Emergence of correlations in the process of thermalization of interacting bosons

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We address the question of the relevance of thermalization and scrambling to the increase of correlations in the quench dynamics of an isolated system with a finite number of interacting bosons. Specifically, we study how, in the process of thermalization, the correlations between occupation numbers increase in time, resulting in the emergence of the Bose-Einstein distribution. Despite the exponential increase of the number of principal components of the wave function, we show, both analytically and numerically, that the two-point correlation function before saturation increases quadratically in time according to perturbation theory. In contrast, we find that the out-of-time-order correlator increases algebraically and not exponentially in time after the perturbative regime and before the saturation. Our results can be confirmed experimentally in traps with interacting bosons and they may be relevant to the problem of black hole scrambling.

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I. INTRODUCTION

In recent years the problem of thermalization in closed systems of interacting fermions and bosons has attracted much attention (see, for example, Refs. [1–7]). An increase of interest in this problem is due to remarkable experimental achievements [8–12] and various theoretical predictions [13–20]. Although the term *thermalization* is not uniquely defined, it is widely used in many-body physics. One of the basic statistical properties of many-body systems is either the Bose-Einstein (BE) or the Fermi-Dirac (FD) distribution that emerges in the thermodynamic limit due to the combinatorics and without interparticle interaction. As for finite isolated systems, the mechanism for the onset of BE and FD distributions is the chaotic structure of many-body eigenstates [5–7,13–15,20–24]. In this case, the interaction between particles plays a crucial role: The fewer particles there are, the stronger the interparticle interaction has to be for the emergence of the statistical properties.

To date, it is understood that the validity of statistical mechanics can be justified not only by averaging over a number of eigenstates with close energies, but also with the use of a single eigenstate if the latter consists of many uncorrelated components in the physically chosen basis. Specifically, it was shown that BE and FD distributions emerge also on the level of individual eigenstates if they are strongly chaotic [21,23–25]. Moreover, for a finite number of particles, the temperature entering these distributions has to be corrected by taking into account the interparticle interaction. This is at variance with the standard derivation obtained for noninteracting particles in the thermodynamic limit. Recent experiments with cold atoms have supported the emergence of the BE distribution in the presence of a strong interaction [26]. The most intriguing point of these studies [13–15,21,23–25] is that both distributions appear even if the number of particles is small; this happens due to the fast growth of the number of components in many-body eigenstates as a function of the number of particles.

Here we address a problem concerning the onset of the BE distribution in the time evolution of a system with few interacting bosons. Our specific interest is to study how the conventional BE distribution emerges *in time* and how this fact is related to the somewhat different problem of the increase of correlations in the process of relaxation of a system to a steady-state distribution. The latter problem is now a hot topic in the literature in view of various applications, such as the evolution of systems with cold atoms, as well as in application to the problem of scrambling in black holes (see [27,28] and references therein).

In our study we consider the quench dynamics described by the Hamiltonian $H = H_0 + V$, where H_0 corresponds to the noninteracting bosons and the interaction is fully embedded in V belonging to the ensemble of matrices with two-body random interaction (TBRI). In this way, by exciting initially a single many-body state of H_0, we explore the evolution of wave packets in the many-body Hilbert space. Recently, it was discovered that for the model parameters for which the many-body eigenstates of H are strongly chaotic, the effective number of components N_{pc} in the wave function increases exponentially in time, before the saturation which is due to the finite many-body space [29]. This time dependence was explained with the use of a phenomenological model that allowed one to obtain simple analytical expressions for the rate of exponential increase of N_{pc} and for its saturation value.

Below, in connection with the results reported in [23,24,29], we show, both analytically and numerically, that the onset of the BE distribution in the TBRI matrix model

occurs on the timescale on which the number of components in many-body eigenstates increases exponentially in time. In order to quantify the onset of the BE distribution, we have studied the correlations between occupation numbers by exploring both the two- and four-point correlators. The latter is just the well known out-of-time-order correlator (OTOC) widely discussed in the literature [30–32]. Specifically, it was predicted that for strongly chaotic systems the OTOC should manifest an exponential time dependence before saturation. One of our main findings is that actually both correlators increase algebraically in time and not exponentially. This result is quite unexpected, as compared with the exponential increase of the number of principal components N_{pc} in the wave packet. Our analytical results are fully confirmed by extensive numerical data.

II. MODEL

The system consists of N identical bosons occupying M single-particle levels specified by random energies ϵ_s with mean spacing $\langle \epsilon_s - \epsilon_{s-1} \rangle = 1$. The Hamiltonian reads ($\hbar = 1$)

$$H = H_0 + V = \sum \epsilon_s a_s^\dagger a_s + \sum V_{s_1 s_2 s_3 s_4} a_{s_1}^\dagger a_{s_2}^\dagger a_{s_3} a_{s_4}. \quad (1)$$

In Eq. (1) a_s and $a_s^\dagger$ are, respectively, the usual annihilation and creation operators in the single-particle energy level s. The two-body matrix elements $V_{s_1 s_2 s_3 s_4}$ are random Gaussian entries with zero mean and variance V^2. The dimension of the Hilbert space generated by the many-particle basis states is $N_H = (N + M - 1)!/N!(M - 1)!$.

