Statistical analysis of articulation points in configuration model networks

Ido Tishby,[1] Ofer Biham,[1] Reimer Kühn,[2] and Eytan Katzav[1]
[1]*Racah Institute of Physics, The Hebrew University, Jerusalem 91904, Israel*
[2]*Department of Mathematics, King's College London, Strand, London WC2R 2LS, United Kingdom*

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An articulation point (AP) in a network is a node whose deletion would split the network component on which it resides into two or more components. APs are vulnerable spots that play an important role in network collapse processes, which may result from node failures, attacks, or epidemics. Therefore, the abundance and properties of APs affect the resilience of the network to these collapse scenarios. Here we present analytical results for the statistical properties of APs in configuration model networks. In order to quantify the abundance of APs, we calculate the probability $P(i \in \mathrm{AP})$ that a random node, i, in a configuration model network with a given degree distribution, $P(K = k)$, is an AP. We also obtain the conditional probability $P(i \in \mathrm{AP}|k)$ that a random node of degree k is an AP and find that high-degree nodes are more likely to be APs than low-degree nodes. Using Bayes's theorem, we obtain the conditional degree distribution, $P(K = k|\mathrm{AP})$, over the set of APs and compare it to the overall degree distribution $P(K = k)$. We propose a centrality measure based on APs: Each node can be characterized by its articulation rank, r, which is the number of components that would be added to the network upon deletion of that node. For nodes which are not APs, the articulation rank is $r = 0$, while for APs it satisfies $r \geqslant 1$. We obtain a closed-form analytical expression for the distribution of articulation ranks, $P(R = r)$. Configuration model networks often exhibit a coexistence between a giant component and finite components. While the giant component is extensive in the network size and exhibits cycles, the finite components are nonextensive tree structures. To examine the distinct properties of APs on the giant and on the finite components, we calculate the probabilities presented above separately for the giant and the finite components. We apply these results to ensembles of configuration model networks with degree distributions that follow a Poisson distribution (Erdős-Rényi networks), an exponential distribution of the form $P(K = k) \sim e^{-\alpha k}$, and a power-law distribution of the form $P(K = k) \sim k^{-\gamma}$ (scale-free networks), where $k \geqslant k_{\min} = 1$. The implications of these results are discussed in the context of common attack scenarios and network dismantling processes.

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I. INTRODUCTION

Network models provide a useful conceptual framework for the study of a large variety of systems and processes in science, technology, and society [1–4]. These models consist of nodes and edges, where the nodes represent physical objects, while the edges represent the interactions between them. Unlike regular lattices in which all the nodes have the same coordination number, network models are characterized by a degree distribution, $P(K = k)$, $k = 0, 1, 2, \ldots$, whose mean is denoted by $\langle K \rangle$. The backbone of a network typically consists of high-degree nodes, or hubs, which connect the different branches and maintain the integrity of the network. In some applications, such as communication networks, it is crucial that the network will consist of a single connected component. However, mathematical models also produce networks which combine a single giant component and small isolated components, as well as fragmented networks which consist of only small components [5].

Networks are often exposed to the loss of nodes and edges, which may severely affect their functionality. Such losses may occur due to inadvertent node failures, propagation of epidemics, or deliberate attacks. Starting from a network which consists of a single connected component, as nodes are deleted some small fragments become disconnected from the giant component. As a result, the size of the giant component decreases until it disintegrates into many small components. The ultimate failure, when the network fragments into small disconnected components, was studied extensively using percolation theory [6–12].

A major factor in the sensitivity of networks to node deletion processes is the fact that the deletion of a single node may separate a whole fragment from the giant component. This fragmentation process greatly accelerates the disintegration of the network. In each network, one can identify the nodes whose deletion would break the component on which they reside into two or more components [13–15]. Such nodes are called articulation points (APs) or cut vertices. In Fig. 1(a), we present a schematic illustration of an AP (marked by a full circle) of degree $k = 3$ in a tree network component. Deletion of the AP would split the network into three separate components. In Fig. 1(b), we show an AP (full circle) of degree $k = 3$, where two of its neighbors reside on a cycle. Deletion of the AP would split the network into two separate components. The node marked by a full circle in Fig. 1(c) is not an AP because each pair of its neighbors share a cycle. As a result, upon deletion of the marked node, all its neighbors remain on the same network component.

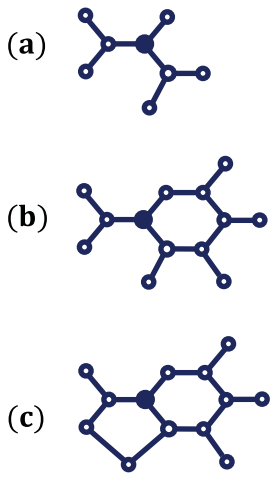

FIG. 1. Schematic illustration of APs and their surrounding network components: (a) An AP (marked by a full circle) of degree $k = 3$ in a tree network component. Deletion of the AP would split the network into three separate components. (b) An AP (full circle) of degree $k = 3$, where two of its neighbors reside on a cycle. Deletion of the AP would split the network into two separate components. (c) Here the node marked by a full circle is not an AP because each pair of neighbors of the marked node share a cycle. As a result, upon deletion of the marked node, all its neighbors remain on the same network component.

Since isolated nodes and leaf nodes cannot be APs, in order for a node i to be an AP, it must be of degree $k \geqslant 2$. Upon deletion of node i of degree k, in order for its k neighbors to remain on the same connected component, each pair of neighbors must be connected to each other via at least one path in the reduced network from which i was removed. In the case that upon deletion of i there is at least one pair of neighbors of i which are not connected to each other, these nodes will end up on different connected components, implying that i is an AP. In tree networks, each pair of nodes are connected by a single path, which means that any node of degree $k \geqslant 2$ is an AP. As the network becomes denser and the number of cycles increases, the abundance of APs tends to decrease. Since the cycles connecting neighbors of a node i may be very long, the determination of whether i is an AP cannot be done locally. It requires access to the complete structure of the whole network in order to identify the cycles which connect all the pairs of neighbors of i [16–20]. The statistical properties of cycles in a network are closely related to those of paths connecting random pairs of nodes [21–30]. More specifically, on tree networks the shortest path is the only path connecting any pair of nodes and thus there are no cycles. In other networks, the distribution of cycle lengths on which a random node resides can be obtained from the distribution of path lengths between pairs of neighbors of a random node, i, in the reduced network from which i is removed. Network components which do not include any APs are called biconnected or two-connected components. In order to split such network components to two or more parts, one needs to simultaneously delete at least two nodes. In such network components, each pair of nodes is connected by at least two disjoint paths [31].

Apart from the loss of nodes, networks are often exposed to the loss of connections between nodes, which can be modeled by edge deletion. In some cases, the deletion of a single edge would break the network component on which it resides into two separate components. Such edges are called bridges or cut edges [32], and in many ways are analogous to the articulation points considered in this paper. In fact, any edge which does not reside on even a single cycle is a bridge. Thus, in network components which exhibit a tree structure, all the edges are bridges. A bridge, like any other edge in a network, provides a connection between two nodes. In the case that a pair of nodes, i and j, are connected by a bridge, each one of them must be either an AP or a leaf node. In the case that the bridge resides on the giant component, at least one of the two nodes at its ends must be an AP. Similarly, each AP is connected to at least one bridge.

The functionality of most networks relies on the integrity of their giant components. Therefore, it is particularly important to study the properties of APs which reside on the giant component. These APs are vulnerable spots in the structure of a network, because the deletion of a single AP may detach an entire branch or several branches from the giant component. This vulnerability is exploited in network attack strategies, which target existing APs and generate new APs via decycling processes [15]. While APs make the network vulnerable to attacks, they are advantageous in fighting epidemics. In particular, the vaccination of APs prevents the spreading of epidemics between the network components connected by these APs. Similarly, in communication networks, the party in possession of an AP may control, screen, block, or alter the communication between the network components connected by this AP. APs are instrumental in the design of efficient algorithms for approximate solutions of difficult computational problems on networks, such as the vertex cover problem [33]. They can also be used to simplify the calculation of determinants of sparse matrices which represent networks that include APs [34].

In this paper, we present analytical results for the statistical properties of APs in Erdős-Rényi (ER) networks [35–37] and configuration model networks with various degree distributions [38–41]. We obtain the probability that a random node in a configuration model network with a given degree distribution $P(K = k)$ is an AP. We also calculate the conditional probability $P(i \in \mathrm{AP}|k)$ that a random node of a given degree k is an AP and the degree distribution $P(K = k|\mathrm{AP})$, conditioned on the APs. The above discussion motivates the introduction of a new AP-based centrality measure: When an AP is deleted, the component on which it resides breaks into two or more components. We denote the number of components, r, which are added to the network upon deletion of a given node, i, as the articulation rank of this node. The articulation rank of a node which is not an AP is $r = 0$, while the articulation ranks of APs satisfy $r \geqslant 1$. We obtain analytical results for the distribution of articulation ranks, $P(R = r)$.

The paper is organized as follows. In Sec. II, we present the configuration model. In Sec. III, we describe relevant properties of the giant component and the finite components of configuration model networks. In Sec. IV, we discuss the main properties of articulation points and present efficient methods for their detection. In Sec. V, we present analytical results for the probability that a random node in a configuration

model network is an articulation point. In Sec. VI, we present analytical results for the degree distribution of articulation points. In Sec. VII, we calculate the distribution of ranks of articulation points. In Sec. VIII, we study the special properties of APs with degree $k = 2$. In Sec. IX, we apply these results to configuration model networks with Poisson degree distributions (ER networks), exponential degree distributions, and power-law degree distributions (scale-free networks). The results are discussed in Sec. X and summarized in Sec. XI. In Appendix A, we present some useful properties of the generating functions which are utilized in the analysis. In Appendix B, we use these inequalities to show that the mean degree of the APs that reside on the giant component is larger than the mean degree of all the nodes of the giant component. In Appendix C, we use the configuration model network with a ternary degree distribution to systematically explore the abundance of APs in the giant component and in the finite components under different conditions.

II. THE CONFIGURATION MODEL

The configuration model is an ensemble of uncorrelated random networks which follow a predefined degree distribution, $P(K = k)$. In analytical studies, one often considers the asymptotic case in which the network size is infinite. In numerical simulations, the network size, N, is finite. In many cases, one bounds the degree distribution from above and below such that $k_{\min} \leqslant k \leqslant k_{\max}$. For example, using $k_{\min} = 1$ eliminates the possibility of isolated nodes, while $k_{\min} = 2$ also eliminates the leaf nodes. Controlling the upper bound is particularly important in the case of degree distributions which exhibit fat tails, such as power-law degree distributions.

The configuration model ensemble is a maximum entropy ensemble under the condition that the degree distribution, $P(K = k)$, is imposed [2,40]. Here, we focus on the case of undirected networks, in which all the edges are bidirectional. In each network instance from an ensemble of configuration model networks of N nodes with a given degree distribution $P(K = k)$, one draws the degrees of all the N nodes independently from $P(K = k)$, producing the degree sequence $k_1, k_2, \ldots, k_N$.

For the computer simulations presented below, we draw random network instances from an ensemble of configuration model networks of N nodes which follow a given degree distribution, $P(K = k)$. For each network instance, we generate a degree sequence of the form $k_1, k_2, \ldots, k_N$, as described above. For the construction process, it is convenient to order the degree sequence in the form $k_1 \geqslant k_2 \geqslant \cdots \geqslant k_N$. It turns out that not every possible degree sequence is graphic, namely admissible as a degree sequence of at least one network instance. Therefore, before trying to construct a network with a given degree sequence, one should first confirm the graphicality of the degree sequence. To be graphic, a degree sequence must satisfy two conditions. The first condition is that the sum of the degrees is an even number, namely $\sum_{i=1}^{N} k_i = 2L$, where L is an integer which represents the number of edges in the network. The second condition is expressed by the Erdős-Gallai theorem, which states that an ordered sequence of the form $k_1 \geqslant k_2 \geqslant \cdots \geqslant k_N$ is graphic if and only if the condition [42,43]

$$\sum_{i=1}^{n} k_i \leqslant n(n-1) + \sum_{i=n+1}^{N} \min(k_i, n) \tag{1}$$

holds for all values of n in the range $1 \leqslant n \leqslant N - 1$.

A convenient way to construct a configuration model network is to prepare the N nodes such that each node, i, is connected to k_i half edges or stubs [2]. Pairs of half edges from different nodes are then chosen randomly and are connected to each other in order to form the network. The result is a network with the desired degree sequence and no correlations. Note that toward the end of the construction the process may get stuck. This may happen in the case that the only remaining pairs of stubs belong to the same node or to nodes which are already connected to each other. In such cases, one may perform some random reconnections in order to enable completion of the construction.

III. THE GIANT COMPONENT AND THE FINITE COMPONENTS

Configuration model networks often consist of multiple connected components. In some cases, the size of the largest component scales linearly with the network size, N. In such cases, the largest component is called the giant component. All the other components are finite, nonextensive components which exhibit tree structures with no cycles. The size of the giant component is determined by the degree distribution, $P(K = k)$. Some families of degree distributions can be parametrized such that the parameter space is separated into two regimes, the dilute network regime in which there is no giant component and the dense network regime in which there is a giant component. On the boundary between these two regimes, there is a percolation transition [1].

Consider a configuration model network of N nodes with a given degree distribution, $P(K = k)$. To obtain the the probability, g, that a random node in the network belongs to the giant component, one needs to first calculate the probability, $\tilde{g}$, that a random neighbor of a random node, i, belongs to the giant component of the reduced network, which does not include the node i. The probability $\tilde{g}$ is determined by [1]

$$1 - \tilde{g} = G_1(1 - \tilde{g}), \tag{2}$$

where

$$G_1(x) = \sum_{k=1}^{\infty} x^{k-1} \widetilde{P}(K = k) \tag{3}$$

is the generating function of the distribution $\widetilde{P}(K = k)$, which is the degree distribution of nodes which are sampled as random neighbors of random nodes. It is given by

$$\widetilde{P}(K = k) = \frac{k}{\langle K \rangle} P(K = k), \tag{4}$$

where

$$\langle K \rangle = \sum_{k=0}^{\infty} k P(K = k) \tag{5}$$

is the mean degree of the nodes in the network. Using $\tilde{g}$, one can then obtain the probability g from the equation

$$g = 1 - G_0(1 - \tilde{g}), \tag{6}$$

where

$$G_0(x) = \sum_{k=0}^{\infty} x^k P(K = k) \tag{7}$$

is the generating function of the distribution $P(K = k)$.

From the definitions of $G_0(x)$ and $G_1(x)$ in Eqs. (7) and (3), respectively, we find that $G_0(1) = 1$ and $G_1(1) = 1$. This means that $x = 1$ is a fixed point for both generating functions. Therefore, $g = \tilde{g} = 0$ is a solution of Eq. (2). This solution corresponds to the case of subcritical networks, in which there is no giant component. In some networks, there are no isolated nodes (of degree $k = 0$) and no leaf nodes (of degree $k = 1$). In such networks, $P(K = k) > 0$ only for $k \geqslant 2$. For these networks, we find that $G_0(0) = 0$ and $G_1(0) = 0$. This implies that in such networks both $x = 0$ and $x = 1$ are fixed points of both $G_0(x)$ and $G_1(x)$. The coexistence of a giant component and finite components appears for degree distributions that support a nontrivial solution of Eq. (2), in which $0 < \tilde{g} < 1$.

The probability that a random node resides on the giant component (GC) is $P(i \in \mathrm{GC}) = g$, and the probability that it resides on one of the finite components (FC) is $P(i \in \mathrm{FC}) = 1 - g$. Similarly, the probabilities that a random neighbor of a random node resides on the giant component is $\widetilde{P}(i \in \mathrm{GC}) = \tilde{g}$, and the probability that it resides on one of the finite components is $\widetilde{P}(i \in \mathrm{FC}) = 1 - \tilde{g}$. A node, i, of degree k resides on the giant component if at least one of its k neighbors resides on the giant component of the reduced network from which i is removed. Therefore,

$$P(i \in \mathrm{GC}|k) = 1 - (1 - \tilde{g})^k \tag{8}$$

and

$$P(i \in \mathrm{FC}|k) = (1 - \tilde{g})^k. \tag{9}$$

The microstructure of the giant component of configuration model networks was recently studied [44]. It was shown that the degree distribution, conditioned on the giant component, is given by

$$P(K = k|\mathrm{GC}) = \frac{1 - (1 - \tilde{g})^k}{g} P(K = k), \tag{10}$$

while the degree distribution, conditioned on the finite components, is given by

$$P(K = k|\mathrm{FC}) = \frac{(1 - \tilde{g})^k}{1 - g} P(K = k). \tag{11}$$

The mean degree of the giant component is

$$\mathbb{E}[K|\mathrm{GC}] = \frac{1 - (1 - \tilde{g})^2}{g} \langle K \rangle \tag{12}$$

or

$$\mathbb{E}[K|\mathrm{GC}] = \frac{\tilde{g}(2 - \tilde{g})}{g} \langle K \rangle, \tag{13}$$

while the mean degree on the finite components is

$$\mathbb{E}[K|\mathrm{FC}] = \frac{(1 - \tilde{g})^2}{1 - g} \langle K \rangle. \tag{14}$$

Using the inequality $(1 - \tilde{g})^2/(1 - g) < 1$ (Appendix A), we find that, as expected, $\mathbb{E}[K|\mathrm{GC}] > \langle K \rangle$ and $\mathbb{E}[K|\mathrm{FC}] < \langle K \rangle$.

