Fundamental solution of unsteady Stokes equations and force on an oscillating sphere near a wall

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We derive the Green's function of unsteady Stokes equations near a plane boundary with no-slip boundary conditions. This provides flow due to an oscillating point force acting on fluid bounded by a wall. Our derivation is different from previous theories and resolves the apparent discrepancies of the reported results. Two-dimensional Fourier transform of the solution with respect to horizontal coordinates is given via elementary functions in a more compact form than by the previous theories. The tensorial Green's function in real space is reduced to two Hankel transforms of order zero. We derive a simple form for the real-space solution in the two limiting cases of a distance to the wall much larger and much smaller than the viscous penetration depth. We demonstrate the applicability of this form by obtaining results for the force exerted on a sphere oscillating near the wall. Using the integral equation on surface traction whose kernel is the fundamental solution, we derive the force in the limits of a distant wall and low frequency. The wall correction to the force decays as the inverse third power of the source to the wall separation distance, much faster than the inverse first power of the classical Lorentz solution for the time-independent problem. Our results significantly extend the range of parameters for which the force admits a simple closed-form solution. Small biological swimmers propelled by inherently unsteady swimming gait generate flows driven by derivatives of the point source and we provide an example of a wall-bounded solution of this type. We demonstrate that frequency expansion is an efficient way of studying the Green's functions in confined geometry that gives the complete series solution for channel geometry.

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I. INTRODUCTION

Fundamental solutions or Green's functions are the main tool of study of linear problems. The particularly important example is viscous hydrodynamics, where the flow equations become linear in the limit of low Reynolds number. One may distinguish the (steady) Stokes equations, where the nonlinear and the unsteady terms of the Navier-Stokes equations are both negligible, and the time-dependent or unsteady Stokes equations, where the time-derivative term must be retained [1]. The unsteady Stokes equations are applied to study time-dependent hydrodynamics, such as fast oscillatory or transient flows. In both steady and unsteady problems the fundamental solution is defined as flow created by a δ-function source, or the point force. The most familiar of these solutions is due to the Oseen tensor occurring in many applications and describing quasisteady low-Reynolds-number flow due to a point force acting on unbounded fluid [1,2]. Green's functions can also be constructed in other problems of low-Reynolds-number hydrodynamics, e.g., the flow due to a point-force in a fluid bounded by a wall [3] and the problem of time-dependent Stokes flow due to a point force in an infinite fluid [1].

The history of Green's function applications in Stokes flows is long. Fundamental solutions allow one to represent flows created by the motion of (or past) bodies as the superposition of flows induced by sources distributed over the bodies' surfaces [1]. Using this representation, one can readily provide a multipole expansion of an arbitrary solution. The expansion gives the far-field asymptotic form taken by flow far from the body as a derivative of the fundamental solution. This in turn can be used to derive the hydrodynamic interactions of well-separated particles in the flow. Green's functions can also be used to recast the flow partial differential equations as boundary integral equations on surface traction and velocity [1,4,5]. Superposition of fundamental solutions and their generalizations with the source given by a derivative of a δ function can provide much more complex solutions (an arbitrary flow is the superposition of an infinite number of these solutions). A familiar example is the superposition of the fundamental solution and its Laplacian providing the flow due to a sphere oscillating in infinite space (this includes steady translational motion of the sphere as a limiting case). For a further list of applications see [1] and references therein. Finally, the fundamental solutions are of great help in numerical computations. Here the boundary integral equation formulation can greatly simplify numerical simulations. In the method of fundamental solutions one attempts to construct the solution of unsteady Stokes equations as a superposition of fundamental solutions with an unknown distribution of sources [6–9]. These two numerical methods may greatly benefit from having a fundamental solution of the time-dependent Stokes equations in confined geometries, since boundaries are inherently present in many relevant applications (e.g., microfluidics). The boundaries can bring qualitative changes in the familiar time-dependent phenomena; see, e.g., the case of Brownian motion in [10]. Already the simplest case of the

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boundaries provided by the infinite plane introduces considerable theoretical difficulties. For instance, the problem of a sphere oscillating near a wall is not separable (see, e.g., [11]).

The literature on flow due to a point force oscillating near a wall (the Green's function of unsteady Stokes equations with a plane boundary) requires a short review, since some results seem to disagree with each other (quantitative description can be found in Secs. VI and VII). This fundamental solution bridges between the fundamental solution of time-independent Stokes equations near a wall [3] and the solution of unsteady Stokes equations in infinite space [1]. The solution in terms of three Hankel transforms of order zero (equivalent to two-dimensional Fourier transform of a radially symmetric function) was provided without derivation by Pozrikidis in [12]. It was claimed that the derivation procedure that had been used by Blake for a similar problem for steady Stokes equations can be generalized to the unsteady case. The book by Pozrikidis [8] mentioned this solution, referring to Ref. [12] and noting that there is "a considerable amount of algebra" (which is not provided in either reference). The two-dimensional Fourier transform of the solution was provided again, in a different form, by Felderhof [10], who seemingly was not aware of [12]. The paper contained a brief description of the derivation using a method developed by Jones for the time-independent problem [13] (below we refer to it as the Felderhof-Jones method; the details can be found in Sec. VII). Only diagonal elements of the Green's function tensor were provided, also in the form of a two-dimensional Fourier transform in the plane parallel to the wall. In a later paper [14] Felderhof acknowledged that [12] derived the Green's function in a form similar to those derived by Sommerfeld and Renner [15,16] for the electromagnetic analog, however, no comparison of the solutions was provided. Reference [14] derived the remaining components of the Green's-function tensor using Jones' method used earlier in [10]. This paper claims that the final result can be obtained as the incompressible limit of [17]. The study in [17] considers the Green's function for the problem of hydrodynamic fluctuations near a single plane boundary (other boundaries are considered as well, however a full treatment is only given for the plane). In the incompressible limit the problem reduces to that considered here. However, the reduction itself is not provided in [14] and, as before, no comparison with the solution in [12] is given. Finally, in the recent effort of Simha *et al.* [11], who studied the force on a sphere oscillating near a wall, it was acknowledged that if the Green's function were available, the performed calculations would simplify. This paper refers to [10], but does not cite [12]. The current situation demands a comparison of different forms of the solution, establishing their equivalence and determining the simplest possible form for prospective theoretical and numerical applications.

In what follows we perform an independent derivation of the solution in the way proposed in [12]. The main difficulty of this approach to the problem, which differs from the Felderhof-Jones method, is that it results in several integrals that cannot be resolved analytically. We have found, however, that a combination of these integrals can be evaluated using a transformation.

The obtained Green's function agrees with the result in [12] up to a constant factor. We demonstrate that our answer, in contrast to that of [12], gives the correct zero-frequency limit in [3]. We also demonstrate that our solution reproduces the boundary conditions and correct force on a sphere oscillating near a wall in the high-frequency limit. We provide a possible explanation for the discrepancy in [12].

We show that our expression for the Green's function, after cumbersome transformations, agrees with that derived by the Felderhof-Jones method in [10,14] except for one of the components of the Green's function tensor. For that component the expression provided in [14] is lacking a factor of the wave number and as a result has wrong dimensions. Our derivation resolves this discrepancy.

The form of Green's function derived in this paper is more compact than that in previous studies (cf. the more cumbersome expression in [10]). We demonstrate that the Green's function in real space reduces to numerical integration of two Hankel transforms of zeroth order (vs three integrals in [12]) and provide reasons why further reduction is impossible. In the high- and low-frequency limits they can be written in terms of elementary functions even in real space. We further demonstrate the applicability of these limiting forms by computing the force acting on the sphere oscillating near a plane rigid boundary.

The problem of a sphere oscillating in unbounded fluid was resolved by Stokes [1,18]. However, the problem of sphere oscillating near a plane boundary is still unresolved. The most recent effort in [11] uses an approximation introduced in [10]. The authors mentioned that the solution procedure is not completely rigorous and that, as mentioned above, it might simplify if the Green's function were available. Here we use the integral equation for the surface traction [1,5], which resembles the method exploited in [11]. That relates the traction and the boundary condition on the particle's surface via the kernel of the integral operator given by the fundamental solution. We provide a shorter version of the integral representation of solutions of unsteady Stokes equations in comparison to that in [1] and the corresponding equation on surface traction. Analytical progress can be made in the limit of a distant wall (whose zeroth-order approximation is the analytically tractable problem of a sphere oscillating in an unbounded liquid [1]) and the limit of low frequencies (whose zeroth-order approximation is the analytically tractable case of quasisteady motion of a sphere near a wall [19,20]). Using the simplified limiting forms of the derived Green's function, the corresponding asymptotic form of the force can be found. There are distinct asymptotic limits corresponding to different hierarchies of the three characteristic length scales of the problem: the radius of the sphere a, the distance H from the sphere center to the wall, and the viscous penetration depth δ. The penetration depth δ gives the characteristic length scale of decay of the flow created by an oscillating plane with given frequency [18]. We derive the expression for the force in the limit of a distant wall, $H \gg a$ and $H \gg \delta$ with an unspecified ratio between H and a. An extra requirement $\delta \ll a$ reproduces the high-frequency result obtained in [11,21]. We revisit this limit showing that, in fact, it materializes under rather strict asymptotic conditions. Our general result, however, is of much wider validity. We also solve the low-frequency limit of $\delta \gg H, a$ and provide a detailed derivation for the case of $H \gg a$. The case of $a \lesssim H$ is not simple even at zero frequency [19,20] and one could only provide the solution as an integral of known functions. Another analytically tractable limit corresponds to the case of a sphere close to the wall, where progress can be made using the lubrication theory [1]. Reference [11] considered the limit of $H \sim \delta \gg a$. These cases are beyond the scope of the present paper.

We also introduce a different approach to the solution based on frequency expansion. This expansion is useless in infinite space, since corrections of all orders to the solution at zero frequency diverge at large distances. However, boundaries regularize the far-field divergence, making the expansion more applicable. We show that the leading-order correction is linear in frequency and that the higher-order corrections are nonanalytic. Moreover, in the problem where the point force is confined between *two* parallel walls the frequency expansion gives regular series whereas all coefficients are finite.

In infinite space translational invariance implies that derivatives of the fundamental solution with respect to the position of the source generate other solutions of (steady or unsteady) Stokes equations. These are solutions with higher-order singularities of the source where the force acting on the fluid is given by the corresponding derivative of the δ function. This is not so in general for problems with boundaries. Derivatives of our solution with respect to lateral coordinates of the source generate new solutions, while the derivatives with respect to transverse (to the boundary) coordinates do not, because they fail to obey the boundary condition on the wall. The solutions with higher-order singularities are of some interest, because they can represent, e.g., force- and torque-free microswimmers, which can exhibit new phenomena in confinement [22]. As an example, we provide the derivation of such a solution driven by the Laplacian of the δ function.

II. PROBLEM FORMULATION

In this section we introduce the fundamental solution starting from the problem of a finite sphere oscillating near the plane wall at $z = 0$. We assume that the Reynolds number is small so that the flow $\boldsymbol{u}(\boldsymbol{x}, t)$ can be described by the unsteady Stokes equations [1]

$$\partial_t \boldsymbol{u} = -\nabla p + \nu\nabla^2\boldsymbol{u}, \quad \boldsymbol{u}(\infty) = 0,$$
$$\boldsymbol{u}(r = a) = \boldsymbol{V}\exp(-i\omega t), \quad \boldsymbol{u}(z = 0) = 0, \tag{1}$$

where p is pressure divided by the density, ν is the kinematic viscosity, and ω and V are the frequency and the amplitude of the oscillation, respectively. Here $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{x}_0$, where $\boldsymbol{x}_0 = (0, 0, H)$ is the coordinate of the center of the sphere. The formulation neglects the sphere's displacements that are assumed to be much smaller than other lengths in the problem (cf. Stokes's formulation in an infinite fluid [1]). The solution can be written in the form

$$\boldsymbol{u}(\boldsymbol{x}, t) = V\tilde{\boldsymbol{u}}\left(\frac{\boldsymbol{x}}{a}\right)\exp(-i\omega t)$$
$$p(\boldsymbol{x}, t) = \frac{\nu V}{a}\tilde{p}\left(\frac{\boldsymbol{x}}{a}\right)\exp(-i\omega t). \tag{2}$$

The dimensionless fields $\tilde{\boldsymbol{u}}$ and $\tilde{p}$ obey

$$\lambda^2\tilde{\boldsymbol{u}} = -\nabla\tilde{p} + \nabla^2\tilde{\boldsymbol{u}}, \quad \tilde{\boldsymbol{u}}(\infty) = 0,$$
$$\tilde{\boldsymbol{u}}(r = 1) = \hat{V}, \quad \tilde{\boldsymbol{u}}(z = 0) = 0, \tag{3}$$

where $\hat{V} = \boldsymbol{V}/V$ and $\lambda^2 = -ia^2/\delta^2$ with $\delta^2 \equiv \nu/\omega$. Here and below the square root of a complex number is defined so that the real part is non-negative, e.g., $\lambda = (1 - i)a/\delta\sqrt{2}$.

In this work we will consider the force applied on the sphere. We remark that there is no lift force due to linearity of the equations, as in the steady motion of a sphere near the wall [2]. Indeed, the reversal of sign of the flow and the pressure produces another solution of the equations with the reverse direction of motion of the sphere. The requirement of invariance of the force under this transformation for the sphere's motion parallel to the plane implies that there is no lift force.

We introduce the integral representation with the help of the fundamental solution. This fundamental solution is introduced similarly to that in an unbounded fluid [1],

$$-\nabla p^k + \nabla^2\boldsymbol{u}^k - \lambda^2\boldsymbol{u}^k = -\hat{x}_k\delta(\boldsymbol{x} - \boldsymbol{x}'), \quad \nabla\cdot\boldsymbol{u}^k = 0,$$
$$\boldsymbol{u}^k(z = 0) = \boldsymbol{u}^k(r \to \infty) = 0, \tag{4}$$

where $\hat{x}_k$ is unit vector in the kth direction. We will occasionally refer to this solution as an unsteady Stokeslet near the wall or the Green's function. The inverse Laplace transform with respect to λ^2 or ω produces the Green's function of time-dependent Stokes equations with the source proportional to $\delta(t)\delta(\boldsymbol{x} - \boldsymbol{x}')$. The solution can be written as

$$u_i^k(\boldsymbol{x}) = \frac{G_{ik}(\boldsymbol{x}, \boldsymbol{x}')}{8\pi}, \quad p^k = \frac{P^k(\boldsymbol{x}, \boldsymbol{x}')}{8\pi}, \tag{5}$$

which is useful for considering the solution as a function of both the spatial coordinate $\boldsymbol{x}$ and the coordinate of the point $\boldsymbol{x}'$ at which the force is applied. The solution for infinite space is obtained by taking the limit $z' \to \infty$ where $\boldsymbol{x}' = (x', y', z')$. We can use the reciprocal theorem to demonstrate the symmetry

$$G_{ik}(\boldsymbol{x}, \boldsymbol{x}') = G_{ki}(\boldsymbol{x}', \boldsymbol{x}). \tag{6}$$

The derivation proceeds as for unbounded fluid [8] (see Appendix A). This relation has the same form as in infinite space [1], however it has more implications because the plane breaks translational and rotational symmetries. Thus the flow $G_{i3}(\boldsymbol{x}, \boldsymbol{x}')$ has axial symmetry around the line defined by the perpendicular from $\boldsymbol{x}'$ to the plane. Correspondingly, $G_{\alpha 3}(\boldsymbol{x}, \boldsymbol{x}')$ [and $G_{33}(\boldsymbol{x}, \boldsymbol{x}')$], where here and below $\alpha = 1, 2$, can be derived from the stream function. The symmetry relation then gives us $G_{3\alpha}(\boldsymbol{x}', \boldsymbol{x})$, which is the component of the flow that is already not axially symmetric. This remarkable reduction will be used below.

The integral representation of the solution of Eqs. (3) is derived as for unbounded fluid [8],

$$\tilde{u}_i(\boldsymbol{x}) = \frac{\lambda^2\hat{V}_k}{8\pi}\int_{|\boldsymbol{x}'-\tilde{\boldsymbol{x}}_0|<1} G_{ik}(\boldsymbol{x}, \boldsymbol{x}')dV'$$
$$-\frac{1}{8\pi}\int_{|\boldsymbol{x}'-\tilde{\boldsymbol{x}}_0|=1} G_{il}(\boldsymbol{x}, \boldsymbol{x}')\tilde{\sigma}_{lk}(\boldsymbol{x}')dS_k' \tag{7}$$

(see Appendix B). This representation is identical to that in infinite space [1]. It is obtained in the standard way by integration of a Lorentz-type identity of unsteady Stokes flows over the volume of the flow. The wall boundary does not contribute to the result of the integration due to the no-slip boundary condition that holds there for both the flow and the fundamental solution (see details in Appendix B).

We observe that it is not necessary to use in the integral representation the volume integration besides the surface integration. We have

$$\int_{|\boldsymbol{x}'-\tilde{\boldsymbol{x}}_0|=1} G_{il}(\boldsymbol{x},\boldsymbol{x}')x_k' dS_l' = \int_{|\boldsymbol{x}'-\tilde{\boldsymbol{x}}_0|<1} G_{ik}(\boldsymbol{x},\boldsymbol{x}')dV', \quad (8)$$

where we used $\nabla_l' G_{il}(\boldsymbol{x},\boldsymbol{x}')=0$ (we designate derivatives with respect to $\boldsymbol{x}'$ by prime), which holds by incompressibility and the symmetry relation given by Eq. (6). Thus we can rewrite Eq. (7) as

$$\tilde{u}_i(\boldsymbol{x}) = \int_{|\boldsymbol{x}'-\tilde{\boldsymbol{x}}_0|=1} G_{il}(\boldsymbol{x},\boldsymbol{x}')(\lambda^2\delta_{lk}(\hat{V}\cdot\boldsymbol{x}') - \tilde{\sigma}_{lk}(\boldsymbol{x}'))\frac{dS_k'}{8\pi}, \quad (9)$$

which can have some advantages for both theoretical and numerical studies.

We take in Eq. (9) the limit of $\boldsymbol{x}$ approaching the surface of the sphere $|\boldsymbol{x}-\tilde{\boldsymbol{x}}_0|=1$, which is a regular limiting process [8]. We find, using the boundary conditions, that

$$\hat{V}_i = \int_{|\boldsymbol{x}'-\tilde{\boldsymbol{x}}_0|=1} G_{il}(\boldsymbol{x},\boldsymbol{x}')[\lambda^2\delta_{lk}(\hat{V}\cdot\boldsymbol{x}') - \tilde{\sigma}_{lk}(\boldsymbol{x}')]\frac{dS_k'}{8\pi}. \quad (10)$$

This (Fredholm-type [8]) integral equation must hold for all $\boldsymbol{x}$ on the sphere, $|\boldsymbol{x}-\tilde{\boldsymbol{x}}_0|=1$. It gives the boundary condition on the sphere as an integral transform of the unknown surface traction $\tilde{\sigma}_{lr}(\boldsymbol{x}')$ with the kernel $G_{ik}(\boldsymbol{x},\boldsymbol{x}')$. The equation defines the traction uniquely [1,8] and is the main tool of our study of the force $\boldsymbol{F}(\lambda)$ applied by the fluid on the sphere. We remark that the first term in brackets on the right-hand side (RHS) is missing in Eq. (2.7.21) of [8], where apparently finite divergence of the stress tensor was disregarded. Similarly to the case of an unbounded fluid [5], the integral becomes one dimensional in the axially symmetric case. A somewhat different integral equation was used for this problem in [11].

