Perturbing the shortest path on a critical directed square lattice

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We investigate the behavior of the shortest path on a directed two-dimensional square lattice for bond percolation at the critical probability p_c. We observe that flipping an edge lying on the shortest path has a nonlocal effect in the form of power-law distributions for both the differences in shortest path lengths and for the minimal enclosed areas. Using maximum likelihood estimation and extrapolation, we find the exponents $\alpha = 1.36 \pm 0.01$ for the path length differences and $\beta = 1.186 \pm 0.001$ for the enclosed areas.

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I. INTRODUCTION

The well established bond percolation model can be generalized on a directed lattice as was done by Redner with the resistor-diode network, which was used implicitly much earlier in the work of Broadbent and Hammersley [1–6].

For bond percolation, the shortest path l is defined as the path connecting two opposite sides of a lattice with the least amount of steps. The length l is also known as the chemical distance [7]. On a randomly directed lattice, the path in addition has to respect the direction of each link. Shortest paths of two-dimensional percolating systems have been investigated extensively over the past 40 years and have been found to be fractal at the percolation threshold [8–13].

We focus on directed networks where all the directions are equiprobable, thus making the system critical on the square lattice [1]. In this special case, the directed lattice falls into the same universality class as standard bond percolation [5,14]. We consider a two-dimensional directed square lattice with side length L and investigate how flipping an edge (i.e., inverting the direction of an edge) affects the shortest path length l. An example of such a situation is shown in Fig. 1. When the shortest path is interrupted by inverting a randomly selected bond on the path, a new distinct shortest path emerges. We use numerical simulations to investigate the difference in length between the new and old shortest path and the area enclosed between them.

We investigate the properties of shortest paths on directed lattices in the context of systems with well defined flow routes such as biochemical pathways, cities with one-way streets, circulatory systems, and porous media in general.

The directed lattice together with the shortest paths shares a lot of similarities with the Lorentz lattice gas model (windtree model) or even more so with the 8-vertex model (icetype model), which also allows for sources and sinks for the paths. However, both models require that for each vertex there are always two edges directed inwards and outwards (four edges in some cases of the 8-vertex model) [15–18]. This is where the analogy breaks down because in the resistor-diode network, a vertex can have three incoming and one outgoing edge or vice versa.

We use the burning method to determine the shortest path that connects one side (top) of the lattice with the other (bottom) [10]. The burning method starts by setting all vertices in the topmost row on "fire." Then, in the following time steps, the fire is propagated along the permissible edges to neighboring sites that are not burnt. At each time step, the newly burnt vertices are labeled with the current time. The procedure stops as soon as the opposite side is reached or earlier in cases where the system does not percolate. Assuming the fire reaches the opposite side, the shortest path length is given by the last time step to be registered. It should be noted that shortest paths on square lattices are usually not unique.

We choose randomly one of the shortest paths determined using the burning method. A randomly selected edge along this path is then flipped and the burning method employed again to find the new shortest path l_{new}. A path that encloses the smallest area together with the original shortest path is selected. In some cases there may arise a choice between two different candidates, one to the left and one to the right of the original shortest path. The choice, however, does not affect the properties investigated. By flipping one of the bonds that lie on the shortest path we introduce an external perturbation with the goal of studying how the system, and in particular, the shortest path responds to it. We want to understand whether these local modifications can result in much larger global changes.

As a first step, we have a look at the distribution of the unperturbed shortest paths $p(l, L)$. It turns out that also in the case of directed networks, $p(l, L)$ is compatible with the same scaling law found for bond percolation [19–22]. From our simulation results we find that

$$p(l, L) = g(l/L^{d_{\min}})/l, \tag{1}$$

where L is the size of the square lattice and $d_{\min} = 1.13$ the fractal dimension of the shortest path [13,21,23]. This comes as no surprise since it has already been conjectured that the model discussed in this paper is in the same

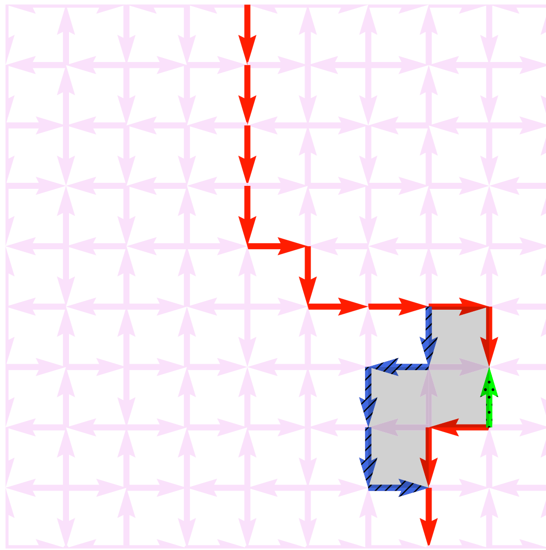

FIG. 1. The original shortest path is in red. After inverting one of the links on the path (green dotted arrow), the shortest path changes. The new shortest path is shown in blue with stripes. It is chosen out of all possible shortest paths such that it minimizes the area of the shaded region.