Below we consider $N = 6$ particles in $M = 11$ levels (dilute limit $N < M$) for which $N_H = 8008$. We choose a considerably "small" Hilbert space since fourth-order time correlations are quite consuming from a computational point of view. Let us also note that increasing only one unit, our data, e.g., taking $N = 7$ particles in $M = 12$ single-particle levels, imply a Hilbert space of dimension $N_H = 31\,824$. Since the algorithms for exact diagonalizations are characterized by a CPU time proportional to N_H^3, this implies a CPU time roughly 60 times larger than that for the case we consider here without changing significantly the physics.

Two-body random matrices (1) were introduced in [33–35] and extensively studied for fermions and bosons (see, for example, [13–15,36–38]). These studies can be considered as paradigmatic also for the case of competing couplings of single-particle levels. Examples of these are the hopping two-body interacting Bose-Hubbard models which have been experimentally implemented [39–41].

The eigenstates $|\alpha\rangle = \sum_k C_k^{(\alpha)} |k\rangle$ of H can be written in terms of the basis states (unperturbed many-body states) $|k\rangle = a^\dagger{}_{k_1} \cdots a^\dagger{}_{k_N} |0\rangle$ of H_0, where

$$H|\alpha\rangle = E^\alpha |\alpha\rangle, \quad H_0 |k\rangle = E_k^0 |k\rangle. \quad (2)$$

An eigenstate $|\alpha\rangle$ of the total Hamiltonian is called chaotic when the number N_{pc} of principal components defined, for instance, via the inverse participation ratio

$$N_{pc} = \frac{1}{\mathrm{IPR}}, \quad \text{with} \quad \mathrm{IPR} = \sum_k \left| C_k^\alpha \right|^4, \quad (3)$$

is sufficiently large, i.e., $\sqrt{N_{pc}} \gg 1$, and the numbers C_k^α can be considered random and noncorrelated (see, for instance, [5–7] and references therein). Note that since the system is isolated and the perturbation V is finite, the eigenstates can fill only a part of the unperturbed basis [5–7] determined by the perturbation V. Specifically, the energy region which is occupied by the eigenstates is restricted by the width of the so-called energy shell [42,43]. The partial filling of the energy shell by an eigenstate can be associated with the many-body localization in the energy representation. In contrast, when an eigenstate fills completely the energy shell the BE distribution emerges on the level of individual eigenstates [21,23–25]. This happens when the perturbation V is sufficiently strong to provide quantum chaos. A detailed discussion of the strength of perturbation necessary to have chaotic eigenstates and thermalization in relation to the system parameters can be found in [23,24]. In what follows we will consider the situation when the latter condition is fulfilled.

III. DYNAMICS IN THE MANY-BODY HILBERT SPACE

In contrast with previous studies [23,24] focused on the thermal properties of individual many-body eigenstates, here we consider the dynamics of the model (1) by exploring two different timescales, before and after the relaxation to a steady state. Specifically, we study the quench dynamics starting from a single many-body state $|k_0\rangle$ of the unperturbed Hamiltonian H_0, after switching on the perturbation V. Given the evolved wave function

$$|\psi(t)\rangle = e^{-iHt} |k_0\rangle,$$

one can express the probability

$$P_k(t) = |\langle k|\psi(t)\rangle|^2$$

to find the system at time t in any unperturbed state $|k\rangle$ as

$$P_k(t) = \sum_{\alpha,\beta} C_{k_0}^{\alpha *} C_k^\alpha C_{k_0}^\beta C_k^{\beta *} e^{-i(E^\beta - E^\alpha)t} \equiv P_{k,k_0}^d + P_{k,k_0}^f(t), \quad (4)$$

where $P_{k,k_0}^d = \sum_\alpha |C_{k_0}^\alpha|^2 |C_k^\alpha|^2$ and $P_{k,k_0}^f(t)$ are the time-independent and time-fluctuating parts, respectively. With this expression, one can analyze the inverse participation ratio

$$\mathrm{IPR} = \sum_k |\langle k|\psi(t)\rangle|^4 = \sum_k \left[P_{k,k_0}^d + P_{k,k_0}^f(t) \right]^2, \quad (5)$$

from which we can extract the number of principal components in the state $|\psi(t)\rangle$ from the relation $N_{pc} = 1/\mathrm{IPR}$.

