Some physical features of the Burr-type-XII distribution

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Aspects related to driven nonequilibrium systems considered in Beck-Cohen superstatistics (BCS) formalism and, in addition, the unbiased maximum entropy density estimation, show the pertinence of considering the three-parameter Burr-type-XII distribution as a suitable way to describe stationary states of complex and nonequilibrium systems. The above is shown following a variant to the procedure presented in the formulation of BCS, which consists of the incorporation of an expanded Boltzmann factor through the Mittag-Leffler function. On the other hand, maximization of Shannon-Boltzmann-Gibbs entropy and other generalized forms of entropy show that the Burr-type-XII distribution may emerge as the least informative distribution for the mentioned systems.

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I. INTRODUCTION

The superstatistics model was presented in [1], proposing that for an inhomogeneous driven nonequilibrium system composed of many spatial cells, locally it behaves according to equilibrium statistical mechanics, with inverse temperature β, whereas globally it obeys another statistic that acts on said parameter. So, stationary distributions $B(E)$ of superstatistical systems can be obtained by superposition of Boltzmann factors $e^{-\beta E}$ weighted with the probability density $f(\beta)$ to observe a particular value of β in a randomly chosen cell, so that

$$B(E) = \int_0^\infty e^{-\beta E} f(\beta) d\beta, \tag{1}$$

where the intensive parameter β is considered approximately constant during the observation. Beck and Cohen have found that according to the function $f(\beta)$ chosen, several data sets respond to one of three superstatistics classes, namely, χ^2 superstatistics, *inverse*-χ^2 superstatistics, and *log-normal* superstatistics. These cases correspond respectively to

$$f(\beta) = \frac{1}{\Gamma(n/2)}\left(\frac{n}{2\beta_0}\right)^{n/2} \beta^{n/2-1} e^{-\frac{n\beta}{2\beta_0}}, \tag{2}$$

$$f(\beta) = \frac{\beta_0}{\Gamma(n/2)}\left(\frac{n\beta_0}{2}\right)^{n/2} \beta^{-n/2-2} e^{-\frac{n\beta_0}{2\beta}}, \tag{3}$$

and

$$f(\beta) = \frac{1}{\sqrt{2\pi} s\beta} e^{-\frac{(\ln\beta/\mu)^2}{2s^2}}. \tag{4}$$

So clearly, $B(E)$ could take different forms depending on the distribution for β considered. Note that basically here it is intended to obtain $B(E) = \langle e^{-\beta E}\rangle$, which could not always be solved in a closed manner.

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The idea of Beck-Cohen superstatistics (BCS) can be followed, assuming that through some dynamical mechanism, a nonequilibrium system could contain subsystems in steady states close to equilibrium, with respective values of each β (inverse of temperature) approximately fixed during a suitable interval of time to perform an observation. The exponential form of the Boltzmann distribution law can be expanded through the Mittag-Leffler function, so a superposition of statistics given by expanded Boltzmann factors can be performed, as will be seen later, allowing us to reach the three-parameter Burr-type-XII distribution.

The Mittag-Leffler function was presented by [2], and generalizations have been introduced by other authors. A known generalized form, presented in [3], is given by

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{(\gamma)_j\, z^j}{j!\Gamma(\beta+\alpha j)}, \tag{5}$$

with $\Re(\beta) > 0$, $\Re(\alpha) > 0$, and $\gamma \neq 0$, where $(\gamma)_j$ denotes the usual Pochhammer symbol defined by $(\gamma)_j = \gamma(\gamma+1)\cdots(\gamma+j-1)$ and $(\gamma)_0 = 1$.

What is remarkable is that the form (5) represents a natural expansion of the exponential function, where, in particular, $E^1_{1,1}(z) = e^z$, so it is possible to get behaviors that move away or approach to the exponential form as much as you want.