IV. ARTICULATION POINTS AND THEIR DETECTION

An articulation point in a network is a node, i, whose deletion would break the network component on which it resides into two or more components. Each one of these components must include at least one neighbor of i. Therefore, the degree of an AP must satisfy the condition $k \geqslant 2$. To determine whether a given node, i, of degree $k \geqslant 2$ is an AP, we first delete it and mark its k neighbors. We randomly choose one of these neighbors and label all the nodes which belong to the connected component on which it resides. This can be done by using either the breadth-first search (BFS) or the depth-first search (DFS) algorithms. If all the other $k - 1$ neighbors of i belong to this connected component, then node i is not an AP. Alternatively, if at least one of these neighbors does not belong to this component, then node i is an AP.

The above approach is useful in order to determine whether a given node is an AP. However, it is not an efficient approach for finding all the APs in the network. This is due to the fact that in this approach one needs to repeat the cluster labeling procedure separately for each node in the network. It turns out that there is a more efficient algorithm, which enables one to detect all the APs in a network using a single DFS run, with respect to a randomly selected reference node [13,14]. Below we demonstrate this algorithm for the giant component of a configuration model network. In order to find all the APs in a configuration model network, we choose a random node, i, on the giant component. We then run a DFS around the root node i over the whole giant component. Using this approach, we essentially determine the shell structure around i. The first shell consists of the neighbors of i. The second shell consist of nodes which are at distance 2 from i, while the ℓth shell consists of nodes which are at distance ℓ from i. For each node in the ℓth shell, we maintain a record of its neighbors in the $(\ell - 1)$th, ℓth, and $(\ell + 1)$th shells.

Consider a node, j, which resides in the ℓth shell and has k' neighbors in the $(\ell + 1)$th shell, denoted by j_m, $m = 1, 2, \ldots, k'$. To determine whether the node j is an AP, we first delete it from the network. We then check for each one of its k' neighbors in the $(\ell + 1)$th shell whether it has a path to the root node, i, in the reduced network from which j was removed. If all these k' neighbors have such paths to node i, then the node j is not an AP. However, if at least one of these k' neighbors does not have such path to i, then the node j is an AP. This procedure should be repeated for all the nodes, j, in the network. The algorithm presented above provides all the APs which reside in the giant component. In the finite tree components, there is no need to apply the algorithm since all the nodes of degrees $k \geqslant 2$ are APs.

V. THE PROBABILITY THAT A RANDOM NODE IS AN ARTICULATION POINT

In this section, we derive a closed form analytical expression for the probability $P(i \in \mathrm{AP})$ that a random node, i, is an AP. To this end, we first consider the conditional probability, $P(i \in \mathrm{AP}|k)$, that a random node of a given degree, k, is an AP. This probability can be expressed by $P(i \in \mathrm{AP}|k) = 1 - P(i \notin \mathrm{AP}|k)$, where $P(i \notin \mathrm{AP}|k)$ is the probability that a random node of degree k is not an AP. Clearly, nodes of degree $k = 0$ or 1 cannot be APs. This is due to the fact that nodes of degree $k = 0$ are isolated from the rest of the network, while nodes of degree $k = 1$ are leaf nodes whose deletion does not affect the connectivity of the rest of the network. Therefore,

$$P(i \in \mathrm{AP}|0) = P(i \in \mathrm{AP}|1) = 0. \tag{15}$$

In order that a node i of degree $k \geqslant 2$ will not be an AP, all its k neighbors must reside on the giant component of the reduced network from which i is removed. This occurs with probability $P(i \notin \mathrm{AP}|k) = \tilde{g}^k$. Therefore,

$$P(i \in \mathrm{AP}|k) = (1 - \tilde{g}^k)\theta(k-2), \tag{16}$$

where $\theta(k)$ is the Heaviside step function, which satisfies $\theta(k) = 1$ for $k \geqslant 0$ and $\theta(k) = 0$ for $k < 0$. Thus, the probability that a random node of unspecified degree is an AP is given by

$$P(i \in \mathrm{AP}) = \sum_{k=2}^{\infty}(1 - \tilde{g}^k)P(K = k). \tag{17}$$

This probability can also be expressed in the form

$$P(i \in AP) = 1 - G_0(\tilde{g}) - (1 - \tilde{g})P(K = 1), \tag{18}$$

where $G_0(x)$ is given by Eq. (7). Note that these results are based on the assumption that the probabilities that different neighbors of a random node, i, reside on the giant component are independent. This assumption is expected to hold in ensembles of uncorrelated networks, such as the configuration model networks.

An important distinction in configuration model networks is between the properties of nodes which reside on the giant component and those which reside on the finite components. In particular, the probability $P(i \in \mathrm{AP})$ can be expressed as a sum of the contributions of the giant and finite components:

$$\begin{aligned} P(i \in \mathrm{AP}) = {} & P(i \in \mathrm{AP}|\mathrm{GC})P(i \in \mathrm{GC}) \\ & + P(i \in \mathrm{AP}|\mathrm{FC})P(i \in \mathrm{FC}), \end{aligned} \tag{19}$$

For nodes which reside on the finite components,

$$P(i \in \mathrm{AP}|\mathrm{FC}) = 1 - P(K = 0|\mathrm{FC}) - P(K = 1|\mathrm{FC}). \tag{20}$$

Using Eq. (11), we find that

$$P(K = 0|\mathrm{FC}) = \frac{P(K = 0)}{1 - g} \tag{21}$$

and

$$P(K = 1|\mathrm{FC}) = \frac{1 - \tilde{g}}{1 - g}P(K = 1). \tag{22}$$

Therefore, the probability that a random node that resides on one of the finite components is an AP is given by

$$P(i \in \mathrm{AP}|\mathrm{FC}) = 1 - \frac{1}{1 - g}P(K = 0) - \frac{1 - \tilde{g}}{1 - g}P(K = 1). \tag{23}$$

Inserting this result into Eq. (19), one can extract the probability that a random node that resides on the giant component is an AP. This probability is given by

$$P(i \in \mathrm{AP}|\mathrm{GC}) = 1 - \frac{1}{g}G_0(\tilde{g}) + \frac{1}{g}P(K = 0). \tag{24}$$

Using Bayes's theorem, the probability that a random AP in the network resides on the giant component can be expressed in the form

$$P(i \in \mathrm{GC}|\mathrm{AP}) = \frac{P(i \in \mathrm{AP}|\mathrm{GC})P(i \in \mathrm{GC})}{P(i \in \mathrm{AP})}. \tag{25}$$

Inserting $P(i \in \mathrm{AP})|\mathrm{GC})$ from Eq. (24), we obtain

$$P(i \in \mathrm{GC}|\mathrm{AP}) = \frac{g - G_0(\tilde{g}) + P(K = 0)}{1 - G_0(\tilde{g}) - (1 - \tilde{g})P(K = 1)}. \tag{26}$$

Thus, the complementary probability that a random AP in the network resides on one of the finite components is given by

$$P(i \in \mathrm{FC}|\mathrm{AP}) = \frac{1 - g - P(K = 0) - (1 - \tilde{g})P(K = 1)}{1 - G_0(\tilde{g}) - (1 - \tilde{g})P(K = 1)}. \tag{27}$$

The probability that a random node of degree k is an AP, given by Eq. (16), can be expressed as a sum of the contributions of the giant and finite components in the form

$$\begin{aligned} P(i \in \mathrm{AP}|k) = {} & P(i \in \mathrm{AP}|\mathrm{GC}, k)P(i \in \mathrm{GC}|k) \\ & + P(i \in \mathrm{AP}|\mathrm{FC}, k)P(i \in \mathrm{FC}|k). \end{aligned} \tag{28}$$

For nodes which reside on the finite components,

$$P(i \in \mathrm{AP}|\mathrm{FC}, k) = \theta(k-2). \tag{29}$$

Inserting Eqs. (8), (9), (16), and (29) in Eq. (28), we obtain

$$P(i \in \mathrm{AP}|\mathrm{GC}, k) = \left[1 - \frac{\tilde{g}^k}{1 - (1 - \tilde{g})^k}\right]\theta(k-2). \tag{30}$$

The probability that a random AP of degree k resides on the giant component is given by

$$P(i \in \mathrm{GC}|\mathrm{AP}, k) = \frac{P(i \in \mathrm{AP}|\mathrm{GC}, k)P(i \in \mathrm{GC}|k)}{P(i \in \mathrm{AP}|k)}. \tag{31}$$

Inserting $P(i \in \mathrm{AP}|\mathrm{GC}, k)$ from Eq. (30), we obtain

$$P(i \in \mathrm{GC}|\mathrm{AP}, k) = 1 - \frac{(1 - \tilde{g})^k}{1 - \tilde{g}^k}, \quad k \geqslant 2. \tag{32}$$

The probability that a random AP of degree k resides on one of the finite components is given by

$$P(i \in \mathrm{FC}|\mathrm{AP}, k) = \frac{P(i \in \mathrm{AP}|\mathrm{FC}, k)P(i \in \mathrm{FC}|k)}{P(i \in \mathrm{AP}|k)}. \quad (33)$$

Inserting $P(i \in \mathrm{AP}|\mathrm{GC}, k)$ from Eq. (30) into Eq. (33), we obtain

$$P(i \in \mathrm{FC}|\mathrm{AP}, k) = \frac{(1-\tilde{g})^k}{1-\tilde{g}^k}, \quad k \geqslant 2. \quad (34)$$

Comparing this result to the expression for $P(i \in \mathrm{FC}|k)$, given by Eq. (9), it is found that for $k \geqslant 2$ and $0 < \tilde{g} < 1$, $P(i \in \mathrm{FC}|\mathrm{AP}, k) > P(i \in \mathrm{FC}|k)$. This result means that an AP of degree $k \geqslant 2$ is more likely to reside on one of the finite components than a random node of the same degree.

In the limit of $\tilde{g} \to 1$, the giant component encompasses the whole network. In this case, for $k \geqslant 2$ the probability $P(i \in \mathrm{GC}|\mathrm{AP}, k) = 1$ and $P(i \in \mathrm{FC}|\mathrm{AP}, k) = 0$. In the opposite limit of $\tilde{g} \to 0$, the network consists of finite components and thus $P(i \in \mathrm{GC}|\mathrm{AP}, k) = 0$ and $P(i \in \mathrm{FC}|\mathrm{AP}, k) = 1$. For $0 < \tilde{g} < 1$, the probability $P(i \in \mathrm{GC}|\mathrm{AP}, k)$ is a monotonically increasing function of k, while $P(i \in \mathrm{FC}|\mathrm{AP}, k)$ is a monotonically decreasing function of k. Thus, most APs of high degrees reside on the giant component while the APs which reside on the finite components tend to be of lower degrees.

VI. THE DEGREE DISTRIBUTION OF THE ARTICULATION POINTS

Using Bayes's theorem, one can express the degree distribution of the APs in the form

$$P(K = k|\mathrm{AP}) = \frac{P(i \in \mathrm{AP}|k)}{P(i \in \mathrm{AP})} P(K = k). \quad (35)$$

Inserting $P(i \in \mathrm{AP})$ from Eq. (18) and $P(i \in \mathrm{AP}|k)$ from Eq. (16). we obtain

$$P(K = k|\mathrm{AP}) = \frac{(1-\tilde{g}^k)\theta(k-2)}{1 - G_0(\tilde{g}) - (1-\tilde{g})P(K=1)} P(K = k). \quad (36)$$

The mean degree of the APs in the network is given by

$$\mathbb{E}[K|\mathrm{AP}] = \sum_{k=2}^{\infty} k P(K = k|\mathrm{AP}). \quad (37)$$

Carrying out the summation, we obtain

$$\mathbb{E}[K|\mathrm{AP}] = \frac{[1 - \tilde{g}G_1(\tilde{g})]}{1 - G_0(\tilde{g}) - (1-\tilde{g})P(K=1)}\langle K\rangle - \frac{(1-\tilde{g})P(K=1)}{1 - G_0(\tilde{g}) - (1-\tilde{g})P(K=1)}, \quad (38)$$

where $G_1(\tilde{g})$ is given by Eq. (3).

The degree distribution $P(K = k|\mathrm{AP})$, given by Eq. (36), can be expressed as a weighted sum of the degree distributions of the giant and finite components, in the form

$$P(K = k|\mathrm{AP}) = P(K = k|\mathrm{AP}, \mathrm{GC})P(i \in \mathrm{GC}|\mathrm{AP}) + P(K = k|\mathrm{AP}, \mathrm{FC})P(i \in \mathrm{FC}|\mathrm{AP}). \quad (39)$$

Since on the finite components all the nodes of degree $k \geqslant 2$ are APs, the degree distribution over the APs which reside on the finite components is given by

$$P(K = k|\mathrm{AP}, \mathrm{FC}) = \frac{\theta(k-2)P(K=k|\mathrm{FC})}{1 - P(K=0|\mathrm{FC}) - P(K=1|\mathrm{FC})}. \quad (40)$$

Inserting the expression for $P(K = k|\mathrm{FC})$, given by Eq. (11), into Eq. (40), we obtain

$$P(K = k|\mathrm{AP}, \mathrm{FC}) = \frac{(1-\tilde{g})^k\theta(k-2)}{1 - g - P(K=0) - (1-\tilde{g})P(K=1)} P(K = k). \quad (41)$$

Inserting this result into Eq. (39), we obtain

$$P(K = k|\mathrm{AP}, \mathrm{GC}) = \left[\frac{1 - \tilde{g}^k - (1-\tilde{g})^k}{g - G_0(\tilde{g}) + P(K=0)}\right] \times \theta(k-2)P(K=k). \quad (42)$$

The expectation values of the degrees of the APs that reside on the giant and finite components are given by

$$\mathbb{E}[K|\mathrm{AP}, \Lambda] = \sum_{k=2}^{\infty} k P(K = k|\mathrm{AP}, \Lambda), \quad (43)$$

where $\Lambda = \mathrm{GC}$ and FC, respectively. Carrying out the summation, we obtain

$$\mathbb{E}[K|\mathrm{AP}, \mathrm{GC}] = \frac{1 - \tilde{g}G_1(\tilde{g}) - (1-\tilde{g})^2}{g - G_0(\tilde{g}) + P(K=0)}\langle K\rangle \quad (44)$$

and

$$\mathbb{E}[K|\mathrm{AP}, \mathrm{FC}] = \frac{(1-\tilde{g})^2\langle K\rangle - (1-\tilde{g})P(K=1)}{1 - g - P(K=0) - (1-\tilde{g})P(K=1)}. \quad (45)$$

In Appendix B, we show that the mean degree of APs on the giant component is larger than the mean degree of all nodes on the giant component, namely

$$\mathbb{E}[K|\mathrm{AP}, \mathrm{GC}] > \mathbb{E}[K|\mathrm{GC}]. \quad (46)$$

This is consistent with the fact that high-degree nodes are more likely to be APs than low-degree nodes. Similarly, the mean degree of APs on the finite components is larger than the mean degree of all nodes on the finite components, namely

$$\mathbb{E}[K|\mathrm{AP}, \mathrm{FC}] > \mathbb{E}[K|\mathrm{FC}]. \quad (47)$$

This can be easily understood as follows. The finite components exhibit tree structures, whose branches terminate by leaf nodes of degree $k = 1$. In some cases, there are also isolated nodes of degree $k = 0$. While isolated nodes and leaf nodes cannot be APs, on tree components all the nodes of degree $k \geqslant 2$ are APs. Therefore, the mean degree of the APs is larger than the mean degree of all the nodes on the finite components.

