\perp}\frac{\partial h_a}{\partial v_{\parallel}} + \frac{1}{2}\big(\nu_{\parallel}^{ab}v_{\perp}^2 + \nu_D^{ab}v_{\parallel}^2\big)\frac{\partial h_a}{\partial v_{\perp}} + \frac{1}{2}\nu_{\parallel}^{ab}v_{\perp}v^2\frac{m_a h_a}{T_b}, \tag{B9}$$

$$J_{ab}^{\parallel,\mathrm{gk}} = \frac{1}{2}\big(\nu_{\parallel}^{ab}(v) - \nu_D^{ab}(v)\big)v_{\parallel}v_{\perp}\frac{\partial h_a}{\partial v_{\perp}} + \frac{1}{2}\big(\nu_{\parallel}^{ab}v_{\parallel}^2 + \nu_D^{ab}v_{\perp}^2\big)\frac{\partial h_a}{\partial v_{\parallel}} + \frac{1}{2}\nu_{\parallel}^{ab}v_{\parallel}v^2\frac{m_a h_a}{T_b}. \tag{B10}$$

1. Equivalence in the drift-kinetic limit

By using the periodicity of $U_{\mu\nu}$ in δ, we can evaluate the gyroaverage in Eq. (22) and obtain the flux densities of the test-particle operator,

$$J_{ab}^{\perp,\mathrm{gk}} = -\frac{\Gamma_{ab}}{m_a}\int d^3v' f'_{b0}\bigg[-U_{\perp\perp}\frac{\partial h_a}{\partial v_{\perp}} + U_{\perp\parallel}\bigg(\frac{m_a}{m_b}\frac{h_a\partial f'_{b0}}{f'_{b0}\partial v'_{\parallel}} - \frac{\partial h_a}{\partial v_{\parallel}}\bigg) + U_{\perp\perp'}\frac{m_a}{m_b}\frac{h_a\partial f'_{b0}}{f'_{b0}\partial v'_{\perp}}\bigg], \tag{B11}$$

$$J_{ab}^{\parallel,\mathrm{gk}} = -\frac{\Gamma_{ab}}{m_a}\int d^3v' f'_{b0}\bigg[-U_{\parallel\perp}\frac{\partial h_a}{\partial v_{\perp}} + U_{\parallel\parallel}\bigg(\frac{m_a}{m_b}\frac{h_a\partial f'_{b0}}{f'_{b0}\partial v'_{\parallel}} - \frac{\partial h_a}{\partial v_{\parallel}}\bigg) + U_{\parallel\perp'}\frac{m_a}{m_b}\frac{h_a\partial f'_{b0}}{f'_{b0}\partial v'_{\perp}}\bigg]. \tag{B12}$$

To prove the equivalence of the drift-kinetic part of the test-particle operator, we only need to show that the flux densities defined in Eqs. (B11) and (B12) are equivalent to Eqs. (B9) and (B10), respectively. The two expressions consist of five pairs of one-to-one correspondence. The proof uses the identity $\boldsymbol{U} = \partial^2 u/\partial\boldsymbol{v}\partial\boldsymbol{v}$ and the definitions of ν_D^{ab} and $\nu_{\parallel}^{ab}$ in terms of the Rosenbluth potentials given in Eqs. (6) and (7). First, for the perpendicular diffusion coefficient we have