On the other hand, it can be found that from the maximum entropy principle, the Burr-type-XII distribution arises as a suitable form to describe nonequilibrium systems, imposing specific moments as constraints. This distribution is given by

$$p(x) = ckx^{c-1}[1+x^c]^{-(k+1)}, \tag{6}$$

with $x > 0$, $c > 0$, and $k > 0$. The Burr-type-XII distribution, presented by [4], has an interesting versatility which is widely known in statistics, as can be seen in several papers as in [5] and [6]. Some of its different forms are shown in Fig. 1.

Reference [7] discusses the statistical and probabilistic properties of the Burr-type-XII distribution, its relationship to other distributions used in reliability analyses, its use as a model for failure data, and methods for graphical estimation,

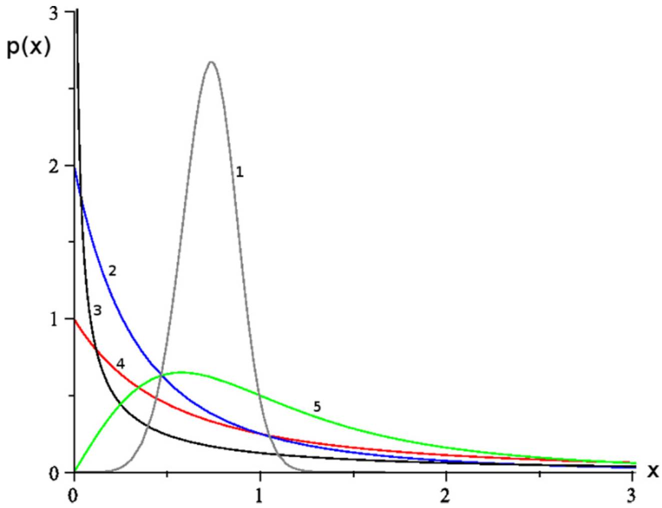

FIG. 1. Some cases for the Burr-type-XII distribution: $c = 6$, $k = 5$ (curve 1), $c = 1, k = 2$ (curve 2), $c = 0.5, k = 1$ (curve 3), $c = 1, k = 1$ (curve 4), and $c = 2, k = 1$ (curve 5).

where it has been found that in comparison to the log-logistic and the log-normal, the Burr's additional flexibility provides a better fitting model for failure data. In [8] there is an application to heavy tailed lifetime data through the three-parameter Burr-type-XII distribution, where results on a real dataset by fitting this distribution to the survival time of breast cancer patients in the Gaza Strip were obtained. The Burr-type-XII contains the q-exponential and q-Weibull distributions as particular cases, and with these a physical description has been established that has allowed the study of some systems of interest, as can be read in [9]. Also, Ref. [10] studies the q-exponential and q-Weibull distributions as particular cases of the Burr-type-XII distribution. In Ref. [11] an advantageous description of the q-Weibull distribution for the analysis of dielectric breakdown in oxides is presented. This distribution is a generalization of the Weibull distribution, just as the q-exponential function generalizes the exponential, and in the mentioned work it has been found that the generalized Weibull distribution leads to a better fit of experimental and simulated data in comparison with the Weibull distribution. This issue, regarding the generalization of the Weibull distribution, was studied in [12], where this distribution can be useful in some situations where neither q-exponential nor Weibull distribution provide satisfactory results, as in the case of the distribution of the highway length. Also, in Ref. [13] a study of the properties of the q-Weibull distribution and the application to data on cancer remission times was carried out, finding that this distribution gives a better fit than the Weibull distribution. Also, results relating to reliability properties, estimation of parameters, and applications in stress-strength analysis were obtained.

Why choose the Mittag-Leffler function?

The Mittag-Leffler function has been chosen because of the interesting properties it has, as well as by the wide applications that have been made to a large number of physical systems and others that are objects of study in the engineering areas and other sciences. It has been found that fractional order integral (or differential) equations provide solutions in terms of the Mittag-Leffler function when studying fractional generalization of the kinetic equation, random walks, levy flights, superdiffusive transport, and in the study of complex systems, explaining the behavior of some phenomena, interpolating between a purely exponential law and a power-law-like behavior.