VII. THE DISTRIBUTION OF ARTICULATION RANKS

The articulation rank, r, of a node i is defined as the number of components which are added to the network upon deletion of i. A node which is not an AP has a rank of $r = 0$, because its deletion does not add any new component to the

network. A node is an AP of rank $r \geqslant 1$ if its deletion breaks the network component on which it resides into $r+1$ parts, thus increasing the number of components by r. The deletion of an AP of articulation rank r which resides on the giant component reduces the size of the giant component while adding r finite components to the network. The deletion of an AP of rank r which resides on one of the finite components breaks this component into $r+1$ fragments. The articulation rank of a node of degree k may take values in the range $r = 0, 1, 2, \ldots, k-1$.

On the finite components all the nodes of degrees $k \geqslant 2$ are APs. Since the finite components are tree networks, the rank of a node of degree k is $r = k-1$. Therefore, the probability that a randomly chosen node of degree k, which resides on one of the finite components is of articulation rank r, is given by

$$P(R=r|\mathrm{FC},k)=\begin{cases}\delta_{r,0} & k=0,1\\ \delta_{r,k-1} & k\geqslant 2,\end{cases} \tag{48}$$

where $\delta_{k,k'}$ is the Kronecker symbol. The probability that a randomly chosen node of degree k which resides on the giant component is of rank r is given by

$$P(R=r|\mathrm{GC},k)=\frac{1}{P(i\in \mathrm{GC}|k)}\binom{k}{r}(1-\tilde{g})^r\tilde{g}^{k-r}, \tag{49}$$

where $P(i \in \mathrm{GC}|k)$ is given by Eq. (8) and $r = 0, 1, \ldots, k-1$. The product $(1-\tilde{g})^r\tilde{g}^{k-r}$ represents the probability that a randomly chosen set of r neighbors of a random node i do not reside on the giant component of the reduced network from which i was removed, while the remaining $k-r$ neighbors do reside on the giant component. The binomial coefficients account for the number of different possibilities to choose a set of r neighbors out of the k neighbors of i. Summing up Eqs. (48) and (49) over all values of $k \geqslant r+1$, we obtain

$$P(R=r|\Lambda)=\sum_{k=r+1}^{\infty}P(R=r|\Lambda,k)P(K=k|\Lambda), \tag{50}$$

where $\Lambda = \mathrm{FC}$ for nodes which reside on one of the finite components and $\Lambda = \mathrm{GC}$ for nodes which reside on the giant component. Carrying out the summation, we obtain

$$P(R=r|\mathrm{FC})=\begin{cases}\frac{1}{1-g}P(K=0)+\frac{1-\tilde{g}}{1-g}P(K=1) & r=0\\ \frac{(1-\tilde{g})^{r+1}}{1-g}P(K=r+1) & r\geqslant 1,\end{cases} \tag{51}$$

and

$$P(R=r|\mathrm{GC})=\frac{(1-\tilde{g})^r}{g}\sum_{k=r+1}^{\infty}\binom{k}{r}\tilde{g}^{k-r}P(K=k). \tag{52}$$

The mean articulation rank of the nodes which reside on the finite components is

$$\mathbb{E}[R|\mathrm{FC}]=\frac{(1-\tilde{g})^2}{1-g}\langle K\rangle-1+\frac{P(K=0)}{1-g}, \tag{53}$$

while the mean articulation rank of the nodes which reside on the giant component is

$$\mathbb{E}[R|\mathrm{GC}]=\frac{\tilde{g}(1-\tilde{g})}{g}\langle K\rangle. \tag{54}$$

Interestingly, using the results shown in Appendix A [Eq. (A13)], it is found that at the percolation threshold, $c = c_0$, the mean articulation rank on the giant component satisfies $\mathbb{E}[R|\mathrm{GC}] = 1$. The distribution of articulation ranks over the whole network can be expressed in the form

$$\begin{aligned}P(R=r)=&\,P(R=r|\mathrm{GC})P(\mathrm{GC})\\ &+P(R=r|\mathrm{FC})P(i\in\mathrm{FC}),\end{aligned} \tag{55}$$

where $P(R=r|\mathrm{GC})$ is given by Eq. (52) and $P(R=r|\mathrm{FC})$ is given by Eq. (51). More explicitly, it takes the form

$$P(R=r)=\begin{cases}P(K=0)+(1-\tilde{g})P(K=1)+\sum_{k=1}^{\infty}\tilde{g}^kP(K=k), & r=0,\\ (1-\tilde{g})^{r+1}P(K=r+1)+(1-\tilde{g})^r\sum_{k=r+1}^{\infty}\binom{k}{r}\tilde{g}^{k-r}P(K=k), & r\geqslant 1.\end{cases} \tag{56}$$

The mean articulation rank of the whole network is

$$\langle R\rangle=(1-\tilde{g})\langle K\rangle-(1-g)+P(K=0). \tag{57}$$

VIII. PROPERTIES OF ARTICULATION POINTS OF DEGREE $k = 2$

Consider an AP of degree $k = 2$ that resides on the GC. One can distinguish between two types of such APs, according to the structural properties of the tree that is detached from the GC upon their deletion. In one type of $k = 2$ APs, referred to as *stubs*, the detached tree does not include any node of degree $k > 2$. Hence, stubs do not bridge between the GC and any branching trees. The other type of $k = 2$ APs, referred to as *tubes*, connect the GC with a branching tree that includes at least one node of degree $k \geqslant 3$. Below we derive a closed form expression for the probability $P(\mathrm{Stub})$ that a random $k = 2$ AP that resides on the GC is a stub. A key observation is that $P(\mathrm{Stub})$ is the probability that the detached tree is a chain of any length that consists of nodes of degree $k = 2$ and a single node of degree $k = 1$ at the end. The probability that the detached tree is a chain of length ℓ is denoted by $P(\mathrm{Stub}, \ell)$. It can be expressed in the form

$$P(\mathrm{Stub},\ell)=\widetilde{P}(K=2|\mathrm{FC})^{\ell-1}P(K=1|\mathrm{FC}), \tag{58}$$

where

$$\widetilde{P}(K=k|\mathrm{FC})=\frac{k}{\mathbb{E}[K|\mathrm{FC}]}P(K=k|\mathrm{FC}) \tag{59}$$

is the probability that a random neighbor of a random node on a finite component is of degree k. The conditioning on a finite component is due to the fact that once the AP is deleted, the detached tree follows the statistical properties of the finite components. Since a stub can be of any length, the probability

$P(\text{Stub})$ is given by

$$P(\text{Stub}) = \sum_{\ell=1}^{\infty} P(\text{Stub}, \ell). \tag{60}$$

Carrying out the summation on the right-hand side, we obtain

$$P(\text{Stub}) = \frac{\widetilde{P}(K=1|\text{FC})}{1-\widetilde{P}(K=2|\text{FC})}. \tag{61}$$

Using Eqs. (59), (11), and (14), we can express $P(\text{Stub})$ in the form

$$P(\text{Stub}) = \frac{P(K=1)}{(1-\tilde{g})[\langle K\rangle - 2P(K=2)]}. \tag{62}$$

The probability that a random $k=2$ AP is a tube is thus $P(\text{Tube}) = 1 - P(\text{Stub})$.

IX. APPLICATIONS TO SPECIFIC NETWORK MODELS

Here we apply the approach presented above to three examples of configuration model networks, with a Poisson degree distribution (ER networks), an exponential degree distribution, and a power-law degree distribution (scale-free networks).

A. Erdős-Rényi networks

The ER network is the simplest kind of a random network and a special case of the configuration model. It is a maximum entropy network under the condition in which only the mean degree, $\langle K\rangle = c$, is constrained. ER networks can be constructed by independently connecting each pair of nodes with probability $p = c/(N-1)$. In the asymptotic limit, the resulting degree distribution follows a Poisson distribution of the form

$$P(K=k) = \frac{e^{-c}c^k}{k!}. \tag{63}$$

Asymptotic ER networks exhibit a percolation transition at $c=1$, such that for $c<1$ the network consists only of finite components, which exhibit tree topologies. The degree distribution and the distribution of shortest path lengths on the finite components of subcritical ER networks were studied in Ref. [27]. For $c>1$, a giant component emerges, coexisting with the finite components. At a higher value of the connectivity, namely at $c = \ln N$, there is a second transition, above which the giant component encompasses the entire network.

ER networks exhibit a special property, resulting from the Poisson degree distribution [Eq. (63)], which satisfies $\widetilde{P}(K=k) = P(K=k-1)$, where $\widetilde{P}(K=k)$ is given by Eq. (4). This implies that for the Poisson distribution, the two generating functions are identical, namely $G_1(x) = G_0(x)$ [25,26]. Using Eqs. (2) and (6), we obtain that for ER networks $\tilde{g} = g$. Thus, for ER networks, Eq. (16) can be replaced by

$$P(i \in \text{AP}|k) = (1-g^k)\theta(k-2). \tag{64}$$

Carrying out the summation in Eq. (3) with $P(K=k)$ given by Eq. (63), one obtains $G_0(x) = G_1(x) = e^{-(1-x)c}$. Inserting this result in Eq. (6), it is found that g satisfies the equation $1-g = e^{-gc}$ [5]. Solving for the probability g as a function of the mean degree, c, one obtains

$$g = 1 + \frac{W(-ce^{-c})}{c}, \tag{65}$$

where $W(x)$ is the Lambert W function [45]. Inserting g from Eq. (65) into Eq. (64), we obtain

$$P(i \in \text{AP}|k) = \left\{1 - \left[1 + \frac{W(-ce^{-c})}{c}\right]^k\right\}\theta(k-2). \tag{66}$$

Using Eq. (18) and the fact that the Poisson distribution satisfies $P(K=1) = ce^{-c}$ and $G_0(g) = e^{-(1-g)c}$, we find that

$$P(i \in \text{AP}) = 1 - e^{-(1-g)c} - (1-g)ce^{-c}, \tag{67}$$

where g is given by Eq. (65). Using Eq. (24), we obtain the probability that a random node on the giant component of an ER network is an AP, which is given by

$$P(i \in \text{AP}|\text{GC}) = 1 - e^{-(1-g)c}. \tag{68}$$

Similarly, from Eq. (23), we obtain the probability that a random node which resides on one of the finite components is an AP, which is

$$P(i \in \text{AP}|\text{FC}) = 1 - e^{-(1-g)c} - ce^{-c}. \tag{69}$$

In Fig. 2(a), we present the probability $P(i \in \text{GC}) = g$ (dashed line) that a random node in an ER network resides on the giant component, obtained from Eq. (65), and the probability $P(i \in \text{FC}) = 1-g$ (dotted line) that such node resides on one of the finite components. In Fig. 2(b), we present analytical results for the probability $P(i \in \text{AP})$ (solid line) that a random node in an ER network is an AP, as a function of c. We also present the probability $P(i \in \text{AP}, i \in \text{GC})$ that a random node is an AP which resides on the giant component (dashed line) and the probability $P(i \in \text{AP}, i \in \text{FC})$ that a random node is an AP which resides in one of the finite components (dotted line). The analytical results are found to be in very good agreement with the results of computer simulations (circles), performed for an ensemble of ER networks of $N = 1000$ nodes. It is found that in the subcritical regime, the probability $P(i \in \text{AP})$ increases monotonically as a function of c. This is due to the fact that as c is increased from 0 to 1, the finite tree components become larger and the fraction of nodes of degrees $k \geqslant 2$ quickly increases. For $c>1$, the contribution of the finite components to $P(i \in \text{AP})$ sharply decreases while the giant component becomes dominant. Just above the percolation threshold, the giant component also includes a large fraction of APs and therefore the probability $P(i \in \text{AP})$ continues to increase. It reaches its maximal value around $c \simeq 1.5$ and then gradually decreases as c is increased further. The decrease in $P(i \in \text{AP})$ is due to the fact that as the network becomes more dense, more cycles are formed, thus reducing the number of APs. These results imply that slightly above the percolation threshold, ER networks are most sensitive to disintegration due to the deletion of APs.

Using Eq. (26), we obtain the probability that a randomly selected AP in an ER network resides on the giant component,

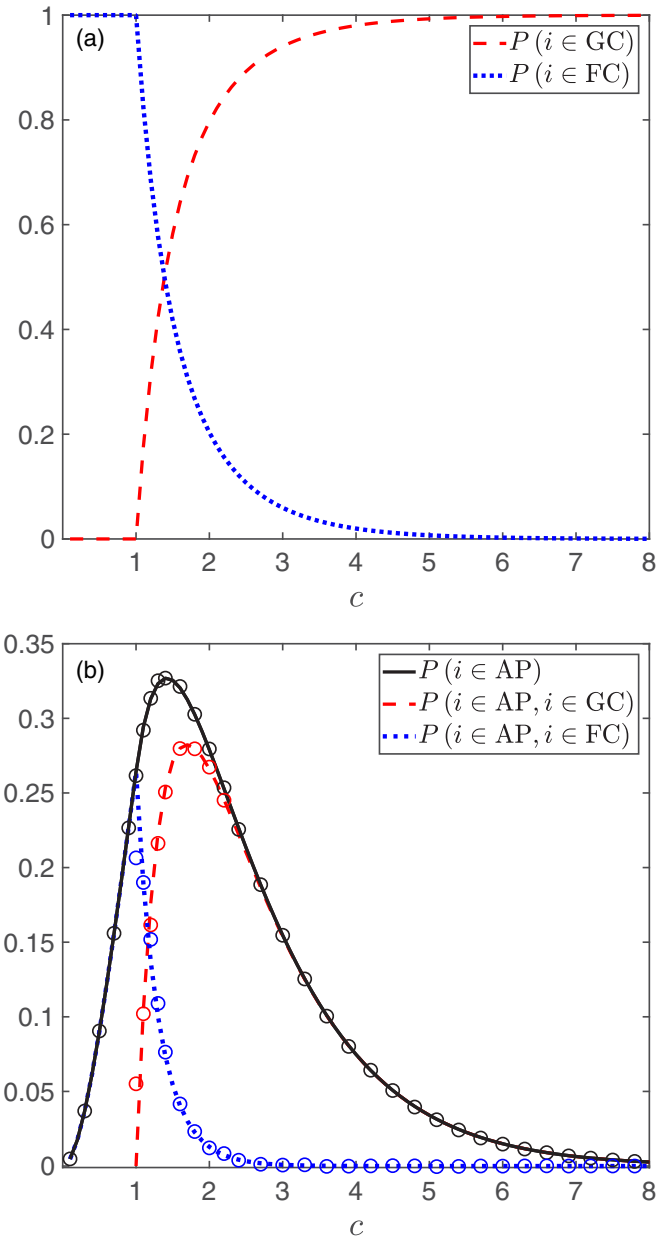

FIG. 2. (a) The probability $P(i \in \mathrm{GC}) = g$ (dashed line) that a random node in an ER network resides on the giant component, obtained from Eq. (65), and the probability $P(i \in \mathrm{FC}) = 1 - g$ (dotted line) that such node resides on one of the finite components. (b) The probability $P(i \in \mathrm{AP})$ that a random node in an ER network is an AP (solid line), as a function of c, obtained from Eq. (67); the probability $P(i \in \mathrm{AP}, i \in \mathrm{GC}) = P(i \in \mathrm{AP}|\mathrm{GC})P(i \in \mathrm{GC})$, obtained from Eq. (68), that a randomly selected node in the network is an AP that resides in the giant component (dashed line); and the probability $P(i \in \mathrm{AP}, i \in \mathrm{FC}) = P(i \in \mathrm{AP}|\mathrm{FC})P(i \in \mathrm{FC})$, obtained from Eq. (69), that a randomly selected node in the network is an AP that resides in one of the finite components (dotted line). The analytical results are found to be in very good agreement with the results of computer simulations (circles), performed for an ensemble of ER networks of $N = 1000$ nodes. It is found that in the limit of sparse networks, the probability $P(i \in \mathrm{AP})$ exhibits a peak around $c \simeq 1.5$, where the network is most sensitive to fragmentation due to the deletion of APs.

which is given by

$$P(i \in \mathrm{GC}|\mathrm{AP}) = \left[\frac{1-g-e^{-c}}{1-g-e^{-c}-(1-g)^2ce^{-c}}\right]g. \quad (70)$$

The complementary probability, $P(i \in \mathrm{FC}|\mathrm{AP}) = 1 - P(i \in \mathrm{GC}|\mathrm{AP})$, is given by

$$P(i \in \mathrm{FC}|\mathrm{AP}) = \left[\frac{1-g-e^{-c}-(1-g)ce^{-c}}{1-g-e^{-c}-(1-g)^2ce^{-c}}\right](1-g). \quad (71)$$

Using Eq. (36), we obtain the degree distribution of APs in ER networks, which is given by

$$P(K=k|\mathrm{AP}) = \left[\frac{(1-g^k)\theta(k-2)}{1-e^{-c(1-g)}-(1-g)ce^{-c}}\right]\frac{e^{-c}c^k}{k!}. \quad (72)$$

Using Eq. (42), we obtain the degree distribution of APs which reside on the giant component, which is given by

$$P(K=k|\mathrm{AP},\mathrm{GC}) = \left[\frac{1-g^k-(1-g)^k}{g-e^{-(1-g)c}+e^{-c}}\right]\frac{e^{-c}c^k}{k!}\theta(k-2). \quad (73)$$

Using Eq. (41), we obtain the degree distribution of APs which reside on the finite components, which takes the form

$$P(K=k|\mathrm{AP},\mathrm{FC}) = -\left[\frac{(1-g)^{k-1}\theta(k-2)}{1-e^{-(1-g)c}-ce^{-c}}\right]\frac{e^{-c}c^k}{k!}. \quad (74)$$

In Fig. 3(a), we present analytical results for the degree distribution $P(K=k)$ of an ER network with $c = 2$ (solid line), the degree distribution $P(K=k|\mathrm{GC})$ of the giant component (dashed line), and the degree distribution $P(K=k|\mathrm{FC})$ of the finite components (dotted line). These results are found to be in very good agreement with the results of computer simulations (circles). In Fig. 3(b), we present analytical results for the degree distribution $P(K=k|\mathrm{AP})$ (solid line) of APs in an ER network with $c = 2$. We also show the degree distribution $P(K=k|\mathrm{AP},\mathrm{GC})$ (dashed line) of APs which reside on the giant component and the degree distribution $P(K=k|\mathrm{AP},\mathrm{FC})$ (dotted line) of APs which reside on the finite components. For a given value of c, nodes of higher degree are more likely to be APs. This is due to the fact that each one of the k neighbors of a node i of degree k may detach from the network component on which i resides upon deletion of node i. In order for node i to be an AP, it is sufficient that one of its neighbors will detach upon deletion of i.

