# Growth kinetics of the two-dimensional Ising model with finite cooling rates

Kangeun Jeong,[1] Bongsoo Kim,[1,2,*] and Sung Jong Lee[2,3]
[1]*Department of Physics, Changwon National University, Changwon 51140, Korea*
[2]*Institute for Soft and Bio Matter Science, Changwon National University, Changwon 51140, Korea*
[3]*Basic Science Institute, Changwon National University, Changwon 51140, Korea*

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We investigate, via spin-flip kinetic Monte Carlo simulations, the entire growth kinetic process of the two-dimensional kinetic Ising model cooled to zero temperature with finite cooling rates. A new slow dynamics, suppressed, and unseen in the instantaneous quench emerges and eventually crosses over to the asymptotic standard coarsening behavior. We present a numerical observation of the excess defect number density that provides direct information on the generated defects due to the critical slowing down. We find that the excess defect density reveals the dynamics of defect generation, which is shown to be in good agreement with the scaling theory pioneered by Kibble and Zurek (KZ). The dynamic spin correlation function reveals that the impulse regime alluded by KZ is characterized by a unique critical coarsening process with the growth law dictated by the Kibble-Zurek mechanism. We determine a new dynamic exponent governing the growth kinetics at the onset times of the zero temperature. The proposed scaling scheme for the excess defect density leads to an analytic expression for this dynamic exponent involving the KZ exponents, indicating the extended influence of the KZ mechanism even down to the kinetics at the onset times of zero temperature. We also perform dynamic simulations of critical heating with finite rates from zero temperature where the power law relaxation (with respect to the inverse cooling rates) of the magnetization can be explained clearly by the KZ exponent.

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## I. INTRODUCTION

Let us take a statistical system which exhibits an order-disorder continuous phase transition. When this system is suddenly brought from an initially disordered phase into the ordered phase via an abrupt change in one of the control parameters of the system (such as the temperature), it does not instantaneously exhibit its full order but rather exhibits a slow time evolution ever toward its final (non)equilibrium stationary state. Locally appearing symmetry-broken "ordered" regions generate proliferating defects, such as domain walls or vortices. The ordering process then usually takes place via annihilations of these topological objects. This so-called phase ordering kinetics [1–3] has long been an important research focus in the area of nonequilibrium (NEQ) statistical physics. It would be a fair assessment that the nature of *most* of the research performed on the subject up to the early 1990s can be characterized with the following several features:

(1) The cooling protocol was an *instantaneous* quench from a disordered state to an ordered state.

(2) The systems under consideration were *not* isolated but in contact with a heat bath.

(3) Related to the second item, except for the case of binary liquids, the underlying dynamics governing the time evolution was a *purely* dissipative one, such as the time-dependent Ginzburg-Landau equation or kinetic Monte Carlo (MC) simulations. No nondissipative (reversible) contributions (for example, models E and F according to the classification of Hohenberg and Halperin [4,5]) were present.

Interesting enough, realizations and developments of ultracold atomic gases have been offering fascinating opportunities for the research going beyond the above paradigmatic but limiting features [6], which are currently being actively pursued [7–30]. As for the first feature, it would be natural to ask how the ordering process is affected when the cooling rate becomes finite, being tuned from fast to slow. Indeed, it is quite interesting that such a study with finite cooling rates was pioneered early on in the context of cosmological phase transitions in the early universe [31] and in another crucial work in the more familiar context of condensed matter physics [32]. However these seminal works by KZ did not focus on the phase ordering kinetics *per se* but rather on how many of the supposedly "frozen" defects are generated after crossing the critical point, offering a generic scaling theory involving the critical slowing down, the so-called KZ mechanism (for a recent review, see Ref. [33]). So, although these two aspects of the dynamics are subsequently occurring in a slowly cooled system, they remain largely unconnected, and as a consequence, a unified view on the entire NEQ process remains lacking, and possible interplay between them remains unexplored. In this regard, recently, there appeared an inspiring work by Biroli, Cugliandolo, and Sicilia (BCS) [14], which, tying together the two aspects of dynamics, attempted to see to the physics of the entire NEQ time evolution ensued by the slow quench. In particular, this work offered a further physical insight into the seemingly frozen KZ time region and proposed a new scaling ansatz that may hold in the entire time span of the ordering process with slow cooling.

*bskim@changwon.ac.kr

 

We, thus, deemed it valuable to revisit and carefully investigate the entire NEQ growth kinetics observed after slow quench. To this end, in the present paper, we looked into the entire NEQ kinetics (encompassing the standard coarsening regime) under a linear cooling protocol with varying cooling rates via spin-flip kinetic MC simulations on the ferromagnetic two-dimensional Ising model (whose phase ordering kinetics under instantaneous quench has been thoroughly studied [1]).

We here briefly state the main findings in our simulation study. Under slow cooling to the zero temperature, in contrast with the case of instantaneous quench, we find that the defect relaxation exhibits a rich NEQ slow dynamics eventually crossing over to the asymptotic coarsening regime. Here, we focus on the dynamics of the *excess* defect number density. Our paper presents a numerical observation and analysis on its dynamics. Onsager's exact solution (for the EQ defect density) allows us to directly measure the *excess* defect density which gives the detailed information on the dynamics of the defects which are generated when the system approaches and passes through the critical temperature. We, thus, find that the excess defect density naturally demarcates the breaking time (that separates the adiabatic and diabatic regimes) as its peak, and its dynamics best reveals the KZ mechanism, confirming its main predictions. We then propose a natural scaling scheme for the dynamics of the excess defect density, which yields a good scaling collapse in the adiabatic, impulse, and final coarsening regimes (except for the crossover regime where the scaling is violated). We also find a new dynamic exponent governing the growth kinetics at the onset times of zero temperature. The aforementioned scaling behavior allows us to get an analytic expression for the new dynamic exponent, which involves the two exponents characterizing the KZ mechanism as well as the usual dynamic exponent for the standard coarsening. This may well indicate that the KZ mechanism affects even the ordering kinetics observed at the onset times of zero temperature.

The present paper is organized as follows. In Sec. II, the kinetic Ising model is defined along with the spin-flip kinetic Monte Carlo simulation method, cooling protocol, and the measured quantities for monitoring the resulting NEQ dynamics. Section III provides a short review on the observed dynamics followed by instantaneous quench with which the slow quench dynamics is compared and contrasted. The main body of the present paper is contained in the Sec. IV where the dynamics observed in the critical region (alluded by the KZ mechanism) as well as the onset zero-temperature dynamics present in the crossover region into the standard coarsening regime. Section V summarizes and concludes the paper.