Let us now define the infinite-time average of an observable $A(t)$ as

$$\overline{A} = \lim_{T\to\infty} \frac{1}{T} \int_0^T dt\, A(t). \quad (6)$$

It is clear that, since in our case we have a nondegenerate spectrum, then $\overline{P_{k,k_0}^f(t)} = 0$ so that

$$\overline{|\langle k|\psi(t)\rangle|^2} = \sum_\alpha \left| C_{k_0}^\alpha \right|^2 \left| C_k^\alpha \right|^2 = P_{k,k_0}^d. \quad (7)$$

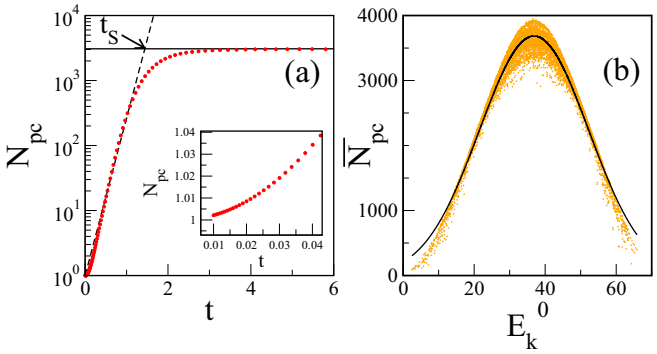

FIG. 1. (a) Number $N_{pc}(t)$ of principal components in time (red circles). The dashed line is the exponential growth with the rate 2Γ, where $\Gamma \approx 2.8$ is the width of the LDOS found numerically from the decay of survival probability (for details see [29]). The horizontal line is the estimate (8). The intersection between the infinite-time average $\overline{N_{pc}}$ and the analytical exponential growth $e^{2\Gamma t}$ defines the saturation time t_S (indicated by an arrow). The inset shows the initial quadratic dependence $N_{pc}(t) \propto 1 + \alpha t^2$. Initially all bosons are placed on the fifth single-particle level so that $|\psi_0\rangle = |k_0\rangle = |00006000000\rangle$. (b) Orange dots represent the infinite-time average number of principal components as a function of the energy E_k^0 of the initial many-body state. The black line is a Gaussian fit. Here $N = 6$, $M = 11$, and $V = 0.4$.

The infinite-time average for the number of principal components can be computed as

$$\overline{\mathrm{IPR}} = \sum_k \left(P^d_{k,k_0}\right)^2 + \overline{\left[P^f_{k,k_0}(t)\right]^2}. \tag{8}$$

The second term on the right-hand side of Eq. (8) can be computed exactly,

$$\overline{\left[P^f_{k,k_0}(t)\right]^2} = \left(P^d_{k,k_0}\right)^2 - \sum_\alpha \left|C^\alpha_{k_0}\right|^4 \left|C^\alpha_k\right|^4, \tag{9}$$

so that

$$\overline{N}_{pc} = 1/\overline{\mathrm{IPR}} = \left[2\sum_k \left(P^d_{k,k_0}\right)^2 - \sum_\alpha \left|C^\alpha_{k_0}\right|^4 \sum_k \left|C^\alpha_k\right|^4\right]^{-1}. \tag{10}$$

This expression determines the asymptotic value reached by $N_{pc}(t)$ after relaxation. It is shown in Fig. 1(a) as a horizontal line. In the same figure we can identify three different regimes: a perturbative one for short time $t \ll 1/\Gamma$ where $N_{pc}(t)$ grows quadratically [see the inset in Fig. 1(a)], a second one characterized by the exponential growth $N_{pc}(t) \simeq \exp(2\Gamma t)$ for $1/\Gamma \lesssim t \lesssim t_S = N/\Gamma$, and a third one (saturation after relaxation) where $N_{pc}(t) \simeq \overline{N}_{pc}$ for $t > N/\Gamma$ (for details see [29]).

Additional important information, which will be used later, is how the stationary value $\overline{N}_{pc}$ depends on the initial state. In Fig. 1(b) we show $\overline{N}_{pc}$ as a function of the unperturbed energy E_k^0 of the initial many-body state $|k_0\rangle$. As one can see, it is quite well approximated (excluding the tails) by a Gaussian shape (see the black solid line).