We observe that besides determining the motion of the sphere, the force also governs the flow at large distances from it, $r \gg a$. We can readily derive the multipole expansion starting from the representation given by Eq. (7). The leading-order term at large distances $r \gg a$ is

$$\tilde{u}_i(\boldsymbol{x}) \approx G_{ik}(\boldsymbol{x},\tilde{\boldsymbol{x}}_0)\left(\frac{\lambda^2\hat{V}_k}{6} - \frac{F_k(\lambda)}{8\pi}\right). \quad (11)$$

This result was obtained for a sphere in [11] by a more complicated procedure [see Eq. (2.20) therein]. This paper conjectured that the result possibly holds for a body of an arbitrary shape. This can be readily shown from our approach. All formulas of this section [apart from Eq. (11)] hold also for a rigid particle of an arbitrary shape with velocity $\hat{V}$ at the surface (with an obvious change of integration domains; rigid rotation can also be included). Designating the volume of the particle by V_p, we find from Eq. (7) that at large distances from the particle,

$$\tilde{u}_i(\boldsymbol{x}) \approx [\lambda^2 V_p\hat{V}_k - F_k(\lambda)]\frac{G_{ik}(\boldsymbol{x},\tilde{\boldsymbol{x}}_0)}{8\pi}. \quad (12)$$

This generalizes Eq. (11) to a particle of arbitrary shape and proves the conjecture (2.20) in [11]. The first term in brackets is of added mass form. We conclude that the properties of the fundamental solution can be used for the study of the flow due to the motion of a finite-size sphere.

III. FREQUENCY EXPANSION

In this section we construct the solution of Eqs. (4) as a series in λ^2. This series can be used for the asymptotic study of the limit of low frequency. The coefficients diverge starting from a certain order, however, resummation of the formally divergent series gives a regular solution.

It is instructive to consider first the Green's function $G^0_{ik}(\boldsymbol{x},\boldsymbol{x}')$ of the infinite fluid. The solution at zero frequency is the Oseen tensor $Y_{ik}(\boldsymbol{r})$,

$$G^0_{ik}(\boldsymbol{x},\boldsymbol{x}',\lambda=0) \equiv Y_{ik}(\boldsymbol{r}) = \frac{\delta_{ik}}{r} + \frac{r_i r_k}{r^3}, \quad (13)$$

where $\boldsymbol{r}=\boldsymbol{x}-\boldsymbol{x}'$ (see, e.g., [1]). One can look for the solution at finite frequency as a series in λ^2 which starts with Y_{ik} as the zeroth-order approximation

$$G^0_{ik}(\boldsymbol{x},\boldsymbol{x}',\lambda) = \sum_{n=0}^{\infty}\lambda^{2n}G^0_{n,ik}(\boldsymbol{r}), \quad G^0_{0,ik}=Y_{ik}, \quad (14)$$

where we use the translational invariance. Substituting this series in Eqs. (4), we find

$$-\nabla_i P^k_n + \nabla^2 G^0_{n,ik} = G^0_{n-1,ik}, \quad G^0_{n,ik}(r\to\infty)=0, \quad (15)$$

where $P^k=\sum_{n=0}^{\infty}\lambda^{2n}P^k_n$. The solution can be written by introducing a linear integral operator $\hat{Y}$ whose kernel is minus the Oseen tensor divided by 8π,

$$G^0_n = \hat{Y}G^0_{n-1} = -\int Y_{il}(\boldsymbol{x}-\boldsymbol{x}'')G^0_{n-1,lk}(\boldsymbol{x}''-\boldsymbol{x}')\frac{d\boldsymbol{x}''}{8\pi}. \quad (16)$$

This gives

$$G^0_{n,ik}(\boldsymbol{x},\boldsymbol{x}') = (\hat{Y}^n Y)_{ik} = (-1)^n\int Y_{ii_1}(\boldsymbol{x}-\boldsymbol{x}_1) \times Y_{i_1i_2}(\boldsymbol{x}_1-\boldsymbol{x}_2)\cdots Y_{i_nk}(\boldsymbol{x}_n-\boldsymbol{x}')\prod_{k=1}^{n}\frac{d\boldsymbol{x}_k}{8\pi}. \quad (17)$$

The resulting series $G^0=\sum_{n=0}^{\infty}\lambda^{2n}\hat{L}^nY$ is readily resummed, giving

$$G^0_{ik}(\boldsymbol{x},\boldsymbol{x}') = \int(1-\lambda^2\hat{Y})^{-1}_{il}(\boldsymbol{x}-\boldsymbol{x}'')Y_{lk}(\boldsymbol{x}''-\boldsymbol{x}')d\boldsymbol{x}'', \quad (18)$$

reducing the problem to finding the kernel of the operator $(1-\lambda^2\hat{Y})^{-1}$ and integration. One might hope that at low frequency the truncation of the infinite series would produce successive approximations to the solution. However, in fact, all terms of the series (besides $n=0$) diverge. We have for the leading-order correction

$$G^0_{1,ik} = -\int Y_{il}(\boldsymbol{x}-\boldsymbol{x}'')Y_{lk}(\boldsymbol{x}''-\boldsymbol{x}')\frac{d\boldsymbol{x}''}{8\pi}, \quad (19)$$

which is divergent at large x''. The divergence signals that the leading-order correction in frequency is of $o(\lambda^2)$. In fact, we can compare the above with the well-known closed-form solution [1]

$$G^0_{ik}(\boldsymbol{x},\boldsymbol{x}') \equiv G^0_{ik}(\boldsymbol{r}) = \frac{4[1-(1+\lambda r)\exp(-\lambda r)]}{\lambda^2 r^3}\frac{r_i r_k}{r^2} + \frac{2}{\lambda^2 r^3}[(1+\lambda r+\lambda^2 r^2)\exp(-\lambda r)-1] \times \left(\delta_{ik}-\frac{r_i r_k}{r^2}\right), \tag{20}$$

where the corresponding pressure is frequency independent, $p^{0k}=2r_k/r^3$ (thus in the time domain it is proportional to the δ function, which corresponds to instantaneous establishment of pressure by incompressibility [14]). The low-frequency limit of $|\lambda|r \ll 1$ is described by

$$G^0_{ik}(\boldsymbol{r}) \approx Y_{ik}(\boldsymbol{r}) - \frac{4\lambda\delta_{ik}}{3} + \frac{r\lambda^2}{4}\left(3\delta_{ik}-\frac{r_i r_k}{r^2}\right). \tag{21}$$

The leading-order correction is linear in λ in accord with the conclusion from the study of the series. This correction is coordinate independent, a fact that has further implications [1,5]. We also observe that λr is independent of a in dimensional units as it must be, since G^0_{ik} is independent of a. The parameter λr is the ratio of the coordinate and δ times a numerical constant [18].

Thus the series solution in infinite space indicates that the correction is of $o(\lambda^2)$, the result of the resummation given by Eq. (18) is a valid form of the solution. The series is rather useful in the presence of the wall. The $\lambda=0$ solution of Eq. (4) is the steady Stokeslet in the vicinity of a no-slip boundary $\tilde{G}_{ik}(\boldsymbol{x},\boldsymbol{x}')$ reported in [3]. The wall produces image sources at $(\boldsymbol{x}')^*=(x',y',-z')$ that compensate for the action of the point force at $\boldsymbol{x}'$ so that $\tilde{G}_{ik}$ decays at large x as x^{-2} and not x^{-1} as in infinite space. This faster decay renders the leading-order correction in the frequency

$$G_{1,ik}(\boldsymbol{x},\boldsymbol{x}') = -\int_{z''>0}\tilde{G}_{li}(\boldsymbol{x}'',\boldsymbol{x})\tilde{G}_{lk}(\boldsymbol{x}'',\boldsymbol{x}')\frac{d\boldsymbol{x}''}{8\pi} \tag{22}$$

finite due to x''^{-4} decay of the integrand at large x''. This form of the correction is found from the series solution $G_{ik}=\sum_{n=0}^{\infty}\lambda^{2n}G_{n,ik}$ as in the infinite fluid [cf. Eq. (19); we can also write the solution in a form similar to Eq. (18)]. We used the symmetry given by Eq. (6). Thus, with the wall the leading-order correction in frequency is quadratic in λ and linear in frequency. However, the next-order term

$$G_2 = \int \tilde{G}_{i_1 i}(\boldsymbol{x}_1,\boldsymbol{x})\tilde{G}_{i_2 i_1}(\boldsymbol{x}_2,\boldsymbol{x}_1)\tilde{G}_{i_2 k}(\boldsymbol{x}_2,\boldsymbol{x}')\frac{d\boldsymbol{x}_1 d\boldsymbol{x}_2}{(8\pi)^2}$$

already diverges logarithmically at large distances. We conclude that (we occasionally write the frequency argument of the Green's function to emphasize the dependence)

$$G_{ik}(\boldsymbol{x},\boldsymbol{x}',\lambda) = \tilde{G}_{ik}(\boldsymbol{x},\boldsymbol{x}') - \lambda^2\int_{z''>0}\tilde{G}_{li}(\boldsymbol{x}'',\boldsymbol{x}) \times \tilde{G}_{lk}(\boldsymbol{x}'',\boldsymbol{x}')\frac{d\boldsymbol{x}''}{8\pi} + O(\lambda^{2+\Delta}), \tag{23}$$

where the exponent obeys $0<\Delta<2$. This asymptotic low-frequency form of the Green's function is confirmed below by the expansion of the full solution in frequency that also gives the explicit form of the integral in the λ^2 term (direct calculation of the integral is formidable).

IV. DERIVATION OF THE SOLUTION

In this section we derive the two-dimensional Fourier transform of the fundamental solution with respect to the horizontal coordinates. We proceed similarly to the solution for the steady Stokeslet near the wall [3]. We consider the flow as a superposition of the flow due to an unsteady Stokeslet in the unbounded fluid, an image of that flow with respect to $z=0$, and a correction. The unsteady Stokeslet flow in the unbounded fluid is given by $u_i^k = G^0_{ik}(\boldsymbol{x},\boldsymbol{x}')/8\pi$, where G^0_{ik} is given by Eq. (20).

We introduce the correction flow $\boldsymbol{w}$,

$$u_i^k(\boldsymbol{x},\boldsymbol{x}') = \frac{G^0_{ik}(\boldsymbol{x}-\boldsymbol{x}')-G^0_{ik}(\boldsymbol{x}-\boldsymbol{x}'^*)}{8\pi} + w_i^k(\boldsymbol{x}-\boldsymbol{x}'^*,x_3'),$$
$$p^k = \frac{r_k}{4\pi r^3} - \frac{R_k}{4\pi R^3} + s^k. \tag{24}$$

Here and in the following the asterisk superscript stands for reflection with respect to the plane $z=0$, so $\boldsymbol{x}'^*=(x_1',x_2',-x_3')$, and $\boldsymbol{R}=\boldsymbol{x}-\boldsymbol{x}'^*$. In this section the vector components are designated by numbers. Notice the translational invariance in the plane: The correction depends on $\boldsymbol{x}-\boldsymbol{x}'^*$ and x_3' and is independent of x_α'. The domain of definition of $\boldsymbol{w}^k(\boldsymbol{R},h)$ is $R_3 \geqslant h$, where $h>0$ is the dimensionless height of the source above the plane and R_3 is the positive vertical component of $\boldsymbol{x}$ plus h. The symmetry relation given by Eq. (6) implies

$$G^0_{ik}(\boldsymbol{x}-\boldsymbol{y}) - G^0_{ik}(\boldsymbol{x}-\boldsymbol{y}^*) + 8\pi w_i^k(\boldsymbol{x}-\boldsymbol{y}^*,y_3) = G^0_{ki}(\boldsymbol{y}-\boldsymbol{x}) - G^0_{ki}(\boldsymbol{y}-\boldsymbol{x}^*) + 8\pi w_k^i(\boldsymbol{y}-\boldsymbol{x}^*,x_3),$$

which must hold for any $\boldsymbol{x}$ and $\boldsymbol{y}$. We observe that $\boldsymbol{x}-\boldsymbol{y}^*=(x_1-y_1,x_2-y_2,x_3+y_3)$ and $\boldsymbol{y}-\boldsymbol{x}^*=(y_1-x_1,y_2-x_2,x_3+y_3)$. Introducing $x=x_1-y_1$ and $y=x_2-y_2$, we have

$$w_i^k(x,y,x_3+y_3,y_3) - w_k^i(-x,-y,x_3+y_3,x_3) = \frac{G^0_{ik}(x,y,x_3+y_3)-G^0_{ki}(-x,-y,x_3+y_3)}{8\pi},$$

which must hold for all x and y and all positive x_3 and y_3. Using Eq. (20), this implies that for all $\boldsymbol{X}=(x,y,z_1+z_2)$ with positive z_α we have

$$w_i^k(x,y,z_1+z_2,z_1) - w_k^i(-x,-y,z_1+z_2,z_2) = (\delta_{i\alpha}\delta_{k3}+\delta_{i3}\delta_{k\alpha})X_\alpha(z_1+z_2)f(X), \tag{25}$$

where $X=|\boldsymbol{X}|$ and

$$f(X) = \frac{3-(3+3\lambda X+\lambda^2X^2)\exp(-\lambda X)}{2\pi\lambda^2X^5}. \tag{26}$$

Below, unless stated otherwise, we consider the fixed position of the source, $\boldsymbol{x}'=(0,0,h)$, and do not write explicitly the dependence of $\boldsymbol{w}$ on h using $\boldsymbol{w}^k(\boldsymbol{R})$ instead of $\boldsymbol{w}^k(\boldsymbol{R},h)$. We solve at $R_3 \geqslant h$ the equation

$$-\nabla_{\boldsymbol{R}}s^k + \nabla^2_{\boldsymbol{R}}\boldsymbol{w}^k - \lambda^2\boldsymbol{w}^k = 0, \quad \nabla_{\boldsymbol{R}}\cdot\boldsymbol{w}^k = 0, \tag{27}$$

where $\nabla_{\boldsymbol{R}}$ is derivative over R_i (cf. [3]). The boundary condition at $R_3=h$ is found using Eqs. (20) and (24),

$$w_i^k(R_1,R_2,h)=\left.\frac{G_{ik}^0(\boldsymbol{R})-G_{ik}^0(\boldsymbol{r})}{8\pi}\right|_{z=0}=(\delta_{k\alpha}\delta_{i3}+\delta_{i\alpha}\delta_{k3})\times hR_\alpha f(R_1,R_2,h). \quad (28)$$

(Throughout the text the greek letters α and β assume the values 1 or 2 only.) The small λ expansion

$$f=\frac{1}{4\pi X^3}-\frac{\lambda^2}{16\pi X}+o(\lambda^2) \quad (29)$$

reproduces the boundary condition of [3] at $\lambda=0$. The absence of the linear term in λ is required to reproduce the result of preceding section showing that the leading-order correction in frequency is quadratic. We perform the Fourier transform

$$\hat{w}_i^k=\int w_i^k(R_1,R_2,R_3)\exp(-iq_1R_1-iq_2R_2)dR_1dR_2,$$

where the circumflex stands for the Fourier transformed field. We further find that ($q^2=q_1^2+q_2^2$)

$$\lambda^2\hat{w}_i^k+iq_\alpha\delta_{\alpha i}\hat{s}^k+\delta_{i3}\frac{\partial\hat{s}^k}{\partial R_3}=\left(\frac{\partial^2}{\partial R_3^2}-q^2\right)\hat{w}_i^k,$$

$$iq_\alpha\hat{w}_\alpha^k+\frac{\partial\hat{w}_3^k}{\partial R_3}=0,\ \left(\frac{\partial^2}{\partial R_3^2}-q^2\right)\hat{s}^k=0. \quad (30)$$

The pressure is a harmonic function that decays at infinity,

$$\hat{s}^k=-B^k\exp(-qR_3), \quad (31)$$

where B^k is independent of R_3. We find

$$\left(\frac{\partial^2}{\partial R_3^2}-q^2-\lambda^2\right)\hat{w}_i^k=q\delta_{i3}B^k\exp(-qR_3)-iq_\alpha\delta_{\alpha i}B^k\exp(-qR_3).$$

The solution reads

$$\hat{w}_i^k=\frac{iq_\alpha\delta_{\alpha i}B^k\exp(-qR_3)-q\delta_{i3}B^k\exp(-qR_3)}{\lambda^2}+B_i^k\exp(-kR_3),\quad k\equiv\sqrt{q^2+\lambda^2}.$$

The coefficients B^k and B_i^k are fixed from the incompressibility condition and the Fourier transform of the boundary condition on $\boldsymbol{w}^k$ at $R_3=h$, which is given by Eq. (28). The former gives

$$0=\frac{\partial\hat{w}_3^k}{\partial R_3}+iq_\alpha\hat{w}_\alpha^k\propto iq_\alpha B_\alpha^k-kB_3^k.$$

Thus we can write

$$\hat{w}_\alpha^k=\frac{iq_\alpha B^k\exp(-qR_3)}{\lambda^2}+B_\alpha^k\exp(-kR_3),$$

$$\hat{w}_3^k=\frac{iq_\alpha B_\alpha^k\exp(-kR_3)}{k}-\frac{qB^k\exp(-qR_3)}{\lambda^2}. \quad (32)$$

The boundary conditions are obtained from Eq. (28),

$$\hat{w}_\alpha^k(R_3=h)=ih\delta_{k3}\frac{\partial\hat{f}(\boldsymbol{q})}{\partial q_\alpha}=\frac{ih\delta_{k3}q_\alpha}{q}\frac{\partial\hat{f}}{\partial q},$$

$$\hat{w}_3^k(R_3=h)=ih\delta_{k\alpha}\frac{\partial\hat{f}(\boldsymbol{q})}{\partial q_\alpha}=\frac{ih\delta_{k\alpha}q_\alpha}{q}\frac{\partial\hat{f}}{\partial q}, \quad (33)$$

where $\hat{f}$ is the Fourier transform

$$\hat{f}(q)=\int f(R_1,R_2,h)\exp(-iq_1R_1-iq_2R_2)dR_1dR_2=2\pi\int_0^\infty f(R)J_0(q\rho)\rho d\rho, \quad (34)$$

where $\rho^2=R_1^2+R_2^2$ and $R^2=\rho^2+h^2$. We find, from Eqs. (32),

$$B^k=\frac{\lambda^2\exp(qh)}{q(q-k)}\left(\hat{w}_3^kk-iq_\alpha\hat{w}_\alpha^k\right)=\frac{ih\lambda^2\exp(qh)}{q-k}\left(\frac{k\delta_{k\alpha}q_\alpha}{q^2}-i\delta_{k3}\right)\frac{\partial\hat{f}}{\partial q}, \quad (35)$$

where in the first line the velocity must be taken at $R_3=h$. Similarly,

$$B_\alpha^k=\exp(kh)\left(\hat{w}_\alpha^k-\frac{iq_\alpha B^k\exp(-qh)}{\lambda^2}\right)=\frac{iq_\alpha kh\exp(kh)}{q^2(k-q)}(q\delta_{k3}+i\delta_{k\beta}q_\beta)\frac{\partial\hat{f}}{\partial q}. \quad (36)$$

We conclude that the flow in Fourier space is

$$\hat{w}_\alpha^k=\frac{hq_\alpha\hat{f}'}{q^2(k-q)}\{\exp[q(h-R_3)](k\delta_{k\beta}q_\beta-iq^2\delta_{k3})-k\exp[k(h-R_3)](\delta_{k\beta}q_\beta-iq\delta_{k3})\},$$

$$\hat{w}_3^k=\frac{h\hat{f}'}{q(k-q)}\{\exp[q(h-R_3)](q^2\delta_{k3}+ik\delta_{k\alpha}q_\alpha)-\exp[k(h-R_3)](q^2\delta_{k3}+iq\delta_{k\alpha}q_\alpha)\}. \quad (37)$$

Thus, to complete the calculation in Fourier space we must find $\hat{f}'(q)$. The cumbersome and nontrivial calculation (see Appendix C for details) finally gives

$$\hat{f}'(q)=\frac{q[\exp(-kh)-\exp(-qh)]}{\lambda^2h}. \quad (38)$$

Thus the velocity in Fourier space is ($z=R_3-h$)

$$\hat{w}_\alpha^\beta=\frac{kq_\alpha q_\beta(e^{-kh}-e^{-qh})(e^{-qz}-e^{-kz})}{\lambda^2q(k-q)},$$

$$\hat{w}_3^\alpha=\frac{iq_\alpha(e^{-kh}-e^{-qh})(ke^{-qz}-qe^{-kz})}{\lambda^2(k-q)}. \quad (39)$$

We have, for the rest of the components,

$$\hat{w}_\alpha^3=\frac{iq_\alpha(e^{-kh}-e^{-qh})(ke^{-kz}-qe^{-qz})}{\lambda^2(k-q)}$$

$$\hat{w}_3^3=\frac{q^2(e^{-kh}-e^{-qh})(e^{-qz}-e^{-kz})}{\lambda^2(k-q)}. \quad (40)$$

These closed-form expressions for w_i^k and their real space form in Eqs. (42) and (43) are one of the main results of the present paper. It is demonstrated in Appendix A that the solution obeys the symmetry of the Green's function.