## II. SIMULATIONS WITH FINITE COOLING RATES

The Ising model takes into account a short-range interaction between the spins on the lattice sites, and its Hamiltonian is defined as

$$H = -J \sum_{\langle i,j \rangle} S_i S_j, \tag{2.1}$$

where $S_i = \pm 1$ is an Ising spin at the site $i$, $\langle i, j \rangle$ denotes the nearest-neighbor sites, and $J$ is the interaction strength which is set to unity (i.e., ferromagnetic interaction) throughout this paper. Onsager's exact solution predicts a continuous transition at the critical temperature $T_c = 2/\ln(1+\sqrt{2}) \simeq 2.269\,185$ (the Boltzmann constant set to unity as well throughout) between the para- and ferromagnetic phases on a square lattice. In order to investigate the NEQ kinetics followed by a temperature quench, we perform spin-flip kinetic MC simulations on the Ising model on a square lattice with linear size $L = 1024$.

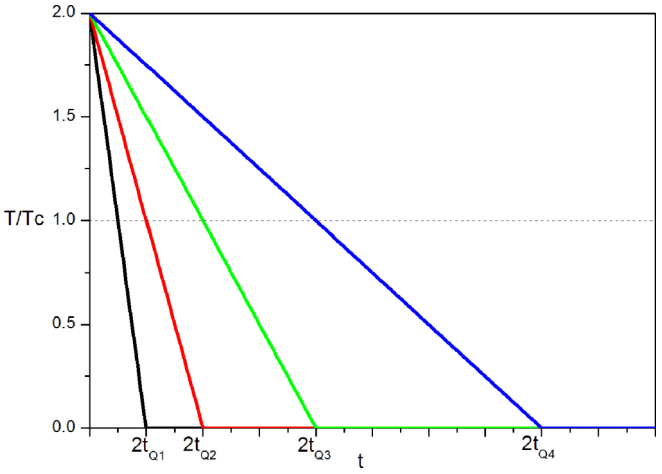


FIG. 1. The linear cooling protocol $T(t) = T_c(2 - t/t_Q)$ used for the present simulations.

The cooling protocol is via a *linear* temperature ramp $T(t)$ (see Fig. 1),

$$T(t) = T_c\left(1 + \frac{t_c - t}{t_Q}\right), \tag{2.2}$$

where $t_c$ is the time at which the temperature reaches the critical temperature $T_c$. In the present simulation, we take $t_c = t_Q$ so that the initial temperature is $T(0) = 2\,T_c$ at which the system is prepared to be at equilibrium. It passes the critical temperature at $t = t_c = t_Q$ and reaches the zero temperature at $t = 2\,t_Q$. Once the system reaches the zero temperature, the temperature remains the same throughout further time evolution. The time $t_Q$ is, thus, the inverse cooling rate; whereas $t_Q \to 0$ leads to an instantaneous quench, $t_Q \to \infty$ annealing. In our simulation, $t_Q$ is varied in the range of $t_Q = 10 \times 2^n$ with $n = 0$–$2, \ldots, 10$ (MCS (Monte Carlo Steps)).

During the time evolution for a given $t_Q$, to quantitatively characterize the evolution process, we measure the dynamic spin correlation and the number density of defects (domain walls):

(a) the dynamic spin correlation function at equal times $t$ defined as

$$C(r,t) = \frac{1}{N}\left\langle \sum_{i=1}^{N} S_i(t) S_{i+r}(t) \right\rangle, \tag{2.3}$$

where $N = L^2$ is the total number of spins.

(b) The number density of defects:

$$n_d(t) = \frac{1}{2N}\left\langle \sum_{i=1}^{N} \sum_{j}^{(i)} \frac{1}{2}[1 - S_i(t) S_j(t)] \right\rangle. \tag{2.4}$$

where $\sum_j^{(i)}$ denotes the sum over nearest-neighbor sites of site $i$. The brackets $\langle \cdots \rangle$ in (2.3) and (2.4) denote the average over different initial states which will evolve with different thermal histories.

It is easy to see from the model Hamiltonian (2.1) that the defect density is related to the average energy density $e(t) = \langle H(t) \rangle / N$ as $n_d(t) = 1 + e(t)/2$. An important quantity to characterize the quench dynamics is the *excess* defect density $\delta n_d(t)$ defined as

$$\delta n_d(t) \equiv n_d(t) - n_d^{\rm eq}[T(t)] = \tfrac{1}{2}\delta e(t), \tag{2.5}$$

where $\delta e(t)$ is the excess energy density $\delta e(t) \equiv e(t) - e^{\rm eq}[T(t)]$, $n_d^{\rm eq}[T(t)]$, and $e^{\rm eq}[T(t)]$ being the equilibrium defect and energy densities, respectively. It is of great advantage that the exact Onsager solution is available for the equilibrium energy (and, hence, defect) density; it takes the expression (with $\beta \equiv 1/T$) [34],

$$e^{\rm eq}(T) = -\coth(2\beta)\Bigg[1 + \frac{2}{\pi}[2\tanh^2(2\beta) - 1] \times K_1\left(\frac{2\sinh(2\beta)}{\cosh^2(2\beta)}\right)\Bigg], \tag{2.6}$$

where $K_1(x) \equiv \int_0^{\pi/2} d\phi / \sqrt{1 - x^2 \sin^2\phi}$ is the complete elliptic integral of the first kind.

The presented results in Sec. IV are the ones with averages over 100 different initial states. We checked that finite size effect is entirely negligible in our simulations on the square lattices with $L = 1024$. Therefore, we believe that we are justified in using Onsager's exact results in computing the excess defect density. As demonstrated below, the dynamics of the excess defect density vividly manifests the KZ mechanism, confirming its main predictions.

## III. REVIEW OF ORDERING KINETICS UNDER INSTANTANEOUS QUENCH

We here briefly review the ordering kinetics for the case of instantaneous temperature quench [1–3]. We intend that our simulation results for the slow quenches in the next section would be contrasted with the known results of the instantaneous quench that corresponds to the limit $t_Q \to 0$.