Of course, one could proceed using another function than the Mittag-Leffler function, but there is nothing to suggest which correct path to follow. Related to this, you can read, for example, Ref. [14], where an alternative expression for the blackbody radiation law is obtained, showing an analysis of the residual monopole spectrum, applying the Bose-Einstein distribution with a dimensionless chemical potential, also applying a formula based on the nonextensive approach and another based on fractional calculus (performing the Mittag-Leffler function).

A very complete survey of the Mittag-Leffler function, generalized Mittag-Leffler functions, and Mittag-Leffler–type functions can be found in [15]. In that work there are special cases, basic properties, recurrence relations, asymptotic expansions, integral representations, relations with Riemann-Liouville fractional calculus operators, generalizations, Laplace transform, Fourier transform, and fractional integrals and derivatives (which are related with the evolution of some interesting physical processes, e.g. [16], where ordinary differential equations of fractional order related to generalized processes of relaxation and oscillation are considered). There are several applications in complex systems through fractional kinetic equations, time (or space) fractional diffusion equations, nonlinear waves, and fractional viscoelastic models. In various cases the asymptotic behavior is an important element for several physics situations. Reference [17] provides descriptions of some physical models, showing the role of the Mittag-Leffler function in fractional modeling. Reference [18] shows the interpolation between a pure exponential and a hyperbolic function performed by the Mittag-Leffler function in the description of relaxation phenomena in complex physical systems within the framework of fractional kinetic equations. On the other hand, the authors of Ref. [19] show that the Mittag-Leffler function can be used as a universal fitting function, explaining the behavior of some cases as monotonic processes, oscillatory behavior, and damped oscillations.

Quoting [15]: "*It is simply said that deviations of physical phenomena from exponential behavior could be governed by physical laws through Mittag-Leffler functions (power-law).... This nonequilibrium statistical mechanics will focus on entropy production, reaction, diffusion, reaction-diffusion, and so forth, and may be governed by fractional calculus....*"

II. A STATIONARY DISTRIBUTION FOR NONEQUILIBRIUM SYSTEMS

A. Superposition of expanded Boltzmann factors

What is proposed below consists of a variant of the BCS model for a nonequilibrium system, providing an alternative expression for a stationary distribution of the system. To get that, here we propose an extended Boltzmann factor for each cell through the Mittag-Leffler function $E_\nu^\gamma(-\beta^\nu\varepsilon^\nu)$, assuming that the system is partitioned into cells, considering each of them locally in a steady state close to equilibrium. (Note that $E_1^1 = E_1 = e^{-\beta\varepsilon}$ provides the classic Boltzmann factor for the equilibrium situation.) In this perspective ν and γ are fixed parameters throughout the system, but parameter β can have any positive value in each cell, and its value is approximately constant for a reasonably large time.

On the other hand, the following probability density function can be obtained (see, for example, [20]) for $x > 0$:

$$p(x) = \nu\gamma x^{\nu-1} E_{\nu,\nu+1}^{\gamma+1}(-\beta^\nu x^\nu), \tag{7}$$

where $0 < \nu \leqslant 1$, $\gamma > 0$, and $\beta > 0$. The interesting thing about this function is that it has been obtained from the cumulative function $F(x) = 1 - E_\nu^\gamma(-\beta^\nu x^\nu)$, incorporating the Mittag-Leffler function with which we can represent the expanded Boltzmann factor $E_\nu^\gamma(-\beta^\nu\varepsilon^\nu)$ associated with the cells, so that we can get the macroscopic behavior of the system through a linear average of the expanded Boltzmann factors over β, where ν and γ are fixed global parameters.

If we take a look into the BCS formalism, χ^2-superstatistics appear as a remarkable superstatistics class. So the probability density (2) for $f(\beta)$ seems important (because the Tsallis nonextensivity distribution is obtained from that case), and some authors (see Refs. [21,22], for example) have justified this choice. Then, we take

$$f(\beta) = \frac{\lambda^{-\nu'}}{\Gamma(\nu')}\beta^{\nu'-1}e^{-\frac{\beta}{\lambda}}, \tag{8}$$

with $\nu' = \nu + 1$ and $\lambda > 0$, $\nu > 0$ parameters.