$$\begin{aligned}\frac{\Gamma_{ab}}{m_a}\int d^3v' f'_{b0}U_{\perp\perp} &= \frac{\Gamma_{ab}}{m_a}\int d^3v' f'_{b0}\frac{\partial^2 u}{\partial\boldsymbol{v}\partial\boldsymbol{v}}:\boldsymbol{e}_{\perp}\boldsymbol{e}_{\perp}\\ &= \frac{\Gamma_{ab}}{m_a}\int d^3v' f'_{b0}\frac{\partial^2 u}{\partial^2 v_{\perp}} = \frac{\Gamma_{ab}}{m_a}\frac{\partial^2 G_0}{\partial v_{\perp}\partial v_{\perp}}\\ &= \frac{\Gamma_{ab}}{m_a}\bigg(\frac{v_{\perp}^2}{v^2}\frac{d^2G_0}{dv^2} + \frac{v_{\parallel}^2}{v^3}\frac{dG_0}{dv}\bigg)\\ &= \frac{1}{2}\big(\nu_{\parallel}^{ab}v_{\perp}^2 + \nu_D^{ab}v_{\parallel}^2\big).\end{aligned} \tag{B13}$$

Similarly, the equivalence of the parallel and cross diffusion coefficients,

$$\frac{\Gamma_{ab}}{m_a}\int d^3v' f'_{b0}U_{\parallel\parallel} = \frac{1}{2}\big(\nu_{\parallel}^{ab}v_{\parallel}^2 + \nu_D^{ab}v_{\perp}^2\big), \tag{B14}$$

$$\frac{\Gamma_{ab}}{m_a}\int d^3v' f'_{b0}U_{\parallel\perp} = \frac{1}{2}\big(\nu_{\parallel}^{ab} - \nu_D^{ab}\big)v_{\parallel}v_{\perp}, \tag{B15}$$

can be demonstrated. Second, for the perpendicular drag coefficient we have

$$\begin{aligned}&-\frac{\Gamma_{ab}}{m_a}\int d^3v' f'_{b0}\bigg(U_{\perp\parallel}\frac{m_a}{m_b}\frac{\partial f'_{b0}}{f'_{b0}\partial v'_{\parallel}} + U_{\perp\perp'}\frac{m_a}{m_b}\frac{\partial f'_{b0}}{f'_{b0}\partial v'_{\perp}}\bigg)\\ &= \frac{\Gamma_{ab}}{T_b}\int d^3v' f'_{b0}\big(U_{\perp\parallel'}v'_{\parallel} + U_{\perp\perp'}v'_{\perp}\big)\\ &= \frac{\Gamma_{ab}}{T_b}\int d^3v' f'_{b0}\big(U_{\perp\parallel}v_{\parallel} + U_{\perp\perp}v_{\perp}\big)\\ &= \frac{\Gamma_{ab}}{T_b}\int d^3v' f'_{b0}\bigg(\frac{\partial^2 u}{\partial v_{\perp}\partial v_{\parallel}}v_{\parallel} + \frac{\partial^2 u}{\partial^2 v_{\perp}}v_{\perp}\bigg)\\ &= \frac{\Gamma_{ab}}{T_b}\bigg(\frac{\partial^2 G_0}{\partial v_{\perp}\partial v_{\parallel}}v_{\parallel} + \frac{\partial^2 G_0}{\partial^2 v_{\perp}}v_{\perp}\bigg)\\ &= \frac{\Gamma_{ab}}{T_b}\frac{d^2G_0}{dv^2}v_{\perp} = \frac{1}{2}\nu_{\parallel}^{ab}v_{\perp}v^2\frac{m_a}{T_b},\end{aligned} \tag{B16}$$

where the identity $\boldsymbol{U}\cdot(\boldsymbol{v}-\boldsymbol{v}') = \boldsymbol{U}\cdot\boldsymbol{u} = 0$ is used. The equivalence of the parallel drag coefficient,

$$\begin{aligned}&-\frac{\Gamma_{ab}}{m_a}\int d^3v' f'_{b0}\bigg(U_{\parallel\parallel}\frac{m_a}{m_b}\frac{h_a\partial f'_{b0}}{f'_{b0}\partial v'_{\parallel}} + U_{\parallel\perp'}\frac{m_a}{m_b}\frac{h_a\partial f'_{b0}}{f'_{b0}\partial v'_{\perp}}\bigg)\\ &= \frac{1}{2}\nu_{\parallel}^{ab}v_{\parallel}v^2\frac{m_a}{T_b},\end{aligned} \tag{B17}$$

can be proved analogously to Eq. (B16). This proof demonstrates that the apparent singularity in ν_D^{ab} and $\nu_{\parallel}^{ab}$ as $v \to 0$ originates from the Landau tensor.