As is shown in [29], the number $N_{pc}(t)$ of principal components in the wave packets increases *exponentially fast* in time,

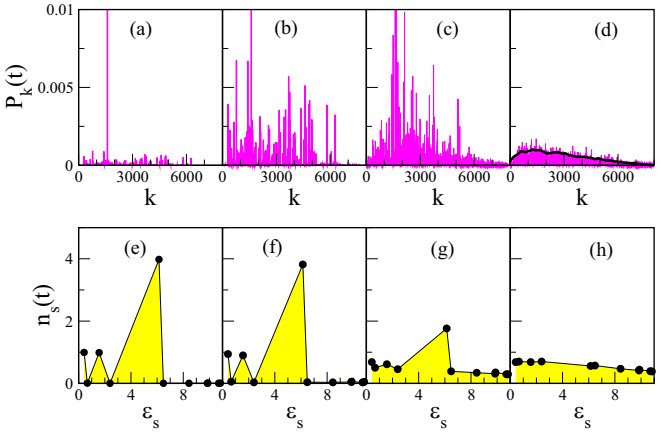

FIG. 2. (a)–(d) Probability $P_k(t)$ at different times t in the unperturbed basis $|k\rangle$: (a) $t = 0.025$, (b) $t = 0.075$, (c) $t = 0.5$, and (d) $t = 5$. (e)–(h) Plot of $n_s(t)$ versus single-particle energies ϵ_s at (e) $t = 0.025$, (f) $t = 0.075$, (g) $t = 0.5$, and (h) $t = 5$. In (d) the envelope of the stationary distribution is shown by the black curve. The initial state is $\Psi_0 = |10104000000\rangle$, where integer numbers are numbers of bosons occupying the s level. The dynamics is shown for $N = 6$, $M = 11$, and $V = 0.4$. For this value of V the eigenstates are strongly chaotic [23,24].

$N_{pc}(t) \sim \exp(2\Gamma t)$ up to some saturation time t_s. The rate of the exponential growth is defined by the width Γ of the local density of states (LDOS),

$$F_{k_0}(E) = \sum_\alpha \left|C^\alpha_{k_0}\right|^2 \delta(E - E^\alpha),$$

obtained by projecting the initial state $|k_0\rangle$ onto the exact eigenstates. In nuclear physics this function is known as the strength function and it describes the relaxation of excited heavy nuclei [44]. Concerning the saturation time t_s, it was found [29] to be proportional to the number of particles, $t_s \approx N/\Gamma$. This time should be treated as the time after which one can speak of a complete thermalization occurring in a system.

IV. ONSET OF THE BOSE-EINSTEIN DISTRIBUTION

The time-dependent occupation number distribution (OND) is defined as

$$n_s(t) = \langle\psi(t)|\hat{n}_s|\psi(t)\rangle = \sum_k n^k_s |\langle k|\psi(t)\rangle|^2, \tag{11}$$

which gives the average number of particles in the single-particle energy level ϵ_s at the time t. Here we take into account that $\langle k|\hat{n}_s|k'\rangle = n^k_s\,\delta_{k,k'}$, where $n^k_s = 0, \ldots, N$. The evolution of $n_s(t)$ in comparison with the wave-packet dynamics $P_k(t)$ is shown in Figs. 2(e)–2(h). These figures demonstrate that when the packet fully occupies the energy shell, the occupation numbers are relaxed to the steady-state distribution. We can estimate analytically both the short- and the long-time behaviors.

Concerning the former, expanding $e^{-iHt} \simeq 1 - iHt + \cdots$, one gets the time dependence for $n_s(t)$ at short times

$$|\langle k|e^{-iHt}|k_0\rangle|^2 \simeq \delta_{k,k_0} + t^2\left[H^2_{k,k_0} - \delta_{k_0,k_0}(H^2)_{k,k_0}\right] + o(t^4), \tag{12}$$

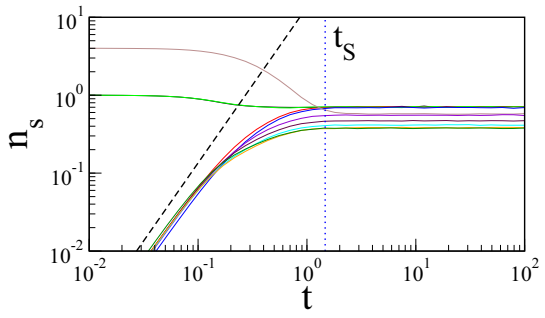

FIG. 3. Evolution of the averaged $n_s(t)$ for all $s = 1, \ldots, M$. The dashed line is the predicted t^2 behavior (13) characteristic of the perturbative regime when $n_s^{k_0} = 0$. The initial state is $|\Psi_0\rangle = |10104000000\rangle$. Here $N = 6$, $M = 11$, and $V = 0.4$ as in Fig. 2. An average over ten realizations of the random potential has been used. The vertical dotted lines indicate the saturation time t_S.

which results in the estimate

$$n_s(t) \simeq n_s^{k_0} + t^2 \sum_{k \neq k_0} \left(n_s^{k_0} - n_s^k\right) H_{k,k_0}^2 + o(t^4). \tag{13}$$

One can see in Fig. 3 that for single-particle s levels which are not initially occupied by particles, $n_s(t)$ grows quadratically in time. As for the saturation values $\overline{n_s}$ after the relaxation time t_s, they can be also obtained analytically by performing an infinite-time average

$$\overline{n_s} = \sum_k n_s^k \overline{|\langle k|\psi(t)\rangle|^2} = \sum_k n_s^k P_{k,k_0}^d. \tag{14}$$

In order to claim that after relaxation the OND is statistically described by a BE distribution, one has to be sure that the fluctuations of n_s follow the standard requirements of statistical mechanics.