In the case of the instantaneous quench from a disordered initial state to an ordered state below $T_c$, the local domains of Ising spins (resulting from the symmetry breaking) coarsen via annihilations of domain walls existing between opposite-spin domains. The most important feature of the domain coarsening is the dynamic scaling phenomenon with respect to the growing length scale $R(t)$ (the average domain size at time $t$), which manifests a statistical similarity of the domain morphologies at different times in the scaling regime. This NEQ scaling phenomenon is manifested in the spin correlation functions and relaxation of the average population of defects. The spin correlation function exhibits the dynamic scaling,

$$C(r,t) = f[r/R(t)], \tag{3.1}$$

where $f(x)$ is called the scaling function and the average domain size $R(t)$ shows a power-law growth in time,

$$R(t) \sim t^{1/z_d}, \tag{3.2}$$

with $z_d$ being a dynamic exponent. The excess defect density $\delta n_d(t)$ should be related to the domain size $R(t)$ as

$$\delta n_d(t) \sim R(t)^{-(d-d_w)}, \tag{3.3}$$

where $d$ is the spatial dimension and $d_w$ is the dimension of the defect according to its shape (point, line, and surfaces). For the ferromagnetic Ising model $(d - d_w) = 1$, regardless of the spatial dimension. The analytic predictions for the form of the scaling function $f(x)$ and the dynamic exponent $z_d$ have been focus issues in the theory community. It is well known that $z_d = 2$ for the spin-flip kinetic Ising model, and hence $R(t) \sim t^{1/2}$ and $\delta n_d(t) \sim t^{-1/2}$ [35].

Instantaneous quench to $T_c$ from a disordered state leads to a nonequilibrium critical dynamics which exhibits a critical coarsening process characterized by the critical dynamic scaling (for $d = 2$),

$$C(r,t) = r^{-\eta} f_c[r/R_c(t)], \tag{3.4}$$

where $\eta$ is an equilibrium critical exponent ($\eta = 1/4$ for the two-dimensional ferromagnetic Ising model) and $f_c(x)$ is the critical scaling function. Here, the average domain size $R_c(t)$ exhibits a power-law growth,

$$R_c(t) \sim t^{1/z}, \tag{3.5}$$

where $z$ is the dynamic critical exponent, which takes $z \simeq 2.16(1)$ for the spin-flip kinetic Ising model [35].

It is also of interest to look at the instantaneous critical heating process from $T = 0$ (the perfect order) to $T_c$. For this case, for a fixed time $t$, the spin correlation function rapidly drops to a constant nonzero value over some distance and then remains over longer distances. Therefore, this constant value is indeed the square of the remaining magnetization $m_{\rm inst}(t)$, i.e., $C(r \to \infty, t) = m_{\rm inst}^2(t)$, which will eventually vanish as $t \to \infty$ in the thermodynamic limit. The distance over which the correlation reaches $m_{\rm inst}^2(t)$ grows in time. The behavior of the correlation reflects the evolution of domain morphology in which there emerges opposite spin domains from the sea of a perfectly ordered spins, and these small domains are growing in time. Here, the dynamic scaling can be expressed as

$$C(r,t) = r^{-\eta} g_c[r/L_c(t)], \tag{3.6}$$

where $L_c(t) \sim t^{1/z}$ is a growing domain size. The scaling function $g_c(x)$ should satisfy $g_c(x) = \text{const}$ for $x \ll 1$ [since in equilibrium $C_{\rm eq}(r) \sim r^{-\eta}$ in two dimensions] and $g_c(x) \sim x^{\eta}$ for $x \gg 1$ [since $C(r,t)$ remains constant for large $r$]. Therefore, considering large $r$ behavior, one can easily obtain the power-law decay of the magnetization,

$$m_{\rm inst}^2(t) = C(r \to \infty, t) \sim L_c^{-\eta}(t) \sim t^{-\eta/z}, \tag{3.7}$$

where $\eta/z \simeq 0.116$ for the two-dimensional Ising model [36].

## IV. KINETICS UNDER SLOW COOLING

### A. The Kibble-Zurek mechanism

In equilibrium, approaching $T_c$, the correlation length $\xi_{\rm eq}(T)$ tends to diverge, which then naturally leads to the

critical slowing down, that is, the power-law divergence of the equilibrium relaxation time $t_{\rm eq}(T)$,

$$\xi_{\rm eq}(T) \sim |\epsilon(T)|^{-\nu}, \quad t_{\rm eq}(T) \sim \xi_{\rm eq}^{z}(T) \sim |\epsilon(T)|^{-\nu z}, \quad (4.1)$$

where $\epsilon(T) \equiv (T - T_c)/T_c$ and $\nu$ is the critical exponent governing the divergence of the equilibrium correlation length ($\nu = 1$ for the two-dimensional Ising model).

For the analytic discussion alluded by KZ, it is essential to set $t_c = 0$ in Eq. (2.2) [hence the temperature ramp is given by $T(t) = T_c(1 - t/t_Q)$] which makes symmetric above and below $T_c$. The time $t$ now starts from $t = -t_Q$ and becomes $t = 0$ at the critical temperature and becomes $t = t_Q$ at the onset of the zero temperature. With a finite cooling rate, a new time scale $t_Q$ enters in the dynamics. When the equilibrium relaxation time changes more slowly than the external time change, that is, $|dt_{\rm eq}/dt| < 1$, the system will maintain its equilibrium during the temperature change, which would happen far from the critical temperature. Approaching $T_c$ since $t_{\rm eq}[T(t)]$ rapidly grows and tends to diverge, at some time the system would inevitably leave the above adiabatic regime and fall out of the equilibrium states. Therefore, this breaking time $|\hat{t}|$ ($-\hat{t}$ above $T_c$) would be demarcated by the relation $|dt_{\rm eq}/dt| \simeq 1$. This relation can be rewritten as

$$\left|\frac{dt_{\rm eq}}{dt}\right| = \left|\frac{dt_{\rm eq}}{dT}\right|\left|\frac{dT}{dt}\right| \simeq 1. \quad (4.2)$$

This immediately leads to

$$|\hat{t}|^{-(1+\nu z)} t_Q^{\nu z} \sim 1 \;\rightarrow\; |\hat{t}| \sim t_Q^{a}, \quad a \equiv \frac{\nu z}{1+\nu z}. \quad (4.3)$$

Thus, in the time region $t \leqslant -\hat{t}$ (recall that $t_c = 0$), the system well equilibrates and the cooling is adiabatic. But leaving the adiabatic regime and passing through $T_c$, the system would not equilibrate, and its NEQ states within the so-called impulse regime (inside the time region $[-\hat{t}, \hat{t}]$) are characterized by the shattered domain structures that is assumed to be almost *frozen* by KZ. KZ predicted the (excess) defect density for this frozen domain morphology. Since the domain structure is assumed to be frozen, it must be an *equilibrium* one at the time $-\hat{t}$, and, hence, the excess defect density $\delta n_d(\hat{t})$ would scale with $t_Q$ as

$$\delta n_d(\hat{t}) \sim \xi_{\rm eq}^{-1}[T(\hat{t})] \sim |\epsilon(\hat{t})|^{\nu} \sim t_Q^{-b}, \quad b \equiv \frac{\nu}{1+\nu z}. \quad (4.4)$$

For the present case, with $z \simeq 2.16(1)$ and $\nu = 1$, the above two exponents characterizing the KZ mechanism are given by $a \simeq 0.684(1)$ and $b \simeq 0.316(1)$ (note that $a = bz$). Equations (4.3) and (4.4) are the main predictions made by KZ [31,32].

