Varieties of scaling regimes in hydromagnetic turbulence

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We revisit the scaling properties of the energy spectra in fully developed incompressible homogeneous turbulence in forced magnetofluids (MHD) in three dimensions (3D), which are believed to be characterized by *universal scaling exponents* in the inertial range. Enumerating these universal scaling exponents that characterize the energy spectra remains a theoretical challenge. To study this, we set up a scaling analysis of the 3D MHD equations, driven by large-scale external forces and with or without a mean magnetic field. We use scaling arguments to bring out various scaling regimes for the energy spectra. We obtain a variety of scaling in the inertial range, ranging from the well-known Kolmogorov spectra in the isotropic 3D ordinary MHD to more complex scaling in the anisotropic cases that depend on the magnitude of the mean magnetic field. We further dwell on the possibility that the energy spectra scales as k^{-2} in the inertial range, where k is a wave vector belonging to the inertial range, and also speculate on unequal scaling of the kinetic and magnetic energy spectra in the inertial range of isotropic 3D ordinary MHD. We predict the possibilities of *scale-dependent anisotropy* and intriguing *weak dynamic scaling* in the Hall MHD and electron MHD regimes of anisotropic MHD turbulence. Our results can be tested in large-scale simulations and relevant laboratory-based and solar wind experiments.

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I. INTRODUCTION

Kinetic energy spectrum in three-dimensional (3D) homogeneous and isotropic turbulence, described by the forced Navier-Stokes (NS) equation, displays universal scaling in the inertial range (that lies intermediate between the large forcing scales and small viscous dissipation scales) for sufficiently large Reynolds numbers [1]. For instance, the celebrated Kolmogorov dimensional analysis (hereafter K41) [2] predicts that the kinetic energy spectrum $E_v(\mathbf{k}) \sim \langle|\mathbf{v}(\mathbf{k},t)|^2\rangle k^2 \sim k^{-5/3}$ (known as the K41 result in the literature [2,3]) in the inertial range of the nonequilibrium turbulent steady states (NESS). Here, $\mathbf{k}$ is a Fourier wave vector belonging to the inertial range and $\mathbf{v}(\mathbf{k},t)$ is the velocity field in the Fourier space; $\langle ... \rangle$ refers to spatiotemporal averages in the NESS [4].

Magnetohydrodynamics (MHD) is the study of the properties of electrically conducting quasineutral fluids in the hydrodynamic limit, valid over huge ranges of spatial scales ranging from centimeters (e.g., laboratory plasmas) to very large scales in astrophysical settings (e.g., solar wind) [5]. A plasma necessarily consists of two electrically charged components—ions and electrons. The dynamical equations for MHD depend crucially on the spatiotemporal scales of interests. For instance, at large spatial (scales larger than the ion Larmor radius) and temporal scales (times larger than ω_{pi}^{-1}), both the ion and electron motions are important, the local relative velocity between the electrons and the ions are negligible compared to the local center-of-mass velocity, for which the ordinary MHD description, which is a one fluid description, suffices [5–7]. In a direct analogy with fluid dynamics, ordinary MHD can be viewed as a coupled dynamics of a velocity $\mathbf{v}$ and a magnetic field $\mathbf{b}$, and the electric field drops out of the dynamics due to the condition of local charge neutrality.

Hall magnetohydrodynamics (HMHD) is again a single-fluid approximation that includes a Hall term in the Ohm's law (see below). This description extends the validity domain of the ordinary MHD system to spatial scales down to a fraction of the ion skin depth or frequencies comparable to the ion gyrofrequency [8]. More specifically, HMHD is a good description when we intend to describe the plasma dynamics up to length scales shorter than the ion inertial length d_i ($d_i = c/\omega_{pi}$, where c is the speed of light and ω_{pi} is the ion plasma frequency) and frequencies smaller than the ion cyclotron frequency ω_{ci} [9]. For example, solar wind at small scales show signatures of HMHD [10]. Eventually at sufficiently small scales $l < c/\omega_{pi}$ *and* at frequencies much higher than ω_{pi}, the ions are effectively frozen due to their larger inertia, and the electrons move in a frozen background of the ions, a regime aptly called electron MHD (EMHD) [11–13]. EMHD phenomenology is believed to be operative in exotic astrophysical contexts like the crust of neutron stars [14], solar corona and magnetotail [15], as well as laboratory experiments [16].

In equilibrium systems, fluctuations near a critical point (or a second-order phase transition) and in the ordered phases of systems with broken continuous symmetries show *universal dynamic scaling* in the long wavelength and long timescale limits [17]. Subsequently, the idea of universal scaling has been extended to nonequilibrium systems as well [18–20]. In

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fully developed fluid turbulence, this notion of universality implies that the scaling of the kinetic energy spectrum and the damping timescale of the velocity fluctuations with wave vectors in the inertial range is independent of the molecular viscosities [20]. The question of scaling of energy spectra, both kinetic and magnetic, in 3D hydromagnetic turbulence, although believed to be universal in the inertial range of fully developed MHD turbulence (i.e., in the large Reynolds number limit), is still not well-settled, either theoretically or experimentally. A major difference between 3D homogeneous fluid turbulence and 3D homogeneous MHD turbulence originates from the possible presence of a mean magnetic field in magnetofluids: A mean fluid velocity, although it makes the system nominally anisotropic, can be removed by a suitable Galilean transformation, thereby restoring full isotropy. In contrast, the magnetic field is invariant under Galilean transformations, and consequently, a mean magnetic field of magnitude B_0 cannot be removed by any Galilean boost and necessarily makes the system genuinely anisotropic. A nonzero B_0 introduces propagating modes in the form of Alfvén waves in ordinary 3D MHD that have no analogues in homogeneous fluid turbulence. Similarly, in HMHD with $B_0 \neq 0$, there are circularly polarized whistler and cyclotron modes [8] that are analogues of the Alfvén waves in ordinary 3D MHD, but have no counterparts in homogeneous and isotropic 3D fluid turbulence. The whistler modes exist in anisotropic 3D EMHD as well [21].

Simple dimensional analysis similar to that for fluid turbulence suggests K41 scaling in the inertial range for both the kinetic and magnetic energy spectra of 3D nonhelical isotropic (i.e., no mean magnetic fields) MHD turbulence. However, some recent studies indicate the possibility of an unexpected k^{-2} scaling of the energy spectra in the inertial range of 3D isotropic MHD [22,23]. In addition, the presence of mean magnetic fields, which gives rise to propagating Alfvén waves, can significantly affect scaling. In general, the effects of propagating waves on the scaling properties of driven systems are still debated. MHD turbulence with a nonzero B_0 or Alfvén waves stands as a very good candidate to study this issue. In Refs. [24,25], it has been argued within a low-order perturbative analysis that the effective or renormalized mean magnetic field B_{0R} (formally defined as the imaginary part of the field propagators at zero frequency) picks up singular corrections in the long wavelength limit. This in turn yields kinetic and magnetic spectra that are anisotropic in magnitude but display spatial scaling same as the K41 prediction. This prediction is different from the results from a 1D model for MHD turbulence [26], where the absence of any singular renormalization of the mean magnetic fields render them irrelevant (in a scaling sense) in comparison with the viscous damping. This too, unexpectedly, yields the K41 scaling for the energy spectra. Numerical studies of Ref. [27] revealed energy spectra closer to those predicted by the Iroshnikov-Kraichnan (IK) scaling of $k^{-3/2}$ [28]. For large B_0, weak turbulence theories for incompressible MHD suggest a $k_\perp^{-2}$ scaling, where $\mathbf{k}_\perp$ is the component of the 3D wave vector $\mathbf{k}$, that is normal to the mean magnetic field [29]. These multitude of predictions for scaling in 3DMHD calls for a generic scaling analysis for both isotropic and anisotropic 3D MHD. For HMHD and EMHD, there are predictions for scale breaking demarcating the long wavelength inertial range and an intermediate wavelength *dispersion range* [13,30]. Recent experimental studies on tabletop laser-plasma [31] reveal various scaling regimes at different ranges of wave vectors. In addition, very little is known about the dynamic scaling in MHD turbulence. Critical examination of dynamic scaling regimes in MHD turbulence [32] would be very useful as well.