2. Equivalence of the gyrodiffusion terms

The gyrodiffusion equivalence can be proved in a similar fashion. Starting with the gyrodiffusion in Landau form given in Eq. (24), we have

$$\begin{aligned}&-\frac{\Gamma_{ab}}{2m_a}\frac{k^2h_a}{\Omega_a^2}\int d^3v' f'_{b0}\big(U_{\perp\perp} + U_{\phi\phi}\big)\\ &= -\frac{\Gamma_{ab}}{2m_a}\frac{k^2h_a}{\Omega_a^2}\int d^3v' f'_{b0}\bigg(\frac{\partial^2 u}{\partial\boldsymbol{v}\partial\boldsymbol{v}}:\boldsymbol{e}_{\perp}\boldsymbol{e}_{\perp} + \frac{\partial^2 u}{\partial\boldsymbol{v}\partial\boldsymbol{v}}:\boldsymbol{e}_{\phi}\boldsymbol{e}_{\phi}\bigg)\\ &= -\frac{\Gamma_{ab}}{2m_a}\frac{k^2h_a}{\Omega_a^2}\bigg\{\frac{\partial^2 G_0}{\partial v_{\perp}^2} + \frac{1}{v_{\perp}}\bigg[\frac{\partial}{\partial\phi}\bigg(\frac{1}{v_{\perp}}\frac{\partial G_0(v)}{\partial\phi}\bigg) + \frac{\partial G_0}{\partial v_{\perp}}\bigg]\bigg\}\\ &= -\frac{\Gamma_{ab}}{2m_a}\frac{k^2h_a}{\Omega_a^2}\bigg(\frac{\partial^2 G_0}{\partial v_{\perp}^2} + \frac{1}{v_{\perp}}\frac{\partial G_0}{\partial v_{\perp}}\bigg)\\ &= \frac{k^2h_a}{4\Omega_a^2}\big[-\nu_D^{ab}(2v_{\parallel}^2 + v_{\perp}^2) - \nu_{\parallel}^{ab}v_{\perp}^2\big].\end{aligned} \tag{B18}$$

APPENDIX C: GYROKINETIC LINEARIZED LANDAU OPERATOR IN SPHERICAL REPRESENTATION

The derivation approach and conclusions of this paper are independent of the coordinate systems, and the operator can be reformulated straightforwardly in different representations.

Here we present the operator in spherical representation without detail derivation. The basis of the spherical coordinate system is $(\boldsymbol{e}_v, \boldsymbol{e}_\xi, \boldsymbol{e}_\phi)$ with $v = |\boldsymbol{v}|$, $\xi = \boldsymbol{e}_v \cdot \boldsymbol{e}_\parallel = \cos\theta$ the pitch-angle coordinate, and ϕ the gyrophase. Analogous to Eqs. (28)–(42), the gyrokinetic Landau operator can be cast into conservative form as

$$C_{ab}^{\mathrm{gk}}(h_a, h_b) = -\nabla \cdot \left(\boldsymbol{J}_{ab}^T + \boldsymbol{J}_{ab}^F\right) + (\text{FLR terms}), \tag{C1}$$

where

$$\boldsymbol{J}_{ab}^T = \Gamma_{ab} \int 2\pi d^2v' \left(\frac{h_a}{m_b} \boldsymbol{I}_E^T \cdot \nabla' f_{b0}' - \frac{f_{b0}'}{m_a} \boldsymbol{I}_D^T \cdot \nabla h_a \right), \tag{C2}$$

$$\boldsymbol{J}_{ab}^F = \Gamma_{ab} \int 2\pi d^2v' \left(\frac{f_{a0}}{m_b} \boldsymbol{I}_E^F \cdot \nabla' h_b' - \frac{h_b'}{m_a} \boldsymbol{I}_D^F \cdot \nabla f_{a0} \right). \tag{C3}$$