The deeper nature of the dynamics within the KZ impulse regime and the dynamics in the crossover regime to the standard coarsening regime and, hence, the dynamics due to a possible interesting interplay between these two regimes remained largely unexplored (the work of BCS is an exception). We here present a simulation study on the NEQ kinetics for both the KZ impulse regime and the interplay regime.

### B. NEQ critical dynamics with the KZ mechanism

Shown in Fig. 2(a) is the relaxation of the defect density for various values of $t_Q$ (under the linear temperature ramp) on a double-logarithmic scale. The relaxation dynamics with finite $t_Q$ shows a striking contrast with that for the zero-temperature instantaneous quench [dotted line in Fig. 2(a)] in which the defect density exhibits a power-law relaxation $\delta n_d(t) = n_d(t) \sim t^{-\phi}$ with $\phi \simeq 0.5$ (in agreement with $\phi = 1/z_d$ with $z_d = 2$) almost immediately after the quench. Figure 2(a) shows that, with finite cooling rates, new slow kinetic regimes appear before crossing into the standard coarsening regime. As the cooling becomes slower ($t_Q$ going larger), the defect relaxation is getting slower. After passing through $T_c$ (at the time $t = t_Q$), the relaxation becomes rather rapid until around the zero temperature (at the time $t = 2t_Q$) and then slowly crosses over to the standard coarsening regime. Note that it takes two more decades of time from the onset time of zero temperature to reach the asymptotic coarsening regime.

A conspicuous feature in Fig. 2(a) is the defect relaxation at the onset times of zero temperature, which, over three decades of time, exhibits a power-law decay with the inverse cooling rate $t_Q$ as

$$n_d(2t_Q) \sim t_Q^{-1/z_0}, \quad 1/z_0 \simeq 0.475(2). \quad (4.5)$$

where $z_0$ is a new dynamic exponent characterizing the kinetics at the onset times of zero temperature. The interesting point here is that the power-law decay shows a small but clear distinction from the standard coarsening one $n_d(t) \sim t^{-1/2}$, that is, $z_0$ is *not* equal to $z_d$. But now we first focus on the KZ mechanism and the associated critical dynamics. After that, we will present a detailed discussion on this new zero-temperature dynamics.

The defect density $n_d(t)$ shown in Fig. 2(a) contains not only the defects generated via the KZ mechanism, but also the equilibrium defects present in the equilibrium states. Thus, in order to directly observe and test the KZ mechanism, it is important to isolate from the entire defect density only the generated defects when the system approaches and passes through $T_c$. A unique feature in the two-dimensional ferromagnetic Ising model is the existence of the Onsager's exact solution. This expression for the equilibrium defect density $n_d^{\rm eq}[T(t)]$ [given in (2.5) and (2.6)] enables us to obtain this excess defect density $\delta n_d[T(t)]$.

The relaxation of the excess defect density is shown in Fig. 2(b). From this figure, we find that the relaxation of the excess defect density reveals the NEQ dynamics envisioned by KZ. Right after the quench follows the adiabatic regime where no defects are generated. The span of this adiabatic regime is, of course, getting wider as the cooling becomes slower (larger $t_Q$). The impulse regime is characterized by the generation of proliferating defects. Leaving the adiabatic regime, the NEQ situation (created by the critical slowing down) starts to generate many defects, and passing $T_c$, the generated defect density becomes maximum at the time which we believe is indeed the breaking time $\hat{t}$. We, thus, see that the excess defect density naturally exhibits the adiabatic regime and nicely locates the breaking time $\hat{t}$ in the impulse regime. After passing the peak time, the defects are generated with much reduced rates, and passing the zero temperature the