In this article, we revisit the universal scaling of the kinetic and magnetic energy spectra in the inertial range in 3D turbulent fully developed forced homogeneous hydromagnetic fluids by employing scaling arguments. We cover (a) ordinary 3D MHD, (b) 3D HMHD, and (c) 3D EMHD. We consider the role of a mean magnetic field in each of the above cases. The scaling theory developed here reveals a variety of scaling regimes. For instance, we find that (i) when forced at the largest scales and assuming nonhelical MHD, the scaling of both the magnetic and kinetic spectra for 3D isotropic ordinary MHD should follow the K41 prediction in the hydrodynamic long wavelength limit. We further discuss the possibility of k^{-2} for the magnetic spectrum; see Ref. [22]. This is obviously a "weak scaling" (where the magnetic and kinetic energy spectra scale differently), as suggested in Ref. [22], since the corresponding kinetic energy spectrum appears to scale very differently. The total energy, as mentioned in Ref. [22], has a backward flux. Hence, unlike the strong scaling K41 spectra (where the magnetic and kinetic energy spectra scale identically), the forward cascade (i.e., from small to large wave numbers) should be that of the other conserved quantity—the cross helicity. Furthermore, we speculate that k^{-2} strong scaling can also be found in systems with large-scale separations between the forcing scale and inertial range; see, e.g., this is possibly connected with Ref. [23]. In contrast, with a finite B_0, i.e., with Alfvén waves present, the scaling generally takes an anisotropic form. Again, with a large-scale forcing and neglecting helicity, assuming the linear propagating Alfvén terms and the nonlinear terms scale in the same way for a finite B_0 the scaling of the energy spectra with $\mathbf{k}_\perp$ follows the K41 result, whereas they scale differently with $k_\parallel$. This scaling behavior in the limit of a very large B_0 that strongly suppresses the nonlinear terms, gives way to a $k_\perp^{-2}$ scaling [33]. Here, $k_\parallel$ is the component of the 3D wave vector $\mathbf{k}$ along $\mathbf{B}_0$. Further, both the velocity and magnetic field fluctuations are characterized by the same dynamic exponent, corresponding to the more commonly found *strong dynamic scaling* [34]. (ii) For 3D HMHD, scale breaking at higher wave vectors are predicted together with anisotropic scaling of the energy spectra for a nonzero B_0. (iii) Last, the scaling of the magnetic energy spectrum in 3D EMHD is argued to be same as 3D HMHD but different from ordinary 3D MHD. Scaling of the energy spectra in 3D anisotropic HMHD and EMHD are found to depend strongly on the magnitude of B_0. For instance, as B_0 rises, the scaling of the magnetic energy spectrum changes from $k_\perp^{-7/3}$ to $k_\perp^{-5/2}k_\parallel^{-1/2}$. We highlight generic *scale-dependent* anisotropy in all regimes of 3D MHD for a nonzero B_0. We further predict the possibilities of unequal dynamic exponents for $\mathbf{v}$ and $\mathbf{b}$ fluctuations in a HMHD scaling regime when the Hall term in HMHD is dominant [35]. Occurrence of weak dynamic scaling is very rare in natural systems. A prominent example is the equilibrium critical dynamics of symmetric binary mixture near its demixing transition (a second-order transition) point. Here, the concentration fluctuations are *distinctly slower* than the velocity fluctuations, reflecting the existence of two distinct dynamic exponents for the concentration and the velocity [17,36–38]. It was also proposed that a model for the ordered phase of the XY model also show similar weak dynamic scaling [39]; this was, however, ruled out later, showing that at 3D there are no weak dynamic scaling in this model [40]. More recently, a nonequilibrium version of Model C is shown to display weak dynamic scaling for certain choices of the model parameters [41]. To our knowledge, 3D HMHD is the first candidate for weak dynamic scaling in the realm of turbulence, which forms a principal prediction from the present study. The rest of this article in organized as follows. In Secs. II B 1 and II B 2 we study scaling in ordinary isotropic and anisotropic 3D MHD, respectively. Next we consider scaling 3D isotropic and anisotropic Hall MHD in Secs. II C 1 and II C 2, respectively. Finally, in Secs. II D 1 and II D 2 we analyze scaling in 3D isotropic and anisotropic EMHD, respectively. In Sec. III we summarize our results.

II. SCALING ANALYSIS

Scaling analysis is a powerful tool that is useful in extracting the dominant scaling behavior in the steady states of a dynamical system. In scaling analysis of a model, first space, time, and the dynamical fields are scaled and next, the scale-invariance of the dynamical equations (invariance of the form of the dynamical equations under rescaling) for appropriate scaling factors for space, time, and the dynamical fields is demanded. For systems with uniform steady states, the dominant scaling behavior in the steady state is ascertained by balancing the most relevant terms (in a scaling sense) in the long wavelength limit and by imposing other conditions that characterize the steady states.

To set up the background, we first revisit scaling analysis of homogeneous and isotropic incompressible fluid turbulence that directly yields the K41 spectrum for the velocity field. Our discussions of scaling in this paper will be based on the premise that the governing equations, i.e., the evolution equations for $\mathbf{v}$ and $\mathbf{b}$ have to be invariant under a scale transformation that scales distances by l and time by $l^{\tilde{z}}$. The *dynamic exponent* $\tilde{z}$ is an unknown which will be determined by some additional constraints. The additional constraint—a crucial ingredient in the scaling analysis—comes from the constancy (scale-independence) of the fluxes of the relevant conserved quantities in the ideal limit (i.e., in the absence of any external forcing or dissipation) at the intermediate scales or inertial range. When the dynamics is that of only one variable, e.g., velocity $\mathbf{v}$ for incompressible fluid turbulence, there can be no ambiguity. If there are more than one dynamical variable (e.g., two for incompressible 3D MHD) or more than one conserved quantity in the ideal limit (again as in 3D MHD, see below), there is yet another additional issue about whether the fields will scale similarly or differently under the spatiotemporal rescaling. The former case turns out to be completely unambiguous. However, for the kind of scaling analysis that we carry out here, the latter cases are of "if...then" variety in some of the physical examples. It should also be noted that spatial anisotropy in the form of an externally imposed magnetic field will lead to the introduction of an additional scale, and the scaling arguments will hold under restrictive conditions which we will be able to specify.

A. 3D fluid turbulence

We revisit the universal scaling of the kinetic energy spectrum in 3D homogeneous and isotropic incompressible fluid turbulence. By using scaling arguments, we reproduce the well-known K41 result. The Navier Stokes equation for an incompressible velocity field $\mathbf{v}$ for an isotropic pure neutral fluid is given by [1,3]

$$\frac{\partial \mathbf{v}}{\partial t} + \lambda_1(\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla p + \nu\nabla^2\mathbf{v} + \mathbf{f}_v, \tag{1}$$

together with the incompressibility condition given by $\nabla\cdot\mathbf{v} = 0$. Here, p and ν are the pressure and kinematic viscosity, respectively; $\mathbf{f}_v$ is a large-scale force needed to sustain fully developed turbulence. Parameter λ_1 takes the value unity, but is formally introduced in conventional renormalization group-based analysis for turbulence as a perturbative expansion parameter [20]. In the inviscid, unforced limit ($\nu = 0, \mathbf{f}_v = 0$), Eq. (1) in 3D conserved the kinetic energy and fluid helicity. The kinetic energy spectrum in 3D is given by $E_v(k) \sim k^2\langle|\mathbf{v}(\mathbf{k},\mathbf{t}|^2\rangle$ in the inertial range. For a nonhelical fluid turbulence, the kinetic energy flux in the steady state cascades from large length scales to small length scales and remains scale-independent in the intermediate inertial range. The physical argument behind this is the fact that energy is injected from outside at the largest (forcing) scales (by the large-scale external forces) and get dissipated at very small scales by the molecular viscosities (viscous scales). In the intervening inertial regime in the steady state, the energy just flows from the large scales to small scales, without any energy injection or dissipation. This ensures that the energy flux is constant in the inertial range [42]. The well-known Kolmogorov dimensional analysis predicts $E_v(k) \sim k^{-5/3}$ [2,3]. We will see below how this result may be recovered from a simple scaling analysis.

To begin with we scale space $\mathbf{x}$ time t and 3D velocity $\mathbf{v}$ as follows:

$$\mathbf{x} \rightarrow l\mathbf{x},\ \ t \rightarrow l^{\tilde{z}}t,\ \ \mathbf{v} \rightarrow l^a\mathbf{v}, \tag{2}$$

where $\tilde{z}$ is the dynamic exponent. Demanding scale invariance, we obtain (in a scaling sense)

$$\frac{\partial \mathbf{v}}{\partial t} \sim \mathbf{v}\cdot\nabla\mathbf{v} \Rightarrow l^{a-\tilde{z}} = l^{2a-1} \Rightarrow a = 1-\tilde{z}, \tag{3}$$

that is consistent with the physical dimension of a velocity. We note that the 3D NS Eq. (1) in the inviscid limit ($\nu = 0$), or the Euler equation is scale-invariant, i.e., its form remains unchanged, with $a = 1-\tilde{z}$ for arbitrary $\tilde{z}$. For $\nu > 0$, this symmetry gets restricted as we show below. Here, we have used that the nonlinear coupling λ_1 *does not* pick up any scale-dependencies under rescaling Eq. (2) that is consistent with its nonrenormalization due to the Galilean invariance of Eq. (1). Viscosity ν is assumed to pick a scale-dependence that is consistent with the value of $\tilde{z}$ (obtained below).

In a mean-field-like approach, the kinetic energy flux or the kinetic energy dissipation rate per unit mass ϵ_v, neglecting intermittency, is assumed to be a constant in the inertial range of the steady states of fully developed fluid turbulence [3], and should not change under rescaling Eq. (2). Thence, demanding scale-independence of ϵ_v we find

$$\epsilon_v \sim \frac{\partial v^2}{\partial t} \sim l^0 \Rightarrow 2a = \tilde{z}. \tag{4}$$

Combining Eqs. (3) and (4) we find

$$a = \tfrac{1}{3}, \quad \tilde{z} = \tfrac{2}{3}, \tag{5}$$

which are in agreement with the results obtained in Ref. [20].