Using Eq. (38), we obtain the mean degree of the APs in ER networks, which is given by

$$\mathbb{E}[K|\mathrm{AP}] = \left[\frac{1-ge^{-c(1-g)}-(1-g)e^{-c}}{1-e^{-c(1-g)}-(1-g)ce^{-c}}\right]c. \quad (75)$$

In Fig. 4, we present analytical results for the mean degree of the APs in ER networks, $\mathbb{E}[K|\mathrm{AP}]$ (solid line), obtained from Eq. (75), as a function of c. We also present analytical results for the mean degree of APs which reside on the giant component, $\mathbb{E}[K|\mathrm{AP},\mathrm{GC}]$ (dashed line), and the mean degree of APs which reside on the finite components, $\mathbb{E}[K|\mathrm{AP},\mathrm{FC}]$

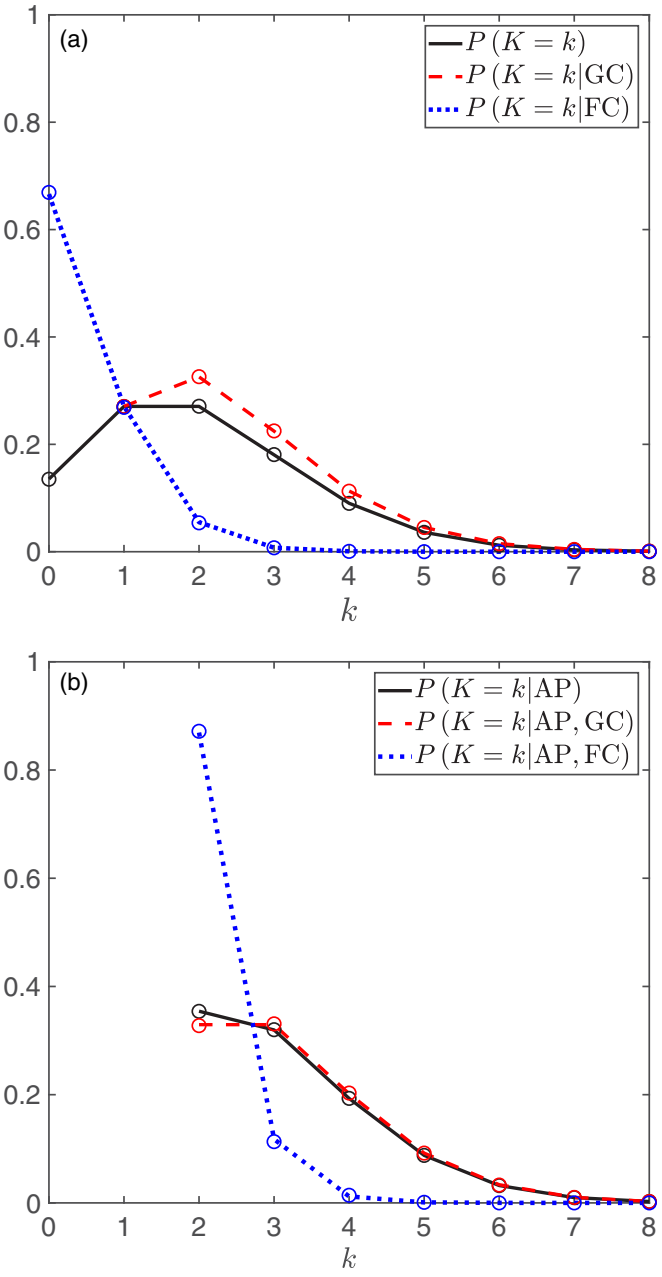

FIG. 3. (a) Analytical results for the degree distribution $P(K=k)$ of an ER network with $c=2$ (solid line), the degree distribution, $P(K=k|\mathrm{GC})$, of the giant component (dashed line), obtained from Eq. (68), and the degree distribution, $P(K=k|\mathrm{FC})$, of the finite components (dotted line), obtained from Eq. (69). (b) Analytical results for the degree distribution $P(K=k|\mathrm{AP})$ (solid line) of APs in an ER network with $c=2$, obtained from Eq. (72); the degree distribution $P(K=k|\mathrm{AP,GC})$ of APs in the giant component (dashed line), obtained from Eq. (73); and the degree distribution $P(K=k|\mathrm{AP,FC})$ of APs in the finite components (dotted line), obtained from Eq. (74). The analytical results are in very good agreement with the results of computer simulations (circles).

(dotted line). These results are obtained from Eqs. (44) and (45), respectively, by inserting the expressions for g, $G_1(x)$ and $P(K=1)$ for ER networks. The analytical results are in very good agreement with the corresponding results obtained from computer simulations (circles).

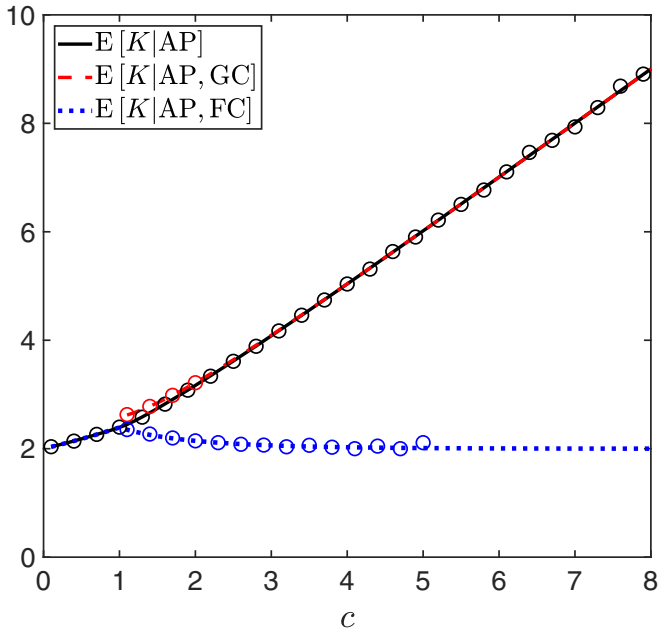

FIG. 4. Analytical results for the mean degree, $\mathbb{E}[K|\mathrm{AP}]$, of APs as a function of c in an ER network (solid line), obtained from Eq. (75); the mean degree, $\mathbb{E}[K|\mathrm{AP,GC}]$, of APs in the giant component (dashed line), obtained from Eq. (44); and the mean degree, $\mathbb{E}[K|\mathrm{AP,FC}]$, of APs in the finite components (dotted line) obtained from Eq. (45). The analytical results are in very good agreement with the results of computer simulations (circles).

The distribution of ranks of random nodes that reside on the giant component of an ER network can be obtained from Eq. (52) by inserting the Poisson distribution for $P(K=k)$ and using the condition $g=\tilde{g}$. Performing the summation, we obtain

$$P(R=r|\mathrm{GC})=\frac{e^{-(1-g)c}[(1-g)c]^r}{r!}. \tag{76}$$

Thus, the rank distribution on the giant component is a Poisson distribution, whose mean is given by

$$\mathbb{E}[R|\mathrm{GC}]=(1-g)c. \tag{77}$$

Using Eq. (51), we obtain the distribution of articulation ranks of random nodes which reside on the finite components, which is given by

$$P(R=r|\mathrm{FC})=\begin{cases} e^{-(1-g)c}+ce^{-c} & r=0,\\ (1-g)^r\frac{e^{-c}c^{r+1}}{(r+1)!} & r\geqslant 1.\end{cases} \tag{78}$$

The overall distribution of articulation ranks is given by

$$P(R=r)=\begin{cases} e^{-(1-g)c}+(1-g)ce^{-c} & r=0,\\ e^{-(1-g)c}\left\{g\frac{[(1-g)c]^r}{r!}+(1-g)\frac{[(1-g)c]^{r+1}}{(r+1)!}\right\} & r\geqslant 1.\end{cases} \tag{79}$$

The mean articulation rank of nodes which reside on the finite components is

$$\mathbb{E}[R|\mathrm{FC}]=e^{-(1-g)c}+ce^{-gc}-1. \tag{80}$$

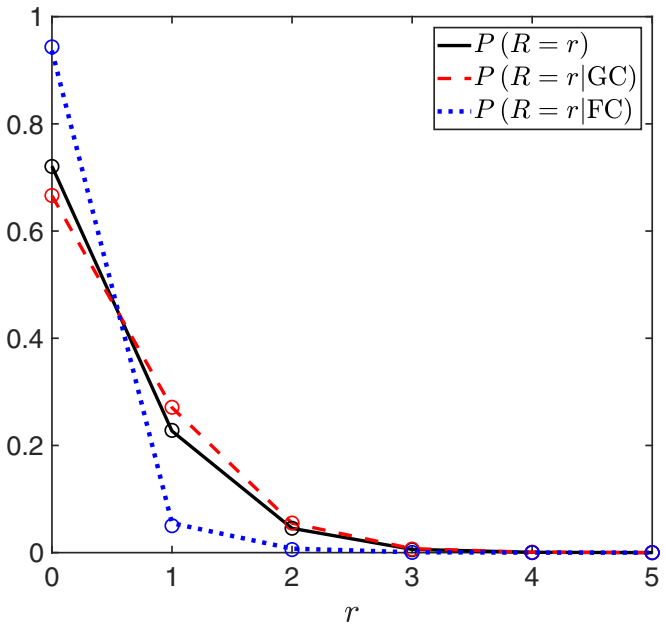

FIG. 5. Analytical results for the distribution $P(R=r)$ of the articulation ranks of nodes in an ER network with $c=2$ (solid line), obtained from Eq. (79); the distribution $P(R=r|\mathrm{GC})$ of the articulation ranks of nodes in the giant component (dashed line), obtained from Eq. (76); and the distribution $P(R=r|\mathrm{FC})$ of the articulation ranks of nodes in the finite components (dotted line), obtained from Eq. (78). The analytical results are in very good agreement with the results of computer simulations (circles).

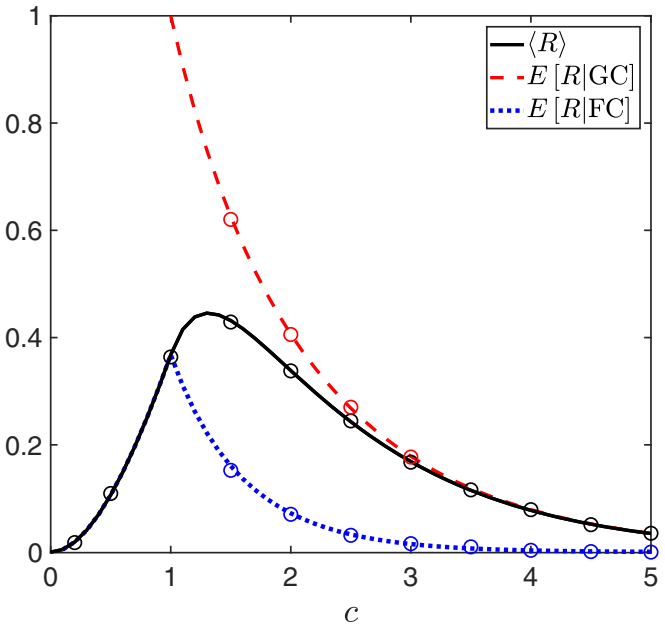

FIG. 6. Analytical results for the mean articulation rank, $\langle R\rangle$, of an ER network, as a function of the mean degree, c (solid line), obtained from Eq. (82). The mean articulation rank $\mathbb{E}[R|\mathrm{GC}]$ of the nodes that reside in the giant component (dashed line), obtained from Eq. (77), and the mean articulation rank $\mathbb{E}[R|\mathrm{FC}]$ of the nodes that reside in the finite components (dotted line), obtained from Eq. (80), are also shown. The analytical results are in very good agreement with the results of computer simulations (circles).

The mean rank of the whole network is given by

$$\langle R\rangle = \mathbb{E}[R|\mathrm{GC}]P(i\in\mathrm{GC}) + \mathbb{E}[R|\mathrm{FC}]P(i\in\mathrm{FC}). \quad (81)$$

Inserting $\mathbb{E}[R|\mathrm{GC}]$ and $\mathbb{E}[R|\mathrm{FC}]$ from Eqs. (77) and (80), respectively, into Eq. (81), we obtain

$$\langle R\rangle = e^{-c} + (c-1)(1-g). \quad (82)$$

In Fig. 5, we present analytical results for the rank distribution $P(R=r)$ (solid line) of an ER network with $c=2$, the rank distribution $P(R=r|\mathrm{GC})$ (dashed line) of the giant component, and the rank distribution $P(R=r|\mathrm{FC})$ (dotted line) of the finite components. We also present the corresponding results obtained from computer simulations (circles), which are in very good agreement with the analytical results.

In Fig. 6, we present analytical results for the mean articulation rank, $\langle R\rangle$, of an ER network (solid line), as a function of the mean degree, c, the mean articulation rank $\mathbb{E}[R|\mathrm{GC}]$ of the nodes which reside in the giant component (dashed line) and the mean articulation rank $\mathbb{E}[R|\mathrm{FC}]$ of the nodes which reside in the finite components (dotted line). The analytical results are in very good agreement with the results of computer simulations (circles). At the percolation threshold ($c=1$), the mean articulation rank of the nodes which reside on the giant component is $\mathbb{E}[R|\mathrm{GC}]=1$, and it decreases monotonically as c is increased. The mean articulation rank of the nodes which reside on the finite components exhibits a cusp at the percolation threshold, and it sharply decreases on both sides. The mean articulation rank of the whole network, which is presented only for the supercritical regime in which the giant and the finite components coexist, increases above the percolation threshold and then decreases as c is increased.

Inserting the properties of ER networks in Eq. (62), we find that the probability that a random $k=2$ AP that resides on the GC is a tube is

$$P(\mathrm{Tube}) = 1 + \frac{ce^{-c}}{(1-ce^{-c})W(-ce^{-c})}. \quad (83)$$

In Fig. 7, we present the probability $P(\mathrm{Tube})$ as a function of the mean degree, c, for ER networks. It is found that $P(\mathrm{Tube})$ is a monotonically decreasing function of c. This implies that as the network becomes more dense, the trees that detach upon deletion of APs become of simpler topologies.