Now we can obtain an alternative form of $B(E)$ [Eq. (1)] by integrating from zero to infinity,

$$p(\varepsilon) = \int_0^\infty \nu\gamma\varepsilon^{\nu-1}E_{\nu,\nu+1}^{\gamma+1}(-\beta^\nu\varepsilon^\nu)f(\beta)d\beta, \tag{9}$$

which can be seen as a type-B superstatistics in the BCS model shown in [1], due to the performance of locally normalized distributions. Then,

$$p(\varepsilon) = \nu\gamma\varepsilon^{\nu-1}\frac{\lambda^{-\nu'}}{\Gamma(\nu')}\sum_{k=0}^\infty \frac{(\gamma+1)_k(-1)^k\varepsilon^{\nu k}}{k!\Gamma(\nu'+\nu k)} \times \int_0^\infty \beta^{\nu k+\nu'-1}e^{-\frac{\beta}{\lambda}}\,d\beta. \tag{10}$$

After integration and algebraic manipulations, it can be written

$$p(\varepsilon) = \nu\gamma\varepsilon^{\nu-1}\lambda^\nu[1 + (\lambda\varepsilon)^\nu]^{-(\gamma+1)}. \tag{11}$$

This is the three-parameter Burr-type-XII distribution for $\varepsilon > 0$, with $\gamma > 0$ and $\nu > 0$ parameters.

B. Two remarkable particular cases

As a particular case, in the limit $\gamma \to \infty$ of (11) and for $\nu = 1$ it is possible to obtain the equilibrium situation

$$\lim_{\gamma\to\infty} p(\varepsilon) = \beta e^{-\beta\varepsilon}, \tag{12}$$

where the substitution $\lambda = 1/(\gamma^{1/\nu}\beta)$ has previously been carried out.

Moreover, with $\nu = 1$, $\gamma + 1 = \frac{1}{q-1}$, and $\lambda(\gamma + 1) = \beta_0$, the expression (11) turns to $p(\varepsilon) = \beta_0(2 - q)[1 + (q - 1)\beta_o\varepsilon]^{-1/(q-1)}$, with application to the so-called nonextensive systems.

In this respect, we can see here that the q-exponential distribution introduced by [23] is simply seen as a particular case of the family (11), a question that is also present in [24], where it has seen that the Burr-type-XII distribution provides a better performance than the q-exponential distribution when applied to fracture roughness data from materials such as $Bi_2Sr_2CaCu_2O_{8+x}$, alumina (Al_2O_3), silicon nitride (Si_3N_4), sialon ($Si_{6-x}Al_xO_x - N_{8-x}$), Pyroceram 9606, and titanium tiboride (TiB_2).

III. MAXIMUM ENTROPY PROBABILITY DISTRIBUTION

It is possible to obtain the Burr-type-XII distribution as the most unbiased probability density function estimation from the Shannon-Boltzmann-Gibbs entropy (SBG) maximization, but it is also possible to obtain it from maximization of generalized forms of entropy widely known today, such as the Tsallis entropy and Mathai's entropy. Regarding that last matter, some authors do not believe it is adequate to deal with generalized forms of entropy, suggesting that the chosen moments used as constraints could allow appropriate distributions to be built that could describe more complex systems, in particular, through the generalization of said moments. This issue has been addressed in several papers, e.g., Refs. [25–28], examining logarithmic moments or with fractional order, which are appropriate when working with a positive random variable with distribution exhibiting fat tails. The mathematical background has been studied in some works such as [29] and [30], where the type of density function and its link with different moments has been extensively discussed, as well as the limitations of the corresponding maximum-entropy approach, the existence conditions for the respective solution, and so on.

Whatever the case may be, the way to obtain the Burr-type-XII distribution through the SBG entropy, or through a generalized form of entropy and their respective maximization process, both will be shown in the following.