The next step is to calculate the scaling of the kinetic energy spectrum from the values of a and $\tilde{z}$, already known as above. We start by noting that

$$\langle \mathbf{v}(\mathbf{k},t)\cdot\mathbf{v}(\mathbf{k}',t)\rangle = F_v(k)\delta(\mathbf{k}+\mathbf{k}'), \tag{6}$$

where $F_v(k)$ is related to $E_v(k)$ (see below). Noting that

$$\mathbf{v}(\mathbf{k},t) \sim \int d^3x \exp(-i\mathbf{k}\cdot\mathbf{x})\mathbf{v}(\mathbf{x},t), \tag{7}$$

under rescaling Eq. (2) we have

$$\mathbf{v}(\mathbf{k},t) \sim l^{a+3} \sim k^{-a-3}, \tag{8}$$

where $k \sim l^{-1}$, in a scaling sense. Next, equating the scale factors on both sides of Eq. (6), we obtain

$$F_v(k) \sim k^{-3-2a}. \tag{9}$$

Now, the kinetic energy spectrum $E_v(k)$ in 3D is given by

$$E_v(k) \sim k^2 F_v(k) \sim k^{-1-2a} \sim k^{-5/3}, \tag{10}$$

in agreement with Ref. [20]. Notice that $\tilde{z} = 2/3$ together with a scale-independent kinetic energy flux implies the effective kinematic viscosity scales as $\nu l^{2-z} \sim \nu l^{4/3}$ [20] that control the relaxation of the $\mathbf{v}$ fluctuations restores the scale-invariance of Eq. (1). That the effective viscosity should be scale-dependent to keep the kinetic energy flux scale-independent has been known ever since the seminal works by Heisenberg [43] and Chandrasekhar [44]. This opens up the distinct possibility that in systems with more than one dynamical variables and independent fluxes, more than one dynamic exponents may be needed to keep the fluxes scale-independent.

B. Ordinary 3D MHD turbulence

Here we first consider homogeneous and isotropic incompressible 3D MHD turbulence, followed by its anisotropic analog.

1. Isotropic 3D MHD turbulence

The ordinary 3DMHD equations for an incompressible homogeneous and isotropic magnetofluid are composed of the generalized Navier-Stokes equation for the velocity field $\mathbf{v}$ and Induction equation for the magnetic field $\mathbf{b}$ [6,7]. These are, respectively,

$$\frac{\partial \mathbf{v}}{\partial t} + \lambda_1(\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla p + \lambda_2(\mathbf{b}\cdot\nabla)\mathbf{b} + \nu\nabla^2\mathbf{v} + \mathbf{f}_v \tag{11}$$

and

$$\frac{\partial \mathbf{b}}{\partial t} + \lambda_1(\mathbf{v}\cdot\nabla)\mathbf{b} = \lambda_1(\mathbf{b}\cdot\nabla)\mathbf{v} + \mu\nabla^2\mathbf{b} + \mathbf{f}_b. \tag{12}$$

The effective pressure p now includes the magnetic contribution $b^2/2$. Furthermore, λ_1, λ_2 are nonlinear coupling constants. Parameters ν and μ are kinematic and magnetic viscosities. We impose incompressibility $\nabla\cdot\mathbf{v} = 0$ and $\nabla\cdot\mathbf{b} = 0$. Functions $\mathbf{f}_v$ and $\mathbf{f}_b$ are external stochastic forces. As for Eq. (1) $\lambda_1 = 1$ and λ_2 just sets the scale of $\mathbf{b}$ with respect to $\mathbf{v}$ [24,25]. Similar to Eq. (1), Eqs. (11) and (12) are invariant under Galilean transformation [24,25]), which ensures nonrenormalization of λ_1 in a RG framework. Further, as pointed out in Refs. [24,25], working in terms of *effective* magnetic fields that leaves Eq. (12) unchanged, leads to nonrenormalization of λ_2 as well. Hence, without any loss of generality, we set $\lambda_1 = \lambda_2 = 1$ in what follows below. The absence of any mean magnetic field implies that $\langle \mathbf{b}(\mathbf{x},t)\rangle = 0$. Functions $\mathbf{f}_v$ and $\mathbf{f}_b$ are external large-scale forces need to maintain fully developed MHD turbulence. Equations (11) and (12) in the inviscid, unforced limit in 3D conserve the total energy $E = \int_x (v^2 + b^2)$, cross helicity $H_c = \int_x \mathbf{v}\cdot\mathbf{b}$ and the magnetic helicity $H_m = \int_x \mathbf{A}\cdot\mathbf{b}$, where $\mathbf{A}$ is the vector potential for $\mathbf{b}$: $\mathbf{b} = \nabla\times\mathbf{A}$.

The scaling of the kinetic and magnetic spectra in the inertial range can be easily obtained by generalizing the analysis developed in Sec. II A. Scaling ansatz Eq. (2) is now to be augmented by the scaling of $\mathbf{b}$:

$$\mathbf{b} \rightarrow l^y \mathbf{b}. \tag{13}$$

As before, we demand scale invariance of Eqs. (11) and (12). We consider large-scale forcings and assume nonhelical MHD turbulence, i.e., $H_c \approx 0$, $H_M \approx 0$. Now balancing the nonlinear terms in Eq. (11) we obtain (in a scaling sense)

$$a = y. \tag{14}$$

Notice that with $\tilde{z} = 1 - a$, the nonlinear terms in Eq. (12) scale in the same way as $\partial\mathbf{b}/\partial t$:

$$\frac{\partial \mathbf{b}}{\partial t} \sim (\mathbf{v}\cdot\nabla)\mathbf{b} \sim (\mathbf{b}\cdot\nabla)\mathbf{v}. \tag{15}$$

Due to the equality $a = y$, scale-independence of the kinetic (magnetic) energy flux automatically ensures scale-independence of the magnetic (kinetic) energy flux. This then corresponds to scale-independence of the total energy flux. Proceeding as in Sec. II A, we then find

$$2a = 2y = \tilde{z}, \tag{16}$$

giving $a = y = 1/3$ and $\tilde{z} = 2/3$. We can now proceed to obtain the scaling of the both kinetic and magnetic energy spectra in the inertial ranges by following the logic outlined in Sec. II A above. Similar to $\mathbf{v}(\mathbf{k},t)$ we define $\mathbf{b}(\mathbf{k},t)$ via

$$\mathbf{b}(\mathbf{k},t) \sim \int d^3x \exp(-i\mathbf{k}\cdot\mathbf{x})\mathbf{b}(\mathbf{x},t), \tag{17}$$

yielding

$$\mathbf{b}(\mathbf{k},t) \sim l^{y+3} \sim k^{-y-3}. \tag{18}$$

Analogous to Eq. (6) we further define

$$\langle \mathbf{b}(\mathbf{k},t)\cdot\mathbf{b}(\mathbf{k}',t)\rangle = F_b(k)\delta(\mathbf{k}+\mathbf{k}'), \tag{19}$$

yielding as for $F_v(k)$

$$F_b(k) \sim k^{-3-2y}. \tag{20}$$

Thus, the magnetic energy spectrum $E_b(k)$ in 3D scales as

$$E_b(k) \sim k^2 F_b(k) \sim k^{-1-2y} \sim k^{-5/3}. \tag{21}$$

The scaling of $E_v(k)$ remains unchanged from what we obtained in Sec. II A. Last, $\tilde{z} = 2/3$ indicates that both ν and μ scale as $l^{2-\tilde{z}}$ in the inertial range. Thus, both $\mathbf{v}$ and $\mathbf{b}$ fluctuations are characterized by the same $\tilde{z}$, or strong dynamic scaling prevails. That both $\mathbf{v}$ and $\mathbf{b}$ must have the same $\tilde{z}$, can also be argued the scale-dependencies of effective ν and μ needed to make the magnetic and kinetic energy spectra must be the same. That we find $a = y$ in nonhelical isotropic 3D MHD is consistent with the discussions in Ref. [23].

So far we have considered scale-independence of only the energy flux (kinetic and magnetic), which straightforwardly leads to K41 scaling for both the energy spectra. This is justified when the total cross helicity and magnetic helicity are zero. Can $E_v(k)$ and $E_b(k)$ ever display non-K41 type inertial range scaling in any situation? Recent studies in Refs. [22,23] suggest that even in 3D isotropic MHD turbulence, the energy spectra can be non-K41 type; Ref. [22] found k^{-2} scaling where as Ref. [23] found both k^{-2} and IK spectra, in addition to K41 spectra. We now discuss possible ways to generalize the scaling theory to allow for non-K41 type inertial range scaling by $E_v(k)$ and $E_b(k)$. This can be then used to study the results in Refs. [22,23]. In Ref. [23] bounds on the scaling exponents of $E_+(k) \sim k^{q+}$ and $E_-(k) \sim k^{q-}$ in the inertial range are discussed, where $E_\pm$ are the energy spectra of the Elsässer variables, which are just linear combinations of $\mathbf{v}$ and $\mathbf{b}$. It has been argued that in the absence of any cross-helicity, $q^+ = q^- \neq 3/2$. This would necessarily mean $a = y$ in our notation and $E_v(k) \sim E_b(k)$. The solution $a = y = 1/3$ corresponding to the K41 spectra satisfies this. However, it is also known that if there is large-scale separation between the forcing scale and the inertial range then effective anisotropy in the inertial range can emerge and the magnetic fields in the forcing scale can play the role of background magnetic fields for the fluctuating magnetic fields in the inertial range [23]. This should naturally generate Alfvén wavelike excitations with linear dispersion (see also below). K41 scaling follows when this is subdominant to the nonlinear cascade. In contrast, when this dominates over the nonlinear interactions in the inertial range, $\tilde{z} = 1$. If we further impose scale-independence of the kinetic and magnetic energy fluxes, then we find $a = y = \tilde{z}/2 = 1/2$. Following the logic outlined above, this yields