The 2×2 tensors for the test-particle part are given by

$$\boldsymbol{I}_E^T = \begin{pmatrix} I_{vv'}^T & I_{v\xi'}^T \\ I_{\xi v'}^T & I_{\xi\xi'}^T \end{pmatrix} \equiv \oint \frac{d\phi}{2\pi} \oint \frac{d\phi'}{2\pi} \begin{pmatrix} U_{vv'} & U_{v\xi'} \\ U_{\xi v'} & U_{\xi\xi'} \end{pmatrix}, \tag{C4}$$

$$\boldsymbol{I}_D^T = \begin{pmatrix} I_{vv}^T & I_{v\xi}^T \\ I_{\xi v}^T & I_{\xi\xi}^T \end{pmatrix} \equiv \oint \frac{d\phi}{2\pi} \oint \frac{d\phi'}{2\pi} \begin{pmatrix} U_{vv} & U_{v\xi} \\ U_{\xi v} & U_{\xi\xi} \end{pmatrix}, \tag{C5}$$

and for field-particle part they are defined as

$$\boldsymbol{I}_E^F = \begin{pmatrix} I_{vv'}^F & I_{v\xi'}^F \\ I_{\xi v'}^F & I_{\xi\xi'}^F \end{pmatrix} \equiv \oint \frac{d\phi}{2\pi} \oint \frac{d\phi'}{2\pi} \cos(k\rho' \sin\phi' - k\rho \sin\phi) \begin{pmatrix} U_{vv'} & U_{v\xi'} \\ U_{\xi v'} & U_{\xi\xi'} \end{pmatrix}, \tag{C6}$$

$$\boldsymbol{I}_D^F = \begin{pmatrix} I_{vv}^F & I_{v\xi}^F \\ I_{\xi v}^F & I_{\xi\xi}^F \end{pmatrix} \equiv \oint \frac{d\phi}{2\pi} \oint \frac{d\phi'}{2\pi} \cos(k\rho' \sin\phi' - k\rho \sin\phi) \begin{pmatrix} U_{vv} & U_{v\xi} \\ U_{\xi v} & U_{\xi\xi} \end{pmatrix}. \tag{C7}$$

The FLR terms can be written as

$$\begin{aligned}(\text{FLR terms})/\Gamma_{ab} = &-\frac{k^2\rho_a^2 h_a}{v^2} \int 2\pi d^2v' \frac{f_{b0}'}{m_a} I_{\text{FLR}}^T - \frac{1}{v^2}\frac{\partial}{\partial v}\left(v^2 \int 2\pi d^2v' \frac{f_{a0}}{m_b} \frac{k\rho_b' h_b'}{v'} I_{\text{FLR}}^{F,1} \right) \\ &- \frac{\partial}{v\partial\xi}\left[\sqrt{1-\xi^2} \int 2\pi d^2v' \frac{f_{a0}}{m_b} \frac{k\rho_b' h_b'}{v'} I_{\text{FLR}}^{F,2} \right] + \frac{k\rho_a}{v} \int 2\pi d^2v' \frac{f_{a0}}{m_b} \left(\frac{\partial h_b'}{\partial v'} I_{\text{FLR}}^{F,3} + \frac{\sqrt{1-\xi'^2}}{v'} \frac{\partial h_b'}{\partial \xi'} I_{\text{FLR}}^{F,4} \right. \\ &\left. - \frac{m_b}{m_a} \frac{h_b' \partial f_{a0}}{f_{a0} \partial v} I_{\text{FLR}}^{F,5} - \frac{\sqrt{1-\xi^2}}{v} \frac{m_b}{m_a} \frac{h_b' \partial f_{a0}}{f_{a0} \partial \xi} I_{\text{FLR}}^{F,6} + \frac{k\rho_b' h_b'}{v'} I_{\text{FLR}}^{F,7} \right),\end{aligned} \tag{C8}$$

