Spin-glass model for the C-dismantling problem

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The C-dismantling (CD) problem aims at finding the minimum vertex set $\mathcal{D}$ of a graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ after removing which the remaining graph will break into connected components with the size not larger than C. In this paper, we introduce a spin-glass model with $C + 1$ integer-value states into the CD problem and then study the properties of this spin-glass model with the belief-propagation (BP) equations under the replica-symmetry ansatz. We give the lower bound ρ_c of the relative size of $\mathcal{D}$ with finite C on regular random graphs and Erdős-Rényi random graphs. We find ρ_c will decrease gradually with growing C, and it converges to ρ_∞ as $C \to \infty$. The CD problem is called the dismantling problem when C is a small finite fraction of $|\mathcal{V}|$. Therefore, ρ_∞ is also the lower bound of the dismantling problem when $|\mathcal{V}| \to \infty$. To reduce the computation complexity of the BP equations, taking the knowledge of the probability of a random selected vertex belonging to a remaining connected component with the size A, the original BP equations can be simplified to one with only three states when $C \to \infty$. The simplified BP equations are very similar to the BP equations of the feedback vertex set spin-glass model [H.-J. Zhou, Eur. Phys. J. B **86**, 455 (2013)]. Finally, we develop two practical belief-propagation-guide decimation algorithms based on the original BP equations (CD-BPD) and the simplified BP equations (SCD-BPD) to solve the CD problem on a certain graph. Our BPD algorithms and two other state-of-art heuristic algorithms are applied on various random graphs and some real-world networks. Computation results show that the CD-BPD is the best of all tested algorithms in the case of small C. But considering the performance and computation consumption, we recommend using SCD-BPD for the network with a small clustering coefficient when C is large.

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I. INTRODUCTION

In a graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ with $N = |\mathcal{V}|$ vertices and $M = |\mathcal{E}|$ undirected edges, there exist some vertices which are crucial to the connectivity of the graph. The set of these vertices $\mathcal{D}$ is called a C-dismantling (CD) set if its removal yields a remaining graph in which the size of each connected component will be equal to or smaller than C [1,2]. In the past few years, researchers have worked on the topic of the CD problem, which asks to find the minimum CD set of a graph, especially in the case of C taking a finite fraction of N, like $C/N = 0.01$, which is also named the dismantling problem [3–10].

For some real-world networks, such as transportation networks and the internet, their robustness and function depend on their scale and connectivity to a large extent [11–13]. On the other hand, we can also stop epidemic (or computer virus) spreading by vaccinating people (or computers) who can divide the infection network to separated components [14–16]. Therefore, as one of the fundamental problems in network science, the CD problem relates to many other important problems and practical applications, ranging from the percolation problem [17], to the information spreading [18,19], and so on.

In 2008 Janson and Thomason proved some useful properties of the CD problem on sparse random graphs [2]. But the problem of finding the minimum CD set or the dismantling set of a certain graph belongs to the nondeterministic polynomial hard (NP-hard) class of computational complexity [6,20,21]. Therefore, researchers are not pinning their hopes on solving this problem with a complete algorithm in time bounded with a polynomial function of M or N but are devoting their efforts to all kinds of heuristic methods. The starting points of these heuristic algorithms, even without a strict proof but whose rationality has been proved by their nice results in solving the dismantling problem, are the possible correlations between network structure and network attacking. Except for some methods based on the vertices' highest degree [11,13,22] or betweenness [23], Morone and Makse considered the information spreading of vertices and proposed the collective information algorithm [7]. The authors claimed that it beats all existing heuristic algorithms at that time. Another algorithm recursively removes vertices having the highest degree from the 2-core of the graph, which is obtained by adaptive removal of all leaves [6]. The CD problem can also be solved with the node explosive percolation algorithms, which start from a completely dismantled graph and then reconstructs the graph by adding the removed vertex back [8,10,24]. Later, some other researchers pointed out that for random graphs the dismantling problem is equivalent to the decycling problem, which is also referred to as the feedback vertex set (FVS) problem and aims to remove as few vertices as possible to break all cycles in the graph [5,14,25–27]. The dismantling algorithms stemming from the FVS problem are characterized by their perfect performance in giving a very

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small dismantling set, and their results are very close to the theoretically optimal value of the decycling problem [3,4,6].

Most of the studies discussed above mainly focus their attention on the dismantling problem, where C will be very large as $N \to \infty$, but they cannot guarantee their performance in the case of small C. In order to have a more comprehensive understanding of the CD problem and to solve this problem efficiently, we focus our attention on the general CD problem with finite C and propose a spin-glass model with $C+1$ states to describe the constraints in the CD problem. Considering that the removal cost of a vertex depends on the protection effort on it, we introduce the spin-glass model considering the situation where each vertex has variable removal costs. By using the belief-propagation (BP) equations under the replica-symmetry (RS) ansatz, we study various properties of the CD problem, including the lower bound of the relative size of minimum $\mathcal{D}$ with C, the probability of a random selected vertex belonging to a connected component with size A, the complexity of the BP equations, and the connection between CD problem and the FVS problem. What is more, we also develop two belief-propagation-guide decimation algorithms (CD-BPD and SCD-BPD) to solve the CD problem on a certain graph. The CD-BPD algorithm is based on BP equations of our spin-glass model, and the SCD-BPD is a coarse-gained algorithm of the CD-BPD in large C limit. Our extensive numerical computations on artificial random graphs and real-world networks exhibit that the CD-BPD has significant advantages in solving the CD problem over others in the case of small C. When C is large, the CD-BPD or SCD-BPD is also the best algorithm for the CD problem on networks with a small clustering coefficient [28].

This paper is organized as follows. In the next section we introduce the spin-glass model for the CD problem and explain how to compute thermodynamic quantities under the RS ansatz. The numerical computation results on random graphs and some real-world networks are given in Sec. III. In the last section, we conclude our work and discuss some possible extensions.

II. SPIN-GLASS MODEL OF THE CD PROBLEM

After a graph has been C-dismantled, the remaining graph will break into numerous connected components not larger than C. Therefore the minimum CD problem is completely equivalent to the maximum C-component set problem asking the maximum set of vertex $\mathcal{S} = \mathcal{V} \setminus \mathcal{D}$, so that the vertices in $\mathcal{S}$ and the edges between them form connected components not larger than C. In the case of $C = 1$, the C-component set problem is equivalent to another NP-hard problem: the vertex cover problem, which has been analyzed with the RS mean-field method extensively [27,29–32]. Inspired by the spin-glass model of the vertex cover problem, in this paper we develop our spin-glass model for the C-component set problem as well as the CD problem and then analyze its properties with the RS mean-field method.