A. Shannon-Boltzmann-Gibbs entropy

We can consider the SBG entropy:

$$S(x) = -k\int_0^\infty f(x)\ln f(x)dx, \tag{13}$$

with k Boltzmann's constant. The application of the principle of maximum entropy requires some constraints for estimating the underlying probability distribution, which gives us the minimum assumptions about the system. We have the normalization condition

$$\int_0^\infty f(x)dx = 1. \tag{14}$$

Additional constraints allow different probability distributions to be built, which can be obtained through different moments, namely,

$$\langle g_i(x)\rangle = \int_0^\infty g_i(x) f(x) dx. \quad (15)$$

From the calculus variational method, we can obtain the Euler-Lagrange equation:

$$\delta\left\{-\int_0^\infty f(x)\ln f(x)dx + \sum_{i=0}\phi_i\left[\langle g_i(x)\rangle - \int_0^\infty g_i(x)f(x)dx\right]\right\} = 0, \quad (16)$$

whose solution is given by

$$f(x) = e^{-\phi_0 - \sum_{i=1}\phi_i g_i(x)}, \quad (17)$$

where ϕ_i are the Lagrange parameters, determined through constraints. By substituting (17) into (14), it can be found that

$$\phi_0 = \ln Z(\phi_1 ... \phi_i), \quad (18)$$

where $Z(\phi_1...\phi_i) = e^{-\sum_i \phi_i g_i(x)}$. Along with this, we also have the Jayne's relations

$$\langle g_i(x)\rangle = -\frac{\partial}{\partial\phi_i}\ln Z, \quad (19)$$

$$\frac{\partial S(x)}{\partial\langle g_i(x)\rangle} = \phi_i. \quad (20)$$

Of course, the form that the function $f(x)$ can acquire will depend at the same time on the form of each of the considered $g_i(x)$. In particular, it can be founded in [31] that the following moments can be chosen:

$$\langle \ln(x)\rangle = \int_0^\infty \ln(x) f(x)dx, \quad (21)$$

$$\langle x_p^\alpha\rangle = \frac{1}{p}\int_0^\infty \ln(1+px^\alpha) f(x)dx, \quad (22)$$

where $x_p^\alpha = \frac{1}{p}\ln(1+px^\alpha)$ has been defined in such a way that it can be seen as a generalization of x^α, which is recovered when $p \to 0$.

The authors of the mentioned paper highlight the fact that the expectation of $\ln x$ given by (21), apart from its relationship to the geometric mean, represents an essential constraint for positively skewed random variables. While the arbitrary generalization given in (22) seems as reasonable as the arbitrary choice of some generalized forms of entropy used by some authors, the aforementioned paper can be read to deepen understanding of the motivation for choosing these moments (which can clearly be discussed as part of different research).

So, with the above we can build the following Euler-Lagrange equation:

$$\frac{\partial}{\partial f}\left\{\int f(x)\ln f(x)dx + \phi_0\left(\int f(x)dx - 1\right) + \phi_1\left(\int \ln(1+px^q)dx - g_1\right) + \phi_2\left(\int \ln x dx - g_2\right)\right\} = 0, \quad (23)$$

which gives the following density function (details can be seen in [32]):

$$f(x) = \frac{1}{q}p^{\frac{\phi_2-1}{q}}\frac{1}{B\left(\frac{1-\phi_2}{q}, -\frac{1-\phi_2}{q}+\frac{\phi_1}{p}\right)}x^{-\phi_2}(1+px^q)^{-\frac{\phi_1}{p}}. \quad (24)$$

After some algebra, the authors present the form

$$f(x) = \frac{\gamma_3}{\beta B(\gamma_1,\gamma_2)}\left(\frac{x}{\beta}\right)^{\gamma_1\gamma_3-1}\left[1+\left(\frac{x}{\beta}\right)^{\gamma_3}\right]^{-(\gamma_1+\gamma_2)}. \quad (25)$$

This fact reveals that a system responding to a power law is a result which is not exclusive of a nonextensive entropy.

The three-parameter Burr-type-XII function arises immediately from (25) with $\gamma_1 = 1$ (there, $B(\cdot,\cdot)$ is the beta function with $\beta > 0$, $\gamma_1 > 0$, $\gamma_2 > 0$, and $\gamma_3 > 0$).