with the additional gyrophase integrals given by

$$\begin{aligned} I_{\text{FLR}}^T = \oint \frac{d\phi}{2\pi} \oint \frac{d\phi'}{2\pi} \Bigg[&\sin\phi \left(\sin\phi U_{vv} - \frac{\xi}{\sqrt{1-\xi^2}} \sin\phi U_{v\xi} + \frac{\cos\phi U_{v\phi}}{\sqrt{1-\xi^2}} \right) \\ &- \frac{\xi \sin\phi}{\sqrt{1-\xi^2}} \left(\sin\phi U_{\xi v} - \frac{\xi}{\sqrt{1-\xi^2}} \sin\phi U_{\xi\xi} + \frac{\cos\phi U_{\xi\phi}}{\sqrt{1-\xi^2}} \right) + \frac{\cos\phi}{\sqrt{1-\xi^2}} \left(\sin\phi U_{\phi v} - \frac{\xi}{\sqrt{1-\xi^2}} \sin\phi U_{\phi\xi} + \frac{\cos\phi U_{\xi\phi}}{\sqrt{1-\xi^2}} \right) \Bigg], \end{aligned} \tag{C9}$$

$$I_{\text{FLR}}^{F,1} = \oint \frac{d\phi}{2\pi} \oint \frac{d\phi'}{2\pi} \sin\left(k\rho' \sin\phi' - k\rho \sin\phi\right) \left(\frac{\xi' \sin\phi' U_{v\xi'}}{\sqrt{1-\xi'^2}} - \sin\phi' U_{vv'} - \frac{\cos\phi' U_{v\phi'}}{\sqrt{1-\xi'^2}} \right), \tag{C10}$$

$$I_{FLR}^{F,2} = \oint \frac{d\phi}{2\pi} \oint \frac{d\phi'}{2\pi} \sin\left(k\rho' \sin\phi' - k\rho \sin\phi\right) \left(\frac{\xi' \sin\phi' U_{\xi\xi'}}{\sqrt{1-\xi'^2}} - \sin\phi' U_{\xi v'} - \frac{\cos\phi' U_{\xi\phi'}}{\sqrt{1-\xi'^2}} \right), \tag{C11}$$

$$I_{\text{FLR}}^{F,3} = \oint \frac{d\phi}{2\pi} \oint \frac{d\phi'}{2\pi} \sin\left(k\rho' \sin\phi' - k\rho \sin\phi\right) \left(\sin\phi U_{vv'} - \frac{\xi}{\sqrt{1-\xi^2}} \sin\phi U_{\xi v'} + \frac{\cos\phi U_{\phi v'}}{\sqrt{1-\xi^2}} \right), \tag{C12}$$

$$I_{\text{FLR}}^{F,4} = \oint \frac{d\phi}{2\pi} \oint \frac{d\phi'}{2\pi} \sin\left(k\rho' \sin\phi' - k\rho \sin\phi\right) \left(\sin\phi U_{v\xi'} - \frac{\xi}{\sqrt{1-\xi^2}} \sin\phi U_{\xi\xi'} + \frac{\cos\phi U_{\phi\xi'}}{\sqrt{1-\xi^2}} \right), \tag{C13}$$

$$I_{\mathrm{FLR}}^{F,5} = \oint \frac{d\phi}{2\pi} \oint \frac{d\phi'}{2\pi} \sin\left(k\rho' \sin\phi' - k\rho \sin\phi\right)\left(\sin\phi U_{vv} - \frac{\xi}{\sqrt{1-\xi^2}} \sin\phi U_{\xi v} + \frac{\cos\phi U_{\phi v}}{\sqrt{1-\xi^2}}\right), \tag{C14}$$