$$E_v(k) \sim E_b(k) \sim k^{-2}, \tag{22}$$

see, e.g., Refs. [22,23]. Furthermore, Ref. [23] has argued than in the presence of cross-helicity $q_+ \neq q_-$, equivalently, $a \neq y$ is possible. The emergence of unequal scaling of v and b effectively implies the existence of an additional dimensional parameter as in the *incomplete self-similarity* discussed in Ref. [45]. We construct such a possibility below. At the outset, we assume $a \neq y$ and set $y = a + \alpha$, where $\alpha \neq 0$ implies scale-breaking. Assuming there is no anomalous scaling of $\mathbf{v}$, $a = 1 - \tilde{z}$, consistent with the dimension of $\mathbf{v}$. Further, assume the cross-helicity flux ϵ_c to be the only relevant (forward) flux in the problem. Demanding scale-independence of ϵ_c, we obtain

$$2a + \alpha = \tilde{z}. \tag{23}$$

Magnetic energy spectrum $E_b(k) \sim k^{-1-2y} \sim k^{-1-2a-2\alpha}$. Thus, $a + \alpha = 1/2$ would give $E_b(k) \sim k^{-2}$. Together with the conditions on a, α and $\tilde{z}$, this implies $\alpha = 1/4$. Of course, for other values of α, the scaling of $E_b(k)$ will change. The scaling analysis cannot, however, precisely evaluate the scaling exponents a, α, and $\tilde{z}$. We note that the magnetic energy per unit volume V

$$\frac{1}{V}\int b^2(\mathbf{x})\, d^3x \propto \int dk k^2 \langle|\mathbf{b}(\mathbf{k})|^2\rangle \equiv \int B(k) dk. \tag{24}$$

Therefore, on dimensional ground

$$B(k) \sim l^{3+2\alpha-2\tilde{z}}. \tag{25}$$

Assuming $B(k)$ is to be constructed from the cross-helicity flux $\epsilon_c \sim \partial(\mathbf{v}\cdot\mathbf{b})/\partial t$, we write

$$B(k) \sim \left[\frac{vb}{t}\right]^\beta l^\gamma \tag{26}$$

for arbitrary $\tilde{z}$ on dimensional ground. Now comparing Eqs. (25) and (26), we find $\beta = 2/3$ and

$$\gamma = \frac{5}{3} + \frac{4\alpha}{3}. \tag{27}$$

The weak turbulence scaling exponent $\gamma = 2$ (i.e., $E_b(k) \sim k^{-2}$) is obtained for $\alpha = 1/4$, as we have found above. In addition, we find $E_b(k)$ scales as $\epsilon_c^{2/3}$, a result that can be tested in directed numerical simulations of 3D MHD equations. The scaling of $E_v(k)$ will be different from $E_b(k)$. We do not comment on that here. Thus, the generalized scaling theory indeed predicts k^{-2} scaling by $E_b(k)$ as one possible solution for scaling but does not rule out other scaling solutions and, more interestingly, generally allows for different scaling by $E_b(k)$ and $E_v(k)$.

2. Alfvén waves: Effects of anisotropy on scaling

Most natural realizations of a plasma usually contain a mean magnetic field, e.g., tokamak plasma and solar wind [46]. Thus, it is pertinent to consider now how a mean magnetic field can alter the scaling behavior elucidated above. We choose the mean magnetic field $\mathbf{B}_0$ to be along the z axis, which leads to additional linear terms in Eqs. (11) and (12):

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla p + (\mathbf{b}\cdot\nabla)\mathbf{b} + B_0\frac{\partial \mathbf{b}}{\partial z} + \nu\nabla^2\mathbf{v} + \mathbf{f}_v \tag{28}$$

and

$$\frac{\partial \mathbf{b}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{b} = (\mathbf{b}\cdot\nabla)\mathbf{v} + B_0\frac{\partial \mathbf{v}}{\partial z} + \mu\nabla^2\mathbf{b} + \mathbf{f}_b. \tag{29}$$

The linear terms in Eqs. (28) and (29) allow for underdamped propagating waves, known as the *Alfvén waves* [5] in the literature, with a dispersion

$$\omega \propto B_0 k_\parallel + O(k^2), \tag{30}$$

in the long wavelength limit, where ω is a Fourier frequency and $k_\parallel$ is the z component of $\mathbf{k}$; $\mathbf{k} = (\mathbf{k}_\perp, k_\parallel)$, $\mathbf{k}_\perp = (k_x, k_y)$.

We continue to assume large-scale forcings and the system to have negligible helicity.

Noting that a nonzero B_0 necessarily makes the system anisotropic, we need to generalize the scaling ansatz to account for anisotropy. In particular, we now expect spatially anisotropic scaling with the length scales in the xy plane to scale different from those along the z axis. Without any loss of generality, we set

$$\mathbf{x} \to l_\perp \mathbf{x},\, z \to l_\parallel z,\; t \to l_\perp^{\tilde{z}},\; \mathbf{v} \to l_\perp^{a}\mathbf{v},\; \mathbf{b} \to l_\perp^{y}\mathbf{b}, \tag{31}$$

where $l_\perp$ is a length scale in the xy plane [47]. We further set length scale along the z axis $l_\parallel \sim l_\perp^{\xi}$ that controls the relative scaling between the xy plane and the z axis; for $\xi \neq 1$, the system is anisotropic. Here $\mathbf{x} = (x, y)$ is the in-plane coordinate. We also define $\nabla_\perp = (\partial_x, \partial_y)$, $\mathbf{v}_\perp = (v_x, v_y)$, $\mathbf{b}_\perp = (b_x, b_y)$. Furthermore, we ignore v_z and b_z, in comparison with $\mathbf{v}_\perp$ and $\mathbf{b}_\perp$, respectively (see below). In addition to introducing anisotropy, for a nonzero B_0, there should be competition between the propagating Alfvén waves and the nonlinear terms in Eqs. (28) and (29). The interplay between this competition and the anisotropy controls the ensuing scaling behavior, as we show below. It is evident from Eqs. (28) and (29) that as B_0 increases, the nonlinear cascades are progressively weakened relative to the strength of the propagating modes. It is thus convenient to introduce a phenomenological dimensionless parameter

$$M = \frac{B_0^2}{s^2}, \tag{32}$$

where s is the typical magnitude of v, b in the inertial range [48]. Depending upon the magnitude of B_0, there are three possible physically distinct regimes: $M \ll 1$, $M \sim \mathcal{O}(1)$ and $M \gg 1$.

For small $M \ll 1$, the nonlinear terms dominate in the inertial range and the system becomes effectively isotropic. Unsurprisingly, the kinetic and magnetic energy spectra should then scale as $k^{-5/3}$ in the inertial range, in accordance with the K41 prediction.

For a stronger B_0, when $M \sim \mathcal{O}(1)$ the linear propagating terms and the nonlinear terms in Eqs. (28) and (29) are comparable in the inertial range, known as strong turbulence. We then balance

$$(\mathbf{v}_\perp \cdot \nabla_\perp)\mathbf{v}_\perp \sim B_0 \partial_z \mathbf{b}_\perp \Rightarrow 2a - 1 = -\xi + y. \tag{33}$$

Next, we balance the in-plane nonlinear terms in Eq. (28):

$$(\mathbf{v}_\perp \cdot \nabla_\perp)\mathbf{v}_\perp \sim (\mathbf{b}_\perp \cdot \nabla_\perp)\mathbf{b}_\perp \Rightarrow a = y. \tag{34}$$

Notice that this automatically gives

$$\frac{\partial \mathbf{b}_\perp}{\partial t} \sim B_0 \frac{\partial}{\partial z}\mathbf{v}_\perp. \tag{35}$$

Using the dispersion relation Eq. (30) we find

$$\tilde{z} = \xi \Rightarrow a = y \tag{36}$$

eventually. Based on the physical arguments enunciated above, we continue to impose scale-independence of the magnetic or kinetic energy flux in the inertial range independent of B_0, which yields

$$2y = \tilde{z} = 2a. \tag{37}$$

Therefore, we get

$$\tilde{z} = \tfrac{2}{3} = \xi,\;\; a = \tfrac{1}{3} = y, \tag{38}$$

see, e.g., Ref. [49]. Notice that the dynamic exponent $\tilde{z}$ is unchanged from its value from the isotropic case ($B_0 = 0$). Also, $\tilde{z} = \xi$ keeps Eq. (30) unchanged under rescaling.

