Statistics of overtake events by a tagged agent

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We consider a minimalist model of overtaking dynamics in one dimension. On each site of a one-dimensional infinite lattice sits an agent carrying a random number specifying the agent's preferred velocity, which is drawn initially for each agent independently from a common distribution. The time evolution is Markovian, where a pair of agents at adjacent sites exchange their positions with a specified rate, while retaining their respective preferred velocities, only if the preferred velocity of the agent on the "left" site is higher. We discuss two different cases: one in which a pair of agents at sites i and $i+1$ exchange their positions with rate 1, independent of their velocity difference, and another in which a pair exchange their positions with a rate equal to the modulus of the velocity difference. In both cases, we find that the net number of overtake events by a tagged agent in a given duration t, denoted by $m(t)$, increases linearly with time t, for large t. In the first case, for a randomly picked agent, m/t, in the limit $t \to \infty$, is distributed uniformly on $[-1, 1]$, independent of the distributions of preferred velocities. In the second case, the distribution is given by the distribution of the preferred velocities itself, with a Galilean shift by the mean velocity. We also find the large time approach to the limiting forms and compare the results with numerical simulations.

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The phenomenon of overtaking is ubiquitous in nature. It occurs naturally in all sorts of traffics, ranging from the vehicular traffic on highways [1] to the transport at the molecular scale by motor proteins [2,3]. Animals in groups overtake each other to move to a less risky position at the center of the group [4]. Overtaking also takes place in sedimentation or electrophoresis of mixtures with polydisperse (different sizes, densities) particles falling (or rising) through a fluid under gravity or electric field [5]. In biological evolution, the population sizes of different genotypes overtake each other depending on their fitness [6–9]. In a completely different context, the real-time correlation functions in quantum interacting many body systems may be understood in terms of overtaking dynamics of particles [10,11].

In spite of its widespread appearances, surprisingly, the statistics of overtake events has not been studied much. In this paper, we investigate the statistics of overtake events for a tagged agent in a simple model of stochastic evolution of self-driven agents (e.g., vehicles, molecular motors, etc.) in one dimension. In an overtake event, an agent with a higher velocity crosses another agent with a lower velocity. We define the *net overtakings* by a tagged agent as the total number of agents that it overtakes minus the total number of agents that overtake it, in a given duration. We study the probability distribution of fluctuations in this quantity. Such statistics provide a useful way to monitor the stochastic system. In particular, in traffic engineering, to obtain flow data, one uses the *moving observer method*, where an observer in a test vehicle moves a fixed distance with a constant speed and counts the number of vehicles that it overtakes and the number of vehicles that overtake it [12]. In these studies, the fluctuations are usually large, but their systematic study is lacking [13].

In this paper, we discuss a minimalist model, consisting of a collection of self-driven agents initially placed at points with constant separation along a line. We associate a real random variable v with each agent, which may be called his preferred velocity. We assume that each agent is assigned the velocity at the beginning, independent of others, from a common probability density function (PDF) $\rho(v)$, which remains unchanged at subsequent times. We ignore the actual positions of agents, and only focus on their relative order along the line. In our model, the configuration at any time t is fully specified by giving the relative order of different agents along the line. The total neglect of actual spatial coordinates is clearly an oversimplification, but this makes the model more tractable, and we will try to show below that this simplified model is still instructive, and has an interesting behavior. Let v_i be the preferred velocity of the ith agent to the right of some reference position on the line. Then, we think of v_i as the preferred velocity of the agent at site i of a one-dimensional lattice, $-\infty < i < +\infty$. The situation roughly corresponds to a crowded highway, where actual spacings between cars are roughly constant. Then, in a frame moving with the mean velocity of traffic, we would see only infrequent and stochastic changes of order of the cars. We assume that this overtaking may be taken as Markovian. As agents overtake each other, they exchange their positions, but they retain their respective preferred velocities with them, which are quenched random variables. In an overtake event, the particle with higher preferred velocity goes to the right of the particle with a lower preferred velocity (see Fig. 1). Therefore, the net overtakings in any time interval by a tagged agent equal the agent's shift in position, denoted by m, in that time interval in the list specifying the configuration.