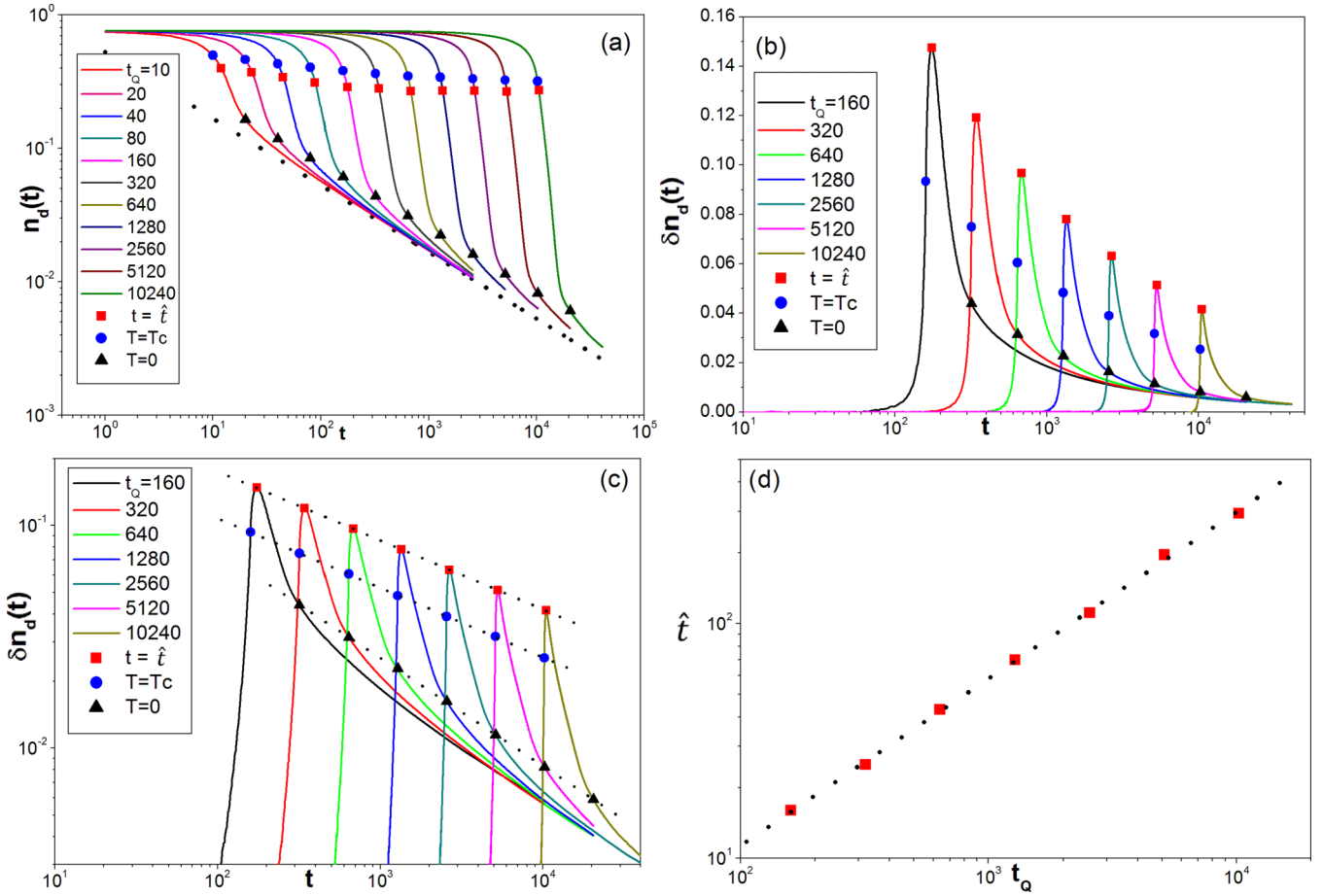


FIG. 2. (a) Relaxation of the defect density $n_d(t)$ vs $t$ for different cooling rates in a double-logarithmic plot. The inverse cooling rate $t_Q$ increases from left to right. The dotted line represents that for the instantaneous zero-temperature quench. In (a)–(c), the blue circles and black triangles, respectively, denote the locations of the critical and the (onset) zero temperatures for different cooling rates. (b) The *excess* defect density $\delta n_d(t)$ vs $t$ for different cooling rates in a semilogarithmic plot. Passing through $T_c$, the excess defect density manifests that the generation of defects increases and culminates at the peak times $\hat{t}$ (red squares). (c) The *excess* defect density $\delta n_d(t)$ vs $t$ for different cooling rates in a double-logarithmic plot. It shows that the excess defect density exhibits power-law relaxations at the critical temperature and at the peak times with the exponents in agreement with the KZ prediction $b = \nu/(1+\nu z) \simeq 0.316$. The onset zero-temperature relaxation shows another power-law decay with a new dynamic exponent (see the text). (d) The peak time $\hat{t}$ (shifted by $t_Q$) vs $t_Q$ in a double-logarithmic plot. It shows a power-law growth $\hat{t} \sim t_Q^{a_p}$ with $a_p \simeq 0.70(1)$, consistent with the KZ prediction $a = \nu z/(1+\nu z) \simeq 0.682$.

defect density slowly crosses over to the standard coarsening regime. Note that this crossover is a slow process: It takes about two more decades of time to enter the asymptotic coarsening regime.

Figure 2(c) shows that the excess defect density both at the critical time (temperature) and at the peak time exhibits power-law decay with $t_Q$ as

$$\delta n_d(t_Q) \sim t_Q^{-b_c}, \quad b_c \simeq 0.320(1),$$
$$\delta n_d(\hat{t}) \sim t_Q^{-b_p}, \quad b_p \simeq 0.350(1). \tag{4.6}$$

Clearly, this result is consistent with the KZ prediction $b = \nu/(1+\nu z) = 0.316$. The location of the peak time induces some numerical inaccuracy, leading to a small discrepancy with the KZ value. Figure 2(d) shows that the peak time $\hat{t}$ (*after* shifting by $t_Q$) increases with $t_Q$ in a power-law fashion,

$$\hat{t} \sim t_Q^{a_p}, \quad a_p \simeq 0.70(1). \tag{4.7}$$

This result is again consistent with the KZ prediction $a = \nu z/(1+\nu z) = 0.684$. The inaccuracy is thought to be related to the uncertainty of reading off the location of the peak time.

It is expected that the excess defect density $\delta n_d(\hat{t})$ would be inversely proportional to the correlation length $\xi(\hat{t})$ corresponding to the domain size $\delta n_d(\hat{t}) \sim \xi^{-1}(\hat{t})$. The latter can be directly obtained from the dynamic spin correlation function $C(r,\hat{t})$. However, since the same critical dynamic behavior with the KZ growth law (with $t_Q$) is expected to be observed for the spin correlation function at $T_c$ (the location of the critical temperature is free of numerical inaccuracy), we here measured the spin correlation function at $T_c$, $C(r,t_Q)$, which is shown in Fig. 3(a). Figure 3(b) shows that the correlation function at $T_c$ indeed satisfies a *critical* dynamic scaling (with $\eta = 1/4$),

$$C(r,t_Q) = r^{-\eta} F_c[r/\xi_c(t_Q)], \tag{4.8}$$

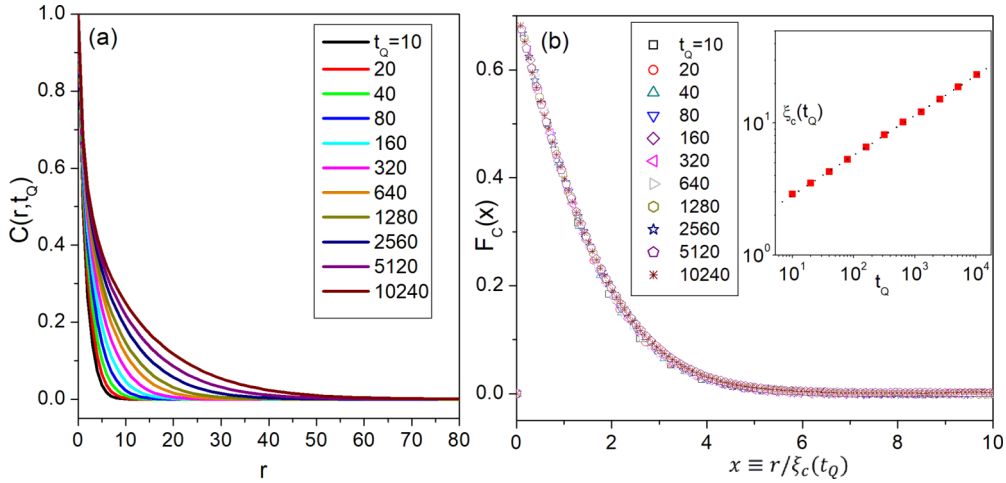


FIG. 3. (a) Dynamic spin correlation functions $C(r, t_Q)$ at the critical temperature for various values of $t_Q$. The inverse cooling rate $t_Q$ increases from left to right. (b) A critical dynamic scaling $C(r, t_Q) = r^{-\eta} F_c[r/\xi_c(t_Q)]$ (with $\eta = 1/4$) for $C(r, t_Q)$ shown in (a). The inset: the critical correlation length $\xi_c(t_Q)$ vs $t_Q$ in a double-logarithmic plot. It shows a power-law growth $\xi_c(t_Q) \sim t_Q^y$ with $y \simeq 0.316(1)$, in good agreement with the KZ exponent $b$.

where the correlation length $\xi_c(t_Q)$ is defined as $[r^\eta C(r, t_Q)]_{r=\xi_c(t_Q)} = 0.4$ for a given $t_Q$. It is observed that the correlation length $\xi_c(t_Q)$ exhibits a power-law scaling with $t_Q$ as $\xi_c(t_Q) \sim t_Q^y$ with $y \simeq 0.316(1)$ as shown in Fig. 3(c). Therefore, we find that this result is in good agreement with the KZ prediction Eq. (4.4) $\delta n_d(\hat{t}) \sim t_Q^{-b}$ with $b = \nu/(1 + \nu z) = 0.316$. We also note that the two scaling functions $F_c(x)$ and $f_c(x)$, the one for the instantaneous critical quench are the same, exhibiting an Ising universal behavior.