Enumeration of the scaling of the energy spectra requires extending the logic outlined above to anisotropic situation. Since $a = y$, we already expect identical scaling by the kinetic and magnetic energy spectra in the inertial range. We define Fourier transforms

$$\mathbf{s}_\perp(\mathbf{k}_\perp, k_\parallel, t) \sim \int \mathbf{s}_\perp(\mathbf{x}_\perp, z, t) d^2x_\perp dz \sim l_\perp^{a+2+\xi} \sim k_\perp^{-a-2-\xi}, \tag{39}$$

where $s = v$ or b. Now define

$$\langle \mathbf{s}_\perp(\mathbf{k}, t) \cdot \mathbf{s}_\perp(\mathbf{k}', t)\rangle = F_a(k)\delta(\mathbf{k}_\perp + \mathbf{k}'_\perp)\delta(k_\parallel + k'_\parallel), \tag{40}$$

where $F_a(k)$ is related to $E_a(k)$ (see below). Under scaling Eq. (31), we obtain

$$F_s(k_\perp, k_\parallel) \sim k_\perp^{-2a-2-\xi}. \tag{41}$$

We can now use $k_\parallel \sim k_\perp^{\xi}$ and obtain the one-dimensional energy spectra as follows:

$$E_v(k_\perp) \sim E_b(k_\perp) \sim k_\perp^{-5/3}, \tag{42}$$

$$E_v(k_\parallel) \sim E_b(k_\parallel) \sim k_\parallel^{-2}, \tag{43}$$

in the inertial range; see, e.g., Ref. [49]. Results Eq. (43) can be obtained as follows: we note that the scaling $\mathbf{v}_\perp \sim l_\perp^{a} \sim l_\parallel^{a/\xi}$, $\mathbf{b}_\perp \sim l_\perp^{y} \sim l_\parallel^{y/\xi}$. Scaling of one-dimensional spectra $E_a(k_\parallel)$ follows from the equality

$$E_{\text{tot},s} = \int E_s(k_\parallel)\, dk_\parallel = \int E_s(k_\perp)\, dk_\perp, \tag{44}$$

where $s = v, b$ and subscript tot implies total energy (kinetic or magnetic). Then dimensionally,

$$E_s(k_\parallel) \sim \left[\frac{E_s(k_\perp)dk_\perp}{dk_\parallel}\right]. \tag{45}$$

This gives $E_v(k_\parallel) \sim k_\parallel^{-2a/\xi-1}$, $E_b(k_\parallel) \sim k_\parallel^{-2y/\xi-1}$, giving Eq. (43) with $a = y = 1/3$, $\xi = 2/3$. Thus, both $E_v(k_\perp)$ and $E_b(k_\perp)$ scale with $k_\perp$ according to the K41 result, but $E_v(k_\parallel)$ and $E_b(k_\parallel)$ scale *differently* with $k_\parallel$.

We note that the result $\xi = 2/3$ can be interpreted as singular renormalization of B_0 in the long wavelength limit: We write the Alfvén wave term

$$B_0 k_\parallel \sim B_0 k_\perp^{2/3} \sim B_0(k_\perp)k_\perp, \tag{46}$$

with $B_0(k_\perp) \sim k_\perp^{-1/3}$. This is reminiscent of the result in Ref. [25].

We now provide *a posteriori* justification for neglecting v_z, b_z the z components of $\mathbf{v}$ and $\mathbf{b}$ in the above analysis. Notice that under scaling Eq. (31)

$$\mathbf{s}_\perp \sim l_\perp^{1/3}, \quad s_z \sim l_\perp^{0}. \tag{47}$$

The latter scaling essentially follows by demanding that different nonlinear terms involving s_z and $\mathbf{s}_\perp$ in Eq. (28) or

Eq. (29) scale in the same way. Thus, in the long wavelength limit $l_\perp \to \infty$, $\mathbf{s}_\perp \equiv (s_x, s_y) \gg s_z$. Hence, E_v and E_b are dominated by $\mathbf{v}_\perp$ and $\mathbf{b}_\perp$ in the long wavelength limit. This justifies our neglecting v_z, b_z in the above analysis.

We further expect the scaling behavior to change substantially for much stronger B_0, i.e., with $M \gg 1$ for which the balances (in a scaling sense) used in Eqs. (33) and (35) should breakdown, with the linear Alfvén wave terms dominating over the nonlinear terms in Eqs. (28) and (29) even in the inertial range. For simplicity we do not distinguish between s_z and $\mathbf{s}_\perp$. We note that in the limit of a very large B_0, the nonlinear terms should be suppressed. This in turn should lead to suppression of the energy fluxes, both kinetic and magnetic. In the limit of very large M, we express the energy fluxes ϵ_s phenomenologically (in a dimensional/scaling sense) as

$$\epsilon_s \sim \frac{\partial s^2}{\partial t}\frac{1}{M} \tag{48}$$

to the leading order in $1/M$; $s = v, b$. Imposing scale-independence of the fluxes then yields

$$\tilde{z} = 4a = 4y. \tag{49}$$

If we ignore anisotropy and consider $\xi = 1$, then the dispersion relation Eq. (30) yields

$$\tilde{z} = 1. \tag{50}$$

This together with Eq. (49) gives

$$a = y = \tfrac{1}{4}. \tag{51}$$

Proceeding as above, this implies

$$E_{v,b}(k) \sim k^{-3/2}, \tag{52}$$

which is the well-known IK spectra [28].

The main criticism of the IK prediction is that despite having $M \gg 1$, anisotropy is ignored, which is not physically acceptable. We will now discuss how the scaling analysis for $M \gg 1$ may be affected by anisotropy. We first note that while for $M \gg 1$ nonlinear terms are expected to be suppressed, dispersion relation Eq. (30) in fact suggests that this suppression is ineffective for $k_\parallel^2 \ll k_\perp^2$. To account for this anisotropic suppression of the fluxes, we phenomenologically modify Eq. (48) for the flux to (again in a dimensional/scaling sense)

$$\epsilon_s \sim \frac{\partial s^2}{\partial t}\frac{l_\parallel^2}{M l_\perp^2}, \tag{53}$$

valid for $k_\perp^2 \lesssim k_\parallel^2$. Now demanding scale-independence of the fluxes and using relevant timescale $\sim l_\parallel$, we find (in a scaling sense)

$$s \sim l_\perp^{1/2} l_\parallel^{-1/4}, \tag{54}$$

$s = v$, b. It is believed that for $M \gg 1$, the nonlinear interactions leading to energy cascades predominantly take place only in the xy plane (i.e., in the plane normal to $B_0\hat{z}$) for very strong B_0 [33], since the propagating Alfvén wave terms dominate along $\hat{z}$ directions. This gives

$$E_{v,b}(k_\perp, k_\parallel) \sim k_\perp^{-2} k_\parallel^{-1/2}, \tag{55}$$

corresponding to *weak turbulence limit*; see Refs. [29,33,50,51].

C. 3D Hall MHD

We now study scaling in Hall MHD (HMHD), first the isotropic case, then the corresponding anisotropic one.

1. Isotropic 3D HMHD

In 3D Hall MHD (HMHD), one generalizes the ordinary 3D MHD equations by including the Hall contribution in the form of the Ohm's law:

$$\mathbf{E} + \mathbf{v} \times \mathbf{b} - \frac{\mathbf{J} \times \mathbf{b}}{\rho_e} = \mu \mathbf{J}, \tag{56}$$

where ρ_e is the electron charge density [8,9]. This generalizes Eq. (12) to

$$\frac{\partial \mathbf{b}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{b}) - d_I \nabla \times [(\nabla \times \mathbf{b}) \times \mathbf{b}] + \mu \nabla^2 \mathbf{b} + \mathbf{f}_b, \tag{57}$$

where d_I is the ion inertial length [8,9]. We consider a vanishing mean magnetic field, i.e., $\langle \mathbf{b} \rangle = 0$. Velocity $\mathbf{v}$ continues to obey Eq. (11). Total energy E remains a conserved quantity in HMHD in its ideal or inviscid limit [9,52]. We ignore helicity for simplicity.

Evidently, for length scales $l \gg d_I$ the d_I-term is irrelevant compared to the first term on the right-hand side of Eq. (57) for a sufficiently large, and the scaling behavior for ordinary isotropic 3D MHD ensues. In the opposite limit, the d_I-term is important. This range of scales is called the dispersion range [13]; this does not exist for ordinary 3D MHD. We focus on the latter case, for which it suffices to ignore the λ_1-term [53]. While there is no symmetry principle that prohibits renormalization of d_I in a perturbative RG framework, considering $d_I \sim \mathcal{O}(1)$ and hence $l \lesssim \mathcal{O}(1)$, any perturbative corrections to d_I stemming from the dispersion range should be "small" and hence ignored in what follows below.

We use scaling *ansatz* as defined by Eqs. (2) and (13). Due to the rather different forms of the nonlinear terms in Eqs. (11) and (57), same scaling of $\mathbf{v}$ and $\mathbf{b}$ is no longer expected. In the dispersion range, $d_I \nabla \times [(\nabla \times \mathbf{b}) \times \mathbf{b}]$ is the dominant nonlinear term in the right-hand side of Eq. (57). The dynamics of $\mathbf{b}$ is essentially controlled by the d_I-nonlinear term in the dispersion range. We then balance

$$\frac{\partial \mathbf{b}}{\partial t} \sim \nabla \times [(\nabla \times \mathbf{b}) \times \mathbf{b}] \Rightarrow y = 2 - \tilde{z}. \tag{58}$$

We continue to use $a = 1 - \tilde{z}$. Between the kinetic and magnetic fluxes, which flux is to be assumed to be scale-independent is crucial. We notice that demanding scale-independent magnetic flux yields

$$2y = \tilde{z} \Rightarrow y = \tfrac{2}{3},\ \tilde{z} = \tfrac{4}{3}. \tag{59}$$

This, however, gives $a = -1/3 < 0$, which is clearly unphysical. We therefore discard this. In contrast, scale-independence of the kinetic energy flux yields

$$2a = \tilde{z} \Rightarrow a = \tfrac{1}{3},\ \tilde{z} = \tfrac{2}{3},\ y = \tfrac{4}{3}. \tag{60}$$