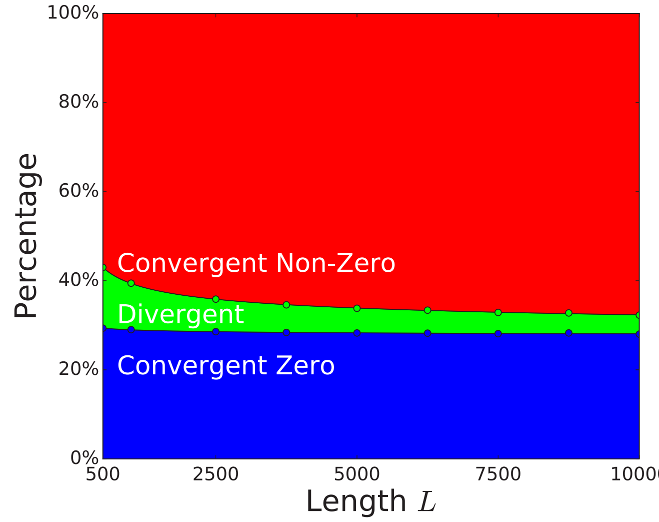

FIG. 2. Ratio of the three possible outcomes for the difference between the old and new shortest path: convergent paths with zero difference in blue, divergent paths in green, and convergent paths with nonzero difference in red.

universality class as standard percolation [5,14]. Using the Leath-Alexandrowicz method, it can be shown that the clusters of sites reached from a seed site in percolation of directed bonds with $p = \frac{1}{2}$ are identically distributed to the clusters of standard percolation at criticality [6,24,25]. We therefore expect the shortest path distribution to be the same. Furthermore, we also confirm the analytical asymptotic results obtained by Hovi and Aharony for the scaling function $g(x)$ (not shown) [22].

II. CHANGE IN SHORTEST PATH LENGTH

On square lattices the length difference between the old shortest path and the new one is always an integer and it can be zero, even, or odd. Two paths sharing the same starting and ending points as shown in Fig. 1 can either have a zero or an even difference in length. This is because the length of any cycle on a square lattice is an even number and thus the difference between any two paths that agree on both ends will be either zero or even. An odd value, on the other hand, implies that at least one of the two end points does not coincide.

The change in length of the shortest path necessarily has to be non-negative. For a new path to be shorter than the previous path, it has to go through the flipped edge because otherwise it would mean that the new path already existed before the change. On the other hand, any new shortest path cannot pass through the flipped edge as illustrated by the following argument. Consider four paths A, B, C, and D. The paths A and B start from the top and go each to one end of the flipped edge e while C and D start each from one end of the flipped edge e and go to the bottom. Let the path $A + e + C$ denote the previous shortest path. Before flipping, both paths $A + D$ and $B + C$ must be as long or longer than $A + e + C$ by definition. In turn, this means that B is longer than A and D is longer than C. When the edge e is now flipped, we observe that $B + e + D$ is longer than both $A + D$ and $B + C$. Therefore, the flipped edge will never be crossed in any new shortest path.

We separate the data obtained from the simulations based on whether the shortest paths coincide on both ends and then further based on whether the difference between them is zero or nonzero. If the two shortest paths do not agree on both ends, we say that they are divergent and otherwise we say that they are convergent. To summarize, the data are classified into divergent paths, convergent paths with zero differences, and convergent paths with nonzero, i.e., even, differences. The ratio of these three categories obtained as a function of system size is shown in Fig. 2. We extrapolate the curves by fitting $\gamma(L) = \gamma_0 + \gamma_s L^{\gamma_e}$ using nonlinear least squares as shown in Fig. 2. For $L \to \infty$ we observe that the samples with convergent paths with zero change in length approach $26.9\% \pm 1.1\%$, those with divergent paths approach $0.0\% \pm 0.2\%$, and those with convergent paths with nonzero change in length approach $72.8\% \pm 0.5\%$. In comparison, if the divergent paths are ignored explicitly, the percentage for zero-difference paths becomes $27.4\% \pm 0.4\%$ and for even differences $72.6\% \pm 0.4\%$.