B. Configuration model networks with exponential degree distributions

Consider a configuration model network with an exponential degree distribution of the form $P(K=k)\sim e^{-\alpha k}$, where $k_{\min}\leqslant k\leqslant k_{\max}$. In case that $k_{\min}\geqslant 2$, it can be shown that $g=\tilde{g}=1$ and there are no APs. Here we consider the case of $k_{\min}=1$ and $k_{\max}=\infty$. In this case, it is convenient to express the degree distribution in the form

$$P(K=k) = \frac{1}{c-1}\left(\frac{c-1}{c}\right)^k, \quad (84)$$

where $c=\langle K\rangle$ is the mean degree. In order to find the properties of APs in such networks, we first calculate the parameters $\tilde{g}$ and g. Inserting the exponential degree distribution of

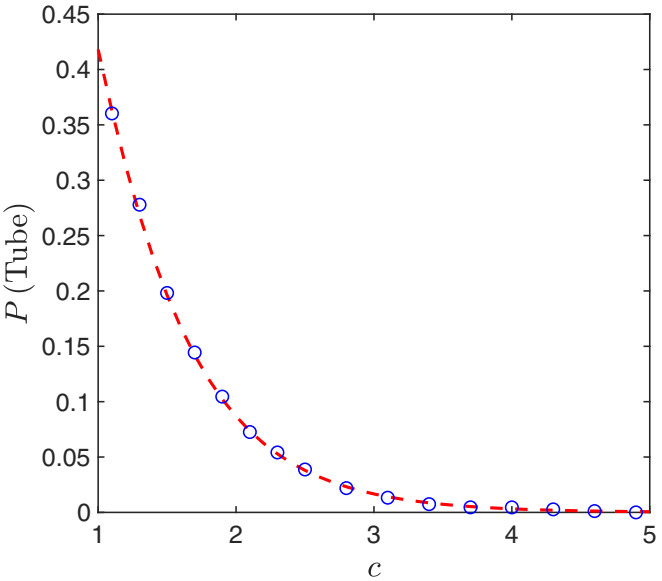

FIG. 7. Analytical results (dashed line) for the probability $P(\text{Tube})$ that a random $k = 2$ AP on the GC of an ER network is a tube, as a function of the mean degree c, obtained from Eq. (83). The analytical results are in excellent agreement with the results obtained from computer simulations (circles).

Eq. (84) into the generating function $G_1(x)$, given by Eq. (3), we obtain

$$G_1(x) = \frac{1}{[c-(c-1)x]^2}. \tag{85}$$

Inserting $G_1(x)$ from Eq. (85) into Eq. (2) and solving for $\tilde{g}$, we find that for $c > 3/2$ there is a nontrivial solution of the form

$$\tilde{g} = \frac{c-3}{2(c-1)} + \frac{1}{2}\sqrt{\frac{c+3}{c-1}}. \tag{86}$$

Inserting this result into Eq. (16), we obtain

$$P(i \in \text{AP}|k) = \left\{1 - \left[\frac{c-3}{2(c-1)} + \frac{1}{2}\sqrt{\frac{c+3}{c-1}}\right]^k\right\}\theta(k-2). \tag{87}$$

Inserting the exponential degree distribution of Eq. (84) into Eq. (7), we obtain

$$G_0(x) = \frac{x}{c-(c-1)x}. \tag{88}$$

Inserting the expression for $\tilde{g}$ from Eq. (86) into $G_0(1-\tilde{g})$ in Eq. (7), we find that for $c > 3/2$

$$g = \frac{3c}{2(c-1)} - \frac{c}{2(c-1)}\sqrt{\frac{c+3}{c-1}}. \tag{89}$$

Thus, it is found that the configuration model network with an exponential degree distribution exhibits a percolation transition at $c_0 = 3/2$, below which $g = \tilde{g} = 0$ and above which $\tilde{g} > 0$ is given by Eq. (86) and $g > 0$ is given by Eq. (89). Inserting this result into Eq. (18), we obtain the probability that a random node is an AP, which is given by

$$P(i \in \text{AP}) = 1 - \frac{\tilde{g}}{c-(c-1)\tilde{g}} - \frac{1-\tilde{g}}{c}. \tag{90}$$

Using Eq. (24), we obtain the probability that a random node on the giant component is an AP, which is given by

$$P(i \in \text{AP}|\text{GC}) = 1 - \frac{\tilde{g}}{[c-(c-1)\tilde{g}]g}. \tag{91}$$

Similarly, using Eq. (23), we obtain the probability that a random node which resides on one of the finite components is an AP, which is

$$P(i \in \text{AP}|\text{FC}) = 1 - \frac{1-\tilde{g}}{(1-g)c}, \tag{92}$$

where $\tilde{g}$ is given by Eq. (86) and g is given by Eq. (89).

The probability that a randomly selected AP resides on the giant component is given by

$$P(i \in \text{GC}|\text{AP}) = \frac{c^2 g(1-\tilde{g}) - c\tilde{g}(1-g)}{(1-\tilde{g})[c^2 - c(1-\tilde{g}) - \tilde{g}]}, \tag{93}$$

while the probability that such node resides on one of the finite components is $P(i \in \text{FC}|\text{AP}) = 1 - P(i \in \text{GC}|\text{AP})$. Using Eq. (36), we obtain the degree distribution of the APs, which is given by

$$P(K = k|\text{AP}) = \left(\frac{1-\tilde{g}^k}{1-\tilde{g}}\right)\left[\frac{c+\tilde{g}-c\tilde{g}}{(c+\tilde{g})(c-1)}\right]\left(\frac{c-1}{c}\right)^{k-1} \times \theta(k-2). \tag{94}$$

The degree distribution of APs which reside on the giant component is given by

$$P(K = k|\text{AP}, \text{GC}) = \frac{c}{(c-1)g}\left[\frac{(c+\tilde{g}-c\tilde{g})(1-\tilde{g}^k)}{(1-\tilde{g})(c+\tilde{g})(c-1)}\right] \times \left(\frac{c-1}{c}\right)^k \theta(k-2), \tag{95}$$

while the degree distribution of APs which reside on the finite components is

$$P(K = k|\text{AP}, \text{FC}) = \frac{c}{(c-1)}\left[\frac{(1-\tilde{g})^k\theta(k-2)}{(1-g)c-(1-\tilde{g})}\right] \times \left(\frac{c-1}{c}\right)^k. \tag{96}$$

From Eq. (38), we obtain the mean degree of the APs, which is given by

$$\mathbb{E}[K|\text{AP}] = 1 + \frac{c(c-1)}{c+\tilde{g}} + \frac{c}{c(1-\tilde{g})+\tilde{g}}. \tag{97}$$

Carrying out the summation in Eq. (52) with $P(K = k)$ given by Eq. (84), we obtain

$$P(R = r|\text{GC}) = \frac{1}{(c-1)g}\left[\frac{(1-\tilde{g})(c-1)}{c}\right]^r \times \left[\left(\frac{c}{c-\tilde{g}c+\tilde{g}}\right)^{r+1} - 1\right]. \tag{98}$$

Inserting the properties of configuration model networks with exponential degree distributions in Eq. (62), we find that the probability that a random $k = 2$ AP that resides on the GC is a tube is

$$P(\text{Tube}) = 1 - \frac{c}{[c^3 - 2(c-1)]\left[1 - \frac{c-3}{2(c-1)} - \frac{1}{2}\sqrt{\frac{c+3}{c-1}}\right]}. \tag{99}$$

It is found that $P(\text{Tube})$ is a monotonically decreasing function of c. This implies that as the network becomes more dense, the trees that detach upon deletion of APs become of simpler topologies.

C. Configuration model networks with power-law degree distributions

Consider a configuration model network with a power-law degree distribution of the form $P(K = k) \sim k^{-\gamma}$, where $k_{\min} \leqslant k \leqslant k_{\max}$. Here we focus on the case of $\gamma > 2$, in which the mean degree, $\langle K \rangle$, is bounded even for $k_{\max} \to \infty$. Power-law distributions do not exhibit a typical scale and are therefore referred to as scale-free networks. The normalized degree distribution is given by

$$P(K = k) = \frac{k^{-\gamma}}{\zeta(\gamma, k_{\min}) - \zeta(\gamma, k_{\max} + 1)}, \tag{100}$$

where $\zeta(\gamma, k)$ is the Hurwitz ζ function [45]. For $\gamma \leqslant 2$, the mean degree diverges when $k_{\max} \to \infty$. For $2 < \gamma \leqslant 3$, the mean degree is bounded while the second moment, $\langle K^2 \rangle$, diverges in the limit of $k_{\max} \to \infty$. For $\gamma > 3$, both moments are bounded. For $\gamma > 2$ and $k_{\min} \geqslant 2$ (where nodes of degrees 0 and 1 do not exist), one can show that $\langle K^2 \rangle > 2\langle K \rangle$ namely the Molloy and Reed criterion, is satisfied and the network exhibits a giant component [38]. Moreover, in this case the giant component encompasses the entire network, namely $g = \tilde{g} = 1$ [28]. In this case, there are no APs in the network and $P(i \in \text{AP}) = 0$. Here we focus on the case of $k_{\min} = 1$, in which there are APs in the network. In this case, for $\gamma > 2$ the mean degree is given by

$$\langle K \rangle = \frac{\zeta(\gamma - 1) - \zeta(\gamma - 1, k_{\max} + 1)}{\zeta(\gamma) - \zeta(\gamma, k_{\max} + 1)}, \tag{101}$$

where $\zeta(\gamma) = \zeta(\gamma, 1)$ is the Riemann ζ function. To obtain $\langle K \rangle$ for $\gamma = 2$, we take the limit $\gamma \to 2^+$ in Eq. (101). We obtain

$$\langle K \rangle_{\gamma=2} = \frac{H_{k_{\max}}}{H^{(2)}_{k_{\max}}}, \tag{102}$$

where H_n is the nth harmonic number and $H^{(m)}_n$ is the generalized nth harmonic number of order m [46]. For $k_{\max} \gg 1$, it satisfies

$$\langle K \rangle_{\gamma=2} \simeq \frac{6}{\pi^2} \ln k_{\max}. \tag{103}$$

The mean degree $\langle K \rangle$ is a monotonically decreasing function of the exponent γ. Therefore, $\langle K \rangle_{\gamma=2}$ is the largest possible value of $\langle K \rangle$ that can be obtained for a given value of $k_{\max}$.

The second moment of the degree distribution is

$$\langle K^2 \rangle = \frac{\zeta(\gamma - 2) - \zeta(\gamma - 2, k_{\max} + 1)}{\zeta(\gamma) - \zeta(\gamma, k_{\max} + 1)}. \tag{104}$$

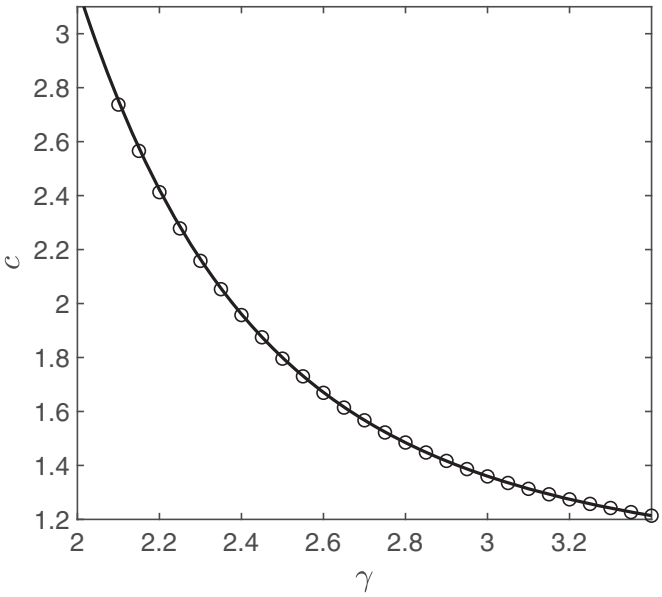

FIG. 8. The mean degree, $c = \langle K \rangle$, as a function of the exponent γ, for a configuration model network with a power-law degree distribution, where $k_{\min} = 1$ and $k_{\max} = 100$. As γ is increased, the tail of the degree distribution decays more quickly and the mean degree decreases. The analytical results (solid line), obtained from Eq. (101), are found to be in excellent agreement with the results of computer simulations (circles) performed for networks of $N = 4 \times 10^4$ nodes.

Inserting the degree distribution of Eq. (100) with $k_{\min} = 1$ into Eqs. (7) and (3), we obtain

$$G_0(x) = \frac{\text{Li}_\gamma(x) - x^{k_{\max}+1}\Phi(x, \gamma, k_{\max} + 1)}{\zeta(\gamma) - \zeta(\gamma, k_{\max} + 1)} \tag{105}$$

and

$$G_1(x) = \frac{\text{Li}_{\gamma-1}(x) - x^{k_{\max}+1}\Phi(x, \gamma - 1, k_{\max} + 1)}{x[\zeta(\gamma - 1) - \zeta(\gamma - 1, k_{\max} + 1)]}, \tag{106}$$

where $\Phi(x, \gamma, k)$ is the Lerch transcendent and $\text{Li}_\gamma(x)$ is the polylogarithm function [47]. The values of the parameters $\tilde{g}$ and g are determined by Eqs. (2) and (6). Unlike the ER network and the configuration model network with an exponential degree distribution, here we do not have closed-form analytical expressions for g and $\tilde{g}$. However, using the expressions above for $G_0(x)$ and $G_1(x)$, the values of g and $\tilde{g}$ can be easily obtained from a numerical solution of Eqs. (2) and (6). To find the percolation threshold, we use the Molloy-Reed criterion, which states that at the transition $\langle K^2 \rangle - 2\langle K \rangle = 0$ [38,39]. Inserting the expressions for $\langle K \rangle$ and $\langle K^2 \rangle$ from Eqs. (101) and (104), for $k_{\max} = 100$ we find that the percolation threshold takes place at $\gamma_0 \simeq 3.37876$, where the mean degree is $c_0 \simeq 1.21946$.

In Fig. 8, we present the mean degree, $c = \langle K \rangle$, as a function of the exponent γ, for a configuration model network with a power-law degree distribution, where $k_{\min} = 1$ and $k_{\max} = 100$. As γ is increased, the tail of the degree distribution decays more quickly and as a results the mean degree decreases.

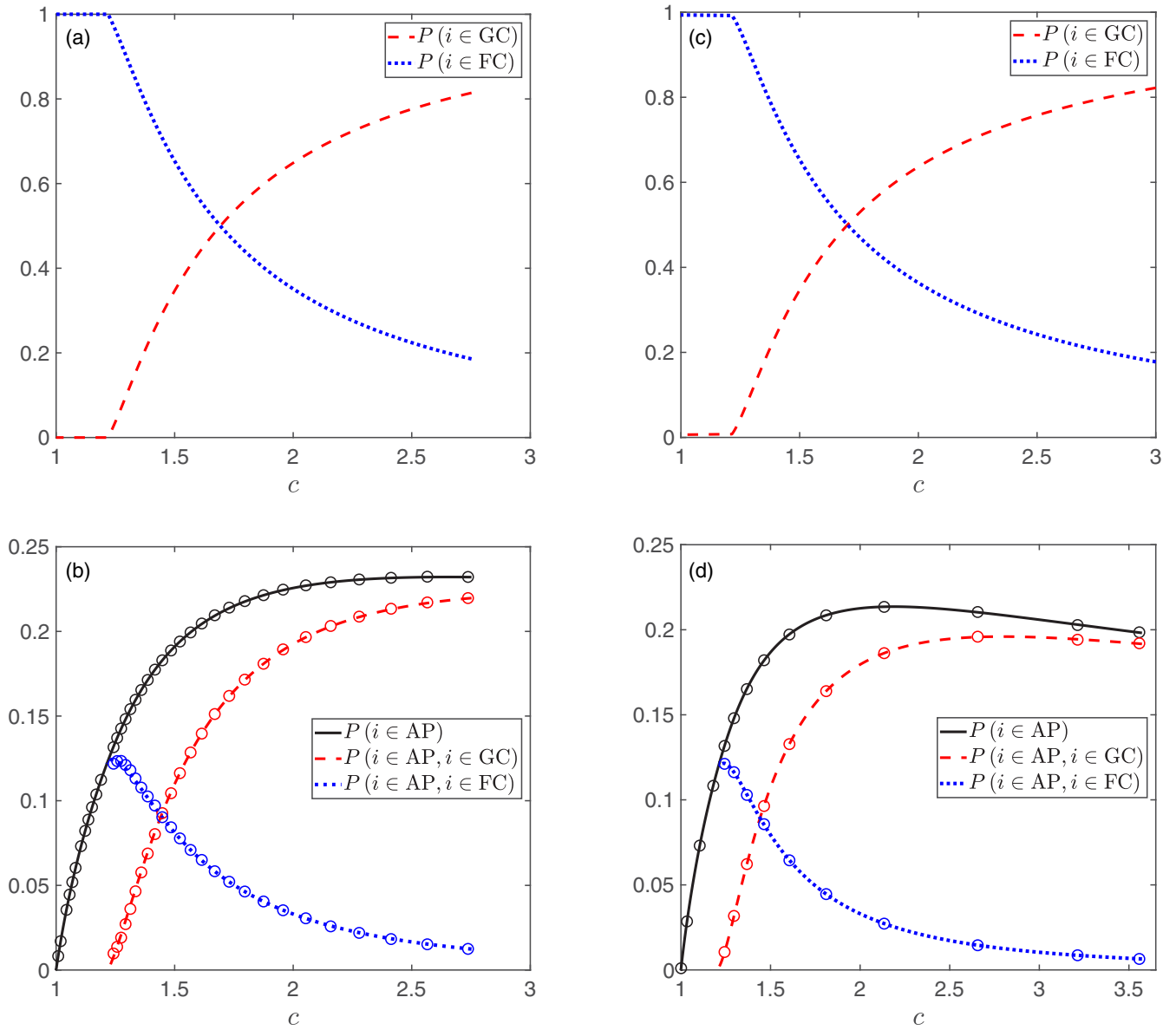

FIG. 9. (a) Analytical results for the probability, $P(i \in \mathrm{GC}) = g$, that a random node in a configuration model network with a power-law degree distribution resides on the giant component (dashed line), as a function of the mean degree, c, where $k_{\min} = 1$ and $k_{\max} = 100$. The complementary probability, $P(i \in \mathrm{FC}) = 1 - g$, that a randomly selected node resides on one of the finite components, is also shown (dotted line); (b) Analytical results for the probability, $P(i \in \mathrm{AP})$, that a randomly selected node, i, is an AP (solid line) in a configuration model network with a power-law degree distribution, as a function of the mean degree c, obtained from Eq. (107). The probability $P(i \in \mathrm{AP}, i \in \mathrm{GC}) = P(i \in \mathrm{AP}|\mathrm{GC})P(i \in \mathrm{GC})$ that a randomly selected node in the network is an AP that resides in the giant component (dashed line), obtained from Eq. (108), and the probability $P(i \in \mathrm{AP}, i \in \mathrm{FC}) = P(i \in \mathrm{AP}|\mathrm{FC})P(i \in \mathrm{FC})$ that a randomly selected node in the network is an AP that resides in one of the finite components (dotted line), obtained from Eq. (109), are also shown. The analytical results are in very good agreement with the results obtained from computer simulations (circles), performed for networks of size $N = 4 \times 10^4$. In panels (c) and (d), we repeat the results shown in panels (a) and (b), respectively, for a larger network of size $N = 10^6$ and $k_{\max} = 1000$.