In a dismantled graph, if vertex $i \in \mathcal{S}$ is in the connected component $\mathcal{C}_\alpha$, we use an integer value $A_i = |\mathcal{C}_\alpha|$ to present the state of vertex i in our spin-glass model, where $|\mathcal{C}_\alpha|$ means the size of $\mathcal{C}_\alpha$ or the number of vertices in $\mathcal{C}_\alpha$. If the vertex $i \in \mathcal{D}$, we say the vertex i is in a connected component with size 0 and $A_i = 0$. In the CD problem, the size of each remaining connected component must be equal to or smaller than C, so A_i can take only $C+1$ different integer values from 0 to C. A microscopic configuration $\underline{A} \equiv \{A_1, A_2, \ldots, A_N\}$ of graph $\mathcal{G}$ is called legitimate if and only if the following constraint is fulfilled:

$$L(\underline{A}) \equiv \prod_i \delta[A_i, \Gamma(i)], \tag{1}$$

where $\Gamma(i)$ returns the size of the connected component containing vertex i in the dismantled graph, and $\delta(x, y)$ is the Kronecker delta function such that $\delta(x, y) = 1$ if $x = y$, and $\delta(x, y) = 0$ if $x \neq y$.

In this spin-glass model, if we consider the removal cost $\omega_i \geqslant 0$ of each vertex i, the CD problem should be pursued for the minimum total removal cost instead of the minimum $\mathcal{D}$. Therefore, the energy of $\underline{A}$ is defined as the total removal cost of all vertices with $A_i = 0$:

$$E(\underline{A}) = \sum_i \omega_i \delta(A_i, 0). \tag{2}$$

We assume the spin-glass system follows the Boltzmann distribution and the probability of observing a legitimate state $\underline{A}$ is

$$p(\underline{A}) = \frac{\exp[-\beta E(\underline{A})]}{Z(\beta)}, \tag{3}$$

where β is the inverse temperature in the canonical ensemble and $Z(\beta)$ is the partition function

$$Z(\beta) = \sum_{\underline{A}} \exp[-\beta E(\underline{A})] L(\underline{A}). \tag{4}$$

Now we consider the marginal probability of a vertex i taking the state A_i, denoted as $q_i^{A_i}$. The value of $q_i^{A_i}$ is strongly influenced by the marginal probabilities of i's nearest neighbors $j \in \partial i$, where ∂i gives the set of nearest-neighbor vertices of i in graph $\mathcal{G}$. After we build a cavity graph $\mathcal{G}_{\setminus i}$ by removing vertex i from $\mathcal{G}$, we can use the Bethe-Peierls approximation to neglect all possible correlations among the marginal probabilities of vertices $j \in \partial i$ [33,34], denoted as $q_{j\to i}^{A_j}$. Then we can have the value of $q_i^{A_i}$ by the following equations:

$$q_i^0 = \frac{e^{-\beta\omega_i}}{z_i}, \tag{5a}$$

$$q_i^{A_i} = \frac{1}{z_i} \sum_{\underline{A}_{\partial i}} \delta\left(\sum_{j\in\partial i} A_j + 1, A_i\right) \prod_{j\in\partial i} q_{j\to i}^{A_j} \quad (A_i \neq 0), \tag{5b}$$

where $\underline{A}_{\partial i} \equiv \{A_j\}_{j\in\partial i}$ is the local configuration of vertex i and the normalization factor z_i is

$$z_i \equiv e^{-\beta\omega_i} + \sum_{\underline{A}_{\partial i}} H\left(C - \sum_{j\in\partial i} A_j - 1\right) \prod_{j\in\partial i} q_{j\to i}^{A_j}, \tag{6}$$

where $H(x)$ is the Heaviside step function such that $H(x) = 1$ if $x \geqslant 0$, and $H(x) = 0$ if $x < 0$.

Equation (5) considers constraints where only when the total size of all neighbor components is smaller than $C-1$

can we accept the vertex $i \in \mathcal{S}$, which will merge all neighbor components into a bigger one with the size $\sum_{j\in\partial i} A_j + 1$. If $\sum_{j\in\partial i} A_j \geqslant C$, the vertex i must be in set $\mathcal{D}$ to prevent forming a connected component whose size exceeds C. $q_{i\to j}^{A_i}$ has the same meaning with $q_i^{A_i}$ except that it is defined on the cavity graph $\mathcal{G}_{\backslash i}$. The self-consistency BP equations of $q_{i\to j}^{A_i}$ are

$$q_{i\to j}^{0} = \frac{e^{-\beta\omega_i}}{z_{i\to j}}, \tag{7a}$$

$$q_{i\to j}^{A_i} = \frac{1}{z_{i\to j}} \sum_{\underline{A}_{\partial i\backslash j}} \delta\left(\sum_{k\in\partial i\backslash j} A_k + 1, A_i\right) \prod_{k\in\partial i\backslash j} q_{k\to i}^{A_k} \quad (A_i \neq 0), \tag{7b}$$

where $\partial i \backslash j$ means the vertex set obtained by deleting vertex j from ∂i, and the normalization factor $z_{i\to j}$ is

$$z_{i\to j} \equiv e^{-\beta\omega_i} + \sum_{\underline{A}_{\partial i\backslash j}} \prod_{k\in\partial i\backslash j} q_{k\to i}^{A_k} H\left(C - \sum_{k\in\partial i\backslash j} A_k - 1\right). \tag{8}$$

For a certain graph instance $\mathcal{G}$, the BP equations can be solved by iterating the equations on edges at a fixed β. After the BP equations are solved, we can obtain thermal dynamical quantities of this spin-glass system under the Bethe-Peierls approximation. We start from the free energy $F = \sum_i f_i - \sum_{(i,j)\in G} f_{ij}$, where the f_i and f_{ij} are the free energy contribution from vertex i and edge (i, j):

$$f_i = -\frac{1}{\beta} \ln z_i, \tag{9}$$

$$f_{ij} = -\frac{1}{\beta} \ln \sum_{A_i, A_j} H(C - A_i - A_j) q_{i\to j}^{A_i} q_{j\to i}^{A_j}. \tag{10}$$

As free energy is an extensive quantity, we are more interested in the free energy density obtained by $f = F/N$. The energy density e of the spin-glass model equals

$$e = \frac{< E >}{N} = \frac{1}{N} \sum_i q_i^0 \omega_i. \tag{11}$$

If all vertices have uniform removal cost and $\omega_i = 1$, the energy density is the relative size of the set $\mathcal{D}$. Finally, we can obtain the entropy density with

$$s = \beta(e - f). \tag{12}$$

III. RESULTS

We now apply the CD spin-glass model on the regular random graphs (RR), Erdős-Rényi (ER) random graphs, scale-free (SF) graphs, and some real-world networks. In the present paper, we generate the SF networks with a static method explained in Ref. [35]. Because we do not have the knowledge of the removal cost, we assume the removal cost $\omega_i = 1$ for all vertices. Actually, the BP equations still hold even if the removal cost is not uniform. Without being specific, the results of artificial random graphs in the following discussion are obtained by averaging over 16 different instances with $N = 2^{17}$.