In view of this very point, we have thoroughly analyzed both temporal fluctuations and "quantum" fluctuations. Concerning the former, they are defined as

$$\Delta n_s^2 = \overline{\langle n_s(t)\rangle^2} - \overline{\langle n_s(t)\rangle}^2 \tag{15}$$

and represent the temporal fluctuations of the classical variable $\langle n_s(t)\rangle$. According to statistical mechanics, (a) the fluctuations have to be small compared to the mean value $\langle n_s\rangle$ and (b) the fluctuations should be Gaussian. Let us first analyze the distribution of the different average values for one single fixed s value. The distributions $P(n_s)$ obtained from the stationary distribution are shown in Fig. 4(a) (for two values of s: $s = 1$ and $s = M$). As one can see, there is very good agreement with a Gaussian fit. The width of these distributions (as given by the second moment of the fitted Gaussians Δn_s^2) weakly depends on the particular chosen s value [see Fig. 4(a)], while the dependence on the initial state is stronger. To this end we compute the relative fluctuations $\Delta n_s/n_s$, choosing as initial states different unperturbed many-body basis states from the whole energy spectrum. In agreement with the results found for Fermi particles [15], we consider in Fig. 4(b) the relative fluctuations $\Delta n_s/n_s$ as a function of the number of principal components of the stationary wave packet (after relaxation) for the correspondent initial states [essentially what is shown in Fig. 1(b)]. As one can see, there is very good agreement

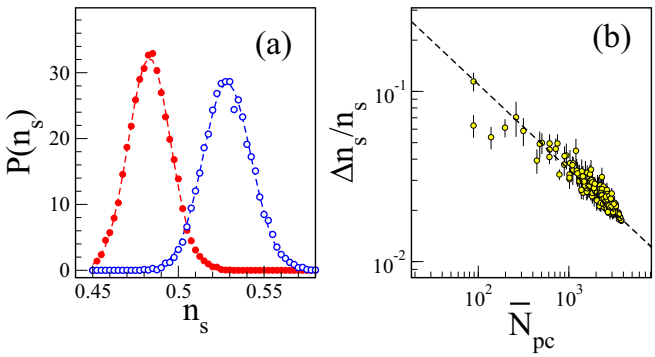

FIG. 4. (a) Probability distribution $P(n_s)$ for two different s values: $s = 1$ (closed red symbols) and $s = 11$ (open blue symbols). Dashed lines represent fits with Gaussian distributions. (b) Relative time fluctuations $\Delta n_s/n_s$ as a function of the correspondent number of principal components $\overline{N}_{pc}$ obtained from the stationary distribution. The dashed line is $1/\sqrt{\overline{N}_{pc}}$. The parameters are $N = 6$, $M = 11$, and $V = 0.4$.

with the dependence $1/\sqrt{\overline{N}_{pc}}$, which is a strong result in view of the requirement of statistical mechanics. Let us stress that the decrease of relative fluctuations occurs not with respect to the number N of particles, but with the number of principal components contained in the stationary distribution. This remarkable result shows that for chaotic systems with few interacting particles the number of principal components in the stationary distribution $\overline{N}_{pc}$ plays the same role as the number of particles N in ordinary statistical mechanics.

An even more interesting point concerns quantum fluctuations. In the grand-canonical ensemble, for *noninteracting* bosons, relative quantum fluctuations satisfy the following relation [45]:

$$\left(\frac{\delta n_s}{n_s}\right)^2 = 1 + \frac{1}{n_s}. \tag{16}$$

In order to compute quantum fluctuations for our finite system in equilibrium, we first perform the infinite-time average for the first two moments,

$$\begin{aligned} \overline{n_s} &= \sum_k n_s^k \overline{|\langle k|\psi(t)\rangle|^2} = \sum_k n_s^k P_{k,k_0}^d, \\ \overline{n_s^2} &= \sum_k \left(n_s^k\right)^2 \overline{|\langle k|\psi(t)\rangle|^2} = \sum_k \left(n_s^k\right)^2 P_{k,k_0}^d, \end{aligned} \tag{17}$$

and from that we obtain

$$\delta n_s^2(k_0) = \sum_k \left(n_s^k\right)^2 P_{k,k_0}^d - \left(\sum_k n_s^k P_{k,k_0}^d\right)^2, \tag{18}$$

where the dependence on the initial basis state $|k_0\rangle$ has been explicitly indicated in Eq. (18).