The last component of the solution, the pressure, is inferred from Eqs. (31), (35), and (38),

$$\hat{s}^3 = \frac{q(e^{-qz-kh} - e^{-q(z+h)})}{k-q},$$
$$\hat{s}^\alpha = \frac{iq_\alpha k(e^{-qz-kh} - e^{-q(z+h)})}{q(k-q)}. \quad (41)$$

This expression seems to be the simplest for the theoretical study. We demonstrate in Appendix D that even pressure produced on the wall by the vertically moving source along the symmetry axis, $s^3(z=0, \rho=0)$, can only be written in quadratures (as an integral). This gives a strong indication that the solution in real space, except at the position of the source [10], can only be written in quadratures. Thus, the use of the solution in real space involves numerical integration. We provide a compact representation of the solution for this purpose.

V. SOLUTION IN REAL SPACE

In this section we consider the solution in real space, which reads

$$u_i^k(\boldsymbol{x}, \boldsymbol{x}') = \frac{G_{ik}^0(\boldsymbol{x}-\boldsymbol{x}') - G_{ik}^0(\boldsymbol{x}-\boldsymbol{x}'^*)}{8\pi} + \int \frac{dq_1 dq_2}{(2\pi)^2}\hat{w}_i^k(z, \boldsymbol{q}, h = x_3') \times \exp[iq_1(x_1-x_1') + iq_2(x_2-x_2')], \quad (42)$$

where $\hat{w}_i^k$ are given by Eqs. (39) and (40) and we used the definition in Eq. (24). The corresponding formula for the pressure is

$$p^k = \frac{r_k}{4\pi r^3} - \frac{R_k}{4\pi R^3} + \int s^k(z, \boldsymbol{q}, h = x_3') \times \exp[iq_1(x_1-x_1') + iq_2(x_2-x_2')]\frac{dq_1 dq_2}{(2\pi)^2}. \quad (43)$$

The solution contains a number of integrals that are not tabulated. The integrals can be found at $z=0$, where we have from Eqs. (39) and (40) that $w_\alpha^\beta(z=0) = w_3^3(z=0) = 0$ and $w_3^\alpha(z=0) = w_\alpha^3(z=0)$ with $(\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{x}')$

$$w_3^\alpha(z=0) = \nabla_\alpha \int \frac{q\,dq}{2\pi}\frac{e^{-kh} - e^{-qh}}{\lambda^2} J_0(qr) = hR_\alpha f(\boldsymbol{R}).$$

Here $f(\boldsymbol{R})$ is defined in Eq. (26) and we used [23]

$$\int q\,dq\, e^{-kh} J_0(qr) = \frac{h(1+\lambda R)\exp(-\lambda R)}{R^3}, \quad (44)$$

with $\boldsymbol{R} = (r_1, r_2, h)$ taken at the wall. This confirms that the solution obeys the boundary conditions at the wall given by Eq. (28).

We further demonstrate that the integrals in Eqs. (42) and (43) can be reduced to two Hankel transforms of order zero. We start from the axially symmetric component of the Stokeslet, $\hat{w}_i^3$. This is described using the stream function ψ defined by

$$w_3^3 = \frac{1}{\rho}\frac{\partial\psi}{\partial\rho}, \quad w_\rho^3 = -\frac{1}{\rho}\frac{\partial\psi}{\partial z}, \quad \psi = \int_0^\rho \rho' w_3^3(\rho', R_3)d\rho', \quad (45)$$

where $w_\alpha^3 = \rho_\alpha w_\rho^3/\rho$. Using Bessel function J_n of order n, these formulas give

$$\hat{w}_3^3 = 2\pi\int_0^\infty w_3^3 J_0(q\rho)\rho\,d\rho = 2\pi\int_0^\infty J_0(q\rho)\frac{\partial\psi}{\partial\rho}d\rho = 2\pi q\int_0^\infty J_1(q\rho)\psi(\rho, R_3)d\rho,$$

where $J_1 = -J_0'$ and $\psi(0) = 0$. Thus ψ can be obtained from $\hat{w}_3^3$ as the inverse Hankel transform of first order,

$$\psi = \rho\int_0^\infty \hat{w}_3^3\frac{J_1(q\rho)dq}{2\pi}.$$

Using Eq. (40), we can write

$$\psi = \rho\frac{\partial}{\partial\rho}\int_0^\infty \frac{q(e^{-qz} - e^{-kz})(e^{-qh} - e^{-kh})J_0(q\rho)dq}{2\pi\lambda^2(k-q)}. \quad (46)$$

This gives a compact representation of the solution for forcing perpendicular to the plane.

The components w_3^α can then be obtained from w_α^3 using the symmetry given by Eqs. (25) and (26). The remaining components of the solution can be obtained using Eq. (39),

$$w_\alpha^\beta = \nabla_\alpha\nabla_\beta\int_0^\infty \frac{k(e^{-qz} - e^{-kz})(e^{-qh} - e^{-kh})J_0(q\rho)dq}{2\pi\lambda^2(k-q)}. \quad (47)$$

Thus the full tensorial Green's function depends on two integrals in Eqs. (46) and (47). These two integrals are similar, but they cannot be reduced to a single integral. The main technical difficulty is an irreducible integral of the type $\int_0^\infty \exp(-qz - kh)J_0(q\rho)dq$. For instance, writing k in the numerator of Eq. (47) as $(k-q)+q$ and taking one of the derivatives yields

$$w_\alpha^\beta = -\nabla_\alpha\nabla_\beta\int\frac{(e^{-kh} - e^{-qh})(e^{-qz} - e^{-kz})J_0(q\rho)dq}{2\pi\lambda^2} + \nabla_\alpha\left(\frac{\rho_\beta\psi}{\rho^2}\right).$$

Given that the integral in the first line can be taken in terms of special or elementary functions, this formula would reduce the calculation of the full Green's-function tensor to one numerical integral in Eq. (46) or (47). Even though the integral cannot be taken analytically, its numerical evaluation would present no difficulty.

An alternative approach to the numerical integration can start from the observation that Eq. (47) implies that the dependence of w_α^β on the indices obeys

$$w_\alpha^\beta=\delta_{\alpha\beta}w+\frac{\rho_\alpha\rho_\beta\partial_\rho w}{\rho},\quad \nabla_\alpha w_\alpha^\beta=-\nabla_z w_3^\beta,$$

where the last condition describes incompressibility and

$$w\equiv\frac{1}{\rho}\frac{\partial}{\partial\rho}\int_0^\infty\frac{k(e^{-qz}-e^{-kz})(e^{-qh}-e^{-kh})J_0(q\rho)dq}{2\pi\lambda^2(k-q)}.$$

Considering w_3^β to be derived from ψ, the function w can be fixed from the resulting ordinary differential equation of second order.

VI. COMPARISON WITH RESULTS OF POZRIKIDIS

In this section we compare our results obtained in the preceding section with those of [12]. This reference (using somewhat confusing notation, whereas the spatial variable serves as a dummy variable for $\boldsymbol{R}$) provided, without derivation, the result which in our notation has the form

$$8\pi w_i^3=-\frac{1}{2\pi\lambda^4}\left(\delta_{i3}\nabla^2-\frac{\partial^2}{\partial x_i\partial z}\right)F_1,\tag{48}$$

where the function F_1 is defined by the integral

$$F_1=\int_0^\infty J_0(\rho q)(k+q)q\,dq[(1-e^{(k-q)h})e^{-k(z+h)}+(1-e^{(q-k)h})e^{-q(z+h)}].\tag{49}$$

This result corresponds to the stream function

$$-8\pi\psi=\rho\frac{\partial}{\partial\rho}\int_0^\infty(e^{-k(z+h)}-e^{-kz-qh}+e^{-q(z+h)}-e^{-qz-kh})\frac{J_0(q\rho)q\,dq}{2\pi\lambda^2(k-q)},\tag{50}$$

where the RHS coincides with Eq. (46). We see that there is a difference of a multiplicative factor -8π between our solution and the solution of [12]. Since we have confirmed in the preceding section that our solution reproduces correctly the boundary condition, we believe that our result is the correct one. We cannot point out the origin of the mistake in [12] as the detailed derivation was not provided. However, [12] tested the result in the limit of $\lambda\to 0$, where Blake's expression for the Stokeslet near a wall [3] must be reproduced. It can observed from Eq. (49) that

$$\lim_{\lambda\to 0}\frac{F_1}{\lambda^4}=\frac{hz}{2}\int_0^\infty J_0(\rho q)e^{-q(z+h)}dq=\frac{hz}{2\sqrt{(z+h)^2+\rho^2}}.$$

Reference [12] claimed that this formula reproduces correctly the result of [3]. However, using Eq. (48), this formula gives

$$\lim_{\lambda\to 0}(-8\pi)w_3^3=\lim_{\lambda\to 0}\frac{1}{2\pi\lambda^4\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial F_1}{\partial\rho}\right)=\frac{hz}{4\pi R^3}\left(\frac{3\rho^2}{R^2}-2\right),\tag{51}$$

where the last line is just w_3^3 of [3]. Therefore, there is a discrepancy (missing multiplicative factor of -8π) between [12] and [3]. This also confirms that our result reduces to that in [3] in the limit of $\lambda\to 0$.

We compare our derivations with the remaining components of the flow provided in [12]. This reference gives

$$8\pi w_i^\alpha=-\frac{1}{2\pi\lambda^4}\frac{\partial}{\partial x_\alpha}\left[\frac{\partial G_1}{\partial x_i}+\left(\delta_{i3}\nabla^2-\frac{\partial^2}{\partial x_i\partial z}\right)G_2\right],$$

where the functions G_i are

$$G_1=\int_0^\infty J_0(\rho q)(k+q)k\,dq\,e^{-q(z+h)}(1-e^{(q-k)h}),$$

$$G_2=\int_0^\infty J_0(\rho q)(k+q)dq\,e^{-k(z+h)}(1-e^{(k-q)h}).\tag{52}$$

Since w_i^α can be obtained from the symmetry of Green's function, we then focus on w_α^β, which obeys

$$8\pi w_\alpha^\beta=-\frac{1}{2\pi\lambda^4}\frac{\partial^2}{\partial x_\beta\partial x_\alpha}\left(G_1-\frac{\partial G_2}{\partial z}\right).\tag{53}$$

We observe that

$$G_1-\frac{\partial G_2}{\partial z}=\int_0^\infty J_0(\rho q)(k+q)k\,dq[e^{-q(z+h)}(1-e^{(q-k)h})+e^{-k(z+h)}(1-e^{(k-q)h})].\tag{54}$$

We see that again the result in Eqs. (53) and (54) deviates from the expressions in Eq. (47) by a multiplicative factor of -8π. Since we show that our formula agrees with [3] in the low-frequency limit [see Eq. (120)], we believe that our derivation is the correct one.

VII. GREEN'S FUNCTION AND FELDERHOF'S RESULT

In this section we compare our results with those in Ref. [10]. The calculation of [10] exploited the Green's function rather than the correction flow defined by Eq. (24). The Fourier transform of Eq. (4) with respect to x and y gives

$$\left(\frac{d^2}{dz^2}-k^2\right)\hat{u}_i^k-\left(iq_\alpha\delta_{\alpha i}+\frac{d}{dz}\delta_{i3}\right)\hat{p}^k=-\delta_{ik}\delta(z-z'),\tag{55}$$

where without loss of generality we assumed that $\boldsymbol{x}'$ is at the z axis using translational invariance in the plane. This representation via the ordinary differential equation was introduced by Jones for the steady Stokes problem [13]. The equation can be readily solved [10]. Since the solution was provided in the form of a Green's function, for comparison with our solution we rewrite our result via the Green's function.

The Fourier transform of the Green's function is

$$\hat{G}_{ik}=\hat{G}^0_{ik}(\boldsymbol{q},z-h)-\hat{G}^0_{ik}(\boldsymbol{q},z+h)+8\pi\hat{w}_i^k(\boldsymbol{q},z,h)$$

[see Eqs. (5) and (24)]. We introduced the two-dimensional Fourier transform of the Green's function for the unbounded fluid given by Eq. (20). This is found by rewriting G^0 as [10]

$$G^0_{ik}=\frac{2\exp(-\lambda r)\delta_{ik}}{r}+\frac{2}{\lambda^2}\nabla_i\nabla_k\frac{1-\exp(-\lambda r)}{r}\tag{56}$$

and using $[\boldsymbol{\rho}=(r_1,r_2)]$ [23]

$$\int\frac{\exp\left(-i\boldsymbol{q}\cdot\boldsymbol{\rho}-\lambda\sqrt{\rho^2+r_3^2}\right)d\boldsymbol{\rho}}{2\pi\sqrt{\rho^2+r_3^2}}=\frac{\exp(-k|r_3|)}{k}.$$

We obtain, for the components of $\hat{G}^0_{ik}(\boldsymbol{q}, r_3)$,

$$\hat{G}^0_{\alpha\beta} = \frac{4\pi}{\lambda^2}\left(\frac{e^{-k|r_3|}(\lambda^2\delta_{\alpha\beta}+q_\alpha q_\beta)}{k} - \frac{q_\alpha q_\beta e^{-q|r_3|}}{q}\right),$$
$$\hat{G}^0_{\alpha 3} = \frac{4\pi i q_\alpha \mathrm{sgn}(r_3)}{\lambda^2}[\exp(-k|r_3|)-\exp(-q|r_3|)],$$
$$\hat{G}^0_{33} = \frac{4\pi q}{\lambda^2 k}[k\exp(-q|r_3|) - q\exp(-k|r_3|)]. \quad (57)$$

The component $\hat{G}^0_{\alpha 3}$, in contrast to the rest of the components, is proportional to $\mathrm{sgn}(r_3)$ and has a discontinuity at $r_3 = 0$. Combining this with Eqs. (39) and (40), we find

$$\hat{G}_{\alpha\beta} = \frac{4\pi\delta_{\alpha\beta}(e^{-k|z-h|}-e^{-k(z+h)})}{k} + \frac{4\pi q_\alpha q_\beta}{\lambda^2} \times\left(\frac{e^{-q(z+h)}-e^{-q|z-h|}}{q} + \frac{e^{-k|z-h|}-e^{-k(z+h)}}{k}\right) + \frac{8\pi k q_\alpha q_\beta (e^{-kh}-e^{-qh})(e^{-qz}-e^{-kz})}{\lambda^2 q(k-q)}.$$

A straightforward, albeit tedious, comparison demonstrates the agreement of this result for $\alpha = \beta$ with that in [10] [up to a multiplicative factor of $32\pi^3$ due to the difference of the respective definitions: 8π comes from the passage between the Green's function and the velocity and $(2\pi)^2$ from different factors in the definition of the Fourier transform]. For $\alpha \neq \beta$ we find

$$\hat{G}_{12} = \hat{G}_{21} = \frac{4\pi q_x q_y}{\lambda^2 q k(q-k)}[k(q+k)e^{-q(z+h)} - k(q-k)e^{-q|z-h|} + q(q-k)e^{-k|z-h|} - 2k^2 e^{-qh-kz} - 2k^2 e^{-kh-qz} - (q^2-2k^2-qk)e^{-k(z+h)}], \quad (58)$$

which is written in a form that simplifies the comparison with Eq. (4.7) in [14]. Notice a missing factor of k in the denominator on the RHS of Eq. (4.7) [14]. This is apparently a typo, since without this factor the dimensions are wrong and do not agree with the other components of the Green's function.

We find, for $\hat{G}_{33}$,

$$\hat{G}_{33} = \frac{4\pi q}{\lambda^2}\left(\frac{q(e^{-k(z+h)}-e^{-k|z-h|})}{k} + e^{-q|z-h|} - e^{-q(z+h)} + \frac{2q(e^{-kh}-e^{-qh})(e^{-qz}-e^{-kz})}{k-q}\right), \quad (59)$$

which agrees with [10]. For the remaining vertical force component of the components of the axially symmetric component we have (we use the fact that $z > 0$ in the physical domain)

$$\hat{G}_{\alpha 3} = \frac{4i\pi q_\alpha}{\lambda^2}\left(e^{-q(z+h)} - e^{-k(z+h)} \pm e^{-k|z-h|} \mp e^{-q|z-h|} + \frac{2(e^{-kh}-e^{-qh})(ke^{-kz}-qe^{-qz})}{k-q}\right), \quad (60)$$

which agrees with [14]. In contrast to the components considered previously, these have a discontinuity: The upper sign is for $z > h$ and the lower sign for $z < h$. Finally, since we proved that the solution obeys the symmetry constraint, we may use $\hat{G}_{3\alpha}(\boldsymbol{q}, z, h) = \hat{G}_{\alpha 3}(-\boldsymbol{q}, h, z)$ for the remaining component $\hat{G}_{3\alpha}$, which reads

$$\hat{G}_{3\alpha} = -\frac{4i\pi q_\alpha}{\lambda^2}\left(e^{-q(z+h)} - e^{-k(z+h)} \mp e^{-k|z-h|} \pm e^{-q|z-h|} + \frac{2(e^{-kz}-e^{-qz})(ke^{-kh}-qe^{-qh})}{k-q}\right), \quad (61)$$

where the upper and lower signs correspond to $z > h$ and $z < h$, respectively.

We conclude that our results agree with those of Felderhof up to a typo in [14]. Below we consider limiting cases where closed-form solutions in real space can be obtained.