B. Tsallis entropy

Nonextensive statistical formalism is a generalization of Boltzmann-Gibbs (BG) statistical mechanics. It was introduced in [23] and yields a new approach to explain the non-BG behavior of systems that are not in equilibrium, establishing that for those systems a general form of entropy can be assigned that allows an adequate description.

Tsallis nonextensive entropy is given by

$$S_q(x) = k\frac{1-\int_0^\infty [f(x)]^q dx}{q-1}. \quad (26)$$

$q \in \Re$ is the entropic index, which represents the degree of nonextensivity according to the property

$$S_q(A+B)/k = [S_q(A)/k] + [S_q(B)/k] + (1-q)[S_q(A)/k][S_q(B)/k], \quad (27)$$

when subsystems A and B are assumed to be probabilistically independent. Taking the limit $q \to 1$ the SBG entropy (13) is obtained.

We can find in [9] the introduction of the fractional moment $\langle x^\alpha\rangle$ and the q-expectation value $\langle x\rangle_q$,

$$\langle x^\alpha\rangle = \int_0^\infty x^\alpha f(x)dx, \quad (28)$$

$$\langle x\rangle_q = \int_0^\infty x\pi(x)dx, \quad (29)$$

where $\pi(x)$ is the escort probability (which can be found in [33]),

$$\pi(x) = \frac{[f(x)]^q}{\int_0^\infty [f(x)]^q dx}. \quad (30)$$

Using (21) and (28) with (29), the moments $\langle x^\alpha\rangle_q$ and $\langle \ln_q x\rangle_q$ can be written as constraints, and in this way the following Euler-Lagrange equation can be obtained:

$$\frac{\partial}{\partial f}\left\{S_q(x) + \phi_0\left(\int_0^\infty \pi(x)dx - 1\right) + \phi_1\left(\int_0^\infty x^\alpha\pi(x)dx - \langle x^\alpha\rangle_q\right) + \phi_2\left(\int_0^\infty \ln_q x\pi(x)dx - \langle\ln_q x\rangle_q\right)\right\} = 0, \quad (31)$$

where $\ln_q x = (x^{1-q} - 1)/(q-1)$ is the generalized logarithmic function, which can be found in [34], with the particular case $\lim_{q\to 1} \ln_q x = \ln x$ (other features are presented in [35]). Then, it is possible to obtain

$$f(x) = \frac{\alpha x^{\alpha-1}}{\langle x^\alpha\rangle_q}\left[1 + \left(\frac{q-1}{2-q}\right)\frac{x^\alpha}{\langle x^\alpha\rangle_q}\right]^{-1/(q-1)}. \tag{32}$$

This result can be checked in [9], where it is used to describe the relaxation dynamics in non-Debye complex systems. Analytically, (32) can be seen as the extended three-parameter Burr-type-XII distribution, an issue reviewed, for example, in Refs. [36,37].

C. Mathai's entropy

In [38] is presented a generalized form of entropy, also associated with Shannon, Boltzmann-Gibbs, Rényi, Tsallis, and Havrda-Charvát entropies. In the continuous case, Mathai's entropy has the form

$$M_\alpha(f) = \frac{\int_{-\infty}^{\infty}[f(x)]^{2-\alpha}dx - 1}{\alpha - 1}, \tag{33}$$

with $\alpha \neq 1$ and $\alpha < 2$, where $1-\alpha$ corresponds to the strength of information content, which appears when a connection with the Kerridge's measure of *inaccuracy* ([39]) is done. The SBG entropy (13) is obtained when the limit $\alpha \longrightarrow 1$ is taken.

The optimization of (33) can be done if it is subject to the following conditions for all x:

(i) $f(x) \geqslant 0$
(ii) $\int_a^b f(x)dx < \infty$
(iii) $\int_a^b x^{(\gamma-1)(1-\alpha)} f(x)dx = fixed$
(iv) $\int_a^b x^{(\gamma-1)(1-\alpha)+\delta} f(x)dx = fixed$
with $\gamma > 0, \delta > 0, \alpha \neq 1, \alpha < 2$.