We have numerically checked this relation; see the data in Fig. 5(a), from which one can see a good correspondence to the above relation in the case when the eigenstates are strongly chaotic. In Fig. 5(b) the same quantity has been plotted for a nonchaotic case. As one can see, quantum fluctuations deviate strongly from the prediction given in Eq. (16). This

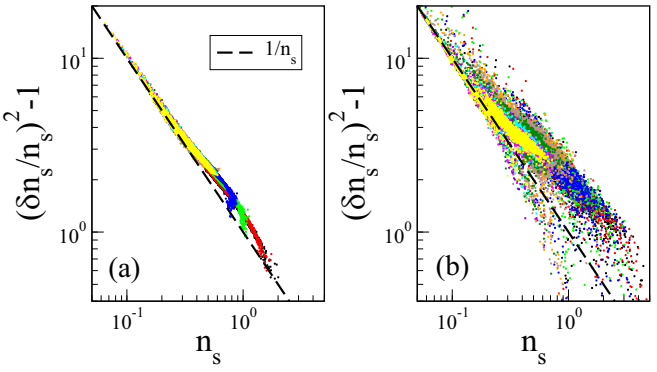

FIG. 5. Relative quantum fluctuations $(\delta n_s/n_s)^2 - 1$. Initial states $|k_0\rangle$ are basis states chosen in the whole energy spectrum. On the x axis the averaged values of n_s are plotted. The dashed line is the theoretical prediction $1/n_s$. Different colors refer to different s values. (a) The $V = 0.4$ case of strong quantum chaos and (b) the $V = 0.04$ case of nonchaotic eigenstates for which Eq. (16) is not valid.

result shows once more that even for a finite number of particles, provided a strong enough interparticle interaction, conventional statistical mechanics works extremely well.

V. TWO-POINT CORRELATION FUNCTION

Let us now study how the onset of the BE distribution is manifested by the emergence of correlations between occupation numbers. First, we start with the two-point correlation function $C_{s,s+1}(t)$ between neighboring occupation numbers

$$C_{s,s+1}(t) = \langle k_0|[\hat{n}_s(t) - \hat{n}_s][\hat{n}_{s+1}(t) - \hat{n}_{s+1}]|k_0\rangle. \tag{19}$$

Initially the correlations are absent, $C_{s,s+1}(0) = 0$, however, they appear, due to the dynamics, in time. The evolution of $C_{s,s+1}(t)$ is shown in Fig. 6(a) for all s values. As one can see, there is a clear relaxation to steady-state values after a time t_S. The negative or positive sign of the asymptotic correlations is related to the particular choice of the initial state. It is also instructive to introduce the global correlator $\mathcal{C}^{(2)}$, which is the sum of the correlators between all neighboring single-particle

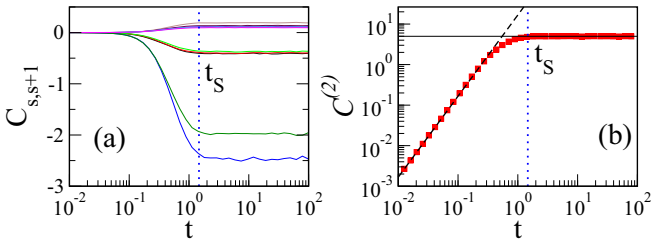

FIG. 6. (a) Correlation function $C_{s,s+1}(t)$ for different s values. (b) Global two-point correlation function $\mathcal{C}^{(2)}(t)$ (red squares). The dashed line is given by Eq. (23). The horizontal line corresponds to Eq. (24). The initial state and parameters are the same as in Fig. 3. The average over ten realizations of the random potential was used. In both (a) and (b) the vertical dotted line indicates the saturation time t_S.

energy levels ϵ_s and ϵ_{s+1},

$$\mathcal{C}^{(2)}(t) = \left|\sum_{s=1}^{M-1} C_{s,s+1}(t)\right|. \tag{20}$$

This correlator is independent of the specific s level and it can be used as a global measure of correlations between occupation numbers of nearest single-particle energy levels. Starting from the initial state $|k_0\rangle$, it can be written as

$$\begin{aligned}\mathcal{C}^{(2)}(t) &= \sum_{s=1}^{M-1} \langle k_0|[\hat{n}_s(t) - \hat{n}_s][\hat{n}_{s+1}(t) - \hat{n}_{s+1}]|k_0\rangle \\ &= \sum_{s=1}^{M-1} \langle k_0|\hat{n}_s(t)\hat{n}_{s+1}(t)|k_0\rangle - n_s^{k_0}\langle k_0|n_{s+1}(t)|k_0\rangle \\ &\quad - n_{s+1}^{k_0}\langle k_0|n_s(t)|k_0\rangle + n_s^{k_0} n_{s+1}^{k_0} \\ &\equiv \sum_{s=1}^{M-1}\sum_k |\langle k|\psi(t)\rangle|^2 W_{k,k_0}^{sr},\end{aligned} \tag{21}$$

where $\hat{n}_s(t) = e^{iHt}\hat{n}_s e^{-iHt}$ and we have defined

$$W_{k,k_0}^{sr} = \left[n_s^k n_r^k + n_s^{k_0} n_r^{k_0} - n_s^{k_0} n_r^k - n_s^k n_r^{k_0}\right]. \tag{22}$$