VIII. DISTANT WALL LIMIT

The fundamental solution of the steady Stokes equations near a wall is characterized by a single intrinsic scale: the distance from the source to the wall, h. A time dependence introduces a scale δ, which introduces two different asymptotic regimes $H \gg \delta$ and $H \ll \delta$. Here we consider the distant wall limit of $H \gg \delta$ or $|\lambda|h \gg 1$. Note that this limiting case does not exist at $\lambda = 0$. The limit of $H \ll \delta$ is considered in Sec. XII.

We provide the Green's function in the distant wall limit through elementary functions. We start from the components derivable from the stream function in Eq. (46). We can assume that $|\lambda| \gg q_c$, where q_c is the characteristic value of q that dominates the integral. Then Eq. (46) becomes

$$\psi = \rho\frac{\partial}{\partial\rho}\int_0^\infty \frac{qJ_0(q\rho)dq}{2\pi\lambda^2}\frac{\exp(-R_3 q)}{k-q}, \quad (62)$$

which holds with exponential accuracy. We see that due to $R_3 \geqslant h$ the exponential factor imposes $q_c \lesssim 1/h$, confirming the self-consistency of the assumption $|\lambda| \gg q_c$ at $|\lambda|h \gg 1$. Considering the leading-order term in the expansion of the denominator in $|\lambda|/q_c$, we find

$$\psi \approx \rho\frac{\partial}{\partial\rho}\int_0^\infty \frac{qJ_0(q\rho)\exp(-R_3 q)dq}{2\pi\lambda^3} = -\rho\frac{\partial^2}{\partial\rho\partial R_3}\int_0^\infty\frac{J_0(q\rho)\exp(-R_3 q)dq}{2\pi\lambda^3} = -\frac{3\rho^2 R_3}{2\pi\lambda^3 R^5}, \quad (63)$$

where $R^2 = \rho^2 + R_3^2$. The correction has power-law smallness in $|\lambda|h$. The corresponding velocity components read

$$w_3^3 = \frac{3R_3(3\rho^2 - 2R_3^2)}{2\pi\lambda^3(\rho^2+R_3^2)^{7/2}}, \quad (64)$$

$$w_\rho^3 = \frac{3\rho(\rho^2 - 4R_3^3)}{2\pi\lambda^3(\rho^2+R_3^2)^{7/2}}. \quad (65)$$

We can consider similarly the high-frequency limit of the rest of the components of w_i^k. We have, from Eq. (39),

$$w_\alpha^\beta = -\nabla_\alpha\nabla_\beta\int\frac{k(e^{-kh}-e^{-qh})(e^{-qz}-e^{-kz})J_0(q\rho)dq}{2\pi\lambda^2(k-q)}.$$

Assuming $|\lambda|h \gg 1$, we obtain that

$$w_\alpha^\beta(\boldsymbol{R}) \approx \nabla_\alpha\nabla_\beta \int \frac{\exp(i\boldsymbol{q}\cdot\boldsymbol{\rho}-R_3q)d\boldsymbol{q}}{(2\pi\lambda)^2 q} = \frac{1}{2\pi\lambda^2}\nabla_\alpha\nabla_\beta\frac{1}{\left(\rho^2+R_3^2\right)^{1/2}} = \frac{3R_\alpha R_\beta - R^2\delta_{\alpha\beta}}{2\pi\lambda^2R^5}.$$

Finally, we have for w_3^α (that can also be derived from w_α^3 using the symmetry), from Eq. (39),

$$w_3^\alpha = \nabla_\alpha \int \frac{\exp(-kh)-\exp(-hq)}{\lambda^2(k-q)}\frac{e^{i\boldsymbol{q}\cdot\boldsymbol{\rho}}d\boldsymbol{q}}{(2\pi)^2} \times [k\exp(-qz) - q\exp(-kz)]. \tag{66}$$

This gives, at $\lambda h \gg 1$, that

$$w_3^\alpha \approx -\nabla_\alpha \int \frac{\exp(i\boldsymbol{q}\cdot\boldsymbol{\rho}-R_3q)d\boldsymbol{q}}{(2\pi\lambda)^2} = \frac{3R_3R_\alpha}{2\pi\lambda^2R^5}. \tag{67}$$

We observe that at $\lambda x_3' \gg 1$ (where x_3' corresponds to h above) the value of $|\boldsymbol{x}-\boldsymbol{x}'^*|$ in the definition

$$G_{ik}(\boldsymbol{x},\boldsymbol{x}') = G_{ik}^0(\boldsymbol{x}-\boldsymbol{x}') - G_{ik}^0(\boldsymbol{x}-\boldsymbol{x}'^*) + 8\pi w_i^k(\boldsymbol{x}-\boldsymbol{x}'^*)$$

obeys $\lambda|\boldsymbol{x}-\boldsymbol{x}'^*| \gg 1$ in the domain of definition $z>0$. Thus we can use the large λr asymptotic form of Eq. (20) given by

$$G_{ik}^0(\boldsymbol{r}) = -\frac{2}{\lambda^2r^3}\left(\delta_{ik}-\frac{3r_ir_k}{r^2}\right) + O(\exp(-\lambda r)) \tag{68}$$

to approximate $G_{ik}^0(\boldsymbol{x}-\boldsymbol{x}'^*)$. We can also use the formulas of this section to approximate $w_i^k(\boldsymbol{x}-\boldsymbol{x}'^*, x_3')$. We find that the leading-order correction to $G_{ik}^0(\boldsymbol{x}-\boldsymbol{x}')$ due to the wall at a large distance is given by

$$G_{i3}(\boldsymbol{x},\boldsymbol{x}') - G_{i3}^0(\boldsymbol{x}-\boldsymbol{x}') \approx \frac{2}{\lambda^2R^3}\left(\delta_{i3}-\frac{3R_iR_3}{R^2}\right),$$
$$G_{\alpha\beta}(\boldsymbol{x},\boldsymbol{x}') - G_{\alpha\beta}^0(\boldsymbol{x}-\boldsymbol{x}') \approx \frac{2}{\lambda^2R^3}\left(\frac{3R_\alpha R_\beta}{R^2}-\delta_{\alpha\beta}\right),$$
$$G_{3\alpha}(\boldsymbol{x},\boldsymbol{x}') - G_{3\alpha}^0(\boldsymbol{x}-\boldsymbol{x}') \approx \frac{6R_\alpha R_3}{\lambda^2R^5}. \tag{69}$$

Thus the magnitude of the correction to the Green's function in the infinite fluid at an indefinitely large distance to the wall decays as the inverse of the third power of the distance to the source's image. It is proportional to the square of the oscillatory penetration length (which is assumed to be much smaller than the distance to the wall). The decay law is the same as for a simple image flow [see Eq. (68)], however, the correction flow w modifies the decay coefficients.

We compare Eqs. (69) with the result for the Green's function of the steady Stokes equations,

$$G_{ik}(\boldsymbol{x},\boldsymbol{x}',\lambda=0) - G_{i3}^0(\boldsymbol{x}-\boldsymbol{x}',\lambda=0) = O\left(\frac{1}{R}\right). \tag{70}$$

In this case also the order of the correction $G_{ik}-G_{ik}^0$ can be obtained from the flow of the simple image where R^{-1} is the law of decay of the Oseen tensor [3]. We see that the decay is much slower: The limits of zero frequency and infinite separation to the wall do not commute.

The result in Eqs. (69) becomes transparent upon comparison with the analogous result for the perfect slip boundary condition in the next section.

IX. COMPARISON WITH PERFECT SLIP BOUNDARY CONDITION

The calculation of the Green's function $G_{ik}^{\text{slip}}(\boldsymbol{x},\boldsymbol{x}')$ for the case of perfect slip boundary condition is much simpler than for the no-slip boundary condition. The result is found using the image point force [14] (the Green's function of infinite fluid is defined there without the factor of 2; cf. [24]),

$$G_{i\alpha}^{\text{slip}}(\boldsymbol{x},\boldsymbol{x}') = G_{i\alpha}^0(\boldsymbol{r}) + G_{i\alpha}^0(\boldsymbol{R}),$$
$$G_{i3}^{\text{slip}}(\boldsymbol{x},\boldsymbol{x}') = G_{i3}^0(\boldsymbol{r}) - G_{i3}^0(\boldsymbol{R}). \tag{71}$$

Indeed, it can be readily seen from Eq. (20) that the normal component of the velocity vanishes, $G_{3\alpha}^{\text{slip}}(z=0) = G_{33}^{\text{slip}}(z=0) = 0$, at the wall by $r_\alpha = R_\alpha$ and $r_3 = -R_3$. The remaining stress condition $\nabla_\alpha G_{3k}^{\text{slip}} + \nabla_3 G_{\alpha k}^{\text{slip}} = 0$ holds by observing that all components $\nabla_\alpha G_{3\beta}^{\text{slip}}$, $\nabla_\alpha G_{33}^{\text{slip}}$, $\nabla_3 G_{\alpha\beta}^{\text{slip}}$, and $\nabla_3 G_{\alpha 3}^{\text{slip}}$ vanish at the wall separately. We find then readily that the leading-order correction to $G_{ik}^0(\boldsymbol{x}-\boldsymbol{x}')$ because of the wall at a large distance is given by

$$G_{i3}^{\text{slip}}(\boldsymbol{x},\boldsymbol{x}') - G_{i3}^0(\boldsymbol{x}-\boldsymbol{x}') \approx \frac{2}{\lambda^2R^3}\left(\delta_{i3}-\frac{3R_iR_3}{R^2}\right),$$
$$G_{\alpha\beta}^{\text{slip}}(\boldsymbol{x},\boldsymbol{x}') - G_{\alpha\beta}^0(\boldsymbol{x}-\boldsymbol{x}') \approx \frac{2}{\lambda^2R^3}\left(\frac{3R_\alpha R_\beta}{R^2}-\delta_{\alpha\beta}\right),$$
$$G_{3\alpha}^{\text{slip}}(\boldsymbol{x},\boldsymbol{x}') - G_{3\alpha}^0(\boldsymbol{x}-\boldsymbol{x}') \approx \frac{6R_\alpha R_3}{\lambda^2R^5}, \tag{72}$$

where we used Eq. (68). We see that the leading correction to G_{ik}^{slip} is the same as that for G_{ik} as given by Eq. (69). The details of the boundary conditions at the wall become irrelevant at large distances.

X. FORCE ON AN OSCILLATING SPHERE FAR FROM A WALL

In this section we demonstrate how the asymptotic properties of the Green's function derived in the preceding section can be used to find the force on the sphere oscillating far from a rigid wall. We recall that the force on the sphere that moves at constant velocity at a large distance h from the wall is given by the famous result by Lorentz,

$$\frac{F_\perp(\lambda=0)}{F_{St}} = 1+\frac{9}{8h}, \quad \frac{F_\parallel(\lambda=0)}{F_{St}} = 1+\frac{9}{16h}, \tag{73}$$

with corrections of $O(h^{-3})$ (see [1,2,25]). Here F_{St} is the Stokes force on a sphere translating in an unbounded fluid and $F_\perp$ and $F_\parallel$ are force components for motion perpendicular and parallel to the wall, respectively (this notation differs from that in [1]). This result, which can be derived using images, can also be obtained from the integral equation, which we use extensively in this work (see Appendix E for the derivation). In this section we use the integral equation on the surface traction to derive the counterpart of this result for the case of $H \gg \delta$ and $H \gg a$ (the relation between δ and a is unconstrained).

We consider the integral equation on the surface traction given by Eq. (10). In the limit of large distances $|\lambda|h \gg 1$ and $h \gg 1$ all points in the integrand of Eq. (10) obey $|\lambda|x'_3 \gg 1$. Thus we can use the approximation given by Eq. (69). On using $R_i \approx 2h\delta_{i3}$ we find in the leading order that $\delta G_{ik}(\boldsymbol{x}, \boldsymbol{x}')$, defined by

$$\delta G_{ik}(\boldsymbol{x}, \boldsymbol{x}') = G_{ik}(\boldsymbol{x}, \boldsymbol{x}') - G^0_{ik}(\boldsymbol{x} - \boldsymbol{x}'), \tag{74}$$

obeys

$$\delta G_{i3}(\boldsymbol{x}, \boldsymbol{x}') \approx -\frac{\delta_{i3}}{2\lambda^2 h^3}, \quad \delta G_{\alpha\beta}(\boldsymbol{x}, \boldsymbol{x}') \approx -\frac{\delta_{\alpha\beta}}{4\lambda^2 h^3}, \tag{75}$$

with $\delta G_{3\alpha}$ of higher order in h^{-1}. Thus, in the leading order $\delta G_{ik}(\boldsymbol{x}, \boldsymbol{x}')$ is a constant diagonal matrix, two of whose eigenvalues are equal because of the planar symmetry. Using Eqs. (74) and (75) in Eq. (10), we find that the equation on the surface traction takes the form

$$\hat{V}_i + \frac{\hat{V}_3\delta_{i3}}{12h^3} + \frac{\hat{V}_\alpha\delta_{i\alpha}}{24h^3} - \frac{\delta_{i3}F_3}{16\pi\lambda^2h^3} - \frac{\delta_{i\alpha}F_\alpha}{32\pi\lambda^2h^3}$$
$$= \int_{|\boldsymbol{x}'-\tilde{\boldsymbol{x}}_0|=1} G^0_{il}(\boldsymbol{x}, \boldsymbol{x}')[\lambda^2 x'_l(\hat{V}\cdot\boldsymbol{x}') - \tilde{\sigma}_{lr}(\boldsymbol{x}')]\frac{dS'}{8\pi}. \tag{76}$$

In the leading order we can use on the left-hand side (LHS) $\boldsymbol{F} = \boldsymbol{F}^\infty$, where $\boldsymbol{F}^\infty$ is the force on a sphere oscillating in infinite space [1],

$$\boldsymbol{F}^\infty = -6\pi\left(1 + \lambda + \frac{\lambda^2}{9}\right)\hat{V}. \tag{77}$$

Thus Eq. (76) becomes

$$\hat{V}_i + \frac{\hat{V}_3\delta_{i3}}{12h^3} + \frac{\hat{V}_\alpha\delta_{i\alpha}}{24h^3} + \left(\frac{3\delta_{i3}\hat{V}_3}{8\lambda^2h^3} + \frac{3\delta_{i\alpha}\hat{V}_\alpha}{16\lambda^2h^3}\right)\left(1 + \lambda + \frac{\lambda^2}{9}\right)$$
$$= \int_{|\boldsymbol{x}'-\tilde{\boldsymbol{x}}_0|=1} G^0_{il}(\boldsymbol{x}, \boldsymbol{x}')[\lambda^2 x'_l(\hat{V}\cdot\boldsymbol{x}') - \tilde{\sigma}_{lr}(\boldsymbol{x}')]\frac{dS'}{8\pi}. \tag{78}$$

The solution of this equation at $h = \infty$ is [1]

$$\tilde{\sigma}_{lr}(\boldsymbol{x})(h = \infty) = -\frac{3}{2}\left((1 + \lambda)\delta_{li} + \frac{\lambda^2}{3}x_l x_i\right)\hat{V}_i, \tag{79}$$

where we used $|\boldsymbol{x}| = 1$. Integration of $\tilde{\sigma}_{lr}(\boldsymbol{x})(h = \infty)$ over the particle surface reproduces Eq. (77). Insertion of this solution in Eq. (78) with $h = \infty$ gives a useful identity

$$\int_{|\boldsymbol{x}'-\tilde{\boldsymbol{x}}_0|=1} G^0_{il}(\boldsymbol{x} - \boldsymbol{x}')[(1 + \lambda)\delta_{lk} + \lambda^2 x'_l x'_k]\frac{3dS'}{16\pi} = \delta_{ik}. \tag{80}$$

We look for the solution of Eq. (78) in the form

$$\tilde{\sigma}_{lr}(\boldsymbol{x}) = \tilde{\sigma}_{lr}(\boldsymbol{x})(h = \infty) + \frac{\delta\sigma_{lr}}{4h^3}, \tag{81}$$

where $\delta\sigma_{lr}$ obeys

$$\frac{\hat{V}_3\delta_{i3}}{3} + \frac{\hat{V}_\alpha\delta_{i\alpha}}{6} + \left(\frac{3\delta_{i3}\hat{V}_3}{2\lambda^2} + \frac{3\delta_{i\alpha}\hat{V}_\alpha}{4\lambda^2}\right)\left(1 + \lambda + \frac{\lambda^2}{9}\right)$$
$$= -\frac{1}{8\pi}\int_{|\boldsymbol{x}'-\tilde{\boldsymbol{x}}_0|=1} G^0_{il}(\boldsymbol{x} - \boldsymbol{x}')\delta\sigma_{lr}(\boldsymbol{x}')dS'. \tag{82}$$

Thus we study the equation

$$g_i = -\frac{1}{8\pi}\int_{|\boldsymbol{x}'-\tilde{\boldsymbol{x}}_0|=1} G^0_{il}(\boldsymbol{x} - \boldsymbol{x}')\delta\sigma_{lr}(\boldsymbol{x}')dS', \tag{83}$$

where g_i is given by the first line of Eq. (82). Isotropy dictates that the solution has a form similar to Eq. (79),

$$\delta\sigma_{lr}(\boldsymbol{x}) = [a(\lambda)\delta_{lk} + b(\lambda)\lambda^2 x_k x_l]g_k, \tag{84}$$

with certain functions $a(\lambda)$ and $b(\lambda)$. Substituting (84) into (83), we find that these functions obey

$$g_i = -\frac{a(\lambda)g_k}{8\pi}\int_{|\boldsymbol{x}'-\tilde{\boldsymbol{x}}_0|=1} G^0_{ik}(\boldsymbol{x} - \boldsymbol{x}')dS'$$
$$- \frac{b(\lambda)\lambda^2 g_k}{8\pi}\int_{|\boldsymbol{x}'-\tilde{\boldsymbol{x}}_0|=1} G^0_{il}(\boldsymbol{x} - \boldsymbol{x}')x'_l x'_k dS'. \tag{85}$$

We find, rewriting the last integral with the help of Eq. (80), that

$$\left(1 + \frac{2b}{3}\right)g_i = \frac{[b(1 + \lambda) - a]g_k}{8\pi}\int_{|\boldsymbol{x}'-\tilde{\boldsymbol{x}}_0|=1} G^0_{ik}(\boldsymbol{x} - \boldsymbol{x}')dS'.$$

Due to isotropy, the last integral has the form

$$\frac{1}{8\pi}\int_{|\boldsymbol{x}'-\tilde{\boldsymbol{x}}_0|=1} G^0_{ik}(\boldsymbol{x} - \boldsymbol{x}')dS' = c(\lambda)\delta_{ik} + d(\lambda)x_i x_k, \tag{86}$$

with certain functions c and d. This gives

$$\left(1 + \frac{2b}{3}\right)g_i = [b(1 + \lambda) - a]cg_i + [b(1 + \lambda) - a]dg_k x_i x_k.$$

We find, assuming that d is nonzero, that $a = -3(1 + \lambda)/2$ and $b = -3/2$. We conclude that

$$\delta\sigma_{lr}(\boldsymbol{x}) = -\tfrac{3}{2}[(1 + \lambda)\delta_{li} + \lambda^2 x_l x_i]\hat{V}_i. \tag{87}$$

We find, combining the above, that the force is given by

$$\boldsymbol{F} = \boldsymbol{F}^\infty - \frac{3\pi}{2h^3}\left(1 + \lambda + \frac{\lambda^2}{3}\right)\boldsymbol{g}. \tag{88}$$

Finally, using the definition of $\boldsymbol{g}$, we find

$$F_\perp = -6\pi\left(1 + \lambda + \frac{\lambda^2}{9}\right)\hat{V}_3$$
$$- \frac{3\pi}{2h^3}\left(1 + \lambda + \frac{\lambda^2}{3}\right)\left[\frac{1}{3} + \frac{3}{2\lambda^2}\left(1 + \lambda + \frac{\lambda^2}{9}\right)\right]\hat{V}_3,$$
$$F_\parallel = -6\pi\left(1 + \lambda + \frac{\lambda^2}{9}\right)\hat{V}_\parallel - \frac{3\pi}{2h^3}\left(1 + \lambda + \frac{\lambda^2}{3}\right)$$
$$\times\left[\frac{1}{6} + \frac{3}{4\lambda^2}\left(1 + \lambda + \frac{\lambda^2}{9}\right)\right]\hat{V}_\parallel. \tag{89}$$

Derivation of the closed-form results for the force in Eqs. (89) is one of the main results of our work. Comparing this result with Eq. (73), we see that the time dependence diminishes the effect of a distant wall. Correction to the force is smaller by orders of magnitude, decaying as the inverse of the third power of the distance, rather than the inverse of the distance. The forces agree asymptotically if Eq. (89), which holds at $|\lambda| \gg 1/h$, is continued to $|\lambda| \sim 1/h$.