In Fig. 9(a), we present the probability $P(i \in \mathrm{GC}) = g$ (dashed line) that a random node in a configuration model network with a power-law degree distribution where $k_{\min} = 1$ and $k_{\max} = 100$ resides on the giant component and the probability $P(i \in \mathrm{FC}) = 1 - g$ (dotted line) that such node resides on one of the finite components, as a function of the mean degree, c. Using Eq. (18), we obtain that the probability that a random node is an AP is given by

$$P(i \in \mathrm{AP}) = 1 - \frac{(1-\tilde{g}) + [\mathrm{Li}_\gamma(\tilde{g}) - \tilde{g}^{k_{\max}+1}\Phi(\tilde{g}, \gamma, k_{\max}+1)]}{\zeta(\gamma) - \zeta(\gamma, k_{\max}+1)}. \quad (107)$$

Using Eq. (24), we express the probability that a random node in the giant component is an AP in the form

$$P(i \in \mathrm{AP}|\mathrm{GC}) = 1 - \frac{\mathrm{Li}_\gamma(\tilde{g}) - \tilde{g}^{k_{\max}+1}\Phi(\tilde{g}, \gamma, k_{\max}+1)}{g[\zeta(\gamma) - \zeta(\gamma, k_{\max}+1)]}. \tag{108}$$

Using Eq. (23), we obtain the probability that a random node in one of the finite components is an AP, which is given by

$$P(i \in \mathrm{AP}|\mathrm{FC}) = 1 - \frac{1-\tilde{g}}{(1-g)[\zeta(\gamma) - \zeta(\gamma, k_{\max}+1)]}. \tag{109}$$

In Fig. 9(b), we present the probability $P(i \in \mathrm{AP})$ (solid line) that a random node in a configuration model network with a power-law degree distribution where $k_{\min} = 1$ and $k_{\max} = 100$ is an AP, as a function of c. We also present the probability $P(i \in \mathrm{AP}, i \in \mathrm{GC})$ (dashed line) that a random node is both in the giant component and is an AP and the probability $P(i \in \mathrm{AP}, i \in \mathrm{FC})$ (dotted line) that a random node resides in one of the finite components and is an AP. The analytical results are found to be in very good agreement with the results of computer simulations (circles), performed for an ensemble of configuration model networks with a power-law degree distribution, which consist of $N = 4 \times 10^4$ nodes. Note that the range of values of c is bounded from above by $\langle K \rangle_{\gamma=2}$, which is determined by $k_{\max}$ according to Eq. (102). In Figs. 9(c) and 9(d), we repeat the results of Figs. 9(a) and 9(b), respectively, for a larger network with $N = 10^6$ and $k_{\max} = 10^3$. Here too the agreement between the analytical results and the simulation results (circles) is very good. Note that the range of values of c is slightly larger due to the larger value of $k_{\max}$. Interestingly, $P(i \in \mathrm{AP})$ exhibits a different qualitative behavior as a function of c in Figs. 9(b) and 9(d); namely it is monotonically increasing for $k_{\max} = 100$ and has a local maximum for $k_{\max} = 1000$.

The behavior of APs on the finite components, expressed by $P(i \in \mathrm{AP}|\mathrm{FC})$, turns out to be qualitatively similar to the corresponding results for ER networks, presented in Fig. 2(b). However, the behavior of APs on the giant component, expressed by $P(i \in \mathrm{AP}|\mathrm{GC})$, is markedly different. In ER networks, the fraction of APs on the giant component reaches a maximum and then starts to decrease as the network becomes more dense. In contrast, in configuration model networks with a power-law degree distribution, $P(i \in \mathrm{AP}|\mathrm{GC})$ increases monotonically as a function of c. This is due to the fact that in a power-law degree distribution there are still many nodes of degree $k = 1$ even when the mean degree, c, is very large. These leaf nodes cannot reside on cycles. As a result, other nodes which reside along the paths leading to the leaf nodes become APs.

The degree distribution of the APs, obtained from Eq. (36), is given by

$$P(K = k|\mathrm{AP}) = \left\{1 - \frac{(1-\tilde{g}) + [\mathrm{Li}_\gamma(\tilde{g}) - \tilde{g}^{k_{\max}+1}\Phi(\tilde{g}, \gamma, k_{\max}+1)]}{\zeta(\gamma) - \zeta(\gamma, k_{\max}+1)}\right\}^{-1} \frac{(1-\tilde{g}^k)k^{-\gamma}}{\zeta(\gamma) - \zeta(\gamma, k_{\max}+1)}\theta(k-2). \tag{110}$$

In Fig. 10(a), we present analytical results for the degree distribution $P(K = k)$ (solid line) of a configuration model network with a power-law degree distribution, where $k_{\min} = 1$, $k_{\max} = 100$, and $c = 2$, the degree distribution $P(K = k|\mathrm{GC})$ (dashed line) of the giant component, and the degree distribution $P(K = k|\mathrm{FC})$ (dotted line) of the finite components. In Fig. 10(b), we present analytical results for the degree distribution $P(K = k|\mathrm{AP})$ (solid line) of APs in a configuration model network with a power-law degree distribution, where $k_{\min} = 1$, $k_{\max} = 100$, and $c = 2$, the degree distribution $P(K = k|\mathrm{AP}, \mathrm{GC})$ (dashed line) of APs in the giant component, and the degree distribution $P(K = k|\mathrm{AP}, \mathrm{FC})$ (dotted line) of APs in the finite components. The analytical results are in very good agreement with the results of computer simulations.

In Fig. 11, we present analytical results for the mean degree of the APs $\mathbb{E}[K|\mathrm{AP}]$ (solid line), in configuration model networks with a power-law degree distribution, where $k_{\min} = 1$ and $k_{\max} = 100$ (a) or $k_{\max} = 1000$ (b), as a function of c. We also present analytical results for the mean degree of APs which reside on the giant component, $\mathbb{E}[K|\mathrm{AP}, \mathrm{GC}]$ (dashed line), and the mean degree of APs which reside on the finite components, $\mathbb{E}[K|\mathrm{AP}, \mathrm{FC}]$ (dotted line). These results are in very good agreement with the results obtained from computer simulations (circles).

The rank distribution is given by

$$P(R = r|\mathrm{GC}) = \frac{(1-\tilde{g})^r}{g[\zeta(\gamma) - \zeta(\gamma, k_{\max}+1)]} \times \sum_{k=r+1}^{\infty} \binom{k}{r} \tilde{g}^{k-r} k^{-\gamma}. \tag{111}$$

In Fig. 12, we present analytical results for the rank distribution $P(R = r)$ (solid line) of a configuration model network with a power-law degree distribution, where $k_{\min} = 1$, $k_{\max} = 100$, and $\gamma = 2.5$. The distribution $P(R = r|\mathrm{GC})$ (dashed line) of the ranks of nodes in the giant component and the distribution $P(R = r|\mathrm{FC})$ (dotted line) of the ranks of nodes in the finite components are also shown. The analytical results are in very good agreement with the results of computer simulations (circles).

In Fig. 13, we present analytical results for the mean articulation rank, $\langle R \rangle$, of a configuration model network with a power-law degree distribution (solid line), as a function of the mean degree, c, the mean articulation rank $\mathbb{E}[R|\mathrm{GC}]$ of the nodes which reside in the giant component (dashed line), and the mean articulation rank $\mathbb{E}[R|\mathrm{FC}]$ of the nodes which reside in the finite components (dotted line). The analytical results are in very good agreement with the results of computer simulations (circles).

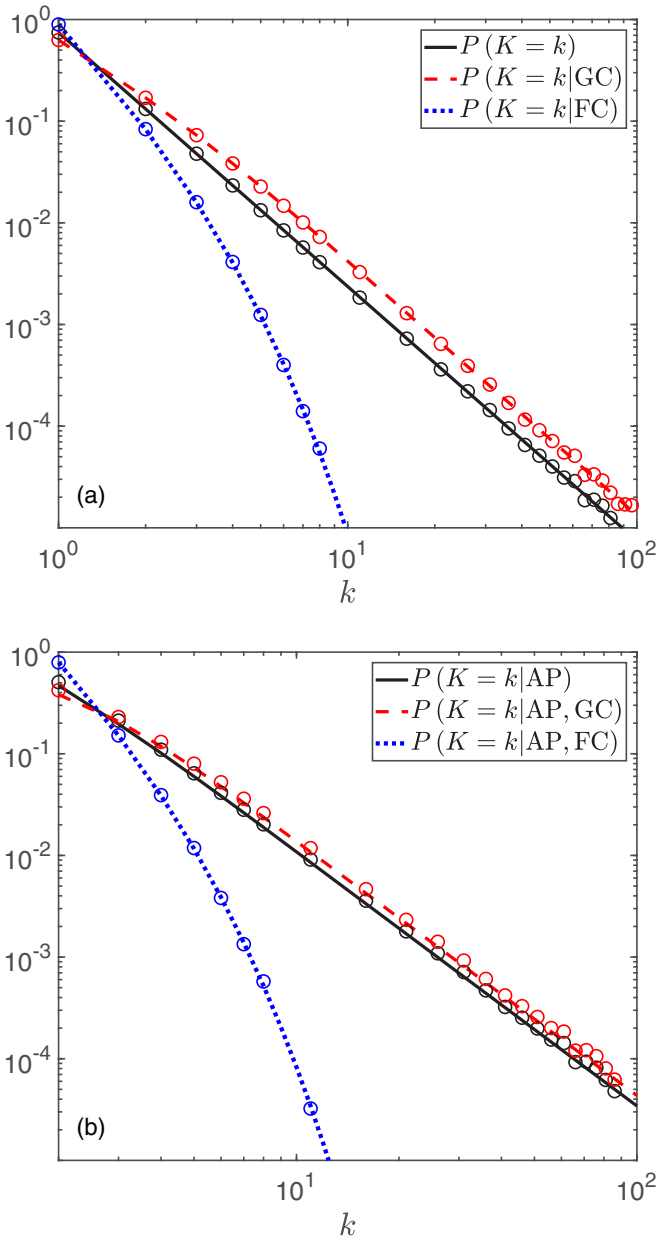

FIG. 10. (a) Analytical results for the degree distribution $P(K=k)$ of a configuration model network with a power-law degree distribution (solid line), where $k_{\min}=1$, $k_{\max}=100$, $\gamma=2.5$ (and $c\simeq 1.8$); the degree distribution $P(K=k|\mathrm{GC})$ of the giant component (dashed line); and the degree distribution $P(K=k|\mathrm{FC})$ of the finite components (dotted line). (b) Analytical results for the degree distribution $P(K=k|\mathrm{AP})$ of APs in a configuration model network with a power-law degree distribution (solid line), where $k_{\min}=1$, $k_{\max}=100$, $\gamma=2.5$ (and $c\simeq 1.8$); the degree distribution $P(K=k|\mathrm{AP,GC})$ of APs in the giant component (dashed line); and the degree distribution $P(K=k|\mathrm{AP,FC})$ of APs in the finite components (dotted line). The analytical results are in very good agreement with the results of computer simulations (circles).

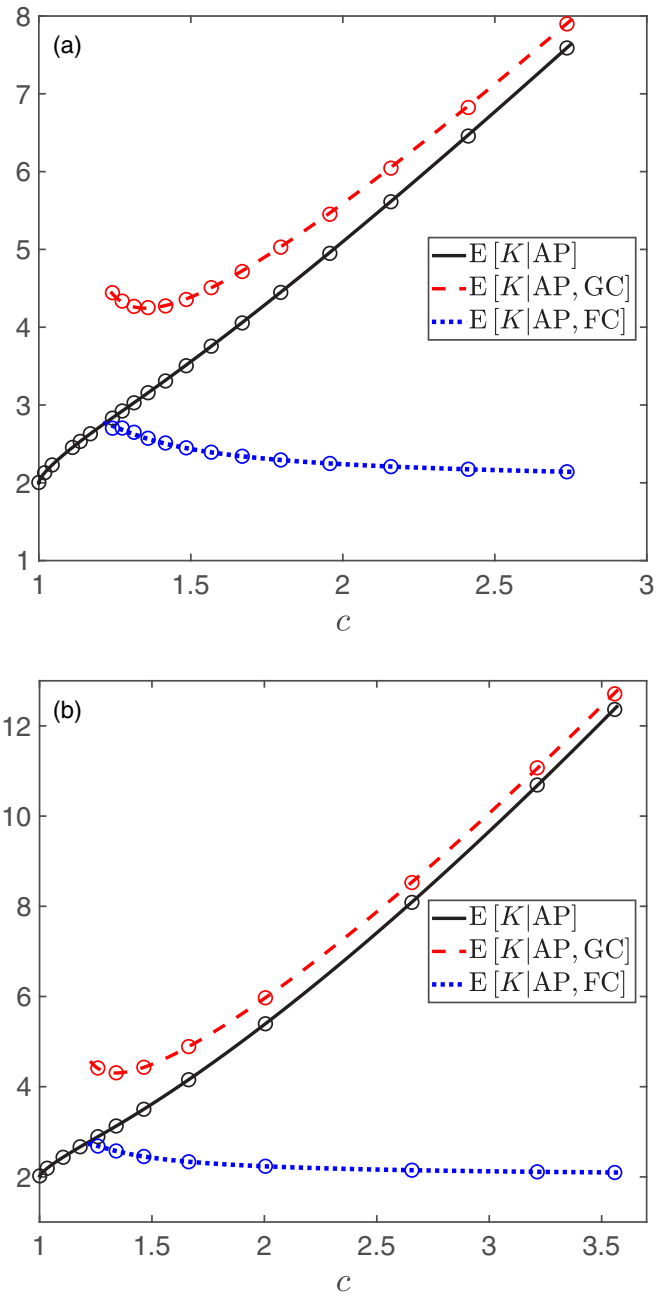

FIG. 11. (a) Analytical results for the mean degree, $\mathbb{E}[K|\mathrm{AP}]$, of APs as a function of c in a configuration model network with a power-law degree distribution (solid line), where $k_{\min}=1$, $k_{\max}=100$; the mean degree, $\mathbb{E}[K|\mathrm{AP,GC}]$, of APs in the giant component (dashed line); and the mean degree, $\mathbb{E}[K|\mathrm{AP,FC}]$, of APs in the finite components (dotted line). The analytical results are in very good agreement with the results of computer simulations (circles) for $N=4\times 10^4$. (b) Similar results for $k_{\max}=1000$ on a larger network of size $N=10^6$.