$$I_{\mathrm{FLR}}^{F,6} = \oint \frac{d\phi}{2\pi} \oint \frac{d\phi'}{2\pi} \sin\left(k\rho' \sin\phi' - k\rho \sin\phi\right)\left(\sin\phi U_{v\xi} - \frac{\xi}{\sqrt{1-\xi^2}} \sin\phi U_{\xi\xi} + \frac{\cos\phi U_{\phi\xi}}{\sqrt{1-\xi^2}}\right), \tag{C15}$$

$$\begin{aligned} I_{\mathrm{FLR}}^{F,7} = {} & \oint \frac{d\phi}{2\pi} \oint \frac{d\phi'}{2\pi} \cos\left(k\rho' \sin\phi' - k\rho \sin\phi\right) \\ & \times \left[\sin\phi\left(\sin\phi' U_{vv'} - \frac{\xi'}{\sqrt{1-\xi'^2}} \sin\phi' U_{v\xi'} + \frac{\cos\phi' U_{v\phi'}}{\sqrt{1-\xi'^2}}\right)\right. \\ & - \frac{\xi \sin\phi}{\sqrt{1-\xi^2}}\left(\sin\phi' U_{\xi v'} - \frac{\xi'}{\sqrt{1-\xi'^2}} \sin\phi' U_{\xi\xi'} + \frac{\cos\phi' U_{\xi\phi'}}{\sqrt{1-\xi'^2}}\right) \\ & \left. + \frac{\cos\phi}{\sqrt{1-\xi^2}}\left(\sin\phi' U_{\phi v'} - \frac{\xi'}{\sqrt{1-\xi'^2}} \sin\phi' U_{\phi\xi'} + \frac{\cos\phi' U_{\phi\phi'}}{\sqrt{1-\xi'^2}}\right)\right]. \end{aligned} \tag{C16}$$

The projection of the Landau tensor $U_{\mu\nu} \equiv \boldsymbol{e}_\mu \cdot \boldsymbol{U} \cdot \boldsymbol{e}_\nu$ with $\mu \in (v, \xi, \phi)$ and $\nu \in (v, \xi, \phi, v', \xi', \phi')$ can be calculated by using $\boldsymbol{u} = v\boldsymbol{e}_v - v'\boldsymbol{e}_{v'}$, $\boldsymbol{e}_v = \xi \boldsymbol{e}_\parallel + \sqrt{1-\xi^2}\boldsymbol{e}_\perp$, $\boldsymbol{e}_\xi = \sqrt{1-\xi^2}\boldsymbol{e}_\parallel - \xi \boldsymbol{e}_\perp$, and relations shown in Fig. 1.