Our finding of the critical dynamic scaling for the spin correlation function at $T_c$ (with the KZ growth law for the correlation length) gives explicit evidence for the argument of BCS [14] that the impulse regime is, instead of being frozen, the one where the system undergoes a critical coarsening process.

### C. Scaling behavior for the defect dynamics

We now discuss the scaling behavior for the dynamics of the excess defect densities for different cooling rates shown in Fig. 2(b). It seems quite natural to make a scaling attempt by collapsing the different peak times on the single point. We, thus, propose

$$\delta n_d(t) = \delta n_d(\hat{t}) \mathcal{F}(t/\hat{t}) \sim t_Q^{-b} \mathcal{F}\left(t/t_Q^a\right), \tag{4.9}$$

where by definition $\mathcal{F}(1) = 1$.

The scaling attempt following Eq. (4.9) is shown in Fig. 4. The proposed scaling scheme is found to yield an excellent scaling collapse for both the adiabatic and the impulse regimes, and for the final coarsening regime as well. But the scaling is found to be violated in the crossover regime. And the scaling function $\mathcal{F}(x)$ takes the limiting forms $\mathcal{F}(x \ll 0) = 0$ and $\mathcal{F}(x \gg 1) \sim x^{-1/z_d}$. The former represents the adiabatic regime, whereas the latter represents the standard coarsening regime.

We mention that the above proposed scaling seems to be qualitatively different from the one proposed by BCS [14]. For one, BCS proposes their scaling ansatz in terms of the equilibrium correlation length $\xi_{\rm eq}[T(t)]$ and equilibrium relaxation time $\tau_{\rm eq}[T(t)]$. One consequence of the BCS scaling is that it predicts the defect relaxation at the onset times of the zero temperature (applied for the two-dimensional Ising model) as $n_d(t_Q) \sim t_Q^{-1/z_d} \sim t_Q^{-1/2}$. On the other hand, our scaling ansatz is based on a scaling of the excess defect density $\delta n_d(t)$ in terms of the peak time $\sim\hat{t}$, where $\hat{t}$ follows the KZ scaling $\hat{t} \sim t_Q^a$. This results in a different scaling exponent for the excess defect density at the onset times of the zero temperature as $n_d(t_Q) \sim t_Q^{-1/z_0}$ where the exponent $1/z_0$ is smaller than $1/z_d$ due to the small difference between $z_d$ and $z$. This will be explained in detail in the next subsection. Moreover, in contrast to the work of BCS, our scaling analysis clearly shows a crossover regime where the scaling is violated.

### D. Dynamics at the onset of zero temperature

We now consider an interesting growth kinetics observed at the onset times $t = 2\,t_Q$ where the system starts to enter the zero temperature. We mentioned earlier that, as observed in Fig. 1, the defect relaxation at zero temperature as a function of $t_Q$ is governed by a new dynamic exponent $z_0$ which is distinct from that in the standard coarsening regime ($z_d = 2$), i.e., $n_d(2t_Q) \sim t_Q^{-1/z_0}$ with $1/z_0 \simeq 0.475(2)$. In order to corroborate this result, we also looked at the dynamic spin correlations at zero temperature, which is shown in Fig. 5(a). As expected, $C(r, 2t_Q)$ exhibits a dynamic scaling [Fig. 5(b)],

$$C(r, 2t_Q) = F_0[r/\xi_0(2t_Q)], \tag{4.10}$$

where the correlation length $\xi_0(2t_Q)$ shows a power-law growth $\xi_0(2t_Q) \sim t_Q^{1/z_0}$ with $1/z_0 \simeq 0.475(2)$ [see the inset of Fig. 5(b)], consistent with an expected result $n_d(2t_Q) \sim \xi_0^{-1}(2t_Q)$. The form of the scaling function $F_0(x)$ is found to be the same as that for the case of the instantaneous quench.

We now show that the dynamic scaling for the defect relaxation, Eq. (4.9), allows us to obtain an analytic expression for the new exponent $1/z_0$ characterizing the zero-temperature

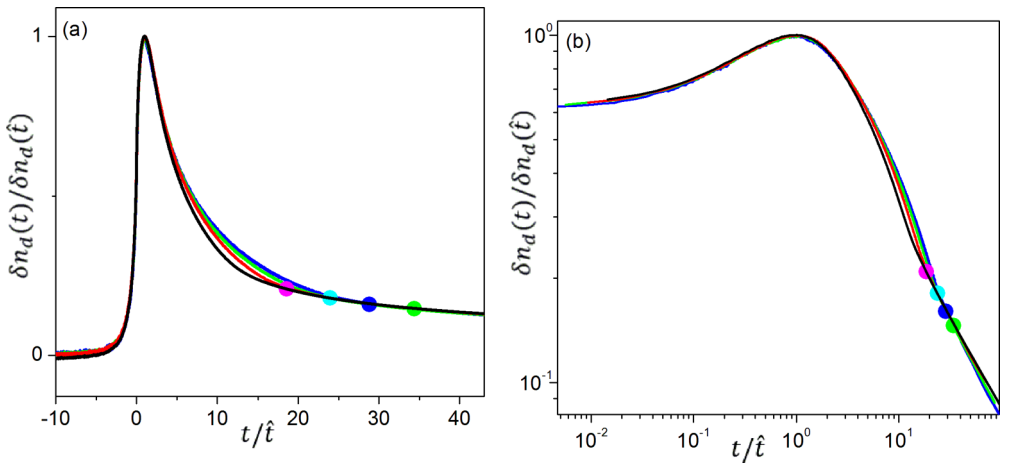


FIG. 4. (a) Scaling plot for the excess defect density, $\delta n_d(t)/\delta n_d(\hat{t})$ vs $t/\hat{t}$. (b) Scaling plot for the excess defect density $\delta n_d(t)/\delta n_d(\hat{t})$ vs $t/\hat{t}$ in a double-logarithmic plot. The filled circles represent the onset times of zero temperatures for $t_Q = 1280, 2560, 5120, 10\,240$ (MCS), respectively (from left to right).