A. Statistical properties of the CD problem on random graphs

Equation (4) tells us that the partition function will be dominated by low-energy configurations when β is large. What is more, in most spin-glass systems including this one, the number of configurations will decrease with decreasing energy, and the entropy density s will also decrease with growing β. As the entropy of a real system must be non-negative, the mean-field result predicts the relative size of the minimum $\mathcal{D}$, denoted by ρ_c, at the inverse temperature $\beta = \beta^*$ where $s(\beta^*) = 0$. For the RR graph, each vertex has the same degree, so we can have the numerical solution of the BP equations (7). For the ER graph, we use population dynamics to investigate ρ_c with various average degree and C [36–39].

From Figs. 1(a) and 1(b), we can see ρ_c decreases gradually with C until it converges to ρ_∞ as $C \to \infty$. Finite-size scaling analysis in Figs. 1(c) and 1(d) exhibits $(\rho_c - \rho_\infty) \propto C^{-\zeta}$. The exponent $\zeta \approx 1$ is almost irrelevant with the degree distribution of random networks. Because of the locally treelike character of random graphs, $\zeta \approx 1$ agrees with the exponent of the tree network [2]. The value of ρ_∞ can also be obtained by extrapolating the result of ρ_c in the large C limit. As discussed above, the dismantling problem can be regarded as the CD problem with infinite C. Therefore, ρ_∞ gives the fraction of the minimum removed vertices in the dismantling problem. The value of ρ_∞ for the RR and ER networks with various degree is presented in Fig. 2: ρ_∞ increases monotonically with growing mean vertex degree in the RR and ER random graph ensembles. We also notice that the difference between the ρ_∞ and the relative size of the minimum FVS predicted in Ref. [5] is inconspicuous, which will be explained in the following discussion.

The mean value of the q_i^A over all vertices, denoted as $\overline{q}^A = \frac{1}{N}\sum_i q_i^A$, is the probability of a randomly selected vertex in a connected component with size A. We compute the value of $\overline{q}^A$ for the RR and ER graphs and present the results in Fig. 3. In the case of small A, $\overline{q}^A$ will decay with growing A quickly. But as A approaches C, all curves in Fig. 3 increase with A/C exponentially. In the large C limit, the exponents of these curves mainly depend on the type and the mean vertex degree of the graph. Moreover, $\overline{q}^A$ reaches its minimum at the point $A_{\min} = \arg\min \overline{q}^A$ and $\lim_{C\to\infty} A_{\min}/C = 0$. Therefore, in the case of $C \to \infty$, there are two types of connected components in a dismantled graph: small connected components with a few vertices and large connected components with the size close to C.

B. The computation complexity of the BP equations

In this subsection, we will discuss the computation complexity of Eqs. (5) and (7). The reader may argue that we must consider all local microscopic configurations $\underline{A}_{\partial i}$ to compute the message $q_{i\to j}^{A_i}$ in Eq. (5), so the computation complexity of the BP iterations must be larger than $(C+1)^{|\partial i|}$. However, only the configurations with $\sum_{j\in\partial i} A_j + 1 < C$ work in