From this expression it is easy to get the short-time behavior

$$\mathcal{C}^{(2)}(t) \simeq t^2 \left|\sum_{s=1}^{M-1}\sum_{r=s+1}^{M}\sum_k H_{k,k_0}^2 W_{k,k_0}^{sr}\right| + o(t^4). \tag{23}$$

The agreement between our numerical results obtained by the dynamics and Eq. (23) marks the range of validity of perturbation theory. As one can see, Eq. (23) does not contain either eigenvalues or eigenfunctions. This means that in order to get the initial spread of the correlator, there is no need to diagonalize the Hamiltonian.

In order to compute the infinite-time average (stationary value), we note that since the number operator $\hat{n}_s$ giving the number of particles in the single-particle energy level ϵ_s is diagonal in the unperturbed many-body basis, i.e., $\langle k|\hat{n}_s|k'\rangle = \delta_{k,k'} n_s^k$, then

$$\overline{\mathcal{C}^{(2)}} = \sum_{s=1}^{M-1}\sum_k P_{k,k_0}^d W_{k,k_0}^{sr}. \tag{24}$$

The time evolution for $\mathcal{C}^{(2)}(t)$ is shown in Fig. 6, together with the analytical predictions. The correspondence between numerical data and analytical predictions is impressive and as one can see, no room is left between the perturbation theory estimate and the equilibration regime where thermalization emerges. This is at variance with the behavior of the four-point correlator that will be studied in the next section. We can conclude that the whole dynamics of $\mathcal{C}^{(2)}(t)$ is fully described by the analytical expressions (23) and (24).

VI. FOUR-POINT CORRELATION FUNCTION (OTOC)

Let us study the four-point correlator between neighboring single-particle energy levels

$$\mathcal{O}_{s,s+1}(t) = \langle k_0|[\hat{n}_s(t), \hat{n}_{s+1}(0)]|^2|k_0\rangle. \tag{25}$$

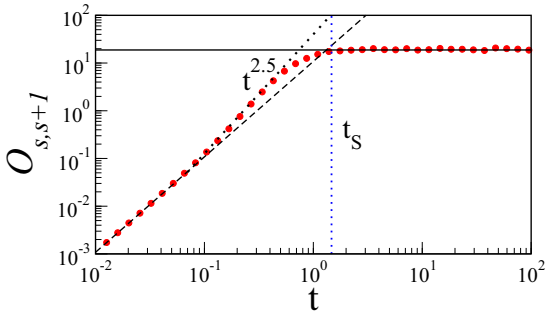

FIG. 7. Evolution of the four-point correlator $\mathcal{O}_{s,s+1}(t)$ for $s = 5$. The dashed line is the analytical prediction (31). The horizontal line corresponds to Eq. (29). The dotted line is the fit for $0.07 < t < 0.5$ (outside the perturbative regime), giving the $t^{2.5}$ dependence. The initial state is $|\Psi_0\rangle = |00006000000\rangle$ and $N = 6$, $M = 11$, and $V = 0.4$. The vertical dotted line indicates the saturation time t_S.

This correlator, also known as the OTOC, has been introduced in the frame of the Sachdev-Ye-Kitaev (SYK) model [46,47] and widely discussed in view of various physical applications (see, e.g., [30,31]).

From the definition it is clear that $\mathcal{O}_{s,s+1}(0) = 0$. In order to compute explicitly Eq. (25), let us insert a completeness relation so that

$$\mathcal{O}_{s,s+1}(t) = \sum_k |\langle k_0|\hat{n}_s(t)|k\rangle|^2 \left(n^k_{s+1} - n^{k_0}_{s+1}\right)^2. \tag{26}$$