The derivation of Eqs. (89) requires the large-distance asymptotic form of the Green's function only. The preceding section then implies that the force for the perfect slip boundary is also given by Eqs. (89) (cf. [11]).

XI. HIGH-FREQUENCY LIMIT AND POTENTIAL FLOW

The derivation of Eqs. (89) involved no constraint on the ratio of a and δ. Restriction to the case $H \gg a \gg \delta$ gives the high-frequency limit where δ is the smallest scale of the problem. The force in the case can be obtained by taking in Eqs. (89) the limit of $|\lambda| \gg 1$. We find

$$F_\perp = -\frac{2\pi\lambda^2}{3}\left(1+\frac{3}{8h^3}\right)\hat{V}_3, \tag{90a}$$

$$F_\parallel = -\frac{2\pi\lambda^2}{3}\left(1+\frac{3}{16h^3}\right)\hat{V}_\parallel. \tag{90b}$$

This result can be interpreted as a distant wall correction to the added mass and it was derived from the potential flow approximation in [11,21]. It differs from what can be inferred from (4.5) of [10] with the reasons for the discrepancy discussed in detail in [11] (see also [21]). Below we rederive Eq. (90) from the potential flow approximation. This gives us the opportunity to present a detailed form of the flow and make refinements.

The authors of [11] observed that in the limit of large frequency the flow is potential everywhere apart from a narrow viscous layer around the sphere with typical thickness δ. The identical situation holds in the infinite fluid [18]. This observation can be proved using the integral representation. For simplicity of the derivation we consider the flow $\tilde{\boldsymbol{u}}^s$ around a sphere oscillating near a wall with perfect slip boundary conditions. It can be verified by repeating the considerations for the no-slip boundary conditions that the integral representation holds,

$$\tilde{u}_i^s(\boldsymbol{x}) = \int_{|\boldsymbol{x}'-\tilde{\boldsymbol{x}}_0|=1} G_{il}^{\text{slip}}(\boldsymbol{x},\boldsymbol{x}')[\lambda^2 x_l'(\hat{V}\cdot\boldsymbol{x}') - \tilde{\sigma}_{lr}(\boldsymbol{x}')]\frac{dS'}{8\pi}. \tag{91}$$

In this case the contribution of the wall boundary in the volume integration of Eq. (B1) of Appendix B vanishes by the vanishing of tangential stresses [cf. Eq. (9)]. We observe that Eq. (68) can be written as

$$G_{ik}^0(\boldsymbol{r}) = -\frac{2}{\lambda^2}\nabla_i\frac{r_k}{r^3} + O(\exp(-\lambda r)), \tag{92}$$

so the flow described by G^0 is potential beyond the distance $\sim\delta$ from the source with exponential accuracy. We find, using the form of G_{il}^{slip} given by Eq. (71), that beyond the viscous layer of width $\sim\delta$ around the sphere $\tilde{\boldsymbol{u}}^s$ is a continuous sum of potential flows [see Eq. (91)]. We conclude that $\tilde{\boldsymbol{u}}^s = \nabla\phi$ holds beyond the narrow layer whose width is of order δ. The potential ϕ obeys the Laplace equation by incompressibility (see the similar proof in [26]). The potential flow obeys the usual boundary conditions on the vanishing normal component of the velocity on the sphere and the wall [18].

We consider the case of velocity perpendicular to the wall, $\hat{V} = \hat{V}_z\hat{z}$. The solution can be written as the sum of potentials of the sphere, the image sphere moving in the opposite direction, and a correction,

$$\phi \approx \frac{1}{2}\hat{V}_z\frac{\partial}{\partial z}\frac{1}{r_0} - \frac{1}{2}\hat{V}_z\frac{\partial}{\partial z}\frac{1}{R_0} + \delta\phi, \tag{93}$$

where we defined $\boldsymbol{r}_0 = \boldsymbol{x}-\boldsymbol{x}_0$ and $\boldsymbol{R}_0 = \boldsymbol{x}-\boldsymbol{x}_0^*$. This corresponds to searching for $\tilde{\boldsymbol{u}}^s$ outside the viscous layer in the form

$$\tilde{\boldsymbol{u}}^s = \hat{V}_z\frac{3r_{0z}\boldsymbol{r}_0 - r_0^2\hat{z}}{2r_0^5} - \hat{V}_z\frac{3R_{0z}\boldsymbol{R}_0 - R_0^2\hat{z}}{2R_0^5} + \nabla\delta\phi, \tag{94}$$

where $\hat{z}$ is a unit vector in the z direction (cf. [18,27]). The first two terms on the RHS produce the zero normal component of velocity on the wall. Thus the correction $\delta\phi$ obeys the Laplace equation with boundary conditions

$$\boldsymbol{r}_0\cdot\left(\nabla\delta\phi - \hat{V}_z\frac{3R_{0z}\boldsymbol{R}_0 - R_0^2\hat{z}}{2R_0^5}\right)\bigg|_{r_0=1} = 0 \tag{95}$$

and $\partial_z\delta\phi(z=0)=0$. The boundary conditions can be simplified by neglecting terms of order higher than h^{-3} as

$$\boldsymbol{r}_0\cdot\left(\nabla\delta\phi - \frac{\hat{V}_z\hat{z}}{8h^3}\right)\bigg|_{r_0=1} = 0, \quad \partial_z\delta\phi(z=0)=0. \tag{96}$$

We can neglect the boundary condition at the wall to order h^{-3}. We find the problem of a sphere that moves at constant velocity $\hat{V}_z\hat{z}/8h^3$. The solution is the first term on the RHS of Eq. (93) with the appropriate change of the velocity. We conclude that up to terms of order h^{-3} the flow has a simple structure

$$\phi = \frac{\hat{V}_z}{2}\left(1+\frac{1}{8h^3}\right)\frac{\partial}{\partial z}\frac{1}{r_0} - \frac{1}{2}\hat{V}_z\frac{\partial}{\partial z}\frac{1}{R_0},$$
$$\tilde{\boldsymbol{u}}^s = \hat{V}_z\left(1+\frac{1}{8h^3}\right)\frac{3r_{0z}\boldsymbol{r}_0 - r_0^2\hat{z}}{2r_0^5} - \hat{V}_z\frac{3R_{0z}\boldsymbol{R}_0 - R_0^2\hat{z}}{2R_0^5}. \tag{97}$$

The force is given by the sum of components $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ due to the first and second terms on the RHS of the preceding equation, respectively. We have [18]

$$\boldsymbol{F}_1 = -\frac{2\pi}{3}\left(1+\frac{1}{8h^3}\right)\frac{d\hat{V}_z}{dt}\hat{z}. \tag{98}$$

We consider the component of the force $\boldsymbol{F}_2$ that derives from the last term on the RHS of Eq. (97) that we designate by ϕ'. We have

$$\phi' = -\frac{1}{2}\hat{V}_z\frac{\partial}{\partial z}\frac{1}{R_0}, \quad \nabla\phi' = -\hat{V}_z\frac{3R_{0z}\boldsymbol{R}_0 - R_0^2\hat{z}}{2R_0^5}. \tag{99}$$

This gives, for the pressure,

$$p = -\frac{(\nabla\phi')^2}{2} + \hat{V}_\alpha\nabla_\alpha\phi' - \hat{V}_z\nabla_z\phi' + \frac{1}{2}\frac{d\hat{V}_z}{dt}\frac{\partial}{\partial z}\frac{1}{R_0} \tag{100}$$

(cf. [18]). The force is determined by the integral of the last term over the sphere

$$(F_2)_i = -\frac{1}{2}\frac{d\hat{V}_z}{dt}\int_{r_0=1} dS\, r_{0i}\frac{\partial}{\partial z}\frac{1}{R_0}. \tag{101}$$

We use the expansion of the last term near $\boldsymbol{x}=\boldsymbol{x}_0$,

$$(F_2)_i \approx -\frac{1}{2}\frac{d\hat{V}_z}{dt}\int_{r_0=1} dS\, r_{0i}r_{0l}\nabla_l\frac{\partial}{\partial z}\frac{1}{R_0}\Big|_{\boldsymbol{x}=\boldsymbol{x}_0} = -\frac{2\pi}{3}\frac{d\hat{V}_z}{dt}\nabla_i\frac{\partial}{\partial z}\frac{1}{R_0}\Big|_{\boldsymbol{x}=\boldsymbol{x}_0} = -\frac{\pi}{6h^3}\frac{d\hat{V}_z}{dt}\hat{z}, \quad (102)$$

where we dropped the terms of order higher than h^{-3}. We find the total force $\boldsymbol{F}=\boldsymbol{F}_1+\boldsymbol{F}_2$ given by

$$\boldsymbol{F} = -\frac{2\pi}{3}\left(1+\frac{3}{8h^3}\right)\frac{d\hat{V}_z}{dt}\hat{z}. \quad (103)$$

This agrees with the result for the ideal flow [27–29] and confirms Eq. (90a). A similar derivation can be readily performed for the particle motion parallel to the wall, confirming Eq. (90b) (cf. [11]).

The derivation of the force in Eq. (103) assumes that the force exerted by the potential flow outside the boundary viscous layer is equal to the full force obtained by the integration of the stress tensor over the particle surface where the flow is nonideal. This equality is simpler for steady Stokes flows [26] and demands care in the unsteady case. We have, using the separation scale Δ obeying $\delta \ll \Delta \ll 1$, that

$$F_i = \int_{|\boldsymbol{x}-\boldsymbol{x}_0|=1}\sigma_{ir}dS = \int_{|\boldsymbol{x}-\boldsymbol{x}_0|=1+\Delta}\sigma_{ir}dS - \int_{1<|\boldsymbol{x}-\boldsymbol{x}_0|<1+\Delta}\nabla_k\sigma_{ik}dV, \quad (104)$$

where the integral of the stress tensor over the particle surface is written as an integral over a sphere outside the viscous layer and the volume integral of the stress tensor divergence. The viscous part of the stress tensor $-p\delta_{ik}+2\nabla_i\nabla_k\phi$ holding outside the viscous layer does not contribute the surface integral on the RHS (this is readily verified by transforming the surface integral in the integral over the volume of the sphere). We find, using $\lambda^2\tilde{u}_i=\nabla_k\sigma_{ik}$, that

$$F_i = F_i^{id} - \lambda^2\int_{1<|\boldsymbol{x}-\boldsymbol{x}_0|<1+\Delta}\tilde{u}_i dV, \quad (105)$$

where F_i^{id} is the ideal flow force given by the RHS of Eq. (103). (This force has a correction of order of δ due to the difference of the particle surface and the sphere outside the boundary layer. This correction is always present due to the neglect of this difference in the boundary conditions for the potential flow.) The last term in Eq. (105) is of order $\lambda^2\hat{V}_i\delta$. The $O(h^{-3})$ correction in Eq. (103) materializes given that $\delta \ll h^{-3}$, which is a more restrictive requirement than the original condition $\delta \ll 1 \ll h$. Thus, our asymptotic result for the force in Eq. (89) holds for a considerably less restrictive range of parameters.

XII. LOW-FREQUENCY LIMIT

In this section we derive the leading-order correction in the frequency using a Taylor expansion of the integrands for the inverse Fourier transform of the fundamental solution in λ. This completes the study started in Sec. III. We confirm the observation of that section that the higher-order terms of the expansion produce divergent integrals signifying the singularity of the asymptotic expansion. The results hold under the condition $H \ll \delta$, thus completing the study of asymptotic forms of the Green's function started in Sec. VIII.

A. Stream function at low frequency

We start the study of the low-frequency limit from the components w_i^3 given by Eq. (45) via the stream function ψ. To find the low-frequency limit of ψ we observe that factors in Eq. (46) obey (up to quadratic order in λ)

$$\frac{\exp[-h(q+\Delta)]-\exp(-hq)}{\Delta} = \frac{h(\lambda^2 h-4q)\exp(-hq)}{4q}, \quad \frac{e^{-z(q+\Delta)}-e^{-zq}}{\lambda^2} = \frac{z(\lambda^2+\lambda^2 qz-4q^2)e^{-zq}}{8q^3}, \quad (106)$$

where we introduced $\Delta = k-q \approx \lambda^2/2q - \lambda^4/8q^3$ (hereafter, smallness of the complex number λ implies that both real and imaginary parts are small). We find

$$\psi = \rho hz\int_0^\infty \frac{J_1(q\rho)\exp(-R_3 q)dq}{16\pi q}[\lambda^2(1+R_3 q)-4q^2]. \quad (107)$$

The zeroth-order term produces

$$\psi(\lambda=0) = \rho hz\frac{\partial}{\partial\rho}\int_0^\infty\frac{J_0(q\rho)\exp(-R_3 q)dq}{4\pi} = -\frac{\rho^2 hz}{4\pi R^3}.$$

This stream function reproduces the stationary Stokeslet flow of [3]

$$w_3^3(\lambda=0) = \frac{3\rho^2 hz}{4\pi R^5} - \frac{2hz}{4\pi R^3} = \frac{hz}{4\pi R^3}\left(\frac{3\rho^2}{R^2}-2\right), \quad w_\rho^3(\lambda=0) = -\frac{1}{\rho}\frac{\partial\psi}{\partial z} = \frac{\rho h}{4\pi R^3}\left(1-\frac{3zR_3}{R^2}\right). \quad (108)$$

The λ^2 term in Eq. (107) is found using [23]

$$\int_0^\infty\frac{J_1(q\rho)\exp(-R_3 q)dq}{q} = \frac{\rho}{R_3+R}, \quad \int_0^\infty J_1(q\rho)\exp(-R_3 q)dq = \frac{\rho}{R(R_3+R)}. \quad (109)$$

We obtain

$$\psi = -\frac{\rho^2 hz}{4\pi R^3} + \frac{\lambda^2\rho^2 hz}{16\pi R} = -\frac{\rho^2 hz}{4\pi R^3}\left(1-\frac{\lambda^2 R^2}{4}\right). \quad (110)$$

We conclude that the flow up to quadratic order in λ is

$$w_3^3 = -\frac{2hz}{4\pi R^3}\left(1-\frac{\lambda^2 R^2}{4}\right) + \frac{3\rho^2 hz}{4\pi R^5}\left(1-\frac{\lambda^2 R^2}{12}\right), \quad w_\rho^3 = \frac{\rho h}{4\pi R^3}\left(1-\frac{\lambda^2 R^2}{4}\right) - \frac{3\rho hzR_3}{4\pi R^5}\left(1-\frac{\lambda^2 R^2}{12}\right). \quad (111)$$

We observe that the correction is purely imaginary, corresponding to the flow correction out of phase with the main term. It can be seen readily that higher-order expansion in λR is singular (discussed below), implying that the correction to this formula can be of order lower than $\lambda^4 R^4$ (cf. Sec. III). We conclude that

$$G_{33}(\boldsymbol{x},\boldsymbol{x}')=\frac{r^2+r_3^2}{r^3}-\frac{R^2+R_3^2}{R^3}+\frac{r\lambda^2}{4}\left(3-\frac{r_3^2}{r^2}\right)-\frac{R\lambda^2}{4}\left(3-\frac{R_3^2}{R^2}\right)-\frac{4x_3'z}{R^3}\left(1-\frac{\lambda^2R^2}{4}\right)+\frac{6\rho^2x_3'z}{R^5}\left(1-\frac{\lambda^2R^2}{12}\right),$$

$$G_{\alpha 3}(\boldsymbol{x},\boldsymbol{x}')=\frac{r_\alpha r_3}{r^3}-\frac{R_\alpha R_3}{R^3}-\frac{\lambda^2 r_\alpha r_3}{4r}+\frac{\lambda^2R_\alpha R_3}{4R}+\frac{2R_\alpha x_3'}{R^3}\left(1-\frac{\lambda^2R^2}{4}\right)-\frac{6R_\alpha x_3'zR_3}{R^5}\left(1-\frac{\lambda^2R^2}{12}\right), \tag{112}$$

where we used Eqs. (21), (24), and (111). We observed that $|\lambda^2(\boldsymbol{x}-\boldsymbol{x}'^*)^2|\ll 1$ implies that $|\lambda^2(\boldsymbol{x}-\boldsymbol{x}')^2|\ll 1$ at $z>0$.