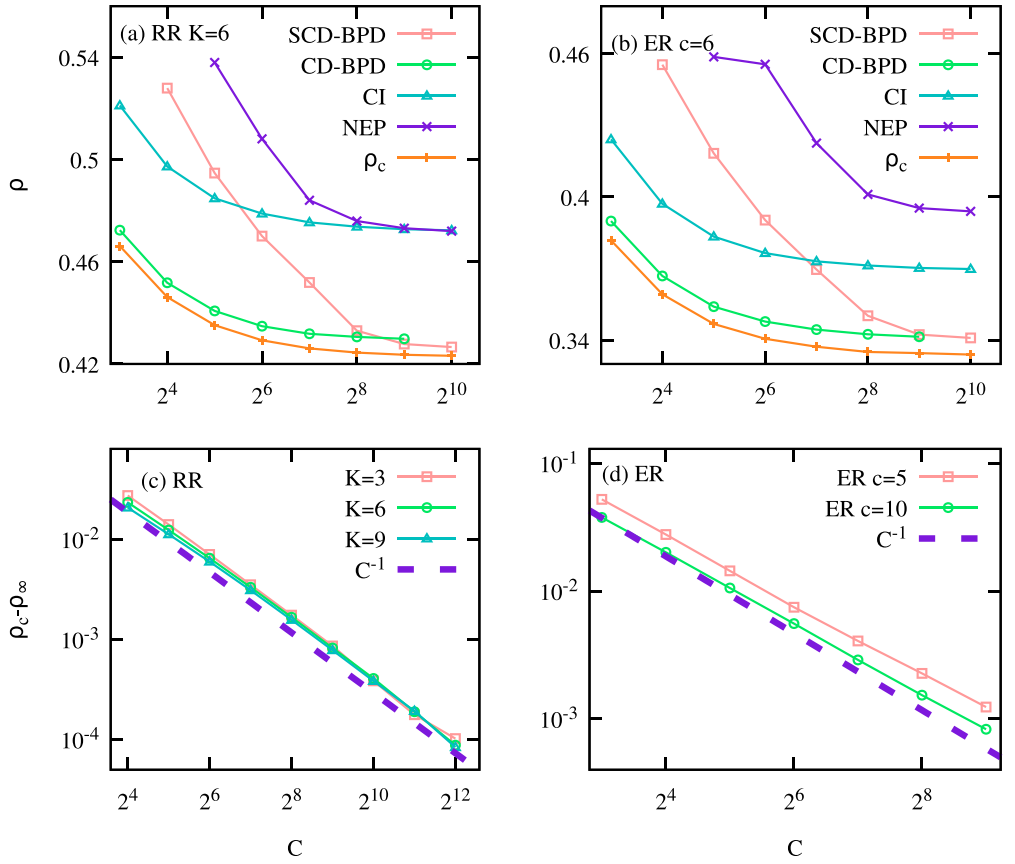

FIG. 1. (a) Fraction of vertices removed as a function of C on RR graph with degree $K = 6$ given by the RS mean-field method (pluses), the CD-BPD (circles), the SCD-BPD (squares) algorithms, the CI (triangles), and the NEP (crosses). (b) The same as panel (a) but on ER graphs with average degree $c = 6$. (c) Scaling plot of the $(\rho_c - \rho_\infty)$ versus C for RR graphs with $K = 3$ (squares) where $\rho_\infty = 0.24236$, $K = 6$ (circles) where $\rho_\infty = 0.42278$, and $K = 9$ (triangles) where $\rho_\infty = 0.519633$. (d) The same as (c) but on ER graphs with degree $c = 5$ (squares) where $\rho_\infty = 0.2785$ and $c = 10$ (circles) where $\rho_\infty = 0.4835$.

the equations, which take a very small fraction of all local configurations. The real computation complexity of the BP equations will also be much smaller than $(C + 1)^{|\partial i|}$.

We start by neglecting all nearest neighbors of vertex i except m and n. If m and n are in connected components with the size A_m and A_n, respectively, in cavity graph $\mathcal{G}_{\setminus i}$

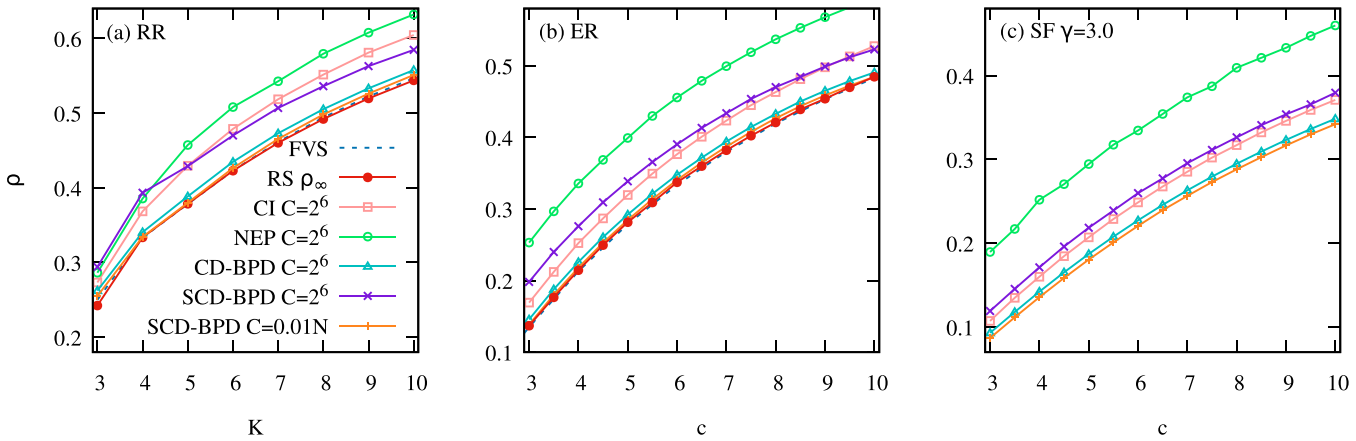

FIG. 2. The relative size ρ of set $\mathcal{D}$ for (a) the RR graph on degree K, (b) the ER graph on mean degree c, and (c) the SF graph on mean degree c with power-law exponent $\gamma = 3.0$ given with the collective influence algorithm with ball radius $\ell = 2$ (CI) (squares) [7], node explosive percolation algorithm with the second score definition in Ref. [8] (NEP) (empty circles), CD-BPD (triangles), and SCD-BPD (crosses) algorithms. ρ_∞ is the value of ρ_c at $C \to \infty$ predicted by the RS mean-field method (solid circles). The dashed lines are the lower bounds of the minimum FVS predicted with the RS mean-field method [5].

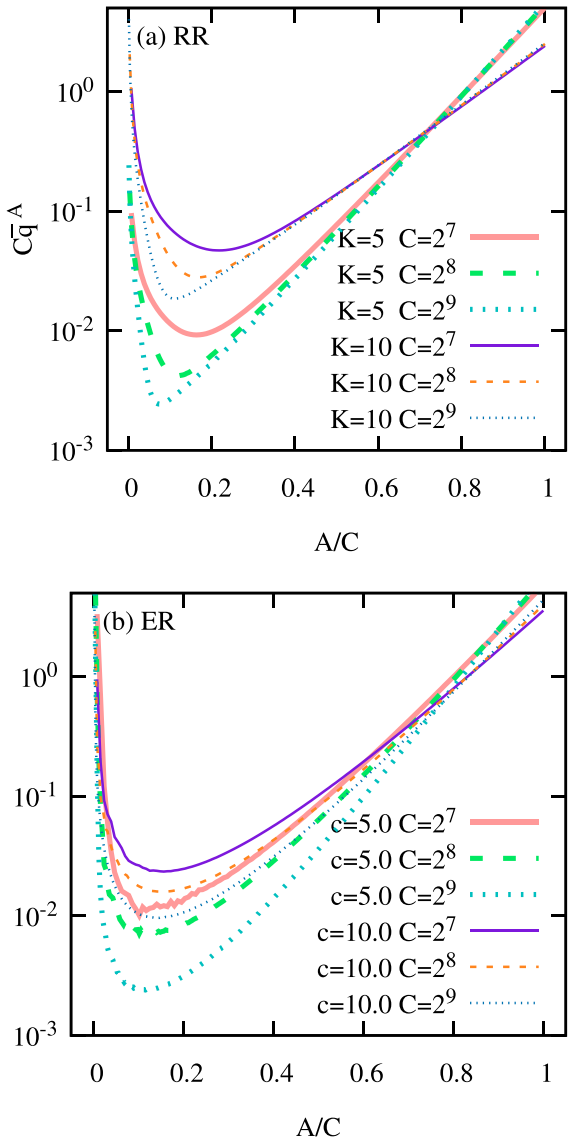

FIG. 3. The mean value of the probability q_i^A over all vertices in (a) RR graphs with degree $K = 5$ and $C = 2^7$ (thick solid line), $C = 2^8$ (thick dashed line), and $C = 2^9$ (thick dotted line) and with degree $K = 10$ and $C = 2^7$ (thin solid line), $C = 2^8$ (thin dashed line), and $C = 2^9$ (thin dotted line); (b) ER graphs with average degree $c = 5$ and $C = 2^7$ (thick solid line), $C = 2^8$ (thick dashed line), and $C = 2^9$ (thick dotted line) and with degree $c = 10$ and $C = 2^7$ (thin solid line), $C = 2^8$ (thin dashed line), and $C = 2^9$ (thin dotted line). To compare the probability distributions with different C easily, these distributions are multiplied by C respectively. $\bar{q}^0$ is not presented in these figures because it is far beyond the others.