kinetics. Note from Eq. (4.9) that

$$\delta n_d(t_Q) \sim t_Q^{-b}\mathcal{F}\left(t_Q/t_Q^a\right) \sim t_Q^{-b}\left(t_Q^{1-a}\right)^{-1/z_d}. \tag{4.11}$$

Since we observe $\delta n_d(2t_Q) \sim t_Q^{-1/z_0}$, Eq. (4.11) gives us the relation,

$$\frac{1}{z_0} = b + \frac{1-a}{z_d} \rightarrow \frac{1}{z_0} = \frac{1}{z_d}\frac{1+\nu z_d}{1+\nu z}. \tag{4.12}$$

With the values of $\nu = 1$, $z_d = 2$, and $z \simeq 2.16(1)$, this relation gives $1/z_0 \simeq 0.475(2)$, which is in good agreement with the simulation result. Note that the above expression for the exponent $z_0$ involves both the KZ exponents and the dynamic exponent $z_d$ for the standard coarsening process. We see that the influence of the KZ mechanism is present even in the growth kinetics observed at the onset times of the zero temperature. Moreover, the appearance of the exponent of the coarsening dynamics in Eq. (4.12) seems to indicate that there exists an interesting interplay between the KZ mechanism and the standard coarsening kinetics.

## V. CRITICAL HEATING

We here consider the case of the *critical heating* from $T = 0$ to $T = T_c$ with the linear ramp $T(t \leqslant t_Q) = T_c\, t/t_Q$ [once $T_c$ is reached, then the temperature remains critical, i.e., $T(t \geqslant t_Q) = T_c$] where the magnetization relaxation reveals the above unique critical coarsening behavior with the KZ mechanism. In this type of reverse quench as in the

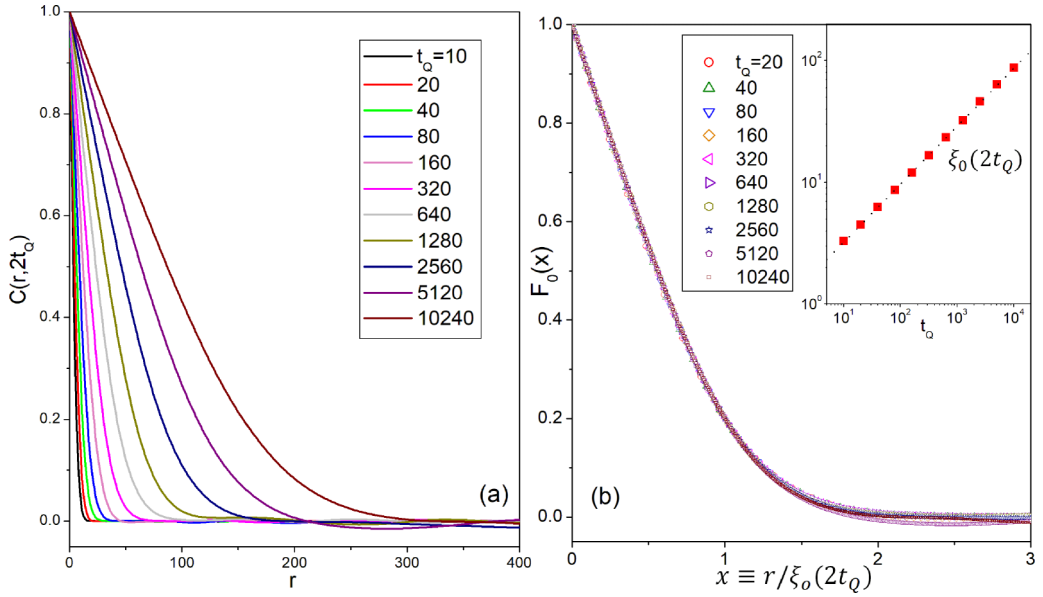


FIG. 5. (a) Dynamic spin correlation functions $C(r, 2t_Q)$ at zero temperature for various cooling rates. The inverse cooling rate $t_Q$ increases from left to right. (b) A dynamic scaling $C(r, 2t_Q) = F_c[r/\xi_0(2t_Q)]$ for the spin correlation function at zero temperature. Here, the zero-temperature correlation length $\xi_0$ is defined as $C(\xi_0, 2t_Q) = 0.2$ for a given $t_Q$. The inset: the zero-temperature correlation length $\xi_0(2t_Q)$ vs $t_Q$ in a double-logarithmic plot. It shows a power-law growth $\xi_0(2t_Q) \sim t^{1/z_0}$ with $1/z_0 \simeq 0.475(1)$.

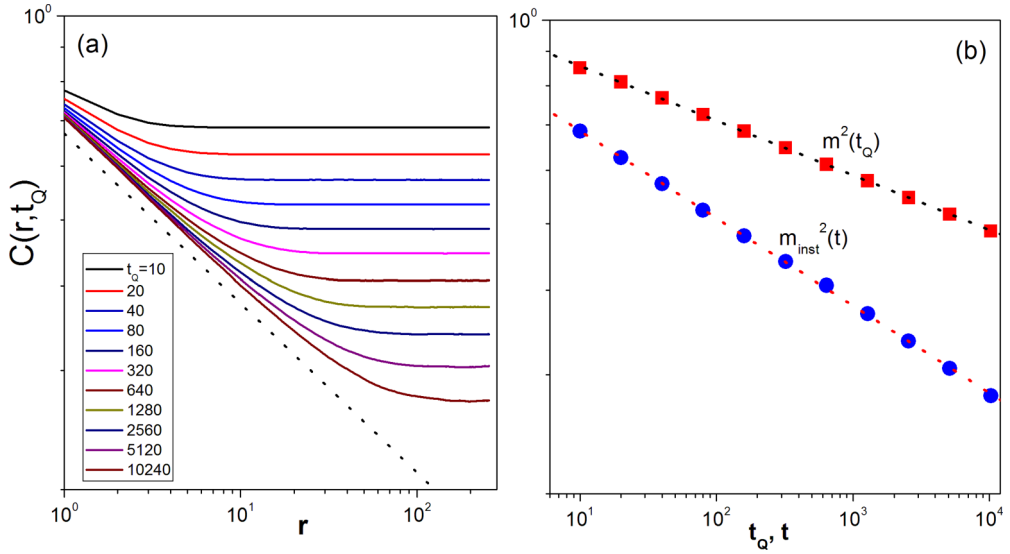


FIG. 6. (a) Dynamic spin correlation functions $C(r, t_Q)$ for the critical heating from $T = 0$ to $T = T_c$ for various finite rates in a double-logarithmic plot. The inverse cooling rate $t_Q$ increases from top to bottom. It reveals the equilibrium critical decay $r^{-\eta}$ with $\eta = 1/4$ (dotted line) and the NEQ relaxation of the magnetization $m^2(t_Q) = C(r \to \infty, t_Q)$ for different rates. (b) Magnetization relaxation $m^2(t_Q)$ vs $t_Q$ for (a). It shows a power-law decay $m^2(t_Q) \sim t_Q^{-b\eta}$ with $b\eta \simeq 0.0794$ (red square), which is to be contrasted with that for the instantaneous critical heating $m^2_{\rm inst}(t) \sim t^{-\eta/z}$ with $\eta/z \simeq 0.116$ (blue circle).