B. Nonsymmetric components

We consider the components of the Green's function that cannot be derived using the stream function. We introduce the representation of the solution as a series in λ^2. Coefficients of the terms of order λ^4 and higher contain nonintegrable singularities at $q=0$ in accord with the predictions of Sec. III. We start with w_α^β, for which we have

$$\hat{w}_\alpha^\beta=\frac{kq_\alpha q_\beta(e^{-kh}-e^{-qh})(e^{-qz}-e^{-kz})}{q\Delta^2(k+q)} \tag{113}$$

[cf. Eq. (39)]. We use that

$$e^{-kh}-e^{-qh}=\sum_{n=1}^{\infty}\frac{(-h\Delta)^n e^{-qh}}{n!}, \tag{114}$$

and similarly for $\exp(-kz)-\exp(-kz)$. We further find

$$\hat{w}_\alpha^\beta=-\frac{q_\alpha q_\beta\exp(-R_3q)}{q}\sum_{n,k=1}^{\infty}\frac{(-1)^{n+k}h^nz^kq^{n+k-2}}{n!k!}\times\frac{\sqrt{1+\epsilon^2}(\sqrt{1+\epsilon^2}-1)^{n+k-2}}{\sqrt{1+\epsilon^2}+1}, \tag{115}$$

where $\epsilon^2=\lambda^2/q^2$. We introduce the Taylor series

$$\frac{\sqrt{1+\epsilon^2}(\sqrt{1+\epsilon^2}-1)^{n+k-2}}{\sqrt{1+\epsilon^2}+1}=\epsilon^{2(n+k-2)}\sum_{l=0}^{\infty}c_{n+k-2}^l\epsilon^{2l}, \tag{116}$$

with certain coefficients c_{n+k-2}^l. Then

$$\hat{w}_\alpha^\beta=-\frac{q_\alpha q_\beta\exp(-R_3q)}{q}\sum_{n,k=1}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{n+k}c_{n+k-2}^lh^nz^k}{n!k!}\times\lambda^{2(n+k+l-2)}q^{2-k-n-2l}. \tag{117}$$

Regrouping the terms gives the solution as a Taylor series in λ,

$$\hat{w}_\alpha^\beta=-\frac{q_\alpha q_\beta\exp(-R_3q)}{q}\sum_{p=0}^{\infty}\lambda^{2p}\times\sum_{n,k=1}^{\infty}\sum_{l=0}^{\infty}\delta_{n+k+l-2,p}\frac{(-1)^{n+k}c_{n+k-2}^lh^nz^k}{n!k!q^{p+l}}$$
$$=-\frac{q_\alpha q_\beta hz\exp(-R_3q)}{q}\left[c_0^0+\frac{\lambda^2(2c_0^1-c_1^0qR_3)}{2q^2}+\lambda^4\left(\frac{c_0^2}{q^4}-\frac{c_1^1R_3}{2q^3}+\frac{c_2^0(2h^2+3hz+2z^2)}{12q^2}\right)+O(\lambda^6)\right]. \tag{118}$$

We see that terms of order λ^4 and higher contain nonintegrable singularities at $q=0$. We will limit the analysis to the leading-order correction that is determined by

$$c_0^0=\tfrac{1}{2},\quad c_0^1=\tfrac{1}{8},\quad c_1^0=\tfrac{1}{4}, \tag{119}$$

where the numerical values are readily confirmed using the definition in Eq. (116). Therefore,

$$\hat{w}_\alpha^\beta\approx-\frac{q_\alpha q_\beta hz\exp(-qR_3)}{2q}\left(1+\frac{\lambda^2(1-qR_3)}{4q^2}\right), \tag{120}$$

where the correction is small provided all the parameters λh, λz, and λ/q are small. This reproduces the solution of [3] at $\lambda=0$ upon the division by 2π due to differences in the definition of the Fourier transform. Then in real space, correspondingly,

$$w_\alpha^\beta\approx\int\frac{q_\alpha q_\beta hz\exp(i\boldsymbol{q}\cdot\boldsymbol{\rho}-qR_3)d\boldsymbol{q}}{8\pi^2q}\left(\frac{\lambda^2(qR_3-1)}{4q^2}-1\right), \tag{121}$$

where $\boldsymbol{\rho}=(R_1,R_2)$. For $\lambda=0$ the integral is found from

$$\int\frac{q_\alpha q_\beta\exp(i\boldsymbol{q}\cdot\boldsymbol{\rho}-qR_3)d\boldsymbol{q}}{2\pi q}=-\nabla_\alpha\nabla_\beta\int_0^\infty J_0(q\rho)dq\exp(-qR_3)=-\nabla_\alpha\nabla_\beta\frac{1}{R}=\frac{R^2\delta_{\alpha\beta}-3R_\alpha R_\beta}{R^5}. \tag{122}$$

To find the finite-λ correction to w_α^β we evaluate (using isotropy)

$$\int\frac{q_\alpha q_\beta\exp(i\boldsymbol{q}\cdot\boldsymbol{\rho}-qR_3)d\boldsymbol{q}}{2\pi q^3}=f_1(\rho)\delta_{\alpha\beta}+\frac{f_2(\rho)R_\alpha R_\beta}{R^2}, \tag{123}$$

where f_i are some functions of ρ. Taking the trace and multiplying with $\rho_\alpha\rho_\beta$ we find

$$\int\frac{\exp(i\boldsymbol{q}\cdot\boldsymbol{\rho}-qR_3)d\boldsymbol{q}}{2\pi q}=2f_1(\rho)+f_2(\rho),$$
$$\int\frac{(\boldsymbol{q}\cdot\boldsymbol{\rho})^2\exp(i\boldsymbol{q}\cdot\boldsymbol{\rho}-qR_3)d\boldsymbol{q}}{2\pi q^3}=\rho^2[f_1(\rho)+f_2(\rho)]. \tag{124}$$

Thus we have

$$2f_1+f_2=\int_0^\infty \exp(-qR_3)J_0(q\rho)dq=\frac{1}{R},$$
$$f_1+f_2=\int\frac{\cos^2\theta\exp(iq\rho\cos\theta-qR_3)d\boldsymbol{q}}{2\pi q^{n-2}}$$
$$=\frac{1}{2}\int_0^\infty dq\exp(-qR_3)[J_0(q\rho)-J_2(q\rho)]$$
$$=\frac{1}{2R}\left(1-\frac{\rho^2}{(R_3+R)^2}\right), \quad (125)$$

where we used the identity

$$\int_0^{2\pi}\cos^2\theta\exp(ix\cos\theta)\frac{d\theta}{2\pi}=-\frac{d^2J_0}{dx^2}=\frac{J_0(x)-J_2(x)}{2}. \quad (126)$$

We find from Eqs. (125),

$$f_1=\frac{R_3^2+R_3R+\rho^2}{R(R_3+R)^2},\quad f_2=-\frac{\rho^2}{R(R_3+R)^2}. \quad (127)$$

Finally,

$$\int\frac{q_\alpha q_\beta\exp(i\boldsymbol{q}\cdot\boldsymbol{\rho}-qR_3)d\boldsymbol{q}}{2\pi q^2}$$
$$=-\partial_{R_3}\int\frac{\exp(i\boldsymbol{q}\cdot\boldsymbol{\rho}-qR_3)d\boldsymbol{q}}{2\pi q^3}q_\alpha q_\beta$$
$$=\frac{\delta_{\alpha\beta}}{R(R_3+R)}-\frac{(R_3+2R)R_\alpha R_\beta}{R^3(R_3+R)^2}, \quad (128)$$

where we evaluated derivatives of f_i given by Eq. (127). By combining the above results we find that w_α^β in Eq. (121) is given by

$$w_\alpha^\beta=\frac{hz(3R_\alpha R_\beta-R^2\delta_{\alpha\beta})}{4\pi R^5}-\frac{\lambda^2hz}{16\pi R(R_3+R)^2}\times\left(\rho^2\delta_{\alpha\beta}+R_\alpha R_\beta\frac{2RR_3-\rho^2}{R^2}\right)+o(\lambda h,\lambda z,\lambda\rho). \quad (129)$$

The correction term is of order lower than λ^4 because of the singularity of the λ^4 term at $q=0$. We find, for the Green's function,

$$G_{\alpha\beta}(\boldsymbol{x},\boldsymbol{x}')x_3'$$
$$\approx\frac{\delta_{\alpha\beta}}{r}+\frac{r_\alpha r_\beta}{r^3}+\frac{r\lambda^2}{4}\left(3\delta_{\alpha\beta}-\frac{r_\alpha r_\beta}{r^2}\right)-\frac{R_\alpha R_\beta}{R^3}$$
$$-\frac{R\lambda^2}{4}\left(3\delta_{\alpha\beta}-\frac{R_\alpha R_\beta}{R^2}\right)+\frac{2zx_3'(3R_\alpha R_\beta-R^2\delta_{\alpha\beta})}{R^5}$$
$$-\frac{\delta_{\alpha\beta}}{R}-\frac{\lambda^2x_3'z}{2R(R_3+R)^2}\left(r^2\delta_{\alpha\beta}+r_\alpha r_\beta\frac{2RR_3-r^2}{R^2}\right). \quad (130)$$

We consider the last remaining component w_3^α directly, rather than deriving it from w_α^3 by symmetry. We have, up to quadratic order in λ^2, that

$$\frac{ke^{-z\Delta}-q}{\lambda^2}=\frac{4q^2(1-qz)-\lambda^2(1+qz+q^2z^2)}{8q^3}. \quad (131)$$

Using this formula and Eq. (106), we find from Eqs. (39) that, up to quadratic order in λ,

$$\hat{w}_3^\alpha=-\frac{iq_\alpha he^{-qR_3}}{2q}\left(1+qz-\frac{\lambda^2(1+R_3q+q^2zR_3)}{4q^2}\right). \quad (132)$$

We have, by differentiation,

$$\frac{\partial\hat{w}_3^\alpha}{\partial R_3}=\frac{iqq_\alpha zh\exp(-R_3q)}{2}\left(1+\frac{\lambda^2(1-qR_3)}{4q^2}\right), \quad (133)$$

confirming, together with Eq. (120), that the incompressibility constraint in Eqs. (30) is obeyed to order λ^2. The inverse Fourier transform of Eq. (133) yields [the direct inverse Fourier transform of Eq. (132) produces a logarithmic divergence at zero that needs special attention]

$$\frac{\partial w_3^\alpha}{\partial R_3}=hz\,\partial_\alpha\int\frac{(\lambda^2(1-qR_3)+4q^2)\exp(i\boldsymbol{q}\cdot\boldsymbol{\rho}-qR_3)d\boldsymbol{q}}{32\pi^2q}$$
$$=\frac{hz}{16\pi}\partial_\alpha\left(\lambda^2+\lambda^2R_3\partial_3+4\partial_3^2\right)\frac{1}{R}. \quad (134)$$

This can be rewritten as

$$\frac{\partial w_3^\alpha}{\partial R_3}=-\frac{hzR_\alpha}{16\pi}\left[\lambda^2\partial_3\left(\frac{R_3}{R^3}\right)+4\partial_3^2\frac{1}{R^3}\right]. \quad (135)$$

Integration of this equation over R_3 up to infinity gives

$$w_3^\alpha=\frac{hR_\alpha}{16\pi}\int_{R_3}^\infty(R_3'-h)dR_3'\left(\lambda^2\partial_3'R_3'+4\partial_3^{2'}\right)\frac{1}{R'^3}.$$

We find

$$w_3^\alpha=\frac{hR_\alpha}{4\pi R^3}\left(1-\frac{\lambda^2R^2}{4}\right)+\frac{3hR_\alpha R_3z}{4\pi R^5}\left(1-\frac{\lambda^2R^2}{12}\right). \quad (136)$$

This reproduces formula (15) of [3] at $\lambda=0$. The corresponding formula for the Green's function is

$$G_{3\alpha}(\boldsymbol{x},\boldsymbol{x}')=\frac{r_3r_\alpha}{r^3}-\frac{R_3R_\alpha}{R^3}-\frac{\lambda^2r_3r_\alpha}{4r}+\frac{\lambda^2R_3R_\alpha}{4R}$$
$$+\frac{2x_3'R_\alpha}{R^3}\left(1-\frac{\lambda^2R^2}{4}\right)$$
$$+\frac{6x_3'R_\alpha R_3z}{R^5}\left(1-\frac{\lambda^2R^2}{12}\right), \quad (137)$$

which is readily seen to agree with $G_{\alpha3}(\boldsymbol{x}',\boldsymbol{x})$ given by Eq. (112).

The results of this section are equivalent to performing the integration of the coefficient of the λ^2 term in Eq. (23). We designate the integral by $G_{ik}'(\boldsymbol{x},\boldsymbol{x}')$, so

$$G_{ik}(\boldsymbol{x},\boldsymbol{x}')=\tilde{G}_{ik}(\boldsymbol{x},\boldsymbol{x}')+\lambda^2G_{ik}'(\boldsymbol{x},\boldsymbol{x}')+O(\lambda^{2+\Delta}). \quad (138)$$

Using the results of this section gives ($x_3'=h$)

$$G_{33}'(\boldsymbol{x},\boldsymbol{x}')=\frac{r}{4}\left(3-\frac{r_3^2}{r^2}\right)-\frac{R}{4}\left(3-\frac{R_3^2}{R^2}\right)+\frac{hz}{R}-\frac{\rho^2hz}{2R^3},$$
$$G_{\alpha3}'(\boldsymbol{x},\boldsymbol{x}')=\frac{R_\alpha R_3}{4R}-\frac{r_\alpha r_3}{4r}-\frac{R_\alpha h}{2R}+\frac{R_\alpha hzR_3}{2R^3},$$

$$G'_{\alpha\beta}(\boldsymbol{x},\boldsymbol{x}')\approx\frac{r}{4}\left(3\delta_{\alpha\beta}-\frac{r_\alpha r_\beta}{r^2}\right)-\frac{R}{4}\left(3\delta_{\alpha\beta}-\frac{R_\alpha R_\beta}{R^2}\right)-\frac{hz}{2R(R_3+R)^2}\left(r^2\delta_{\alpha\beta}+r_\alpha r_\beta\frac{2RR_3-r^2}{R^2}\right),$$
$$G'_{3\alpha}(\boldsymbol{x},\boldsymbol{x}')=\frac{R_3R_\alpha}{4R}-\frac{r_3r_\alpha}{4r}-\frac{hR_\alpha}{2R}-\frac{hR_\alpha R_3z}{2R^3}. \quad (139)$$

The derived expressions can be used to study the force.

XIII. FORCE IN THE LOW-FREQUENCY LIMIT

In this section we study the force on a sphere oscillating with a small frequency near a wall. We assume that the viscous penetration depth is the largest scale of the problem, $H\ll\delta$ and $a\ll\delta$ (in the dimensionless form $|\lambda|h\ll 1$ and $|\lambda|\ll 1$). The zeroth-order approximation for the flow $\boldsymbol{u}_0$ is the steady Stokes problem of a sphere moving at constant velocity $\boldsymbol{v}$ near the wall,

$$\nabla p_0=\nabla^2\boldsymbol{u}_0,\quad \boldsymbol{u}_0(|\boldsymbol{x}-\boldsymbol{x}_0|=1)=\hat{V},\quad \boldsymbol{u}_0(z=0)=0 \quad (140)$$

[cf. Eq. (3)]. This problem was solved in [19,20], however the solution takes a rather complex form of an infinite series. Thus we first explore the simpler limit of a distant wall, i.e., $a\ll H\ll\delta$. We consider the integral equation on the surface traction $\sigma^0_{kr}(\boldsymbol{x})$ of Eq. (140) in Appendix E. It can be shown that in the leading order in the distance to the wall, the traction, which is constant for the Stokes problem of a sphere in an unbounded fluid, also remains constant (this could also be obtained using the method of reflections [2]),

$$\sigma^0_{kr}=-\frac{3}{2}\left(1+\frac{9}{8h}\right)\delta_{kz}\hat{V}_z-\frac{3}{2}\left(1+\frac{9}{16h}\right)\delta_{k\alpha}\hat{V}_\alpha. \quad (141)$$

This reproduces the Lorentz result for the force given by Eq. (73) multiplied with the surface area 4π. We derive the leading-order correction in λ^2 similarly to the analysis in Sec. X. We use the approximation

$$G_{ik}(\boldsymbol{x},\boldsymbol{x}')\approx\tilde{G}_{ik}(\boldsymbol{x},\boldsymbol{x}')+\lambda^2G'_{ik}(\boldsymbol{x},\boldsymbol{x}'), \quad (142)$$

where G'_{ik} obeys the integral representation given by Eq. (23) and has the explicit form determined in the preceding section. In the limit of low frequencies, Eq. (10), up to the quadratic order in frequency, takes the form

$$\hat{V}_i+\frac{\lambda^2\sigma^0_{lr}}{8\pi}\int_{|\boldsymbol{x}'-\tilde{\boldsymbol{x}}_0|=1}G'_{il}(\boldsymbol{x},\boldsymbol{x}')dS'-\frac{\lambda^2\hat{V}_k}{8\pi}\int_{|\boldsymbol{x}'-\tilde{\boldsymbol{x}}_0|<1}\tilde{G}_{ik}(\boldsymbol{x},\boldsymbol{x}')dV'=-\frac{1}{8\pi}\int_{|\boldsymbol{x}'-\tilde{\boldsymbol{x}}_0|=1}\tilde{G}_{il}(\boldsymbol{x},\boldsymbol{x}')\tilde{\sigma}_{lr}(\boldsymbol{x}')dS', \quad (143)$$

where we use the volume integral form of the $\int_{|\boldsymbol{x}'-\tilde{\boldsymbol{x}}_0|=1}G_{il}(\boldsymbol{x},\boldsymbol{x}')x'_lx'_kdS'$ term. We consider the coefficients on the LHS in the leading order in h. From Eq. (139), using $R_3\approx 2h$, $z\approx h$, and $R_\alpha=r_\alpha\ll h$, it follows that

$$G'_{33}(\boldsymbol{x},\boldsymbol{x}')\approx-\frac{h}{2},\quad G'_{\alpha\beta}(\boldsymbol{x},\boldsymbol{x}')\approx-\frac{3\delta_{\alpha\beta}h}{2}, \quad (144)$$

while other components are $O(1)$ at most. Thus the correction is a constant diagonal matrix. The term in the second line and the $1/h$ correction to σ^0_{lr} in Eq. (143) can be neglected altogether, giving

$$\hat{V}_i+\frac{3h\lambda^2\hat{V}_l}{8}(\delta_{i3}\delta_{l3}+3\delta_{i\alpha}\delta_{l\alpha})=-\frac{1}{8\pi}\int_{|\boldsymbol{x}'-\tilde{\boldsymbol{x}}_0|=1}\tilde{G}_{il}(\boldsymbol{x},\boldsymbol{x}')\tilde{\sigma}_{lr}(\boldsymbol{x}')dS'. \quad (145)$$

The expression on the LHS gives the surface traction $\tilde{\sigma}_{lr}$ of the sphere that moves near the wall at constant velocity. The force is given by Eq. (73), which, in the leading order, reads

$$\frac{F_\perp}{F_{St}}=1+\frac{9}{8h}+\frac{3h\lambda^2}{8},\quad \frac{F_\parallel}{F_{St}}=1+\frac{9}{16h}+\frac{9h\lambda^2}{8}. \quad (146)$$

The λ^2 correction is smaller than $1/h$ Lorentz correction by a factor $\lambda^2h^2=(H/\delta)^2$. If this parameter is larger than $(a/H)^2$, that is, $H^2>a\delta$, then the former correction dominates over the steady Stokes flow $(a/H)^3$ correction to Eq. (73).