and $i \in \mathcal{S}$, two connected components will combine together into a new one with the size $A_m + A_n + 1$. The probability of $A_m + A_n$ can be described by a new introduced probability distribution $\tilde{q}^A$ in which $A \in \{0, \ldots, C\}$ and

$$\tilde{q}^A = \sum_{A_m + A_n < C} q_{m\to i}^{A_m} q_{n\to i}^{A_n} \delta(A_m + A_n, A) \quad (A < C), \tag{13a}$$

$$\tilde{q}^C = \sum_{A_m + A_n \geqslant C} q_{m\to i}^{A_m} q_{n\to i}^{A_n}. \tag{13b}$$

Because the size of the connected component cannot be larger than C, we are concerned only about the situations of $A_m + A_n < C$ and $A = A_m + A_n$ in this case. For $A = C$, we sum all probabilities of $A_m + A_n \geqslant C$ together. In our later discussion, Eq. (13) is abbreviated as $\tilde{q}^A = q_{m\to i}^{A_m} \otimes q_{n\to i}^{A_n}$.

Now we consider all nearest neighbors of vertex i. In the same way, we can use a product of $\otimes$ to compute the probability of $\tilde{q}_i^{A_i}$, which means the probability distribution of $A_i = \sum_{j\in\partial i} A_j$ when $A_i < C$ and $A_i = C$ when $\sum_{j\in\partial i} A_j \geqslant C$:

$$\tilde{q}_i^{A_i} = \bigotimes_{k\in\partial i} q_{k\to i}^{A_k}. \tag{14}$$

In the following, we can obtain the value of $q_i^{A_i}$ easily from $\tilde{q}_i^{A_i}$:

$$q_i^0 = e^{-\beta\omega_i}/z_i, \tag{15a}$$

$$q_i^{A_i} = \tilde{q}_i^{A_i - 1}/z_i \quad (A_i \neq 0), \tag{15b}$$

where

$$z_i \equiv e^{-\beta\omega_i} + \sum_{A=0}^{C-1} \tilde{q}_i^A. \tag{16}$$

Similarity, we can also compute the cavity message $q_{i\to j}^{A_i}$ from $\tilde{q}_{i\to j}^{A_i}$:

$$q_{i\to j}^0 = e^{-\beta\omega_i}/z_{i\to j}, \tag{17a}$$

$$q_{i\to j}^{A_i} = \tilde{q}_{i\to j}^{A_i - 1}/z_{i\to j} \qquad (A_{i\to j} \neq 0), \tag{17b}$$

where

$$\tilde{q}_{i\to j}^{A_i} = \bigotimes_{k\in\partial i\setminus j} q_{k\to i}^{A_k}, \tag{18}$$

$$z_{i\to j} \equiv e^{-\beta\omega_i} + \sum_{A=0}^{C-1} \tilde{q}_{i\to j}^A. \tag{19}$$

The complexity of the operation $\otimes$ is on the order of $(C+1)^2$. For a vertex i with $|\partial i| = k$, we will use the operator $\otimes$ $3k-2$ times to update all messages $\{q_{i\to j}^{A_i}\}_{j\in\partial i}$ (see Appendix A for more details). Therefore, the computation time of updating all messages on graph $\mathcal{G}$ will be on the order of NKC^2 or MC^2, where K is the average degree of each vertex. In Fig. 4 we present the computation time of updating all messages on various random graphs with different C. The computation time is proportional to the N, to the mean degree c in ER graph, or to the degree K in RR graph and to C^2. What is more, the computation time is irrelevant to the type of random graph ensembles as long as they have the same average degree.

C. The dismantling problem in the large N limit

In this subsection, we will discuss the dismantling problem in the large N limit, where either $C \to \infty$ and we keep the

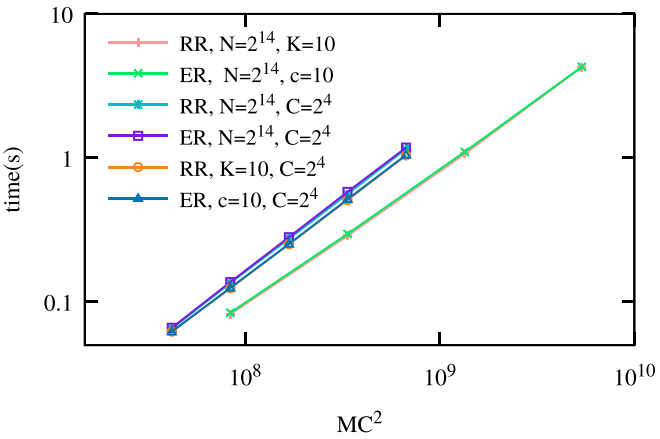

FIG. 4. The real computation time of updating all messages on a random graph. All messages are updated synchronously and in parallel on a desktop computer (AMD-2700X, dual channel memory at 2666 MHz). When $N = 2^{14}$ and $K = 10$ in RR graphs (pluses) or $c = 10$ in ER graphs (crosses), we run $C = 2^5, 2^6, 2^7, 2^8$. When $N = 2^{14}$ and $C = 2^4$, we run $K = 20, 40, 80, 160, 320$ in RR graphs (stars) and $c = 20, 40, 80, 160, 320$ in ER graphs (squares). When $K = 10$ in RR graphs (circles) or $c = 10$ in ER graphs (triangles) and $C = 2^4$, we run $N = 2^{15}, 2^{16}, 2^{17}, 2^{18}, 2^{19}$.

following limitation:

$$\lim_{N\to\infty} \frac{C}{N} = 0.01. \tag{20}$$

In that case, the computation time of BP iterations will be proportional to N^3, and both computation time and memory usage will become unaffordable. Therefore, we hope Eq. (7) can be simplified further.