instantaneous quench, the zero-temperature perfect order is getting destroyed, and, hence, the magnetization $m(t_Q)$ (at the onset times of the critical temperature) will exhibit a power-law decay with $t_Q$. This power-law decay of the order parameter can be characterized by a critical dynamic scaling. For this, the dynamic spin correlation functions $C(r, t_Q)$ are measured as the function of the distance $r$ for different values of $t_Q$, which are shown in Fig. 6(a). As seen in this figure, at long distances, the spins become uncorrelated, and hence, we have $C(r \to \infty, t_Q) = m^2(t_Q)$. A critical dynamic scaling is expected to hold for the spin correlation function,

$$C(r, t_Q) = r^{-\eta} G_c[r/\xi_c(t_Q)]. \tag{5.1}$$

The scaling function $G_c(x)$ would behave as $G_c(x) = \text{const.}$ for $x \ll 1$ since the correlation should recover the equilibrium critical behavior $C_{\rm eq}(r) \sim r^{-\eta}$ as $t_Q \to \infty$. Also, since $C(r, t_Q)$ becomes independent of $r$ for long distances, $G_c(x) \sim x^{\eta}$ for $x \gg 1$. This scaling behavior leads to

$$m^2(t_Q) \sim r^{-\eta}[r/\xi_c(t_Q)]^{\eta} \sim \xi_c^{-\eta}(t_Q). \tag{5.2}$$

Recall that, for the instantaneous reverse quench (from $T = 0$ to $T = T_c$),

$m^2_{\rm inst}(t) \sim \xi_c^{-\eta}(t) \sim t^{-\eta/z}$ with the exponent being $\eta/z \simeq 0.115$. In contrast, for finite quench rates, as shown in Fig. 6(b), $m^2(t_Q)$ extracted from the spin correlation functions is observed to obey the power-law relaxation,

$$m^2(t_Q) \sim t_Q^{-x}, \quad x \simeq 0.079(4). \tag{5.3}$$

The measured value of the exponent $x \simeq 0.079(4)$ is clearly distinct from the value of $\eta/z \simeq 0.115$ observed for the instantaneous quench. The underlying physical explanation for the observed numerical value of $x \simeq 0.079(4)$ is to resort to the KZ mechanism. That is, the correlation length $\xi_c(t_Q)$ should exhibit a power-law growth in accord with the KZ mechanism,

$$m^2(t_Q) \sim \xi_c^{-\eta}(t_Q) \sim t_Q^{-b\eta}, \quad b\eta = \nu\eta/(1+\nu z) \simeq 0.0794. \tag{5.4}$$

All these discussions amply demonstrate that, as first asserted in the work of BCS, the KZ's impulse regime is actually the one where the system is undergoing a *critical coarsening* process with the correlation length being power-law growing in time with the characteristic KZ exponent $b = \nu/(1+\nu z)$.

## VI. SUMMARY AND CONCLUSION

In the present paper, via spin-flip kinetic Monte Carlo simulations, we investigated the NEQ growth kinetics of the ferromagnetic Ising model on a square lattice, ensued after slow quench under the linear cooling protocol. We focus on the entire time evolution encompassing the standard coarsening process not just on the KZ impulse regime.

In agreement with the assertion by BCS, we demonstrated that, instead of being viewed as frozen domain morphology, the KZ impulse regime is characterized by the critical coarsening kinetics where the spin correlation function obeys a critical dynamic scaling of the form $C(r, t) = r^{-\eta} F_c[r/\xi_c(t)]$ with $t = \hat{t}$ and $t = t_Q$. The interesting feature here is that the spin correlation length $\xi(t)$ exhibits a unique power-law growth (with $t_Q$, the inverse cooling rates) $\xi(t) \sim t_Q^b$ with the growth exponent dictated by the KZ mechanism, i.e., $b = \nu/(1+\nu z)$. This critical behavior is in striking contrast with the one for the instantaneous critical quench $\xi(t) \sim t^{1/z}$. The excess defect density exhibits the corresponding power-law relaxation $\delta n_d(t) \sim \xi^{-1}(t_Q) \sim t_Q^{-b}$. The same unique critical dynamic behavior is also at work in the relaxation of the order parameter [the magnetization $m(t_Q)$] for the slow critical heating from $T = 0$ to $T_c$ via the linear temperature ramp. The magnetization relaxation manifests the KZ mechanism

as it shows a power-law relaxation $m^2(t_Q) \sim \xi^{-\eta}(t_Q) \sim t_Q^{-b\eta}$, which is again distinct from $m^2_{\rm inst}(t) \sim t^{-\eta/z}$ for the instantaneous critical heating (from $T = 0$ to $T = T_c$).

The zero temperature demarcates the early stage of the crossover regime (into the standard coarsening one). The defect relaxation at zero temperatures (for different cooling rates) manifests a new dynamic exponent $z_0$, i.e., $n_d(2t_Q) \sim t_Q^{-1/z_0}$ with $z_0$ distinct from $z_d$, the dynamic exponent for instantaneous zero-temperature quench. We presented a simple derivation (from the scaling behavior for the defect dynamics) for the relation between the two dynamic exponents $z_0 = z_d(1 + \nu z)/(1 + \nu z_d)$, giving $z_0 \simeq 2.10$. Hence, the domain growth becomes a subdiffusive one. This new zero-temperature dynamics appears to result from an interesting interplay between the KZ mechanism and the standard coarsening kinetics. The distinct zero-temperature dynamics (and, hence, the interplay between the impulse and the standard coarsening regimes) may well be a universal phenomenon that can be found in other classical and quantum systems with continuous phase transitions. It would, thus, be interesting to look into the nature of the zero-temperature dynamics in various systems, classical as well as quantum.

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