The scaling exponents Eq. (60) keeps the whole of Eq. (57) scale-invariant, but breaks the scale-invariance of Eq. (11). More importantly, what are the physical dynamic exponents for **v** and **b** here that control the renormalization of ν and μ? Imposition of the scale-invariance of the kinetic energy flux ensures that the kinematic viscosity ν indeed picks up a scale-dependence $\sim l^{2-\tilde{z}}$. Since this choice of $\tilde{z}$ does not keep the magnetic flux scale-independent, we cannot say the same for the magnetic viscosity μ, leaving the question of the physical dynamic exponent for **b** unresolved. This can, however, be settled by allowing for *two* different dynamic exponents $\tilde{z}_v$ and $\tilde{z}_b$, respectively, for **v** and **b**, such that scale-independence can be imposed on each of the kinetic and magnetic energy fluxes separately. This automatically yields

$$2a = \tilde{z}_v \Rightarrow a = \tfrac{1}{3},\ \tilde{z}_v = \tfrac{2}{3},$$
$$2y = \tilde{z}_b \Rightarrow y = \tfrac{2}{3},\ \tilde{z}_b = \tfrac{4}{3}. \tag{61}$$

Thus, we obtain *weak dynamic scaling* [35]. That scale-independence of the kinetic and magnetic energy fluxes should imply two dynamic exponents $\tilde{z}_v$ and $\tilde{z}_b$ is in agreement with Ref. [44]. To our knowledge, there has been no systematic measurements of the timescales of **v** and **b** fluctuations. This may, however, be measured in numerically, e.g., by calculating time-dependent correlation functions of **v** and **b** in pseudospectral methods [54]. With $\tilde{z}_b > \tilde{z}_v$, magnetic fluctuations are longer lived. Hence, at sufficiently long timescales larger than $1/k^{\tilde{z}_v}$ but smaller than $1/k^{\tilde{z}_b}$, the **v** fluctuations die out, effectively making the **b** fluctuations autonomous. However, at shorter timescales, the magnetic field fluctuations will appear frozen in time, and **v** effectively fluctuates in a given background of spatially nonuniform but frozen in time **b**.

Following the logic outlined in Sec. II B 1 we can now obtain the scaling of the kinetic and magnetic energy spectra valid over length scales smaller than d_I. We find

$$E_v(k) \sim k^{-1-2a} \sim k^{-5/3},\ E_b(k) \sim k^{-1-2y} \sim k^{-7/3}. \tag{62}$$

Thus, in this length-scale the magnetic energy spectrum is distinctly steeper than that kinetic energy spectra. Last, in the inertial range with length scale $l \gg d_I$, unsurprisingly the scaling of $E_v(k)$ and $E_b(k)$ are identical to those in 3D MHD.

2. Anisotropy effects

We now study the effects of spatial anisotropy brought in by a mean magnetic field B_0, assumed to be along the z direction. Equation (57) now generalizes to

$$\frac{\partial \mathbf{b}}{\partial t} + \lambda_1(\mathbf{v}\cdot\nabla)\mathbf{b} = d_I(\mathbf{b}\cdot\nabla)\mathbf{v} - d_I\nabla\times[(\nabla\times\mathbf{b})\times\mathbf{b}] - d_I B_0\partial_z\nabla\times\mathbf{b} + \mu\nabla^2\mathbf{b} + \mathbf{f}_b. \tag{63}$$

Velocity **v** follows Eq. (28). Similar to the Alfvén waves, Eqs. (28) and (63) have circularly polarized whistler and cyclotron modes having dispersion of the form

$$\omega \propto kk_\parallel \tag{64}$$

for a large enough d_I [9].

We notice that with increasing B_0, the nonlinear terms in Eq. (63) are progressively suppressed. Thus, as in 3D anisotropic ordinary MHD, we expect different scaling behavior for small or large B_0. We begin considering the situation when the linear and the nonlinear terms balance. This is the direct analog of the strong limit of 3D anisotropic ordinary MHD turbulence. Given our results obtained in Secs. II B 2 and II C 1, we anticipate both anisotropy and weak dynamic scaling for length scales smaller than d_I with $B_0 \neq 0$. We proceed by using the scaling *ansatz* defined in Eq. (31). Similar to Sec. II B 2, we ignore v_z and b_z in what follows below. First, we balance

$$\frac{\partial \mathbf{b}_\perp}{\partial t} \sim d_I\nabla\times[(\nabla\times\mathbf{b}_\perp)\times\mathbf{b}_\perp] \sim \partial_z\nabla\times\mathbf{b}_\perp \Rightarrow y = 2-\tilde{z},\ \tilde{z} = 1+\xi. \tag{65}$$

Here we have implicitly assumed that $\xi < 1$, and hence $k_\perp$ dominates over $k_\parallel$ in the dispersion regime. For strong B_0 following the logic developed above and from Eq. (11)

$$(\mathbf{v}_\perp\cdot\nabla_\perp)\mathbf{v}_\perp \sim (\mathbf{v}_\perp\cdot\nabla_\perp)\mathbf{v}_\perp \Rightarrow a = 1-\tilde{z}, \tag{66}$$

$$\partial_t\mathbf{v}_\perp \sim B_0\partial_z\mathbf{b}_\perp \Rightarrow a-\tilde{z} = y-\xi. \tag{67}$$

This is the analog of the strong limit of 3D anisotropic MHD. Assuming scale-independent kinetic energy flux, we obtain by using Eq. (66)

$$a = \tfrac{1}{3},\ \tilde{z} = \tfrac{2}{3},\ y = \tfrac{4}{3}. \tag{68}$$

This unexpectedly gives $\xi = \tilde{z} - 1 < 0$, which is clearly unphysical. However, if we use Eq. (67) we get

$$\xi = \tilde{z} - a + y = \tfrac{5}{3} > 1 \Rightarrow \tilde{z} > 2, \tag{69}$$

which is rather unexpected. Inspired by our scaling analysis for isotropic HMHD, we try to resolve this by assuming two dynamic exponents $\tilde{z}_v$ and $\tilde{z}_b$, respectively, for **v** and **b**. As in the isotropic case, we obtain $\tilde{z}_v$ by imposing scale-independence of the kinetic energy spectrum. We find

$$\tilde{z}_v = \tfrac{2}{3},\ a = \tfrac{1}{3}. \tag{70}$$

However, using Eq. (65) together with the condition of scale-independent magnetic energy spectrum, we get

$$y = \tfrac{2}{3},\ \tilde{z}_b = \tfrac{4}{3},\ \xi = \tfrac{1}{3}. \tag{71}$$

Thus, weak dynamic scaling is obtained. Further $\xi > 0$ justifies our neglecting v_z and b_z in the above analysis; see discussions in Sec. II B 2. The existence of two dynamics exponents $\tilde{z}_v$ and $\tilde{z}_b$ is actually consistent with the original idea of Chandrasekhar [44], which ensures scale-independence of both the kinetic and magnetic energy flux. As in isotropic 3D HMHD, $\tilde{z}_b > \tilde{z}_v$, implying magnetic fields to fluctuate independent of the velocity fields for sufficiently large timescales. Similar to its isotropic analog, it would be interesting to verify this numerically. It is now straightforward to obtain the scaling of the energy spectra. We obtain

$$E_v(k_\perp) \sim k_\perp^{-5/3}, \tag{72}$$

$$E_b(k_\perp) \sim k_\perp^{-7/3}; \tag{73}$$

see also Ref. [51]. Analogously, we find $E_v \sim k_\parallel^{-3}$, $E_b \sim k_\parallel^{-5}$. Thus, in the dispersion range both $E_v(k)$ and $E_b(k)$ scale with $k_\perp$ in ways same their respective scaling with k in the isotropic case, where as their scaling with $k_\parallel$ are markedly different.

For very large B_0, for which the d_I-nonlinear term in Eq. (63) is strongly suppressed, the scaling of $E_b(k_\perp)$ is expected to change from Eq. (73). Several other possibilities for the scaling of $E_b(k_\perp)$ can then exist. For instance, for strong B_0 if we assume that the propagating mode sets the dynamic exponent $\tilde{z}_b$ and $k_\parallel$ and $k_\perp$ scale the same way, then $\tilde{z}_b = 2$. The condition of scale-independence of the magnetic flux yields $2y = \tilde{z}_b$, giving $y = 1$. This then yields $E_b(k_\perp) \sim k_\perp^{-3}$, assuming the energy cascade is confined to the plane normal to $\mathbf{B}_0$; see, e.g., Ref. [55].

If we now account for spatial anisotropy (should be important for strong B_0) and phenomenologically express the scale-independence of the magnetic energy flux as

$$\epsilon_b \sim \frac{\partial b^2}{\partial t} \frac{l_\parallel^2}{M_b l_\perp^2} \sim l^0, \tag{74}$$

then $4y = \tilde{z}_b$, where $M_b = B_0^2/b^2$ with b being the typical magnitude of the magnetic fields in the dispersion regime. Noting that the timescale $\tau \sim l_\perp l_\parallel$, being controlled by the propagating mode, we have the anisotropic scaling of $\mathbf{b}$:

$$\mathbf{b} \sim l_\perp^{3/4} l_\parallel^{-1/4}. \tag{75}$$

This in turn gives

$$E_b(k) \sim k_\perp^{-5/2} k_\parallel^{-1/2}, \tag{76}$$

see Refs. [51,55].

D. 3D electron MHD

We now analyze the scaling behavior of 3D EMHD—first the isotropic case, then the anisotropic version.

1. Isotropic 3D EMHD

We first study the scaling in 3D isotropic electron MHD (EMHD). The EMHD equation for the magnetic field in the absence of any mean magnetic field is [11,12]

$$\frac{\partial}{\partial t}(\mathbf{b} - \lambda_e^2 \nabla^2 \mathbf{b}) = -g \nabla \times \left[(\nabla \times \mathbf{b}) \times \left(\mathbf{b} - \lambda_e^2 \nabla^2 \mathbf{b}\right)\right] + \mu \nabla^2 \mathbf{b} - \frac{\nu_e c^2}{\omega_{pe}^2} \nabla^4 \mathbf{b} + \mathbf{f}_b. \tag{77}$$

Here, $\nu_e c^2/\omega_{pe}^2$ is a hyperviscosity. Although there are no symmetry arguments that prevent renormalization of the coupling g, we ignore such issues here considering the fact that Eq. (77) is expected to be valid for sufficiently small scales for which fluctuation corrections should be small. We ignore fluctuations in the density for simplicity.