From Fig. 3 we know that $\overline{q}_i$ increases exponentially with growing A/C as long as $A/C > 0$. Therefore, a vertex belonging to a connected component with very large size but smaller than $C/2$ is almost impossible and can be neglected. Under this assumption, we can introduce another discrete probability distribution with only three states: 0, $\mathbb{I}$ and $\mathbb{X}$, which mean a vertex i in set $\mathcal{D}$, in a connected component with finite size and in a connected component with infinite size larger than $C/2$. $\hat{q}_i^0$, $\hat{q}_i^{\mathbb{I}}$, and $\hat{q}_i^{\mathbb{X}}$ are marginal probabilities of vertex i in states 0, $\mathbb{I}$, and $\mathbb{X}$, respectively. Then, in the same way, we can define messages $\hat{q}_{i\to j}^0$, $\hat{q}_{i\to j}^{\mathbb{I}}$, and $\hat{q}_{i\to j}^{\mathbb{X}}$ in the cavity graph $\mathcal{G}_{\backslash i}$. If all neighbor vertices $j \in \partial i$ in state $\mathbb{I}$ or 0, we can add vertex i to set $\mathcal{S}$, and the vertex i will be in state $\mathbb{I}$. If there is only one nearest-neighbor vertex in state $\mathbb{X}$, and all other neighbor vertices are in state $\mathbb{I}$ or 0, adding vertex i to set $\mathcal{S}$ will generate a new connected component with infinite size but still satisfying the limitation 20. However, if there are more than one neighbor vertices in state $\mathbb{X}$, the size of the new generated component will be larger than $0.01N$, which means the vertex i must be in set $\mathcal{D}$. Now, we can have the following self-consistent equations of these probability distributions:

$$\hat{q}_{i\to j}^0 = \frac{e^{-\beta\omega_i}}{\hat{z}_{i\to j}}, \tag{21a}$$

$$\hat{q}_{i\to j}^{\mathbb{I}} = \frac{1}{\hat{z}_{i\to j}} \prod_{k\in\partial i\backslash j} \left(\hat{q}_{k\to i}^0 + \hat{q}_{k\to i}^{\mathbb{I}}\right), \tag{21b}$$

$$\hat{q}_{i\to j}^{\mathbb{X}} = \frac{1}{\hat{z}_{i\to j}} \sum_{k\in\partial i\backslash j} \hat{q}_{k\to i}^{\mathbb{X}} \prod_{m\in\partial i\backslash j,k} \left(\hat{q}_{m\to i}^0 + \hat{q}_{m\to i}^{\mathbb{I}}\right), \tag{21c}$$

where

$$\hat{z}_{i\to j} \equiv e^{-\beta\omega_i} + \prod_{k\in\partial i\backslash j} \left(\hat{q}_{k\to i}^0 + \hat{q}_{k\to i}^{\mathbb{I}}\right) + \sum_{k\in\partial i\backslash j} \frac{\hat{q}_{k\to i}^{\mathbb{X}}}{\hat{q}_{k\to i}^0 + \hat{q}_{k\to i}^{\mathbb{I}}} \prod_{k\in\partial i\backslash j} \left(\hat{q}_{k\to i}^0 + \hat{q}_{k\to i}^{\mathbb{I}}\right). \tag{22}$$

And the marginal probability of each vertex $\hat{q}_i$ can also be computed by

$$\hat{q}_i^0 = \frac{e^{-\beta\omega_i}}{\hat{z}_i}, \tag{23a}$$

$$\hat{q}_i^{\mathbb{I}} = \frac{1}{\hat{z}_i} \prod_{j\in\partial i} \left(\hat{q}_{j\to i}^0 + \hat{q}_{j\to i}^{\mathbb{I}}\right), \tag{23b}$$

$$\hat{q}_i^{\mathbb{X}} = \frac{1}{\hat{z}_{i\to j}} \sum_{j\in\partial i} \hat{q}_{j\to i}^{\mathbb{X}} \prod_{k\in\partial i\backslash j} \left(\hat{q}_{k\to i}^0 + \hat{q}_{k\to i}^{\mathbb{I}}\right), \tag{23c}$$

where

$$\hat{z}_i \equiv e^{-\beta\omega_i} + \left(1 + \sum_{j\in\partial i} \frac{\hat{q}_{j\to i}^{\mathbb{X}}}{\hat{q}_{j\to i}^0 + \hat{q}_{j\to i}^{\mathbb{I}}}\right) \prod_{j\in\partial i} \left(\hat{q}_{j\to i}^0 + \hat{q}_{j\to i}^{\mathbb{I}}\right). \tag{24}$$

Comparing the equations above with the BP iterations in the FVS spin-glass model discussed in Refs. [4,5,27], we find there is only one very small difference between them: Eqs. (21c) and (23c) use $\hat{q}_{j\to i}^{\mathbb{X}}$ instead of $\hat{q}_{j\to i}^{\mathbb{X}} + \hat{q}_{j\to i}^{\mathbb{I}}$. Actually, both $\hat{q}_{i\to j}^{\mathbb{I}}$ in Eq. (21) and $q_{i\to j}^i$ in Eq. (20) of Ref. [5] are very small in their respective iteration equations. This result confirms the connection between the dismantling problem and the FVS problem under the thermodynamic limit and explains why the lower bounds of the FVS problem and ρ_∞ are close to each other and why decycling algorithms work so well in dismantling problems.

D. The CD-BPD algorithm and SCD-BPD algorithm

In this subsection, we develop two belief-propagation-guide decimation (BPD) algorithms based on Eqs. (7) (denoted as CD-BPD) and (21) (denoted as SCD-BPD), respectively, to solve the CD problem on a certain graph. The details and pseudocode of the two BPD algorithms are given in Appendix B. We compare the performance of the two BPD algorithms with two other heuristic methods [collective influence algorithm with ball radius $\ell = 2$ (CI) [7] and node explosive percolation algorithm with the second score definition in Ref. [8] (NEP)] on ER, RR, and SF graphs with various average degree and present all results in Figs. 1 and 2. In Fig. 1 we find the CD-BPD gives near optimal $\mathcal{D}$ very close to the result predicted by the RS mean-field method, and it is far better than other algorithms. Because the approximation of Eq. (21) holds only in the large C limit, it is not surprising that the results of the SCD-BPD approach that of CD-BPD gradually with growing C and even outperform CD-BPD a little in the RR graph with $C = 512$. Actually, if $m, n \in \partial i$ and they are in the same loop with the length shorter than