Setting

$$\langle k_0|\hat{n}_s(t)|k\rangle = \sum_q \mathcal{F}_{k,q}(t)\mathcal{F}^*_{k_0,q}(t) n^q_s, \tag{27}$$

where we have defined

$$\mathcal{F}_{k,q}(t) = \langle q|e^{-iHt}|k\rangle = \sum_\alpha C^\alpha_q C^\alpha_k e^{-iE^\alpha t}, \tag{28}$$

the infinite-time average can be written as

$$\overline{\mathcal{O}_{s,s+1}} = \sum_k \left(n^k_{s+1} - n^{k_0}_{s+1}\right)^2 \left\{ \left[\sum_\alpha C^\alpha_k C^\alpha_{k_0} \mathcal{N}^{\alpha,\alpha}_s \right]^2 + \sum_{\alpha\neq\beta} \left|C^\alpha_k\right|^2 \left|C^\beta_{k_0}\right|^2 \left(\mathcal{N}^{\alpha,\beta}_s\right)^2 \right\}, \tag{29}$$

where we have defined the matrix

$$\mathcal{N}^{\alpha,\beta}_s = \sum_k C^\alpha_k C^\beta_k n^k_s. \tag{30}$$

After some algebra and following the previous procedure, one can obtain the short-time behavior

$$\mathcal{O}_{s,s+1}(t) \simeq t^2 \sum_{k\neq k_0} H^2_{k,k_0} \left(n^k_s - n^{k_0}_s\right)^2 \left(n^k_{s+1} - n^{k_0}_{s+1}\right)^2, \tag{31}$$

i.e., the correlator $\mathcal{O}_{s,s+1}(t)$ increases in time quadratically on a short timescale, whose validity defines the perturbative regime. Numerical data for $\mathcal{O}_{s,s+1}(t)$ are shown in Fig. 7 together with the expressions (31) and (29). Our results demonstrate that beyond the perturbative regime, characterized by a quadratic growth in time, another more interesting regime appears where a clear deviation from the t^2 dependence occurs. Although the time window in which deviations from the quadratic growth appear is too small to draw definite conclusions, the standard power-law fitting gives the approximate relation $t^{2.5}$, which is in agreement with the results recently found in [48]. This should indicate that the TBRI model approaches the thermal regime as a slow scrambler at variance with the SYK model, where a fast (exponential) scrambling is expected. Of course, there are many differences between the two models which do not permit a conclusive statement: (i) The TBRI model conserves the number of particles while the SYK model does not, (ii) here we consider a small number of particles N, while in the SYK model $N \to \infty$, and (iii) observables are different. However, our data indicate that the exponential increase of the OTOC may not occur for a finite number of particles even in the presence of strong chaos and thermalization.

VII. CONCLUSION AND DISCUSSION

We addressed the question of how the conventional Bose-Einstein distribution emerges in time in an isolated system with a finite number of interacting bosons. Since this process is accompanied by an increase of strong correlations between the occupation numbers $n_s(t)$, a large part of our study was devoted to the details of the time dependence of these correlations.

For our analysis we have used the well known model (1) describing bosons interacting with each other via two-body random matrix elements. By exploring the quench dynamics, we showed that the BE distribution emerges on the same timescale t_S on which the number of principal components in the wave function increases exponentially in time in the many-body Hilbert space [29]. This timescale t_S is proportional to the number N of bosons and defines the time after which one can speak of a complete thermalization in the system.

In order to confirm the true statistical behavior of the occupation numbers, we have carefully studied the fluctuations of $n_s(t)$ after the relaxation. In accordance with standard statistical mechanics, our data manifest that the fluctuations are of the Gaussian type and that they are small compared to the mean values of $n_s(t)$. It was also shown that relative quantum fluctuations $\delta n^2_s / n^2_s$ are also in agreement with the Bose statistics (see [45]).

In order to reveal how the process of thermalization is related to the onset of correlations, we have studied, both analytically and numerically, two correlators. One is the standard two-point correlator between nearest occupation numbers n_s and n_{s+1} and the other is the out-of-time-order correlator discussed in the literature. We have found that the two-point correlator increases in time quadratically before the saturation, in agreement with perturbation theory. As for the OTOC, beyond the initial perturbative quadratic growth and before the saturation, our numerical data demonstrate an algebraic dependence $t^{2.5}$. Remarkably, the exponent found characterizing the polynomial growth is in agreement with recent findings [48] but at variance with the exponential prediction obtained for a very large number of particles in closely related random models [30,31].

Our results also show how the information initially encoded in a local unperturbed state spreads over the whole

system and transforms onto global correlations specified by the BE distribution of occupation numbers. Although the dynamics is completely reversible due to the unitarity of the evolution operator, it is practically impossible to extract the information about the initial state by measuring the correlations between the components of the wave function. The full information about the initial state can be extracted only if there is additionally complete knowledge of the random operator V. In this sense, one can indeed speak of a practical loss of information due to scrambling. On the other hand, the process of this loss is accompanied by the emergence of global (thermodynamic) correlations, as demonstrated by the data reported here.

We hope that our study can help to understand the relation between thermalization and scrambling from one side and the onset of correlations in the evolution of chaotic systems from the other one. Since the TBRI matrix model (1) has been proved to manifest generic statistical properties occurring in realistic physical systems (see, for example, [49]), the obtained results can be confirmed experimentally by studying interacting bosons in optical traps. Our study may also be important in view of the problem of black hole scrambling (see [27,28] and references therein).

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