Inserting the properties of configuration model networks with a power-law degree distributions in Eq. (62), we find that the probability that a random $k=2$ AP that resides on the GC is a tube is

$$P(\mathrm{Tube}) = 1-\frac{1}{(1-\tilde{g})[\zeta(\gamma-1)-\zeta(\gamma-1,k_{\max}+1)-2^{1-\gamma}]}. \tag{112}$$

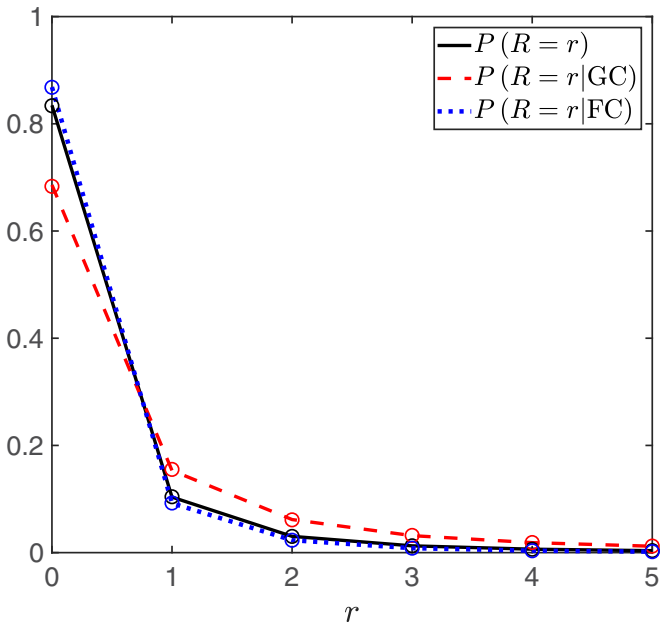

FIG. 12. Analytical results for the distribution $P(R=r)$ of the ranks of nodes in a configuration model network with a power-law degree distribution (solid line), where $k_{\min}=1$, $k_{\max}=100$, $\gamma=2.5$, and $c\simeq 1.8$. The distribution $P(R=r|\mathrm{GC})$ of the ranks of nodes in the giant component (dashed line) and the distribution $P(R=r|\mathrm{FC})$ of the ranks of nodes in the finite components (dotted line) are also shown. The analytical results are in very good agreement with the results of computer simulations (circles).

In Fig. 14, we present the probability $P(\mathrm{Tube})$ as a function of the mean degree, c, for scale-free networks with $k_{\max}=100$. It is found that as the network becomes more dense, the trees that detach upon deletion of APs become of simpler topologies.

X. DISCUSSION

Transportation, communication, and many other networks consist of a single connected component, in which there is at least one path connecting any pair of nodes. This property is essential for the functionality of these networks. Individual nodes in such networks may lose functionality because of inadvertent failures or intentional attacks. The failure of a node disrupts the local processes taking place in that node as well as the communication between this node and all the other nodes in the network. In addition, such failure disconnects those paths connecting other pairs of nodes, which go through that node. In the case that all the paths between nodes j and j' go through the failed node, i, these two nodes end up in two disconnected components of the network. In such case, the failed node i is an AP, namely a node whose deletion would break the network component on which it resides into two or more components. Networks which do not include any APs are called biconnected networks. In such networks, each pair of nodes j and j' is connected to each other by at least two nonoverlapping paths. These two nonoverlapping paths form a closed loop, whose length is equal to the sum of the lengths of the two paths. Thus, in biconnected networks, each pair of

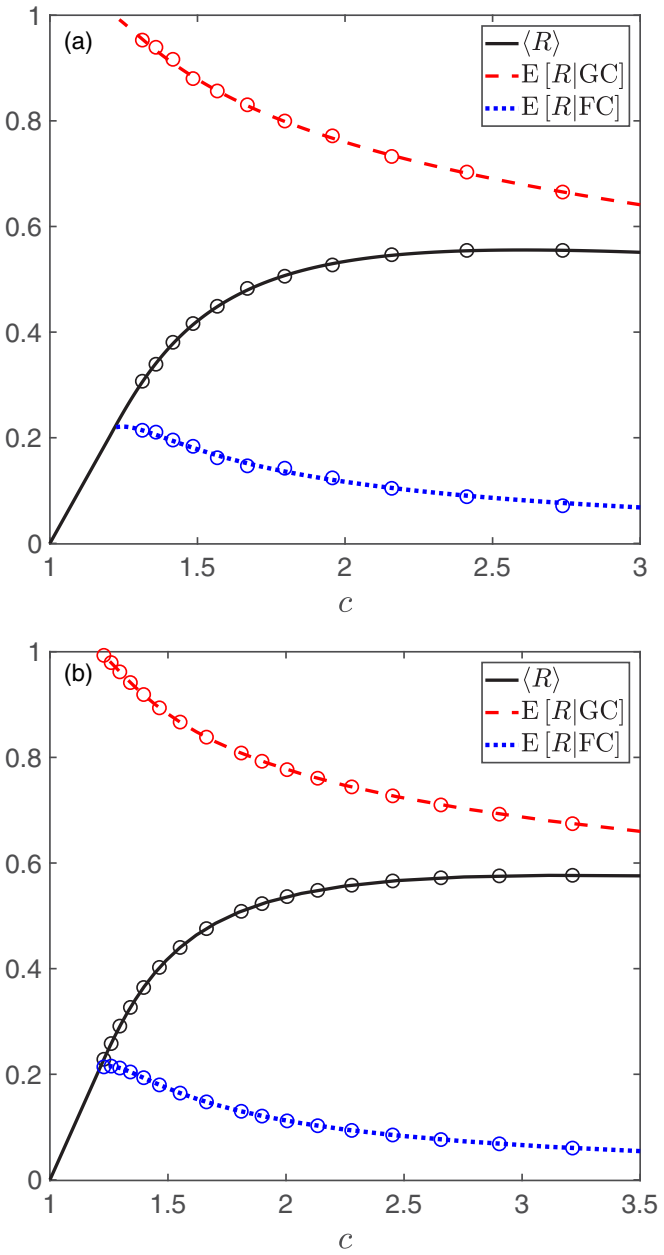

FIG. 13. (a) Analytical results for the mean articulation rank, $\langle R\rangle$, of a configuration model network with a power-law degree distribution (solid line) with $k_{\max}=100$, as a function of the mean degree, c. The mean articulation rank $\mathbb{E}[R|\mathrm{GC}]$ of the nodes which reside in the giant component (dashed line) and the mean articulation rank $\mathbb{E}[R|\mathrm{FC}]$ of the nodes which reside in the finite components (dotted line) are also shown. The analytical results are in very good agreement with the results of computer simulations (circles) for $N=4\times 10^4$. (b) Similar results for $k_{\max}=1000$ on a larger network of size $N=10^6$.

nodes resides on at least one common cycle. Thus, the deletion of any single node will leave at least one path between any pair of other nodes intact. While biconnected networks are resilient to the deletion of a single node, they are still vulnerable to multiple node deletions. This is due to the fact that each node deletion may disconnect one of the nonoverlapping path between a pair of nodes j and j'. Moreover, the deletion of

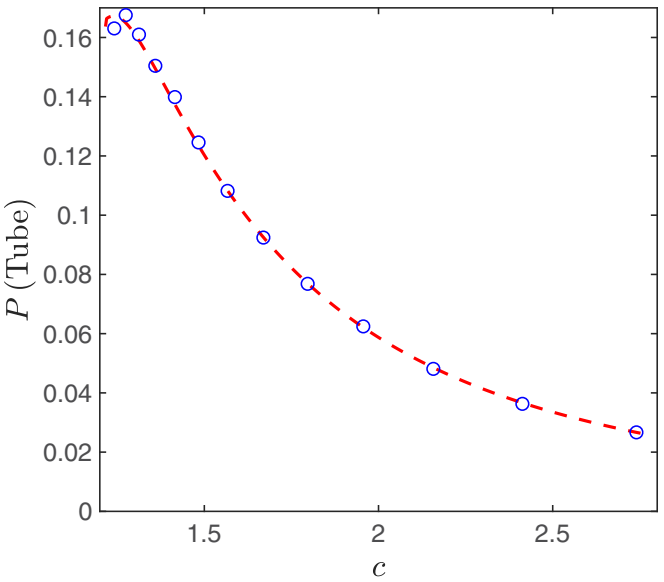

FIG. 14. Analytical results (dashed line) for the probability $P(\text{Tube})$ that a random $k = 2$ AP on the GC of a configuration model network with a power-law degree distribution is a tube, as a function of the mean degree c, obtained from Eq. (112). The analytical results are in excellent agreement with the results obtained from computer simulations (circles).

a node, i, often turns some other non-AP nodes into APs. These newly formed APs are nodes that share a cycle with i and do not reside on any other cycle. The replenishing of APs accelerates the dismantling of the network because each time an AP is deleted it disconnects additional nodes from the networks. The properties of APs are utilized in optimized algorithms of network dismantling [11,12]. The first stage of these dismantling processes is the decycling stage in which one node is deleted in each cycle, transforming the network into a tree network. In tree networks, all the nodes of degrees $k \geqslant 2$ are APs and thus the deletion of such nodes efficiently breaks the network into many small components.

The sensitivity of a network to different types of attacks depends on properties such as the degree distribution, correlations, and the statistics of cycle lengths. For example, it was shown that scale-free networks are resilient to random attacks but vulnerable to preferential attacks which target the high-degree nodes [8]. Since high-degree nodes are more likely to be APs than low-degree nodes, the deletion of a high-degree node is more likely to break the network into two or more components than the deletion of a low-degree node.

The role of APs in the resilience of complex networks to various random and targeted attacks was recently studied using a combination of model networks and a large set of empirical networks [15]. Two AP-based attack scenarios were introduced. In the AP-targeted attack, at each time step one deletes the most destructive AP, namely the AP whose deletion removes the largest number of nodes from the giant component. It was found that in the case that only a small fraction of the nodes are deleted, this procedure leads to the fastest reduction in the size of the giant component for a wide range of real-world networks. In the greedy AP-removal scenario, at each time step one simultaneously deletes all the APs that exist in the network at that time. Following the deletion, new APs emerge, and the process continues until no new APs are formed and the remaining network becomes biconnected. The remaining network is referred to as the residual giant bicomponent (RGB). It was found that the fraction of nodes which are APs, $P(i \in \text{AP})$, and the fraction of nodes which reside in the RGB, $P(i \in \text{RGB})$, provide a useful characterization of the network. To identify the topological characteristics that determine these two quantities, the probabilities $P(i \in \text{AP})$ and $P(i \in \text{RGB})$ were compared between each empirical network and its randomized counterpart. Using a complete randomization which only maintains the number of nodes, N, and the number of edges, L, it was found that such randomization completely alters $P(i \in \text{AP})$ and $P(i \in \text{RGB})$ and thus eliminates the topological characteristics that determine them. In contrast, using degree-preserving randomization, which rewires the links while keeping the degree sequence unchanged, it was found that these two quantities are not altered significantly [15]. This means that they are essentially encoded in the degree distribution $P(K = k)$. This implies that configuration model networks provide a good description of the statistical properties of APs in complex empirical networks with the same degree distribution. Another useful property of the family of configuration model network ensembles is that it is closed under random node deletion and preferential node deletion processes. This means that when a configuration model network loses nodes via random deletion or preferential deletion, its degree distribution is modified accordingly, but it remains a configuration model network and does not form degree-degree correlations.

XI. SUMMARY

We presented analytical results for the statistical properties of articulation points in configuration model networks. We obtained closed form expressions for the probability $P(i \in \text{AP})$ that a random node in a network is an AP and for the probability $P(i \in \text{AP}|k)$ that a random node of a given degree, k, is an AP. It is found that high-degree nodes are more likely to be APs than low-degree nodes. Using Bayes's theorem, we obtained the degree distribution $P(K = k|\text{AP})$ of APs in the network. It is found that APs of high degrees are more likely to reside on the GC while APs of low degrees are more likely to reside on the FCs. Apart from its degree, each node can be characterized by its articulation rank, r, which is the number of components that would be added to the network upon deletion of that node. Clearly, the articulation rank of a node of degree k may take the values $r = 0, 1, 2, \ldots, k-1$. For nodes which are not APs, the articulation rank is $r = 0$, while the articulation ranks of APs satisfy $r \geqslant 1$. We obtained a closed-form expression for the distribution of articulation ranks, $P(R = r)$. To examine the distinct properties of APs on the giant and finite components, we evaluated the probabilities presented above separately for the giant and the finite components. We applied these results to ensembles of configuration model networks with degree distributions which follow a Poisson distribution (ER networks), an exponential distribution of the form $P(K = k) \sim e^{-\alpha k}$, where $k \geqslant k_{\min} = 1$, and a power-law distribution of the form $P(K = k) \sim k^{-\gamma}$ (scale-free networks), where $k_{\min} \leqslant k \leqslant k_{\max}$, where $k_{\min} = 1$.

It is found that for the Poisson and exponential degree distributions, as the mean degree c is increased, the fraction of APs in the network increases in the sparse network regime, reaches a maximum value, and then declines as the dense network regime is approached. In contrast, for the power-law distribution, the behavior of $P(i \in \mathrm{AP})$ depends on the value of $k_{\max}$. Note that in scale-free networks the maximal value of c is bounded from above by $k_{\max}$ according to Eq. (102).

APPENDIX A: PROPERTIES OF THE GENERATING FUNCTIONS

Using the properties of the generating functions $G_0(x)$ and $G_1(x)$, one can obtain some inequalities between the values of g and $\tilde{g}$. The relation between g and $\tilde{g}$ is expressed by Eq. (6), which takes the form

$$1-g=\sum_{k=0}^{\infty}(1-\tilde{g})^k P(K=k). \tag{A1}$$

For $0<\tilde{g}<1$, one can replace all the powers $(1-\tilde{g})^k$ with $k\geqslant 1$ by $1-\tilde{g}$ and obtain

$$1-g<P(K=0)+\sum_{k=1}^{\infty}(1-\tilde{g})P(K=k). \tag{A2}$$

Expressing the sum on the right-hand side in terms of $P(K=0)$, we obtain the inequality

$$1-g<(1-\tilde{g})+\tilde{g}P(K=0). \tag{A3}$$

Therefore,

$$g>\tilde{g}[1-P(K=0)]. \tag{A4}$$

Similarly, by replacing all the powers $(1-\tilde{g})^k$ with $k\geqslant 2$ by $(1-\tilde{g})^2$, we obtain a stronger constraint of the form

$$1-g<(1-\tilde{g})^2+\tilde{g}(2-\tilde{g})P(K=0)+\tilde{g}(1-\tilde{g})P(K=1). \tag{A5}$$

Expressing g in terms of $\tilde{g}$, $P(K=0)$ and $P(K=1)$, we obtain a lower bound for g, which is given by

$$g>\tilde{g}(2-\tilde{g})-\tilde{g}(2-\tilde{g})P(K=0)-\tilde{g}(1-\tilde{g})P(K=1). \tag{A6}$$

To obtain an upper bound for g, we multiply Eq. (2) by $1-\tilde{g}$ and obtain

$$(1-\tilde{g})^2=\sum_{k=0}^{\infty}(1-\tilde{g})^k\widetilde{P}(K=k). \tag{A7}$$

Comparing the right-hand sides of Eqs. (A1) and (A7), one observes that these equations provide the mean of $(1-\tilde{g})^k$ under $P(K=k)$ and $\widetilde{P}(K=k)$, respectively. Since $\widetilde{P}(K=k)$ gives more weight to large values of k, while $(1-\tilde{g})^k$ decreases monotonically as a function of k, we conclude that for $0<\tilde{g}<1$

$$\frac{(1-\tilde{g})^2}{1-g}<1. \tag{A8}$$

Therefore, the upper bound of g is given by

$$g<\tilde{g}(2-\tilde{g}). \tag{A9}$$

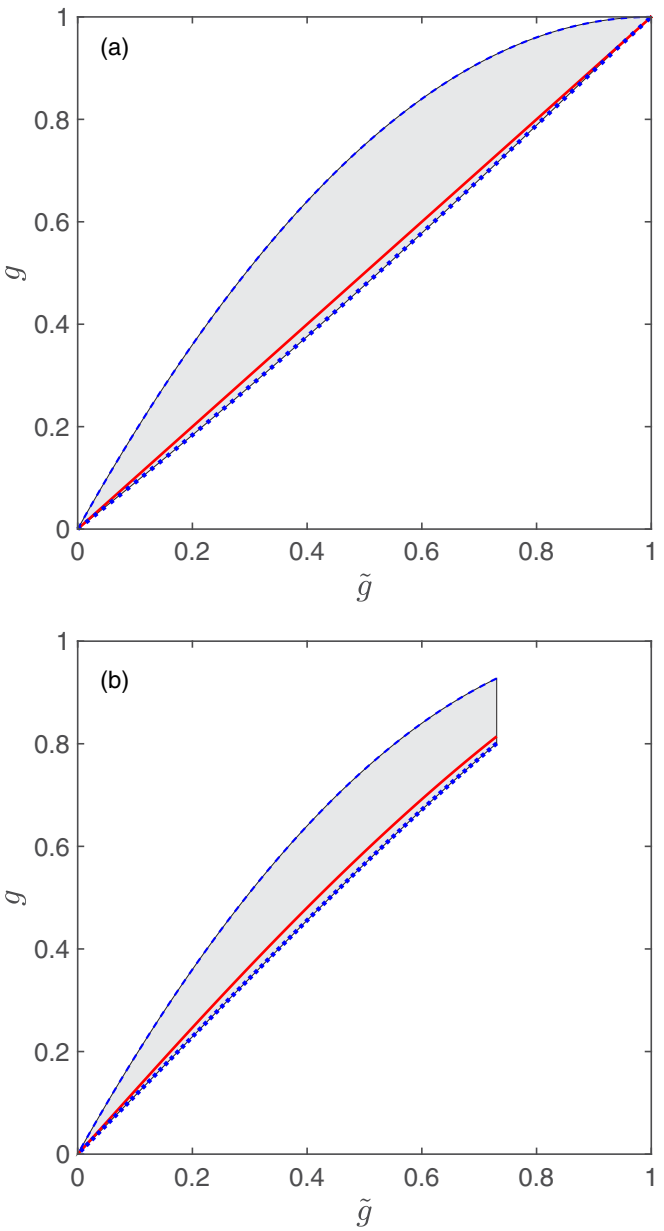

FIG. 15. The probability g that a random node resides in the giant component as a function of the probability $\tilde{g}$ that a random neighbor of a random node resides in the giant component for (a) ER networks and (b) configuration model networks with a power-law degree distribution. The shaded area shows the range of values of g between the lower bound (dotted line), given by Eq. (A6), and the upper bound (dashed line), given by Eq. (A9). In both networks, the lower bound is very close to the actual result, while the upper bound is significantly higher. For the power-law degree distribution, it is found that for $k_{\max}=100$ the value of $g=g(\tilde{g})$, obtained when the exponent γ is lowered toward $\gamma=2$, is around 0.8. In order to obtain larger values of g, one needs to further increase the value of the upper cutoff, $k_{\max}$.