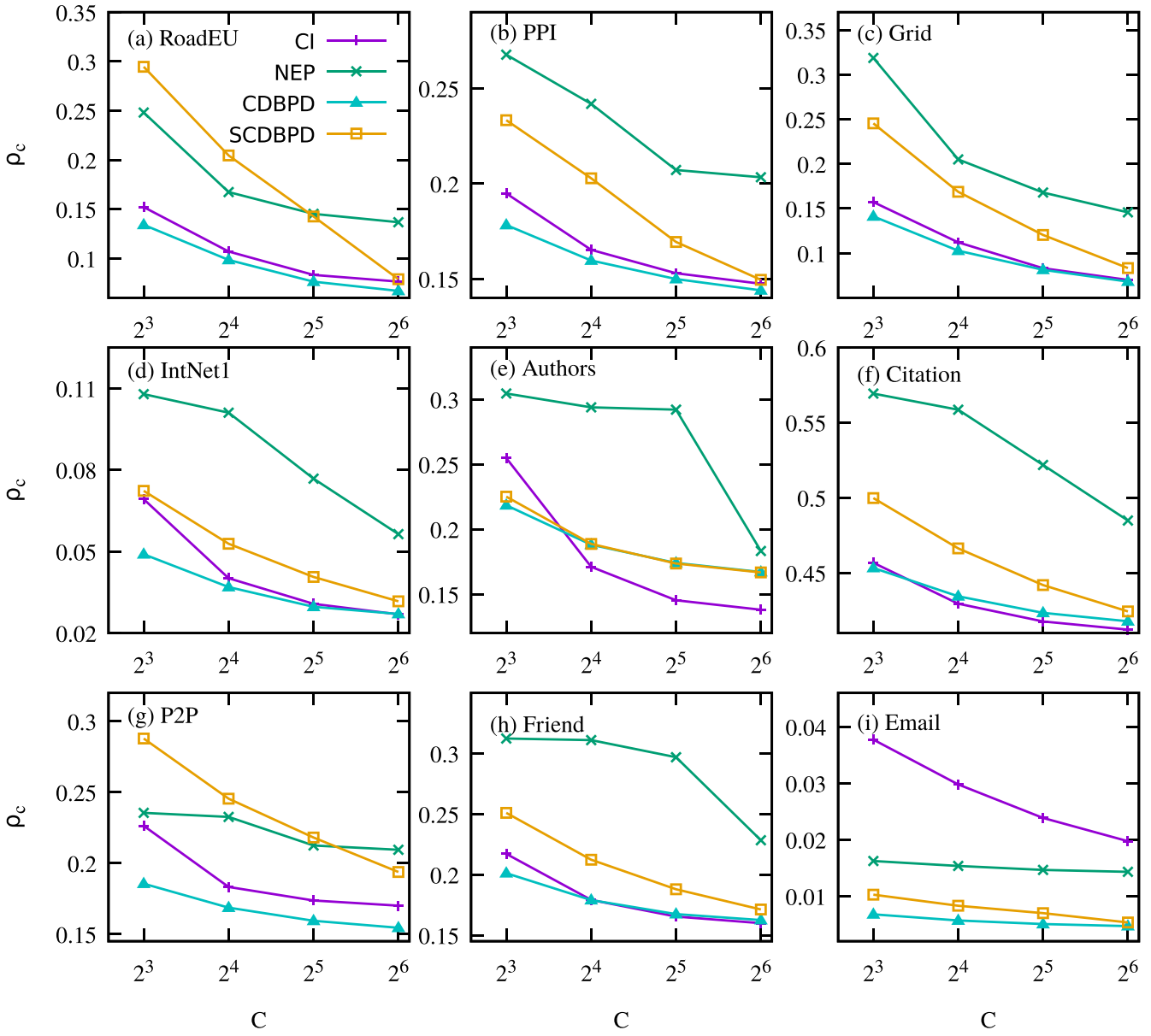

FIG. 5. The relative size ρ of set $\mathcal{D}$ for some real-world networks as a function of C given by CI (pluses), NEP (crosses), CD-BPD (triangles), and SCD-BPD (squares) algorithms. The numbers in the brackets are the clustering coefficients of the corresponding graph. (a) RoadEU (0.0671) [40], (b) PPI (0.1301) [41], (c) Grid (0.0801) [28], (d) IntNet1 (0.2522) [42], (e) Authors (0.6334) [43], (f) Citation (0.2848) [42], (g) P2P (0.0055) [42], (h) Friend(0.2367) [42], (i) Email (0.0671) [43].

C, it is possible that they are also in the same connected component and $A_m = A_n$. In that case, the Bethe-Peierls approximation is invalid, and extra vertices will be added to $\mathcal{D}$. The random graphs used in our computation have a finite number of vertices, so the length of typical cycles is also limited (on the order of $\ln N$) [44–46]. Therefore, in the case of C larger than the length of typical cycles, the SCD-BPD may surpass CD-BPD not only in computation efficiency but also in computation results.

Figure 2 concerns the performance of these algorithms in all kinds of artificial random graphs. We can see that the CD-BPD still gives the best results for the CD problem with $C = 64$. Actually, as ρ_c approaches ρ_∞ in the order of C^{-1}, the results given by the CD-BPD with $C = 64$ is already very close to the ρ_∞ and also close to the result given by the SCD-BPD in the dismantling problem.

Finally, we apply these algorithms in some real-world networks, which contain many communities, local loops, and hierarchical levels. The value of ρ with various C are presented in Fig. 5. Except for the Authors, Citations, and Friends networks, where the CI obtains better results when $C > 8$, the CD-BPD gives the minimum $\mathcal{D}$ in all tested algorithms. We find the clustering coefficients of Authors, Citations, and Friends networks are relatively large, which means there exists a mass of short loops in them. We believe that is the main reason why the CD-BPD does not work well in these instances. Another network with conspicuous clustering coefficient is the IntNet1, where the CI performs as well as

the CD-BPD when $C \geqslant 32$. The results of the SCD-BPD also approach that of the CD-BPD with growing C in these networks.

IV. CONCLUSION AND DISCUSSION

In this paper, we propose a spin-glass model for the CD problem and study its properties by the RS mean-field method. We also develop two BPD algorithms to solve the CD problem in a certain network. The CD-BPD gives the best result in all state-of-art algorithms for the CD problem with small C, and the SCD-BPD consumes less computation resource when C is large. Both of them work well for the large C as long as the clustering coefficients of the network is small.

Although the loops in a random graph may lead to a negative effect for the CD spin-glass mode, the values of ρ_c and ρ_∞ are not overestimated because the number of loops with finite length in a random graph is also finite even in the thermodynamic limit [44–46]. On the other hand, the length of typical loops in a random graph will be larger than C as $N \to \infty$, and the RS mean-field method neglects these possible long-range correlations. Considering both aspects together, we say the CD spin-glass model provides lower bounds ρ_c and ρ_∞ of the CD and the dismantling problem in the random graphs.