When variational calculus is performed, Euler-Lagrange equations lead to

$$\frac{\partial}{\partial f}\left\{\frac{\int_{-\infty}^{\infty}[f(x)]^{2-\alpha}dx - 1}{\alpha-1} + \phi_1\left(\int_a^b x^{(\gamma-1)(1-\alpha)} f(x)dx - g_1\right) + \phi_2\left(\int_a^b x^{(\gamma-1)(1-\alpha)+\delta} f(x)dx - g_2\right)\right\} = 0. \tag{34}$$

Then we get the function that maximizes (33), which is

$$f(x) = \varsigma x^{\gamma-1}[1 - \beta(1-\alpha)x^\delta]^{\frac{1}{1-\alpha}}, \tag{35}$$

with $\beta > 0$, $1 - \beta(1-\alpha)x^\delta > 0$, and ς is the normalization constant.

So, we have three cases: $\alpha < 1$, where we get a family of generalized β density functions of type 1; $\alpha > 1$, where a family of generalized β density functions of type 2 is obtained; and $\alpha \longrightarrow 1$, where a generalized Γ density can be found. A Burr-type-XII function is contained in the case $\alpha > 1, \delta = \gamma$ with

$$\varsigma = \frac{\delta\Gamma\left(\frac{1}{\alpha-1}\right)\beta(\alpha-1)}{\Gamma\left(\frac{1}{\alpha-1} - 1\right)}. \tag{36}$$

IV. FINAL REMARKS

Burr-type-XII distribution was presented as a suitable function to describe stationary distributions of complex and nonequilibrium systems. It can be seen as an alternative to explain systems described by the superstatistical model presented by [1], where it remains to determine with more precision the scope that one model has in relation to the other. In a work recently accepted in a different journal (which may be revised in [40]) the application to traffic delay, medicine, and turbulence explained by χ^2 superstatistics, inverse-χ^2 superstatistics, and log-normal superstatistics, respectively, were treated with Eq. (11). (All those cases can be seen in [41].)

Expression (11) has a construction inspired by the procedure followed by Beck and Cohen but with a simpler and more intuitive variant based in the great versatility shown by the Mittag-Leffler distribution. On the other hand, by assigning a SBG entropy to the types of systems mentioned, or even associating them with a more general form of entropy than the SBG form, it is possible to find the Burr-type-XII distribution as the least informative form, estimated with constraints imposed through generalized forms of different moments, which have been widely used in descriptions of non-Boltzmann-type systems. But there could be other alternatives. For example, if we think that the principle of maximum entropy must be applied through the usual constraints (which lead to the Boltzmann distribution via the SBG entropy), then, according to [42], there exists an expression that allows us to obtain a form that shows us the appearance of the entropy of complex systems when considering some positive and deformed function of the exponential function. The authors show in this interesting research that the procedure uses the appropriate physical context, lending reliability to its results, since it is also able to address the thermodynamics of the system. In another paper (see [43]), considering three of the four Shannon-Khinchin axioms, the same authors present a two-parameter family entropy form for complex systems, which can be associated with functions of the Lambert-W exponentials type when it is required to find the corresponding distribution function. Clearly, we could check that here, but we still cannot rule out that the use of generalized constraints cannot be incorporated in the process of maximization of the SBG entropy or another generalized forms of entropy, in order to characterize a complex system.

It should be mentioned that other relations of the Burr-type-XII distribution with some aspects of complex physical systems have already been presented by other authors, among which is Ref. [44], where a stochastic point of view about the Burr-type-XII distribution is given, containing interesting physical and statistical interpretations. Quoting Beck in Ref. [1]: "*...In general, complex nonequilibrium problems may require different types of superstatistics. Tsallis statistics is just one example of many possible new statistics. There is no a priori reason to expect that other superstatistics would not be present in nature....*"

Therefore, the possibility that the Burr-type-XII function is a more adequate way, than some existing ones, to explore stationary states of certain types of complex and nonequilibrium systems should be taken into account.

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