Although one may conclude from the results above that the percentage of divergent paths vanishes in the asymptotic limit of infinite size, we want to point out that this is not the case. The fact is that there exists a very small, albeit nonvanishing probability of having more than one percolating cluster at the critical point [26]. The exact value of this probability has been calculated by Cardy for two-dimensional systems in the limit of infinitely large system sizes [27]. According to Cardy's formula, it turns out that for $L \to \infty$, the probability of having two spanning clusters in two dimensions is $P \sim \exp(-2\pi) \simeq 0.001\,87$, which is less than the error. In order to show that that the percentage of divergent paths for infinite system sizes coincides with the value calculated by Cardy, we would have to make much more precise simulations, such as the one proposed in Ref. [26].

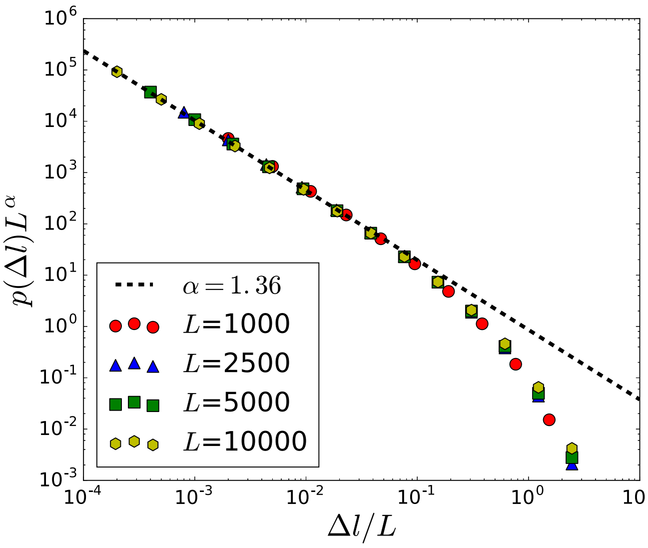

FIG. 3. Data collapse for the distribution of nonzero differences in shortest path lengths $p(\Delta l)$ for convergent paths using data for different lattice sizes L. The dashed line represents a power-law fit to the data with exponent $\alpha = 1.36$.

If we consider only the convergent paths with nonzero differences, the distribution $p(\Delta l)$ of the differences in length between old and new shortest paths shows scale-free behavior with an exponent estimated to be $\alpha = 1.36 \pm 0.01$. Figure 3 shows an excellent data collapse for the distribution $p(\Delta l)$ that obeys scaling of the form $p(\Delta l) = L^{-\alpha} f(\Delta l/L)$. Without showing the results, we mention here that we obtained the same scaling exponent (within error bars) for the path difference distribution in the case of standard bond percolation at $p_c = \frac{1}{2}$. In the case of divergent paths, again only with nonzero differences, the results are quite different as can be seen in Fig. 4. A truncated power law is observed, but with an

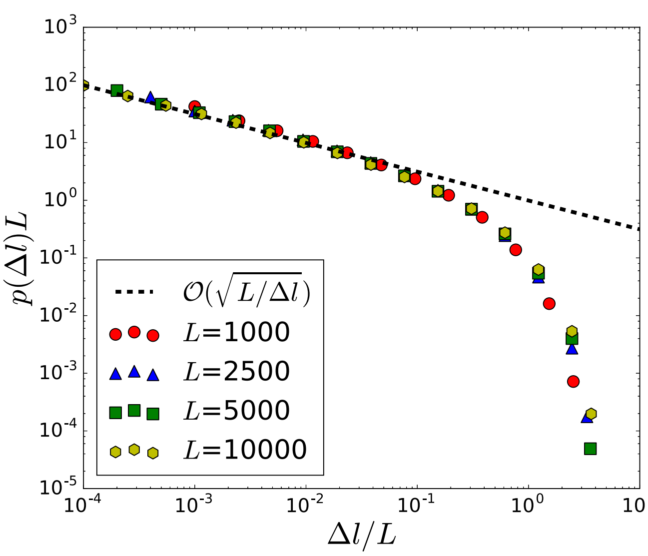

FIG. 4. Data collapse for the distribution of nonzero differences in shortest path lengths $p(\Delta l)$ for divergent paths using data for different lattice sizes L. The dashed line represents a power-law curve with exponent $\alpha = 0.5$.