We now return to the case of the unconstrained ratio a/H with $\delta\gg a,H$. In this case the force can be obtained as an integral over known functions. By linearity we can write the surface traction $\sigma^0_{kr}(\boldsymbol{x})$ for Eq. (140) at the point $\boldsymbol{x}$ of the sphere surface as

$$\sigma^0_{kr}(\boldsymbol{x},\lambda=0)=F_{ik}(\boldsymbol{x})\hat{V}_i, \quad (147)$$

with a tensor $F_{ik}(\boldsymbol{x})$ whose form at large h is readily obtained from Eq. (141). At $h\sim 1$ this tensor has a complex form. Considering $F_{ik}(\boldsymbol{x})$ at $h\lesssim 1$, we can express the force $F_i(\{\boldsymbol{v}(\boldsymbol{x})\})$ on the sphere on which a given surface velocity distribution $\boldsymbol{v}(\boldsymbol{x})$ holds. From the reciprocal theorem we have

$$F_i(\{\boldsymbol{v}(\boldsymbol{x})\})=\int_{|\boldsymbol{x}-\tilde{\boldsymbol{x}}_0|=1}F_{ik}(\boldsymbol{x})v_k(\boldsymbol{x})dS. \quad (148)$$

The leading-order correction in frequency to the force $\boldsymbol{F}^0(\hat{V})$ can be determined by Eq. (140). This can be written as a series with the help of the solution of [19,20]. In the limit of low frequencies, Eq. (10), up to quadratic order in frequency, takes the form

$$\hat{V}_i+\frac{\lambda^2\hat{V}_k}{8\pi}\int_{|\boldsymbol{x}'-\tilde{\boldsymbol{x}}_0|=1}G'_{il}(\boldsymbol{x},\boldsymbol{x}')F_{kl}(\boldsymbol{x}')dS'-\frac{\lambda^2\hat{V}_k}{8\pi}\int_{|\boldsymbol{x}'-\tilde{\boldsymbol{x}}_0|<1}\tilde{G}_{ik}(\boldsymbol{x},\boldsymbol{x}')dV'=-\frac{1}{8\pi}\int_{|\boldsymbol{x}'-\tilde{\boldsymbol{x}}_0|=1}\tilde{G}_{il}(\boldsymbol{x},\boldsymbol{x}')\tilde{\sigma}_{lr}(\boldsymbol{x}')dS'. \quad (149)$$

We found the equation on the surface traction of a sphere that moves in the vicinity of the wall with a velocity distribution given by the LHS of the equation. We conclude from Eq. (148)

and linearity that the force is

$$F_i = F_i^0(\hat{V}) + \frac{\lambda^2 \hat{V}_k}{8\pi}\int_{|\boldsymbol{x}-\tilde{\boldsymbol{x}}_0|=1} F_{ip}(\boldsymbol{x})dS \times\left(\int_{|\boldsymbol{x}'-\tilde{\boldsymbol{x}}_0|=1} dS' G'_{pl}(\boldsymbol{x},\boldsymbol{x}')F_{kl}(\boldsymbol{x}') - \int_{|\boldsymbol{x}'-\tilde{\boldsymbol{x}}_0|<1}\tilde{G}_{pk}(\boldsymbol{x},\boldsymbol{x}')dV'\right). \quad (150)$$

This reduces the computation of the force to the numerical integration of known functions. Besides the already considered limit of $H \gg a$, this formula can be simplified in the limit where the sphere is close to the wall using the lubrication theory [1]. A detailed consideration of this limiting case is beyond the scope of the present paper.

XIV. FLOW IN THE LOW-FREQUENCY LIMIT

The leading-order correction to the flow in frequency can be readily obtained via the regular perturbation theory as in Sec. III. Looking for the solution of Eq. (3) in the form $\tilde{\boldsymbol{u}} = \boldsymbol{u}^0 + \lambda^2\boldsymbol{u}'$, where $\boldsymbol{u}^0$ is given in Eq. (140), we find

$$u'_i(\boldsymbol{x}) = -\int_{z'>0}\tilde{G}_{il}(\boldsymbol{x},\boldsymbol{x}')u_l^0(\boldsymbol{x}')\frac{d\boldsymbol{x}'}{8\pi}. \quad (151)$$

The series solution of [19,20] for $\boldsymbol{u}^0$ can be used to represent $\boldsymbol{u}'$ as a series of integrals. The flow can also be used as an alternative way for the study of the force considered in the preceding section.

XV. HIGHER-ORDER SOLUTION

In this section we describe the procedure for determining Green's functions with higher-order singularities, whose source is a derivative of the δ function. We will demonstrate that the solution for the source given by the Laplacian of the δ function can be obtained from the fundamental solution with the help of a rather simple correction.

The fundamental solution of unsteady Stokes equations in infinite space can serve to produce higher-order singular solutions. Taking the derivative of the solution gives another solution of the Stokes equations with the source given by the corresponding derivative of the δ function. The set of solutions obtained by taking all possible derivatives of the fundamental solution is complete: We can write an arbitrary solution as a superposition of the singular solutions. In our case derivatives over the lateral coordinates x or y would also produce a higher-order singular solutions; however, z derivatives do not satisfy the boundary condition at the plane. Therefore, these derivatives require a separate analysis. In this section we derive one such solution involving z derivatives. This is the solution of the time-dependent Stokes equations in the presence of the wall, whose source is the Laplacian of the δ function:

$$-\nabla\tilde{p}^k + \nabla^2\tilde{\boldsymbol{u}}^k - \lambda^2\tilde{\boldsymbol{u}}^k = -\hat{x}_k\nabla^2\delta(\boldsymbol{x}-\boldsymbol{x}'), \quad \nabla\cdot\tilde{\boldsymbol{u}}^k = 0,$$
$$\tilde{\boldsymbol{u}}^k(z=0) = \tilde{\boldsymbol{u}}^k(r\to\infty) = 0. \quad (152)$$

We write the solution in the form

$$\tilde{u}_i^k = \nabla^2\left(\frac{G_{ik}^0(\boldsymbol{x},\boldsymbol{x}') - G_{ik}^0(\boldsymbol{x},\boldsymbol{x}'^*)}{8\pi}\right) + \tilde{w}_i^k,$$
$$\tilde{p}^k = \nabla^2\left(\frac{r_k}{4\pi r^3} - \frac{R_k}{4\pi R^3}\right) + \tilde{s}^k, \quad (153)$$

where the correction fields $\tilde{w}_i^k$ and $\tilde{s}^k$ obey Eq. (27) with a different boundary condition

$$\tilde{w}_i^k(R_1,R_2,h) = \nabla^2\left(\frac{G_{ik}^0(\boldsymbol{R}) - G_{ik}^0(\boldsymbol{r})}{8\pi}\right)\Bigg|_{z=0}. \quad (154)$$

We observe that for any function $h(r)$ that depends only on the distance $r=|\boldsymbol{r}|$ we have

$$\nabla^2(hr_ir_k) = 2h\delta_{ik} + \frac{r_ir_k}{r^6}\frac{d}{dr}\left(r^6\frac{dh}{dr}\right). \quad (155)$$

We find, using this identity and Eq. (20), that

$$\nabla^2 G_{ik}^0(\boldsymbol{r}) = \frac{2\delta_{ik}(1+\lambda r+\lambda^2r^2)\exp(-\lambda r)}{r^3} - \frac{2r_ir_k(3+3\lambda r+\lambda^2r^2)\exp(-\lambda r)}{r^5}. \quad (156)$$

It is readily confirmed that at $\lambda=0$ this reproduces the Laplacian of the Oseen tensor [1]. We find, using this equation and from Eq. (154), that

$$\tilde{w}_i^k(R_1,R_2,h) = -(\delta_{k\alpha}\delta_{i3}+\delta_{i\alpha}\delta_{k3}) \times\frac{hR_\alpha}{2\pi R^5}(3+3\lambda R+\lambda^2R^2)\exp(-\lambda R), \quad (157)$$

whereas as before α and β can only take the value of 1 or 2. Thus $\tilde{w}$ and w obey identical equations with similar boundary conditions. We can write the solution as

$$\tilde{\boldsymbol{w}}^k = \lambda^2\boldsymbol{w}^k - \boldsymbol{v}^k, \quad \tilde{s}^k = s^k - S^k, \quad (158)$$

where $\boldsymbol{v}^k$ and S^k satisfy

$$-\nabla_{\boldsymbol{R}}S^k + \nabla_{\boldsymbol{R}}^2\boldsymbol{v}^k - \lambda^2\boldsymbol{v}^k = 0, \quad \nabla_{\boldsymbol{R}}\cdot\boldsymbol{v}^k = 0, \quad (159)$$

with the boundary condition

$$v_i^k(R_1,R_2,h) = \frac{3(\delta_{k\alpha}\delta_{i3}+\delta_{i\alpha}\delta_{k3})hR_\alpha}{2\pi R^5}. \quad (160)$$

We find, comparing this problem with the problem on $\boldsymbol{w}$ and using Eqs. (37), (34), and (C3), that

$$\hat{v}_\alpha^k = -\frac{q_\alpha}{q(k-q)}[e^{q(h-R_3)}(k\delta_{k\beta}q_\beta - iq^2\delta_{k3}) - ke^{k(h-R_3)}(\delta_{k\beta}q_\beta - iq\delta_{k3})],$$
$$\hat{v}_3^k = -\frac{1}{k-q}[e^{q(h-R_3)}(q^2\delta_{k3}+ik\delta_{k\alpha}q_\alpha) - e^{k(h-R_3)}(q^2\delta_{k3}+iq\delta_{k\alpha}q_\alpha)]. \quad (161)$$

Further study proceeds similarly to the study of the fundamental solution.

XVI. CONCLUSION AND FUTURE WORK

In this paper we provided an alternative derivation of the fundamental solution for unsteady Stokes equations near a plane wall. The derived solution has a more compact form than the previously reported solutions [10,12,14]. We verified our derivation by demonstrating that it reproduces known results in various asymptotic limits. Based on our derivation, we were able to resolve the apparent discrepancies between the existing theories correcting various typos in [10,12,14].

As opposed to the solution for steady Stokes equations [3], the number of images required to construct the solution of unsteady Stokes equations is infinite [12]. The solution includes two spatial scales: the viscous penetration depth δ and the distance to wall H. This results in two asymptotic limits $H \gg \delta$ and $H \ll \delta$, in which the Green's function in real space can be written via elementary functions (in a general case only the Fourier-space solution is given via elementary functions, while the real-space solution demands finding two Hankel transforms). These limits are also provided.

We exploited simplifications of the Green's function at $H \gg \delta$ and $H \ll \delta$ to study the force exerted on a sphere oscillating near the wall by the viscous liquid. The limit of the distant wall is of universal applicability since there is in practice always a boundary. This limit for steady Stokes flow is a classical problem that was considered by Lorentz [2]. The force is given by the (Stokes) force in unbounded fluid plus a correction due to the boundary that decays as the inverse of the distance to the wall. This law of decay coincides with the law of decay of Stokes flow far from the sphere. Translations parallel and perpendicular to the wall represent two independent problems with different friction coefficients [2]. We provided an analogous derivation for an unsteady Stokes flow. Here the limit of the distant wall pertains to the case, whereas the distance to the wall H is much larger than both δ and the sphere radius a. The ratio a/δ could be arbitrary. The rigorous asymptotic results for the force are provided in Eqs. (89). The force is given by the corresponding expression for the unbounded fluid plus a correction due to the boundary that decays as the third power of the distance to the wall. This decay law is intuitive since the flow due to a sphere oscillating in infinite fluid decays as the third power of the distance to the sphere (cf. with the case of steady Stokes flow). Oscillations parallel and perpendicular to the wall are decoupled independent motions with different friction coefficients as for the steady flow. The inverse Fourier transform of the derived expression can be used to study arbitrary time-dependent motion of a sphere far from the wall at relatively short times (the assumption $\delta \ll H$ implies a lower bound on the frequency or an upper bound on time). The H^{-3} decay law is quite fast, so in practice unsteadiness can provide a cutoff for particle-wall interactions.

Taking the limit $a \gg \delta$ (so that $H \gg a \gg \delta$) reproduces the known result for the high-frequency limit derived in [11,21]. This is the limit where the vorticity is concentrated in the narrow boundary layer at the sphere surface. The width of this layer is $O(\delta)$, while the flow outside the layer is potential. These arguments are typically semiheuristic (see, e.g., [18]), however they can be rigorously derived using an integral representation of the solution and by noting that the Stokeslet flow becomes potential at distances from the source larger than δ (cf. [26]). The force in the high-frequency limit can be interpreted as a distant wall correction to the added mass. The derivation of the correction from the potential flow approximation in [11,21] used previous results of [27,28] (see also [29]). Our result for $H \gg \delta$ and unconstrained a/δ is simpler that the corresponding derivation in [11,21].

The authors of [11,21] criticized [10], where an assumption on the form of far-field flow was made. That assumption would lead to an incorrect result for the force in the high-frequency limit. We provided an independent derivation of far-field flow which for a sphere agrees with the form in [11,21]. Our approach allowed us to prove the conjecture of [11,21] that a similar result holds for particles of arbitrary shape. The complete multipole expansion can be derived from the integral equation and the Green's function. It would be interesting to reexamine the approach of [11] using the rigorous Green's function derived here.

Simplification of the Green's function at $\delta \gg H$ made it possible to analyze the limit $\delta \gg H \gg a$, where the force is given by Eq. (146). The contribution of unsteadiness to the force is significant in comparison to the steady Stokes flow correction provided that $H^2 > a\delta$. The force at $\delta \gg H, a$ and arbitrary H/a is given in quadratures by Eq. (150).

We also derived the solution with the source given by the Laplacian of the δ function to illustrate the general scheme to derive the solution with an arbitrary source singularity. The derivation can also be used for the study of the flow caused by a sphere oscillating near a wall. We recall that in infinite space the flow due to the oscillating sphere can be obtained by superposition of solutions with $\delta(\boldsymbol{x})$ and $\nabla^2\delta(\boldsymbol{x})$ sources [1]. Similarly, we can consider the flow near a wall generated by superposition of solutions with $\delta(\boldsymbol{x})$, $\nabla_\perp^2\delta(\boldsymbol{x})$, and $\nabla^2\delta(\boldsymbol{x})$ sources. Here the flow with the source $\nabla_\perp^2\delta(\boldsymbol{x})$ can be obtained by applying $\nabla_\perp^2 = \nabla_x^2 + \nabla_y^2$ on the fundamental solution. The superposition obeys the boundary conditions on the wall and is a good starting point for the perturbative approach. The study of the flow derived in this way is left for future work.

We demonstrated that at large distances between the wall and the source (much larger than the viscous penetration depth) the Green's functions for the no-slip and full-slip (i.e., free surface) boundary conditions coincide. This implies that solutions to the problems of oscillating sphere with no-slip and full-slip boundary at distant wall agree. This is quite useful, since for the full-slip boundary the method of reflections can be used [11]. Similar facts are well known for steady Stokes flows, where the reflected flow satisfies the full-slip rather than no-slip boundary conditions, while it is used to construct the solution to the problem with the no-slip boundary [2].

It is remarkable that the Green's function for a free plane boundary is much simpler than for the no-slip wall. The solution is provided by just one rather than an infinite number of images (see [14] and Sec. IX). This implies that problems of particle motion near a full-slip boundary are much simpler (cf. [11]). This type of problem can occur in practice for unsteady motion of biological swimmers near the water surface. The solution of this problem could be useful for the study of hydrodynamic interactions of organisms swimming under the water surface.

It was demonstrated in [14] that an inverse Fourier transform of the Green's function with respect to the frequency can be found in a closed form. It would be of interest to study the corresponding transform of the solution derived here, which is advantageous towards the numerical calculation of the inverse Fourier transform with respect to the wave number.

The derivation of the fundamental solution of the unsteady Stokes equations as a series in frequency can become regular for other geometries. It is remarkable that the expansion in powers of λ^2 holds as a regular convergent series for the fundamental solution $G^{ch}_{ik}(\boldsymbol{x},\boldsymbol{x}',\lambda)$ of unsteady Stokes equations in a slit geometry, i.e., for the fluid confined between two parallel walls. The Green's function of the steady Stokes equations in the channel $\tilde{G}^{ch}_{ik}(\boldsymbol{x},\boldsymbol{x}')$ was derived in [30] and in a different form in [13]. This function decays quadratically with the distance $|\boldsymbol{x}-\boldsymbol{x}'|$, which in this case is unbounded only in horizontal directions. We find that, in this case,

$$G^{ch}_{ik}(\boldsymbol{x},\boldsymbol{x}')=\sum_{n=0}^{\infty}\lambda^{2n}\int_{0<z_k<H}\tilde{G}^{ch}_{i_1 i}(\boldsymbol{x}_1,\boldsymbol{x})\tilde{G}^{ch}_{i_2 i_1}(\boldsymbol{x}_2,\boldsymbol{x}_1)\times\tilde{G}^{ch}_{i_3 i_2}(\boldsymbol{x}_3,\boldsymbol{x}_2)\cdots\tilde{G}^{ch}_{i_n i_{n-1}}(\boldsymbol{x}_n,\boldsymbol{x}_{n-1})\times\tilde{G}^{ch}_{i_n k}(\boldsymbol{x}_n,\boldsymbol{x}')\prod_{k=1}^{n}\frac{d\boldsymbol{x}_k}{8\pi},$$

where H is the distance between the walls. The terms of this series are finite, so the above provides a valid asymptotic expansion for the study of the limit of low frequency. It is likely that the full solution can also be derived; some properties were provided in [31]. Another confined geometry where the series solution would apply is a circular pipe. Steady Stokes flow due to point force in a pipe was obtained in [32,33]; some properties of similar unsteady flow were derived in [34].

Due to innumerable uses of the fundamental solutions in various problems of viscous hydrodynamics, we believe that the form of unsteady Stokeslet near a wall derived here will find many applications involving time-dependent confined flows.

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APPENDIX A: SYMMETRY OF THE GREEN'S FUNCTION

Here we derive the symmetry of the Green's function as given by Eq. (6). We consider the flows

$$u_i(\boldsymbol{x})=\frac{G_{ik}(\boldsymbol{x},\boldsymbol{x}_1)f_k}{8\pi},\quad u_i'(\boldsymbol{x})=\frac{G_{ik}(\boldsymbol{x},\boldsymbol{x}_2)g_k}{8\pi},\tag{A1}$$

with pressures p and p', respectively, and stress tensors

$$\sigma_{il}=-p\delta_{il}+\nabla_l u_i+\nabla_i u_l,\quad \sigma_{il}'=-p'\delta_{il}+\nabla_l u_i'+\nabla_i u_l'.$$

We observe that integration of the identity

$$\nabla_k(u_i\sigma_{ik}')-\nabla_k(u_i'\sigma_{ik})=\boldsymbol{u}'\cdot\boldsymbol{f}\delta(\boldsymbol{x}-\boldsymbol{x}_1)-\boldsymbol{u}\cdot\boldsymbol{g}\delta(\boldsymbol{x}-\boldsymbol{x}_2)$$

over $\boldsymbol{x}$ gives, on using the boundary conditions,

$$G_{ik}(\boldsymbol{x}_1,\boldsymbol{x}_2)g_k f_i=G_{ik}(\boldsymbol{x}_2,\boldsymbol{x}_1)f_k g_i.\tag{A2}$$

This implies Eq. (6) since this holds for arbitrary $\boldsymbol{f}$ and $\boldsymbol{g}$. We confirm that the solution given by Eqs. (39) and (40) obeys the symmetry constraint

$$\hat{w}^3_\alpha(\boldsymbol{q},z_1+z_2,z_1)-\hat{w}^\alpha_3(-\boldsymbol{q},z_1+z_2,z_2)=\frac{iq_\alpha\{\exp[-k(z_1+z_2)]-\exp[-(z_1+z_2)q]\}}{\lambda^2}$$

obtained by the Fourier transform of Eq. (25). We used Eq. (38) and restored the dependence of $\boldsymbol{w}^k$ on the vertical distance z_i between the source and the plane. Indeed, the solution given by Eqs. (39) and (40) gives

$$\hat{w}^3_\alpha(\boldsymbol{q},z_1+z_2,z_1)=[\exp(-kz_1)-\exp(-z_1q)]\times\frac{iq_\alpha[k\exp(-kz_2)-q\exp(-qz_2)]}{\lambda^2(k-q)},$$
$$\hat{w}^\alpha_3(-\boldsymbol{q},z_1+z_2,z_2)=[\exp(-kz_2)-\exp(-z_2q)]\times\frac{iq_\alpha[q\exp(-kz_1)-k\exp(-qz_1)]}{\lambda^2(k-q)},\tag{A3}$$

which obeys Eq. (A3).