Additionally, our CD spin-glass model connects the vertex cover problem with $C = 1$ and the FVS problem with $C \to \infty$. We also notice that the FVS problem shares many features with the vertex cover problem, such as how the critical temperatures change with the mean vertex degree in a nonmonotonic way [27,47]. So we speculate that the first-step replica-symmetry-breaking phase transition of the CD problem will belong to the same universal class with the FVS and the vertex cover problem [37,38,48]. We will confirm this speculation in a separate paper.

The SCD-BPD algorithm solves the CD and dismantling problems in a more straightforward way than the FVS-BPD algorithm discussed in Ref. [4], although they have similar message-passing equations. There are three stages in decycling algorithms [3,4]: finding the minimum FVS, breaking the remaining tree, and introducing some possible cycles. The SCD-BPD algorithm can give the dismantling set directly without the latter two stages.

Our numerical computations on real-world networks reveal that the CD-BPD and SCD-BPD do not work well in networks with numerous short loops. This drawback can be made up for by considering the effects of local short loops. We can start this work from the simplest triangle structure and then extend it to more complex local structures. We believe that better results for these real-world networks can be obtained in our future work.

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APPENDIX A: PSEUDOCODE OF UPDATE OF ALL MESSAGES ON GRAPH $\mathcal{G}$ WITH EQS. (13) AND (17)

Here we explain how to update all messages on graph $\mathcal{G}$ with Eqs. (13) and (17). For each vertex, if we update all messages $\{q^{A_i}_{i\to j}\}_{j\in\partial i}$ together, we need to use the $\otimes$ operation only $3k-2$ times, where $k = |\partial i|$. Then the computation can be simplified further.

Algorithm 1: Update all messages on graph $\mathcal{G}$ with Eqs. (13) and (17).

Generate a random order of all vertices $\mathcal{V}$: $\{t_1, t_2, \ldots, t_N\}$;
for $r = 1, \ldots, N$ **do**
  Select the vertex i in the order of $\{t_1, t_2, \ldots, t_N\}$, then $i = t_r$;
  Define a sequence $\{j_1, j_2, \ldots, j_k\}$ for the nearest neighbors ∂i;
  Set message $q'_{j_1} = \{1, 0, \ldots, 0\}$ and $q''_{j_k} = \{1, 0, \ldots, 0\}$;
  for $s = 2, \ldots, k$ **do**
    Compute $q'_{j_s} = q'_{j_{s-1}} \otimes q_{j_{s-1}\to i}$;
  end for
  for $s = k-1, \ldots, 1$ **do**
    Compute $q''_{j_s} = q''_{j_{s+1}} \otimes q_{j_{s+1}\to i}$;
  end for
  for s=1, ..., k **do**
    Compute $\tilde{q}_{i\to j_s} = q'_{j_s} \otimes q''_{j_s}$;
    Update $q_{i\to j_s}$ from $\tilde{q}_{i\to j_s}$ by Eq. (17);
  end for
end for

APPENDIX B: CD-BPD ALGORITHM AND SCD-BPD ALGORITHM

For a given graph $\mathcal{G}$, the BP equations cannot only estimate the size of the minimum set $\mathcal{D}$, but also give a near-optimal solution of the CD problem with the BPD algorithm. In this paper, we introduce BP equations to study the properties of CD problem with finite C and then simplify it in the large C limit. Therefore, we develop two different BPD algorithms based on the original BP equations and the simplified BP equations, which are denoted as CD-BPD and SCD-BPD, respectively.

At the beginning of each BPD algorithm, we set $\beta > \beta^*$ and randomly initial all messages $\{q_{i\to j}\}$ or $\{\hat{q}_{i\to j}\}$ on graph $\mathcal{G}$. Then we empty the set $\mathcal{S} = \emptyset$ and $\mathcal{D} = \emptyset$. In each round of the BPD algorithms, the BP equation (7) or (21) are performed enough times so that every message can spread its information to the entire connected component. The BP equations may not reach its fixed point in the BPD algorithm, but it does not prevent us from computing the marginal probability of each vertex q_i^0 or $\hat{q}_i^0$ by Eq. (5) or (23). Then a small fraction of vertices with the largest q_i^0, $\hat{q}_i^0$ are added into set $\mathcal{D}$. At the same time, we will also remove these vertices and their adjacent edges from the graph. During this process, it is possible that the remaining graph breaks to many connected components, some of which will be equal to or smaller than C. Because these small connected components satisfy the constraint of the CD problem, we can remove the entire connected component

away from the remaining graph to avoid unnecessary vertex attacking. The vertices in these small connected components are added to the set $\mathcal{S}$. After that, we can iterate BP equations for the next round until all vertices are removed from the graph and $\mathcal{D} \bigcup \mathcal{S} = \mathcal{V}$. Here we present the pseudocode of two BPD algorithms in the following algorithm.

Algorithm 2. The CD-BPD and SCD-BPD algorithm based on the original BP equations and the simplified BP equations.

For a graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ with $N = |\mathcal{V}|$ vertices, inverse temperature β, a small fraction f, and iteration number T, initial all messages $\{q_{i\to j}\}$ or $\{\hat{q}_{i\to j}\}$ on the graph $\mathcal{G}$ randomly;
Empty the set $\mathcal{D}$ and $\mathcal{S}$: $\mathcal{D} = \emptyset$, $\mathcal{S} = \emptyset$;
while $\mathcal{D} \bigcup \mathcal{S} \neq \mathcal{V}$ **do**
  for$t = 1, \ldots, T$ **do**
    Try to find out the solution of BP equations by updating messages $\{q_{i\to j}\}$ or $\{\hat{q}_{i\to j}\}$;
  end for
  for$i = 1, \ldots, N$ **do**
    Compute the value of q_i^0 or $\hat{q}_i^0$ by their corresponding RS cavity equations;
  end for
  for s=*1*, $\ldots$, fN **do**
    Find the vertex i with the largest q_i^0 or $\hat{q}_i^0$ in the remaining graph;
    Add vertex i into set $\mathcal{D}$;
    Delete vertex i with its adjacent edges from the remaining graph;
    if There are connected components with the size not larger than C in the remaining graph **then**
      Add all vertices in these connected components into set $\mathcal{S}$;
      Remove the entire connected components from the remaining graph;
    end if
  end for
end while

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