In fact, this argument shows that for any $0<x<1$

$$xG_1(x)<G_0(x)-P(K=0). \tag{A10}$$

In Fig. 15, we show the probability g that a random node resides on the giant component (solid line) as a function of the

probability $\tilde{g}$ that a random neighbor of a random node resides on the giant component for ER networks [Fig. 15(a)] and for configuration model networks with a power-law degree distribution [Fig. 15(b)]. The shaded area shows the range of values of g between the lower bound (dotted line), given by Eq. (A6), and the upper bound (dashed line), given by Eq. (A9). It is found that in both networks the lower bound is very close to the actual result, while the upper bound is much higher. In the case of the power-law degree distribution, it is found that for $k_{\max} = 100$ the value of $g = g(\tilde{g})$, obtained when the exponent γ is lowered toward $\gamma = 2$ is around 0.8. In order to obtain larger values of g, one needs to increase the value of the upper cutoff, $k_{\max}$.

Just above the percolation transition, where $\tilde{g} \ll 1$, one can expand the term $(1-\tilde{g})^{k-1}$ in Eq. (2) to second order in $\tilde{g}$. Solving the resulting equation, we obtain that in this limit

$$\tilde{g} = 2\frac{\langle K^2\rangle - 2\langle K\rangle}{\langle K^3\rangle - 3\langle K^2\rangle + 2\langle K\rangle}. \tag{A11}$$

Using a similar expansion for $(1-\tilde{g})^k$ in Eq. (6), we find that just above the percolation transition, they are related by

$$g = \langle K\rangle\tilde{g} + \tfrac{1}{2}(\langle K^2\rangle - \langle K\rangle)\tilde{g}^2. \tag{A12}$$

Thus, in the limit of $c \to c_0{}^+$, where c_0 is the mean degree at the percolation transition, $0 < g, \tilde{g} \ll 1$ and

$$\lim_{c\to c_0{}^+}\left(\frac{\tilde{g}}{g}\right) = \frac{\tilde{g}(1-\tilde{g})}{g} = \frac{1}{\langle K\rangle}. \tag{A13}$$

Combining this results with Eq. (13), it is found that just above the percolation transition, the mean degree on the giant component is

$$\lim_{c\to c_0{}^+}\mathbb{E}[K|\mathrm{GC}] = 2. \tag{A14}$$

This result implies that unlike the value of the percolation threshold c_0 which depends on the degree distribution, the mean degree of the giant component upon its formation is universally $\mathbb{E}[K|\mathrm{GC}] = 2$.

APPENDIX B: THE MEAN DEGREE OF THE ARTICULATION POINTS THAT RESIDE ON THE GIANT COMPONENT

In this Appendix, we show that the mean degree of APs on the giant component, given by Eq. (44), is larger than the mean degree of all nodes on the giant component, given by Eq. (12). To this end, we express Eq. (44) in the form

$$\mathbb{E}[K|\mathrm{AP,GC}] = \frac{\tilde{g}(2-\tilde{g}) - \tilde{g}G_1(\tilde{g})}{g - [G_0(\tilde{g}) - P(K=0)]}\langle K\rangle. \tag{B1}$$

From Eq. (A10), we obtain

$$G_0(\tilde{g}) - P(K=0) > \tilde{g}G_1(\tilde{g}). \tag{B2}$$

Inserting this result into Eq. (B1), we obtain

$$\mathbb{E}[K|\mathrm{AP,GC}] > \frac{\tilde{g}(2-\tilde{g}) - \tilde{g}G_1(\tilde{g})}{g - \tilde{g}G_1(\tilde{g})}\langle K\rangle. \tag{B3}$$

From Eq. (A9), we know that

$$\frac{\tilde{g}(2-\tilde{g})}{g} > 1. \tag{B4}$$

Therefore,

$$\frac{\tilde{g}(2-\tilde{g}) - x}{g - x} > \frac{\tilde{g}(2-\tilde{g})}{g} \tag{B5}$$

for any value of x in the range $0 < x < \min\{g, \tilde{g}(2-\tilde{g})\}$. The subtracted term $x = \tilde{g}G_1(\tilde{g})$ clearly satisfies $\tilde{g}G_1(\tilde{g}) < \tilde{g}(2-\tilde{g})$. This is due to the fact that $G_1(\tilde{g}) < 1$ while $2 - \tilde{g} > 1$. Since $\mathbb{E}[K|\mathrm{AP,GC}] > 0$, one concludes that the condition $\tilde{g}G_1(\tilde{g}) < g$ is also satisfied. Therefore,

$$\frac{\tilde{g}(2-\tilde{g}) - \tilde{g}G_1(\tilde{g})}{g - \tilde{g}G_1(\tilde{g})} > \frac{\tilde{g}(2-\tilde{g})}{g}. \tag{B6}$$

We thus conclude that

$$\mathbb{E}[K|\mathrm{AP,GC}] > \mathbb{E}[K|\mathrm{GC}]. \tag{B7}$$

APPENDIX C: ARTICULATION POINTS IN THE TERNARY NETWORK

The statistical properties of APs in configuration model networks are sensitive to the abundance of nodes of low degrees, particularly nodes of degree $k = 1$ (leaf nodes) and $k = 2$. In contrast, nodes of degree $k = 0$ are excluded from the giant component and are isolated from other finite components. Thus, the isolated nodes can be discarded, with a suitable renormalization of the degree distribution. In order to perform a systematic analysis of the statistical properties of APs, we consider a configuration model network with a ternary degree distribution of the form [2]

$$P(K=k) = p_1\delta_{k,1} + p_2\delta_{k,2} + p_3\delta_{k,3}, \tag{C1}$$

where $\delta_{k,n}$ is the Kronecker δ and $p_1 + p_2 + p_3 = 1$. Since there are three nonzero probabilities and one normalization condition, this model exhibits two free parameters. It thus provides more flexibility than the ER network and the configuration model networks with an exponential and a power-law degree distribution, whose degree distributions are governed by a single parameter. The mean degree of ternary network is given by

$$\langle K\rangle = p_1 + 2p_2 + 3p_3. \tag{C2}$$

The generating functions are

$$G_0(x) = p_1x + p_2x^2 + p_3x^3 \tag{C3}$$

and

$$G_1(x) = \frac{p_1 + 2p_2x + 3p_3x^2}{p_1 + 2p_2 + 3p_3}. \tag{C4}$$

Solving Eq. (2) for $\tilde{g}$, with $G_1(x)$ given by Eq. (C4), we find that

$$\tilde{g} = \begin{cases} 0 & p_3 \leqslant \frac{p_1}{3}, \\ 1 - \frac{p_1}{3p_3} & p_3 > \frac{p_1}{3}. \end{cases} \tag{C5}$$

Using Eq. (6) for g, where $G_0(x)$ is given by Eq. (C3), we find that

$$g = \begin{cases} 0 & p_3 \leqslant \frac{p_1}{3}, \\ 1 - \left(\frac{p_1}{3p_3}\right)p_1 - \left(\frac{p_1}{3p_3}\right)^2 p_2 - \left(\frac{p_1}{3p_3}\right)^3 p_3 & p_3 > \frac{p_1}{3}. \end{cases} \tag{C6}$$

Thus, the percolation threshold is located at $p_3 = p_1/3$, independently of p_2. Using the normalization condition, we find that for any given value of p_2, a giant component exists for

$$p_3 > \frac{1-p_2}{4}. \tag{C7}$$

For convenience, we define the parameters

$$q_1 = \frac{p_1}{3p_3} \tag{C8}$$

and

$$q_2 = \frac{p_2}{3p_3}. \tag{C9}$$

From Eqs. (C5) and (C6), one can see that a necessary and sufficient condition for the existence of a giant component is that q_1 will be in the range $0 \leqslant q_1 < 1$. In contrast, there is no such condition on q_2; namely a giant component may exist for any value of $q_2 \geqslant 0$.

The degree distribution of the giant component is given by

$$P(K=k|\mathrm{GC}) = \frac{1-q_1^k}{1-q_1p_1-q_1^2p_2-q_1^3p_3}P(K=k), \tag{C10}$$

where $k = 1, 2, 3$ and $P(k)$ is given by Eq. (C1). The degree distribution on the finite components is given by

$$P(K=k|\mathrm{FC}) = \frac{q_1^k}{q_1p_1+q_1^2p_2+q_1^3p_3}P(K=k). \tag{C11}$$

Thus, the mean degree on the giant component is given by

$$\mathbb{E}[K|\mathrm{GC}] = \frac{q_1^2}{1-q_1p_1-q_1^2p_2-q_1^3p_3}\langle K\rangle, \tag{C12}$$

while the mean degree on the finite components is given by

$$\mathbb{E}[K|\mathrm{FC}] = \frac{q_1^2}{q_1p_1+q_1^2p_2+q_1^3p_3}\langle K\rangle. \tag{C13}$$

Using Eq. (18), we obtain the probability that a random node is an AP, which is given by

$$P(i\in\mathrm{AP}) = q_1[(2-q_1)(p_2+p_3)+(1-q_1)^2p_3]. \tag{C14}$$

From Eq. (24), we obtain the probability that a random node which resides on the giant component is an AP, which is

$$P(i\in\mathrm{AP}|\mathrm{GC}) = \frac{3q_1(2q_2+1)}{1+4q_1+3q_2+q_1^2+3q_1q_2}, \tag{C15}$$

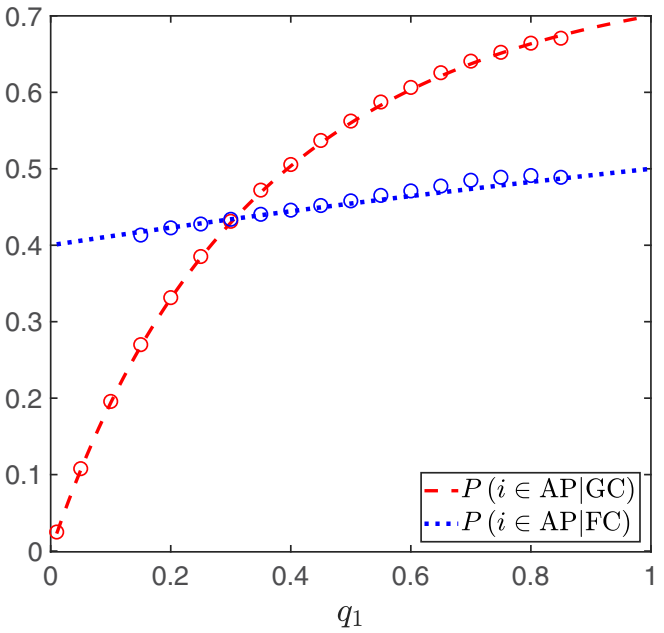

FIG. 16. The probability $P(i\in\mathrm{AP}|\mathrm{GC})$ that a random node in the giant component of a configuration model network with a ternary degree distribution is an AP (dashed line) and the probability $P(i\in\mathrm{AP}|\mathrm{FC})$ that a random node in one of the finite components is an AP (dotted line), as a function of q_1, for $q_2 = 0.6$. It is found that for small values of q_1 the two probabilities satisfy $P(i\in\mathrm{AP}|\mathrm{GC}) < P(i\in\mathrm{AP}|\mathrm{FC})$, while for large values of q_1 they satisfy $P(i\in\mathrm{AP}|\mathrm{GC}) > P(i\in\mathrm{AP}|\mathrm{FC})$. It implies that when leaf nodes are scarce, APs are more abundant on the finite components, and when leaf nodes are abundant, APs are more more abundant on the giant component.

while from Eq. (23) we obtain the probability that a random node which resides on one of the finite components is given by

$$P(i\in\mathrm{AP}|\mathrm{FC}) = \frac{q_1+3q_2}{3+q_1+3q_2}. \tag{C16}$$

An interesting question that arises is whether APs are more abundant on the giant component or on the finite components, namely whether $P(i\in\mathrm{AP}|\mathrm{GC})$ is larger or smaller than $P(i\in\mathrm{AP}|\mathrm{FC})$. On the one hand, the giant component includes nodes of higher degrees, which are more likely to be APs, and a smaller fraction of leaf nodes, which cannot be APs. On the other hand, in the finite components all the nodes of degrees $k \geqslant 2$ are APs, unlike the giant component which exhibit cycles which reduce the number of APs. From inspection of Eqs. (C15) and (C16), we conclude that in the limit of $q_1 \to 1$, one obtains $P(i\in\mathrm{AP}|\mathrm{GC}) > P(i\in\mathrm{AP}|\mathrm{FC})$. In contrast, in the limit of $q_1 \ll 1$, we obtain $P(i\in\mathrm{AP}|\mathrm{GC}) < P(i\in\mathrm{AP}|\mathrm{FC})$.

In Fig. 16, we present the probability $P(i\in\mathrm{AP}|\mathrm{GC})$ that a random node in the giant component of a configuration model network with a ternary degree distribution is an AP (dashed line) and the probability $P(i\in\mathrm{AP}|\mathrm{FC})$ that a random node in one of the finite components is an AP (dotted line), as a function of q_1, for $q_2 = 0.6$. It is found that for small values of q_1, the two probabilities satisfy $P(i\in\mathrm{AP}|\mathrm{GC}) < P(i\in$

AP|FC), while for large values of q_1 $P(i \in \text{AP}|\text{GC}) > P(i \in \text{AP}|\text{FC})$. This implies that the relative abundances of APs in the giant component and in the finite components depend on the parameters of the network.

The probability that a randomly selected AP resides on the giant component is given by

$$P(i \in \text{GC}|\text{AP}) = \frac{3(1-q_1)(2q_2+1)}{(2-q_1)(3q_2+1)+(1-q_1)^2}, \quad \text{(C17)}$$

while the probability that a randomly selected AP resides on one of the finite components is given by

$$P(i \in \text{FC}|\text{AP}) = \frac{q_1(q_1+3q_2)}{(2-q_1)(3q_2+1)+(1-q_1)^2}. \quad \text{(C18)}$$

From inspection of Eqs. (C17) and (C18), we find that in the limit of $q_2 \gg 1$ and $q_1 \ll 1$ the probability that a randomly selected AP resides on the giant component satisfies $P(i \in \text{GC}|\text{AP}) \to 1$, while the probability that such node resides on one of the finite components vanishes.

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