APPENDIX B: INTEGRAL REPRESENTATION

To derive the integral representation for the oscillating sphere flow given by Eq. (7) we use the identity

$$\nabla_l(\tilde{u}_i\sigma^k_{il})=\nabla_l(u^k_i\tilde{\sigma}_{il})-\tilde{u}_k\delta(\boldsymbol{x}-\boldsymbol{x}'),\tag{B1}$$

where we introduced the stress tensor of the Stokeslet σ^k_{il},

$$\sigma^k_{il}=-p^k\delta_{il}+\nabla_l u^k_i+\nabla_i u^k_l.\tag{B2}$$

Integration over the volume of the flow gives ($dS_k=\hat{r}_k dS$ where $\hat{r}=\boldsymbol{r}/r$)

$$\int_{|\boldsymbol{x}-\tilde{\boldsymbol{x}}_0|=1}\frac{G_{ik}(\boldsymbol{x},\boldsymbol{x}')\tilde{\sigma}_{il}(\boldsymbol{x})dS_l}{8\pi}+\tilde{u}_k(\boldsymbol{x}')=\int_{|\boldsymbol{x}-\tilde{\boldsymbol{x}}_0|=1}\tilde{u}_i(\boldsymbol{x})\sigma^k_{il}(\boldsymbol{x},\boldsymbol{x}')dS_l,\tag{B3}$$

where $\boldsymbol{x}'$ is assumed to be outside the sphere and $\tilde{\boldsymbol{x}}_0=(0,0,H/a)$. Using the constancy of $\tilde{\boldsymbol{u}}$ on the sphere and

$$\int_{|\boldsymbol{x}-\tilde{\boldsymbol{x}}_0|=1}\sigma^k_{il}(\boldsymbol{x},\boldsymbol{x}')dS_l=\lambda^2\int_{|\boldsymbol{x}-\tilde{\boldsymbol{x}}_0|<1}u^k_i(\boldsymbol{x},\boldsymbol{x}')dV,\tag{B4}$$

we find Eq. (7) upon using Eq. (6) and renaming the variables.

APPENDIX C: CALCULATION OF THE FOURIER TRANSFORM

We consider calculation of integrals appearing in $\hat{f}(q)$ defined by Eqs. (26) and (34). Taking derivative of the integral [23,35–37]

$$\int_0^\infty\frac{J_0(q\rho)\rho d\rho}{(\rho^2+h^2)^{3/2}}=\frac{\exp(-qh)}{h}\tag{C1}$$

over h, we find

$$\int_0^\infty\frac{J_0(q\rho)\rho d\rho}{(\rho^2+h^2)^{5/2}}=\frac{(qh+1)\exp(-qh)}{3h^3}.\tag{C2}$$

Thus we can write

$$\lambda^2 \hat{f}(q) = \frac{(qh+1)\exp(-qh)}{h^3} - 3I_5 - 3\lambda I_4 - \lambda^2 I_3, \quad \text{(C3)}$$

where we introduced

$$I_n = \int \frac{\exp(-iq_1R_1 - iq_2R_2 - \lambda R)dR_1dR_2}{2\pi R^n}. \quad \text{(C4)}$$

The integrals I_n are not tabulated for n of interest. Our attempts at the calculation failed. However, the combination $3I_5 + 3\lambda I_4 + \lambda^2 I_3$ that enters $f(q)$ can be found. We consider I_n as a function of h and introduce the representation

$$I_n = \lim_{\epsilon\to 0}\int \exp(iq_3h)\tilde{I}_n' \frac{dq_3}{2\pi},$$
$$\tilde{I}_n' = \int \exp\left(-i\boldsymbol{s}\cdot\boldsymbol{R} - \lambda R - \frac{\epsilon}{R}\right)\frac{d\boldsymbol{R}}{2\pi R^n}. \quad \text{(C5)}$$

Here we introduced $\boldsymbol{s} = (q_1, q_2, q_3)$ and the convergence factor ϵ. We use that [23]

$$\int \exp\left(-i\boldsymbol{s}\cdot\boldsymbol{R} - \lambda R - \frac{\epsilon}{R}\right)\frac{d\boldsymbol{R}}{4\pi R^n}$$
$$= \int_0^\infty \exp\left(-\lambda R - \frac{\epsilon}{R}\right)\frac{\sin(sR)dR}{sR^{n-1}}$$
$$= \frac{2}{s\epsilon^{n/2-1}}\mathrm{Im}[(\lambda - is)^{n/2-1}K_{n-2}(2\sqrt{\epsilon(\lambda - is)})], \quad \text{(C6)}$$

where $K_{2-n}(x) = K_{n-2}(x)$, Im stands for the imaginary part, and $s^2 = q^2 + q_3^2$. We find

$$I_n = \lim_{\epsilon\to 0}\int \mathrm{Im}[(\lambda - is)^{n/2-1}K_{n-2}(2\sqrt{\epsilon(\lambda - is)})]$$
$$\times \frac{2\cos(q_3h)dq_3}{\pi s\epsilon^{n/2-1}}. \quad \text{(C7)}$$

We are interested in $h \neq 0$ when we can write

$$I_n = \lim_{\epsilon\to 0}\mathrm{Im}\int \frac{2\cos(q_3h)dq_3}{\pi s}\tilde{I}_n, \quad \text{(C8)}$$

where $\tilde{I}_n$ is defined by

$$\tilde{I}_n = \frac{1}{\epsilon^{n/2-1}}\Bigg((\lambda - is)^{n/2-1}K_{n-2}(2\sqrt{\epsilon(\lambda - is)})$$
$$+ \frac{\epsilon^{n/2-1}(is-\lambda)^{n-2}}{(n-2)!}\left[\ln(\sqrt{\epsilon}) - \frac{\psi(1)+\psi(n-1)}{2}\right]$$
$$- \frac{1}{2}\sum_{k=0}^{n-3}\frac{(n-k-3)!(is-\lambda)^k}{k!\epsilon^{n/2-1-k}}\Bigg). \quad \text{(C9)}$$

Indeed, the introduced terms are a linear combination of $\delta(h)$ and its derivatives (after applying Im). The introduced terms make the integrand of I_n have a finite limit $\epsilon \to 0$,

$$\lim_{\epsilon\to 0}\tilde{I}_n = \frac{(-1)^{n+1}}{2(n-2)!}(\lambda - is)^{n-2}\ln(\lambda - is). \quad \text{(C10)}$$

We cannot however interchange the order of the limit and the integration in Eq. (C8) because the resulting integral diverges. Thus we introduce further regularization

$$I_n = \lim_{\delta\to 0}\lim_{\epsilon\to 0}\mathrm{Im}\int \frac{2\cos(q_3h)\exp(-\delta|q_3|)dq_3}{\pi s}\tilde{I}_n. \quad \text{(C11)}$$

We find

$$I_n = \frac{(-1)^{n+1}}{(n-2)!}\mathrm{Im}\lim_{\delta\to 0}\int \cos(q_3h)\left(\lambda - i\sqrt{q^2+q_3^2}\right)^{n-2}$$
$$\times \ln\left(\lambda - i\sqrt{q^2+q_3^2}\right)\frac{\exp(-\delta|q_3|)dq_3}{\pi\sqrt{q^2+q_3^2}}. \quad \text{(C12)}$$

This representation agrees at $h \neq 0$ with the relation $I_{n-1} = -\partial_\lambda I_n$ implied by the definition in Eq. (C4). We consider the combination $3I_5 + 3\lambda I_4 + \lambda^2 I_3$ entering $f(q)$ in Eq. (C3). We find, combining the terms,

$$3I_5 + 3\lambda I_4 + \lambda^2 I_3 = \mathrm{Re}\lim_{\delta\to 0}\int \ln\left(\lambda - i\sqrt{q^2+q_3^2}\right)$$
$$\times \cos(q_3h)\frac{\exp(-\delta|q_3|)dq_3}{2\pi}(k^2+q_3^2)$$
$$= \lim_{\delta\to 0}\int_0^\infty \frac{dq_3}{2\pi}\cos(q_3h)\exp(-\delta|q_3|)$$
$$\times (k^2+q_3^2)\ln(k^2+q_3^2). \quad \text{(C13)}$$

We start the calculation with the integral [23]

$$\int_0^\infty dq_3 e^{iq_3h-\delta|q_3|}\ln(k^2+q_3^2)$$
$$= \frac{2}{\delta - ih}[\ln k - \mathrm{ci}(x)\cos x - \mathrm{si}(x)\sin x],$$
$$x = k(\delta - ih), \quad \text{(C14)}$$

where $\mathrm{ci}(x)$ and $\mathrm{si}(x)$ are the integral cosine and sine, respectively. We have, at $\delta \to 0$, from series formulas for the functions,

$$\mathrm{Re}\lim_{\delta\to 0}\frac{2\,\mathrm{ci}(x)\cos x}{\delta - ih} = -\mathrm{Im}\lim_{\delta\to 0}\frac{2\cosh(kh)\mathrm{ci}(x)}{h}$$
$$= \frac{\pi\cosh(kh)}{h},$$
$$\mathrm{Re}\lim_{\delta\to 0}\frac{2\,\mathrm{si}(x)\sin x}{\delta - ih} = -\frac{\pi\sinh(kh)}{h}. \quad \text{(C15)}$$

We conclude that

$$\lim_{\delta\to 0}\int_0^\infty \cos(q_3h)\exp(-\delta|q_3|)\ln(k^2+q_3^2)dq_3$$
$$= \lim_{\delta\to 0}\mathrm{Re}\int_0^\infty dq_3 e^{iq_3h-\delta|q_3|}\ln(k^2+q_3^2) = -\frac{\pi e^{-kh}}{h}. \quad \text{(C16)}$$

We find that

$$3I_5 + 3\lambda I_4 + \lambda^2 I_3$$
$$= \frac{\partial^2}{\partial h^2}\left(\frac{\exp(-kh)}{2h}\right) - \frac{k^2\exp(-kh)}{2h}$$
$$= \frac{(1+kh)\exp(-kh)}{h^3}. \quad \text{(C17)}$$

We conclude from Eq. (C3) that

$$\lambda^2 \hat{f}(q) = \frac{(1+qh)e^{-qh}}{h^3} - \frac{(1+kh)e^{-kh}}{h^3}. \tag{C18}$$

Taking the derivative over q, we find Eq. (38) from the main text.

APPENDIX D: PRESSURE AND INTEGRABILITY

Here we consider in more detail the pressure field, which is the simplest component of the solution. In real space it can only be found in quadratures. We demonstrate that already the simplest quantity, which is the value of the pressure at the particular point on the wall beneath the source, introduces integrals that are not writable via tabulated special functions. The value can however be written in terms of the derivative of a special function with respect to the order.

We observe that as a harmonic function $s(\boldsymbol{x})$ must be writable in terms of its values at $z=0$,

$$s(\boldsymbol{x}) = \int \exp(iq_x x + iq_y y - qz) s(q_x, q_y, z=0) \frac{d\boldsymbol{q}}{(2\pi)^2},$$
$$s(q_x, q_y, z=0) = \int \exp(-iq_x x - iq_y y) \times s(x, y, z=0) dx\, dy. \tag{D1}$$

It can be readily seen that the solution given by Eqs. (41) fits this general form. We first consider the pressure on the wall $p^k(z=0) = -h\delta_{k3}/2\pi(\rho^2+h^2)^{3/2} + s^k(z=0)$, where $\rho^2 = x^2+y^2$ [see Eq. (37)]. The pressure for the vertical forcing is obtained from Eq. (41) as

$$\lambda^2 \hat{s}^3(z=0) = q(k+q)(e^{-kh} - e^{-qh}). \tag{D2}$$

We find, using that for a radially symmetric function $l(\rho)$ in two dimensions, we have $l(\rho) = \int_0^\infty l(q) J_0(q\rho) q\, dq/2\pi$, where $l(q)$ is the two-dimensional Fourier transform of $l(\rho)$ that $[\nabla_\perp^2 = \rho^{-1}\partial_\rho(\rho\partial_\rho)]$

$$\lambda^2 s^3(z=0) = \nabla_\perp^2 \frac{\partial}{\partial h} \int_0^\infty (e^{-kh} - e^{-qh}) \frac{J_0(q\rho) dq}{2\pi} + \nabla_\perp^2 \int_0^\infty (ke^{-qh} - qe^{-kh}) \frac{J_0(q\rho) dq}{2\pi}. \tag{D3}$$

We have, for three of the integrals [23],

$$\int_0^\infty q J_0(q\rho) e^{-kh} dq = \frac{h(1+\lambda\sqrt{h^2+\rho^2}) e^{-\lambda\sqrt{h^2+\rho^2}}}{(h^2+\rho^2)^{3/2}},$$
$$\int_0^\infty J_0(q\rho) e^{-hq} dq = \frac{1}{\sqrt{h^2+\rho^2}},$$
$$\int_0^\infty J_0(q\rho) e^{-kh} dq = -\frac{\partial}{\partial h} \int_0^\infty \frac{dq}{k} J_0(q\rho) e^{-kh} = -\frac{\partial [I_0(z_-) K_0(z_+)]}{\partial h} = \frac{z_+ I_0(z_-) K_1(z_+) + z_- I_1(z_-) K_1(z_+)}{\sqrt{h^2+r^2}}, \tag{D4}$$

where $2z_\pm = \lambda(\sqrt{h^2+\rho^2} \pm h)$. We used that the modified Bessel functions of order ν, $I_\nu(z)$ and $K_\nu(z)$, obey $I_0' = I_1$ and $K_0' = -K_1$. We did not find in tables the integral remaining for the complete finding of $s(z=0)$,

$$\int_0^\infty k J_0(qr) \exp(-hq) dq. \tag{D5}$$

We conclude that seemingly (unless the integral above can be found by a certain transformation) we can only write down the real-space pressure at $z=0$ in quadratures. We reinforce the conclusion by considering $p(\rho=0, z=0)$, which can be written via special functions using

$$\int_0^\infty k \exp(-hq) dq = \frac{\lambda^2 L_1 + L_2}{2} - \frac{1}{h^2}, \tag{D6}$$

where the integrals L_i are defined below. We have, from the integral in [23] using l'Hôspital's rule,

$$L_1 \equiv \int_0^\infty \frac{\exp(-hq) dq}{k} = \frac{\partial}{\partial \nu}[J_{-\nu}(\lambda h) - \boldsymbol{J}_{-\nu}(\lambda h)]|_{\nu=0} = \frac{\pi[\boldsymbol{H}_0(\lambda h) - Y_0(\lambda h)]}{2}, \tag{D7}$$

where Y_ν, $\boldsymbol{J}_\nu$, and $\boldsymbol{H}_\nu$ are Weber's (Bessel function of the second kind), Anger's, and Struve's functions, respectively. The other integral in [23] gives

$$L_2 = \int_0^\infty \left(2k + 2q - \frac{\lambda^2}{k}\right) \exp(-hq) dq = \lambda^2 \frac{\partial}{\partial \nu}[J_{-\nu}(\lambda h) - \boldsymbol{J}_{-\nu}(\lambda h)]|_{\nu=2}. \tag{D8}$$

We can write the first derivative with respect to the order via the special functions

$$\left.\frac{\partial J_\nu(x)}{\partial \nu}\right|_{\nu=-2} = -\left.\frac{\partial J_\nu(x)}{\partial \nu}\right|_{\nu=2} + \pi Y_2(x) = \frac{\pi Y_2(x)}{2} - \frac{2[J_0(x) + x J_1(x)]}{x^2}. \tag{D9}$$

However, the derivative of $\boldsymbol{J}_\nu$ with respect to the order can only be written in quadratures [38]. Thus, with the possible exception of special points, such as the position of the source [10], pressure can only be written in quadratures.

APPENDIX E: LORENTZ PROBLEM VIA THE INTEGRAL EQUATION

Here we derive the stress tensor for the problem of steady motion of a sphere near the wall in the leading order in H. We have, from [3] or Sec. XII, at $\lambda=0$ that the Green's function of the steady Stokes problem obeys

$$\tilde{G}_{ik}(\boldsymbol{x}, \boldsymbol{x}') = Y_{ik}(\boldsymbol{r}) + \delta\tilde{G}_{ik}(\boldsymbol{x}, \boldsymbol{x}'), \tag{E1}$$

where

$$\delta\tilde{G}_{3\alpha}(\boldsymbol{x}, \boldsymbol{x}') = -Y_{3\alpha}(\boldsymbol{R}) + \frac{2hR_\alpha}{R^3} + \frac{6hR_\alpha R_3 z}{R^5},$$
$$\delta\tilde{G}_{\alpha\beta}(\boldsymbol{x}, \boldsymbol{x}') = -Y_{\alpha\beta}(\boldsymbol{R}) + \frac{2zh(3R_\alpha R_\beta - R^2\delta_{\alpha\beta})}{R^5},$$

$$\delta\tilde{G}_{33}(\boldsymbol{x},\boldsymbol{x}') = -Y_{33}(\boldsymbol{R}) - \frac{4hz}{R^3} + \frac{6\rho^2 hz}{R^5},$$
$$\delta\tilde{G}_{\alpha 3}(\boldsymbol{x},\boldsymbol{x}') = -Y_{\alpha 3}(\boldsymbol{R}) + \frac{2R_\alpha h}{R^3} - \frac{6R_\alpha hzR_3}{R^5}. \quad \text{(E2)}$$

The integral equation on the surface traction is obtained by setting $\lambda = 0$ in Eq. (10),

$$\hat{V}_i = -\frac{1}{8\pi}\int_{|\boldsymbol{x}'-\tilde{\boldsymbol{x}}_0|=1} \tilde{G}_{il}(\boldsymbol{x},\boldsymbol{x}')\tilde{\sigma}_{lr}(\boldsymbol{x}')dS'. \quad \text{(E3)}$$

When the sphere is far from the wall the points entering the surface integral obey $r_\alpha = R_\alpha \ll h$, $z \approx h$, and $R_3 \approx 2h$, which gives

$$\delta\tilde{G}_{\alpha\beta}(\boldsymbol{x},\boldsymbol{x}') \approx -\frac{3\delta_{\alpha\beta}}{4h}, \quad \delta\tilde{G}_{33}(\boldsymbol{x},\boldsymbol{x}') = -\frac{3}{2h}, \quad \text{(E4)}$$

where the omitted components are much smaller. The integral equation becomes, in the leading order

$$\hat{V}_i - \frac{3\hat{V}_l}{16\pi}\int_{|\boldsymbol{x}'-\tilde{\boldsymbol{x}}_0|=1} \delta\tilde{G}_{il}(\boldsymbol{x},\boldsymbol{x}')dS' = -\frac{1}{8\pi}\int_{|\boldsymbol{x}'-\tilde{\boldsymbol{x}}_0|=1} Y_{il}(\boldsymbol{x}-\boldsymbol{x}')\tilde{\sigma}_{lr}(\boldsymbol{x}')dS', \quad \text{(E5)}$$

where we used that, in the leading order, the surface traction is that of the Stokes problem $\tilde{\sigma}_{ir} = -3\hat{V}_i/2$ (it is uniform and gives the Stokes force $-6\pi\hat{V}_i$ on multiplication with the surface area 4π). Using, in Eq. (E5), $\delta\tilde{G}_{il}$ from Eq. (E4), we find the equation on surface traction of a sphere moving at constant velocity in infinite space. The solution is the constant surface traction of the corresponding Stokes problem given by Eq. (141).

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