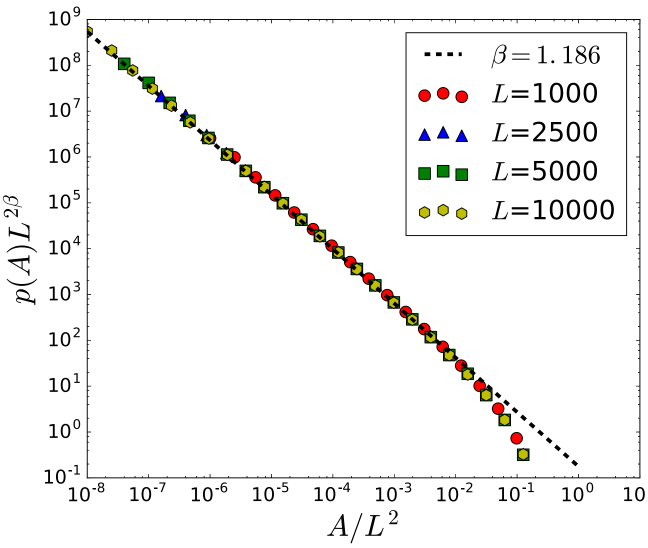

FIG. 5. Data collapse for the distribution of enclosed areas $p(A)$ using data for different lattice sizes L. The dashed line represents a power-law fit to the data with exponent $\beta = 1.186$.

exponent of approximately $\alpha \simeq 0.5$, which is different from the scaling exponent of unity used to obtain the data collapse.

III. AREA BETWEEN SHORTEST PATHS

The distribution $p(A)$ of the size of the minimal area A enclosed between the old and new shortest paths also shows scale-free behavior with an exponent estimated to be $\beta = 1.186 \pm 0.001$. Figure 5 shows an excellent data collapse for the distribution $p(A)$ that obeys scaling in the form $p(A) = L^{-2\beta} f(A/L^2)$. The samples used only contain convergent paths. Again, without showing the results, we mention here that we obtained the same scaling exponent (within error bars) for the area distribution in the case of standard bond percolation at $p_c = \frac{1}{2}$. The exponent β agrees within error bars with the exponent $\beta = 1.16 \pm 0.03$ for the size distribution of the areas enclosed by watersheds from landscapes that only differ slightly at one location reported in Ref. [28] in the case of uncorrelated landscapes with Hurst exponent $H = -1$.

IV. EXPONENT ESTIMATION

The exponents have been estimated using maximum likelihood estimation together with a discrete exponentially truncated power-law based on the method used in Ref. [29]. The use of maximum likelihood estimation is motivated by the fact that we look at probability distributions. The estimated exponents for different lattice sizes L have been observed to converge algebraically to some limit value γ_0 as $L \to \infty$. This limit is extrapolated by fitting the estimated exponents $\gamma(L)$ against $\gamma(L) = \gamma_0 + \gamma_s L^{\gamma_e}$ using nonlinear least squares.

V. CONCLUSION

We have investigated the effect of flipping one edge in planar directed networks (diode networks) at the critical probability p_c and found power-law behaviors for, both, the probability distribution of the differences in shortest path lengths and for the minimal enclosed areas between the old and new shortest paths. This implies that very small perturbations made to the shortest path can cause nonlocal changes in the system. We found that the number of divergent paths almost vanishes in the thermodynamic limit, in accordance with the existence of multiple percolation clusters. The differences in shortest path lengths are thus expected to be overwhelmingly even valued (or zero) in the thermodynamic limit. We also give estimates for the exponents of the observed power laws using maximum likelihood estimates based on a truncated power law and extrapolation. The exponents are $\alpha = 1.36 \pm 0.01$ for the differences in shortest path lengths and $\beta = 1.186 \pm 0.001$ for the minimal enclosed areas. It would be interesting also to study the shape of the enclosed areas and see if it scales with the exponents of directed percolation.

Finally, we found that the value we obtain for the exponent β agrees within the error bars with the exponent $\beta = 1.16 \pm 0.03$ reported for the case of perturbed watersheds in Ref. [28]. Watersheds are fractal nonintersecting lines that separate adjacent drainage basins on random landscapes. Although we cannot make a direct connection between shortest paths of percolating clusters and watersheds, we can offer a few arguments. First of all, water flow routes and drainage systems on random surfaces can be mapped to the resistor-diode network model considered in this paper. Moreover, both cases deal with a distribution that is the result of a small perturbation of a fractal path. In both cases, a new path emerges after the perturbation, creating an enclosed area that is distributed according to the same power law. This suggests that the resistor-diode network and watersheds might be related to each other through some general properties of landscapes.

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