We introduce the following scaling:

$$\mathbf{x} \to l\mathbf{x}, \;\; t \to l^{\tilde{z}} t, \;\; \mathbf{b} \to l^a. \tag{78}$$

For $\lambda_e^2/l^2 \ll 1$, the g-nonlinear term in Eq. (77) reduces to the standard Hall term in Eq. (57). We therefore focus on the opposite limit $\lambda_e^2/l^2 \gg 1$ and ignore the fourth-order hyperviscosity term in Eq. (77). Equation (77) then reduces to

$$\frac{\partial}{\partial t} \lambda_e^2 \nabla^2 \mathbf{b} = g \nabla \times [\nabla^2 \mathbf{b} \times (\nabla \times \mathbf{b})] + \mu \nabla^2 \mathbf{b} + \mathbf{f}_b. \tag{79}$$

Noting that EMHD description applies in small scales, demanding scale-invariance we balance

$$\frac{\partial}{\partial t} \nabla^2 \mathbf{b} \sim \nabla \times [\nabla^2 \mathbf{b} \times (\nabla \times \mathbf{b})] \tag{80}$$

in the steady state, giving $y = 2 - \tilde{z}$. Constancy of the magnetic energy flux then implies

$$2y = \tilde{z} \Rightarrow y = \tfrac{2}{3}, \;\; \tilde{z} = \tfrac{4}{3}, \tag{81}$$

unchanged from isotropic HMHD. It is now straightforward to obtain

$$E_b(k) \sim k^{-7/3}, \tag{82}$$

valid for length scales appropriate for EMHD, and identical to the scaling of $E_b(k)$ in the dispersion range of 3D HMHD.

2. Anisotropic effects

In the presence of a mean magnetic field B_0 along the z direction, Eq. (79) takes the form

$$\frac{\partial \mathbf{b}}{\partial t} = -g \nabla \times [(\nabla \times \mathbf{b}) \times \mathbf{b}] + g B_0 \partial_z \nabla \times \mathbf{b} + \mu \nabla^2 \mathbf{b} + \mathbf{f}_b. \tag{83}$$

This leads to a dispersion (ignoring the viscous term)

$$\omega \sim k k_\parallel. \tag{84}$$

Now, introduce scaling

$$\mathbf{x} \to l_\perp \mathbf{x}, \, z \to l_\parallel z, \; t \to l_\perp^{\tilde{z}}, \; \mathbf{b} \to l_\perp^y \mathbf{b}, \tag{85}$$

where $l_\perp$ is a length scale in the xy plane. As before, we further set length scale along the z axis $l_\parallel \sim l_\perp^\xi$ that controls the relative scaling between the xy plane and the z axis; for $\xi \neq 1$, the system is anisotropic. It is now easy to extract the scaling exponents by directly following the logic outlined for 3D anisotropic HMHD. We study the strong B_0 case when the linear and the nonlinear terms balance in the dispersion range. We find

$$E_b(k_\perp) \sim k_\perp^{-7/3}, \;\; E_b(k_\parallel) \sim k_\parallel^{-5}, \; \xi = \tfrac{1}{3}, \; \tilde{z}_b = \tfrac{2}{3}. \tag{86}$$

It is not a surprise that the above scaling in Eq. (86) is identical to the scaling obtained for $E_b(k)$, given the similarity between Eqs. (79) and (63) with $\mathbf{v} = 0$. Our results are actually quite close to those found in other studies. For instance, Refs. [13,56] indeed found the magnetic energy spectra to scale as $k_\perp^{-7/3}$; the anisotropy exponent $\xi = 1/3$ and $\tilde{z}_b = 4/3$. This gives credence to our scaling analysis.

Similar to anisotropic 3D HMHD, the magnetic energy spectrum $E_b(k)$ in anisotropic 3D EMHD can display scaling k^{-3} and $k_\perp^{-5/2} k_\parallel^{-1/2}$ under similar conditions.

III. SUMMARY AND OUTLOOK

We have here revisited the scaling of the magnetic and kinetic energy spectra in the various regimes of incompressible

3D MHD by developing a scaling theory. We obtain $k^{-5/3}$ scaling for both the kinetic and magnetic energy spectra in ordinary isotropic 3D MHD. We further discuss the possibility of k^{-2} spectra in isotropic 3D MHD. The scaling theory predicts that the nature of scaling of the energy spectra in the anisotropic 3D MHD can be diverse, depending upon the strength of the mean magnetic field B_0, a feature that persists in anisotropic 3D Hall MHD and anisotropic 3D EMHD as well. For instance, when the magnitude of B_0 is such that the linear Alfvén wave terms balance the nonlinear terms in the inertial range, the scaling of both the kinetic and magnetic energy spectra with respect to $k_\perp$ is still given by the K41 result but takes a different power law when expressed in terms of $k_\parallel$. This is associated with an anisotropy exponent $\xi = 2/3$ that relates the scaling of $k_\parallel$ with $k_\perp$. The scaling analysis also yields the IK scaling for strong B_0 if the spatial anisotropy is ignored. Interestingly, however, if one uses the scale-dependent version of B_0 in the IK scaling, one immediately gets back the K41 result. For very large B_0, for which the linear Alfvén wave terms dominate over the nonlinear terms in the inertial range, we find $E_v(k_\perp, k_\parallel) \sim E_b(k_\perp, k_\parallel) \sim k_\perp^{-2} k_\parallel^{-1/2}$. These are in agreement with the existing results. Independent of any spatial isotropy, we always get the same dynamic exponent for **v** and **b** in ordinary MHD, corresponding to strong dynamic scaling. High resolution numerical studies should complement the scaling results, settle the controversies surrounding scaling in 3D MHD.

For HMHD, we predict that the scaling of $E_b(k)$ in the dispersion range should be steeper than that for $E_v(k)$. This holds with or without a mean magnetic field. More interestingly, we predict two different dynamic exponents for **v** and **b**. Since we find $\tilde{z}_b > \tilde{z}_v$, **v** fluctuations decay much faster than the **b** fluctuations. As a result, **b** fluctuations effectively appear as frozen fields in the dynamics of **v** over length scales belonging to the dispersion range, where as for sufficiently large timescales, the dynamics of **b** fluctuations should be independent of the **v** fluctuations in the same regime, and hence reduces to 3D EMHD. This conclusion remains true whether or not there is a mean magnetic field. For 3D EMHD, the predictions from our analysis for **b** agrees with the same in 3D HMHD, which are again observed in relevant numerical studies [13,56]. Our scaling analysis predicts $-7/3$ scaling for $E_b(k)$ in isotropic 3D EMHD and in the dispersion regime of isotropic 3D HMHD. In the corresponding anisotropic cases, $E_b(k_\perp)$ still scales as $k_\perp^{-7/3}$, but $E_b(k_\parallel)$ scales as $k_\parallel^{-5}$ with respect to $k_\parallel$. We also identify an anisotropy exponent $-1/3$, different from its value in 3D anisotropic MHD. Notice that the the anisotropy exponent in 3D HMHD is half of that in 3D ordinary MHD. This means that the anisotropic effects and effective two-dimensionalization is stronger in HMHD than ordinary MHD. In the limit of very strong magnetic B_0, we obtain $E_b(k_\perp, k_\parallel) \sim k_\perp^{-5/2} k_\parallel^{-1/2}$, which is the analog of weak MHD turbulence in 3D HMHD. Scaling of the magnetic energy spectrum in 3D EMHD is found to be same as in 3D HMHD. In this context, we note that a recent study on tabletop laser plasma revealed a $k^{-7/3}$ for high wave vectors at late times, indicative of an EMHD or HMHD like behavior [31]. It may be noted that Refs. [57,58] predicted somewhat different scaling for the energy spectra. It will be interesting to see how our scaling approach may be extended or modified appropriately to account for the results in Refs. [57,58].

It is now well-accepted that the universal properties of fully developed turbulence—fluid or MHD—cannot be characterized by the two-point correlation functions (equivalently by the energy spectra) alone. Instead, one needs to calculate a hierarchy of *multiscaling* exponents for different order structure functions (including the two point ones) [3,13,59,60]. Our scaling analysis is of course not adequate to capture multiscaling. Nonetheless, different *multiscaling universality classes* of 3D MHD should be associated to different scaling regimes (e.g., ordinary MHD or HMHD) of MHD turbulence elucidated here. Thus, our scaling analysis should be helpful in delineating or classifying the possible universal multiscaling properties of MHD turbulence.

The arguments behind our scaling analysis are sufficiently general, and should be applicable to a wider range of systems. Indeed, it will be interesting to apply these to related systems, e.g., compressible turbulence, rotating turbulence, turbulence in a binary fluid above and below the miscibility transition point, two-dimensional fluid, and MHD turbulence. We hope our work will trigger new studies for these systems along the lines developed here.

Note added in proof. We recently came to know of the works by Dallas and Alexakis on freely evolving MHD turbulence, where they reported a variety of scaling behavior by the energy spectra, e.g., k^{-2} and $k^{-5/3}$. See, e.g., Ref. [61]. These generally complement our studies for forced MHD turbulence.