[1] L. Landau, Phys. Z. Sowjetunion (USSR) **10**, 154 (1936).
[2] M. N. Rosenbluth, W. M. MacDonald, and D. L. Judd, Phys. Rev. **107**, 1 (1957).
[3] P. Helander and D. J. Sigmar, *Collisional Transport in Magnetized Plasmas* (Cambridge University Press, Cambridge, 2002).
[4] L. Chacón, D. C. Barnes, D. A. Knoll, and G. H. Miley, J. Comput. Phys. **157**, 618 (2000).
[5] L. Chacón, D. C. Barnes, D. A. Knoll, and G. H. Miley, J. Comput. Phys. **157**, 654 (2000).
[6] A. Pataki and L. Greengard, J. Comput. Phys. **230**, 7840 (2011).
[7] W. T. Taitano, L. Chacón, A. N. Simakov, and K. Molvig, J. Comput. Phys. **297**, 357 (2015).
[8] P. Degond and B. Lucquin-Desreux, Numer. Math. **68**, 239 (1994).
[9] C. Buet and S. Cordier, J. Comput. Phys. **179**, 43 (2002).
[10] E. S. Yoon and C. S. Chang, Phys. Plasmas **21**, 032503 (2014).
[11] R. Hager, E. S. Yoon, S. Ku, E. F. D'Azevedo, P. H. Worley, and C. S. Chang, J. Comput. Phys. **315**, 644 (2016).
[12] E. Hirvijoki and M. F. Adams, Phys. Plasmas **24**, 032121 (2017).
[13] P. J. Catto and K. T. Tsang, Phys. Fluids **20**, 396 (1977).
[14] E. A. Frieman and L. Chen, Phys. Fluids **25**, 502 (1982).
[15] H. Sugama and W. Horton, Phys. Plasmas **5**, 2560 (1998).
[16] A. J. Brizard and T. S. Hahm, Rev. Mod. Phys. **79**, 421 (2007).
[17] X. Q. Xu and M. N. Rosenbluth, Phys. Fluids B **3**, 627 (1991).
[18] I. G. Abel, M. Barnes, S. C. Cowley, W. Dorland, and A. A. Schekochihin, Phys. Plasmas **15**, 122509 (2008).
[19] S. P. Hirshman and D. J. Sigmar, Phys. Fluids **19**, 1532 (1976).
[20] H. Sugama, T.-H. Watanabe, and M. Nunami, Phys. Plasmas **16**, 056101 (2009).
[21] A. M. Dimits and B. I. Cohen, Phys. Rev. E **49**, 709 (1994).
[22] P. J. Catto and D. R. Ernst, Plasma Phys. Controlled Fusion **51**, 062001 (2009).
[23] Z. Lin, W. M. Tang, and W. W. Lee, Phys. Plasmas **2**, 2975 (1995).
[24] W. X. Wang, N. Nakajima, M. Okamoto, and S. Murakami, Plasma Phys. Controlled Fusion **41**, 1091 (1999).
[25] B. Li and D. R. Ernst, Phys. Rev. Lett. **106**, 195002 (2011).
[26] J. D. Huba, *NRL Plasma Formulary* (Naval Research Laboratory, Washington, DC, 2016).
[27] F. L. Hinton, Phys. Plasmas **15**, 042501 (2008).
[28] C. Canuto, M. Hussaini, A. Quarteroni, and T. A. Zang, *Spectral Methods—Fundamentals in Single Domains*, 2nd ed. (Springer-Verlag, Berlin, 2006).
[29] M. Landreman and D. R. Ernst, J. Comput. Phys. **243**, 130 (2013).
[30] F. L. Hinton and M. N. Rosenbluth, Plasma Phys. Controlled Fusion **41**, A653 (1999).
[31] Y. Xiao, P. J. Catto, and K. Molvig, Phys. Plasmas **14**, 032302 (2007).
[32] J. Lee, F. I. Parra, and M. Barnes, Nucl. Fusion **54**, 022002 (2014).
[33] M. Landreman and D. R. Ernst, Plasma Phys. Controlled Fusion **54**, 115006 (2012).
[34] O. Sauter, C. Angioni, and Y. R. Lin-Liu, Phys. Plasmas **6**, 2834 (1999).
[35] S. K. Wong and V. S. Chan, Plasma Phys. Controlled Fusion **53**, 095005 (2011).
[36] B. C. Lyons, S. C. Jardin, and J. J. Ramos, Phys. Plasmas **19**, 082515 (2012).
[37] E. A. Belli and J. Candy, Plasma Phys. Controlled Fusion **54**, 015015 (2012).
[38] F. Merz, Gyrokinetic Simulation of Multimode Plasma Turbulence, Ph.D. thesis, Universität Münster, 2008.
[39] H. Doerk, Gyrokinetic Simulation of Microtearing Turbulence, Ph.D. thesis, Universität Ulm, 2013.
[40] P. Crandall, D. Jarema, H. Doerk, Q. Pan, G. Merlo, T. Görler, A. Bañón-Navarro, D. Told, M. Maurer, and F. Jenko (unpublished).