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[1] K. R. Sreenivasan and R. A. Antonia, Annu. Rev. Fluid Mech. **29**, 435 (1997); S. K. Dhar *et al.*, Pramana J. Phys. **48**, 325 (1997).
[2] A. N. Kolmogorov, C. R. Acad. Sci. USSR **30**, 301 (1941).
[3] U. Frisch, *Turbulence: The Legacy of A. N. Kolmogorov* (Cambridge University Press, Cambridge, 1995).
[4] Detailed experiments and numerical studies indicate small correction to the K41 result. We ignore this issue here.
[5] D. Montgomery, in *Lecture Notes on Turbulence*, edited by J. R. Herring and J. C. McWilliam (World Scientific, Singapore, 1989); D. Biskamp, in *Nonlinear Magnetohydrodynamics*, edited by W. Grossman *et al.* (Cambridge University Press, Cambridge, 1993).
[6] J. D. Jackson, *Classical Electrodynamics*, 2nd ed. (Wiley Eastern, New Delhi, 1975).
[7] A. Raichaudhurim, *The Physics of Fluids and Plasmas* (Cambridge University Press, Cambridge, 1998).

[8] F. Sahraoui, S. Galtier, and G. Belmont, J. Plasma Phys. **73**, 723 (2007).
[9] S. Galtier, J. Plasma Phys. **72**, 721 (2006).
[10] V. Krishan and S. M. Mahajan, Phys. Scr. **T116**, 128 (2005).
[11] D. Biskamp *et al.*, Phys. Plasmas **6**, 751 (1999).
[12] B. Eliasson and P. K. Shukla, Phys. Rev. Lett. **99**, 205005 (2007).
[13] J. Cho and A. Lazarian, Astrophys. J. **701**, 236 (2009).
[14] D. A. Shalybkov and V. A. Urpin, Astron. Astrophys. **321**, 685 (1997); R. Hollerbach and G. Rüdiger, Mon. Not. R. Astron. Soc. **347**, 1273 (2004); J. A. Pons and U. Geppert, Astron. Astrophys. **470**, 303 (2007); R. Perna and J. A. Pons, Astrophys. J. **727**, L51 (2011); Y. Kojima and S. Kisaka, Mon. Not. R. Astron. Soc. **421**, 2722 (2012); D. Viganò, J. A. Pons, and J. A. Miralles, Comput. Phys. Commun. **183**, 2042 (2012); K. N. Gourgouliatos and A. Cumming, Phys. Rev. Lett. **112**, 171101 (2014); P. Marchant, A. Reisenegger, J. Alejandro Valdivia, and J. H. Hoyos, Astrophys. J. **796**, 94 (2014); T. S. Wood and R. Hollerbach, Phys. Rev. Lett. **114**, 191101 (2015).
[15] M. E. Mandt, R. E. Denton, and J. F. Drake, Geophys. Res. Lett. **21**, 73 (1994); K. Avinash, S. V. Bulanov, T. Esirkepov, P. Kaw, F. Pegoraro, P. V. Sasorov, and A. Sen, Phys. Plasmas **5**, 2849 (1998); X. H. Deng and H. Matsumoto, Nature **410**, 557 (2001).
[16] A. V. Gordeev and L. I. Rudakov, J. Exp. Theor. Phys. **28**, 1226 (1969); M. Yamada *et al.*, Nat. Comm. **5**, 4774 (2014).
[17] P. C. Hohenberg and B. I. Halperin, Rev. Mod. Phys. **49**, 435 (1977).
[18] B. Schmittmann and R. K. P. Zia, *Phase Transitions and Critical Phenomena*, edited by C. Domb and J. L. Lebowitz (Academic Press, London, 1995), Vol. 17.
[19] M. Kardar, G. Parisi, and Y.-C. Zhang, Phys. Rev. Lett. **56**, 889 (1986).
[20] V. Yakhot and S. A. Orszag, Phys. Rev. Lett. **57**, 1722 (1986).
[21] M. Lyutikov, Phys. Rev. E **88**, 053103 (2013).
[22] A. Brandenburg, T. Kahniashvili, and A. G. Tevzadze, Phys. Rev. Lett. **114**, 075001 (2015).
[23] J. D. Gibbon, A. Gupta, G. Krstulovic, R. Pandit, H. Politano, Y. Ponty, A. Pouquet, G. Sahoo, and J. Stawarz, Phys. Rev. E **93**, 043104 (2016).
[24] M. K. Verma, Phys. Rep. **401**, 229 (2004).
[25] A. Basu and J. K. Bhattacharjee, J. Stat. Mech. (2005) P07002.
[26] A. Basu, J. K. Bhattacharjee, and S. Ramaswamy, Eur. Phys. J. B **9**, 725 (1999).
[27] J. C. Perez, J. Mason, S. Boldyrev, and F. Cattaneo, Phys. Rev. X **2**, 041005 (2012).
[28] P. S. Iroshnikov, Soviet Astronomy **7**, 566 (1964) [Astron. Zh. **40**, 742 (1963)]; R. H. Kraichnan, Phys. Fluids **8**, 1385 (1965).
[29] S. Galtier, S. V. Nazarenko, A. C. Newell, and A. Pouquet, J. Plasma Physics **63**, 447 (2000).
[30] M. L. Goldstein, R. T. Wicks, S. Perri, and F. Sahraoui, Phil. Trans. Royal Soc. A **373**, 20140147 (2015).
[31] G. Chatterjee *et al.*, Nat. Comm. **8**, 15970 (2017).
[32] R. Pandit, S. S. Ray, and D. Mitra, Eur. Phys. J B **64**, 463 (2008); S. S. Ray, D. Mitra, and R. Pandit, New J. Phys. **10**, 033003 (2008).
[33] C. S. Ng and A. Bhattacharjee, Phys. Plasmas **4**, 605 (1997).
[34] A. Basu, Europhys. Lett. **65**, 505 (2004).
[35] D. Das, A. Basu, M. Barma, and S. Ramaswamy, Phys. Rev. E **64**, 021402 (2001).
[36] E. D. Siggia, B. I. Halperin, and P. C. Hohenberg, Phys. Rev. B **13**, 2110 (1976).
[37] L. P. Kadanoff and J. Swift, Phys. Rev. **166**, 89 (1968); K. Kawasaki, Ann. Phys. (N.Y.) **61**, 1 (1970).
[38] U. Täuber, *Critical Dynamics* (Cambridge University Press, Cambridge, 2014).
[39] C. De Dominicis and L. Peliti, Phys. Rev. B **18**, 353 (1978).
[40] V. Dohm and R. A. Ferrell, Phys. Lett. **67**, 387 (1978).
[41] V. K. Akkineni and U. C. Täuber, Phys. Rev. E **69**, 036113 (2004).
[42] We ignore intermittency of local energy dissipation that is intimately connected to the multiscaling phenomena.
[43] W. Heisenberg, Z. Phys. **124**, 628 (1948); Proc. Roy. Soc. A **195**, 402 (1948).
[44] S. Chandrasekhar, Phys. Rev. **75**, 896 (1949).
[45] C. Connaughton, S. Nazarenko, and A. C. Newell, Physica D **184**, 86 (2003).
[46] C. H. K. Chen, S. D. Bale1, C. S. Salem, and B. A. Maruca, Ap. J **770**, 125 (2013); K. H. Kiyani, S. C. Chapman, F. Sahraoui, B. Hnat, O. Fauvarque, and Yu. V. Khotyaintsev, ApJ **763**, 10 (2013).
[47] Scaling *ansatz* Eq. (31) for the fields $\mathbf{v}$ and $\mathbf{b}$ can also be equivalently expressed in terms of $l_\perp$ by using the scaling relation between $l_\perp$ and $l_\parallel$.
[48] By not distinguishing between v and b in the definition of M, we implicitly assume $v \sim b$ (in a scaling sense), giving $a = y$ that is borne out by the results obtained below.
[49] P. Goldreich and S. Sridhar, Ap. J. **438**, 763 (1995).
[50] J. C. Perez and S. Boldyrev, Ap. J. **672**, L61 (2008).
[51] S. Galtier, J. Phys. A: Math. Theor. **51**, 293001 (2018).
[52] I. V. Khalzov, F. Ebrahimi, D. D. Schnack, and V. V. Mirnov, Phys. Plasmas **19**, 012111 (2012).
[53] We essentially focus on intermediate length-scales much larger than the dissipative scales, but smaller than the scales at which ordinary MHD prevails.
[54] D. O. Gómez, Adv. Plasma Astrophys. **6**, 433 (2010).
[55] R. Meyrand, K. H. Kiyani, Ö. D. Gürcan, and S. Galtier, Phys. Rev. X **8**, 031066 (2018).
[56] J. Cho and A. Lazarian, Astrophys. J. **615**, L41 (2004).
[57] P. Goldreich and S. Sridhar, Ap. J. **485**, 680 (1997).
[58] S. Boldyrev and J. C. Perez, Ap. J. **758**, L44 (2012).
[59] A. Basu, A. Sain, S. K. Dhar, and R. Pandit, Phys. Rev. Lett. **81**, 2687 (1998).
[60] D. Banerjee, S. S. Ray, G. Sahoo, and R. Pandit, Phys. Rev. Lett. **111**, 174501 (2013).
[61] V. Dallas and A. Alexakis, Phys. Rev. E **88**, 053014 (2013); **88**, 063017 (2013); Astrophys. J. **788**, L36 (2014).