Here we consider two different cases for the exchange rates. In the first case, we set the rate of exchange for agents between two neighboring lattice sites i and $i+1$ as $r = \theta(v_i - v_{i+1})$, where $\theta(v) = 1$ for $v > 0$ and 0 for $v \leqslant 0$, is the

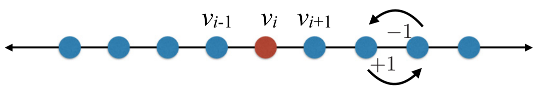

FIG. 1. Each site of a one-dimensional lattice is occupied by an agent with a certain velocity, and two neighboring agents interchange their positions with a specific rate. During the exchange, the net overtakings of the faster (slower) moving agent increases (decreases) by unity. The red filled circle indicates the tagged agent.

Heaviside theta function. This case is more appropriate for a crowded scenario where what matters is that an agent tries to overtake, but the actual difference in their preferred velocities is less important. Note that changing v to any other monotonic increasing function of v leaves this dynamics unchanged. In this case, m scales linearly with time t, and hence, it is useful to consider the scaled random variable $c = m/t$. Here c gives the mean change in the position of the tagged agent per unit time on the discrete lattice, which is our configuration space. This case is related to the totally asymmetric simple exclusion process (TASEP), with infinitely many classes of particles, and has been studied much in the literature [14–19]. We show below that this limiting value c is a function of the preferred velocity of the tagged agent. If the tagged agent is picked at random, it becomes a random variable. Interestingly, the probability distribution of this random variable c is independent of the initial distribution of velocities, and in the limit $t \to \infty$, it is uniformly distributed on $[-1, 1]$, for all continuous distributions of the velocities.

For cars on a highway at a lower density, the rate of overtaking between two agents is higher, if the difference between their velocities is larger. This may be approximated in our lattice model, which makes no mention of spacings between cars, by taking the Markovian overtaking rate to be $r = \theta(v_i - v_{i+1})\,(v_i - v_{i+1})$. In this case also, m again scales linearly with t. However, the limiting PDF of $c = m/t$ is given by the PDF of v itself, but shifted by the mean velocity $\langle v\rangle$. For brevity, by writing in general $r = \theta(v_i - v_{i+1})\,(v_i - v_{i+1})^{\alpha}$, we refer to the above two cases in the following as $\alpha = 0$ and 1 respectively. The PDF of c, at any time t, can be written as

$$p(c,t) = \int_{-\infty}^{\infty} p_1(c,t|v_0)\,\rho(v_0)\,dv_0, \qquad (1)$$

where $p_1(c,t|v_0)$ is the conditional PDF of c for a tagged agent having a given preferred velocity v_0.

Let us first consider the case $\alpha = 0$. If we consider a single tagged agent (say in "gray," for convenience), having the quenched velocity v_0, the rest of the agents can be divided into two groups: the agents (say in "black") whose velocities are greater than v_0 and the agents (say in "white") whose velocities are less than v_0. The dynamics of agents within the black group as well as within the white group is invisible to the gray agent. From the point of view of the gray agent, a black agent can overtake a white agent from the left and not the other way around—this is then equivalent to a TASEP with the black agents as particles and the white agents as holes. The gray agent can overtake a white agent (hole) and a black agent (particle) can overtake the gray agent, whereas the reverse moves are not allowed. A black agent cannot distinguish between the gray and a white agent. Similarly, a white agent cannot distinguish the gray agent from a black agent. In TASEP language, the gray agent is known as a *second class* particle [14]. Therefore, the motion of this tagged (gray) agent is same as that of a single second class particle in a TASEP, starting on an initial uncorrelated background of particles (also known as first class particles) with density $\rho_+(v_0) = \int_{v_0}^{\infty} \rho(v)$ and holes with density $\rho_-(v_0) = 1 - \rho_+(v_0)$. For this second class particle, it has been shown [14] that, in the $t \to \infty$ limit, the scaled displacement on the lattice $c = m/t$, converges almost surely to $\bar{c}(v_0) = 1 - 2\rho_+(v_0)$. This result holds for our tagged (gray) agent with a given v_0—which itself is random that varies from one realization to another. Evidently, $\bar{c}(v_0)$ is bounded by ± 1, with $\bar{c} \to \pm 1$ for $v_0 \to \pm\infty$ (or the upper and the lower supports respectively) and $\bar{c}(v_0^*) = 0$ for $\rho_+(v_0^*) = \rho_-(v_0^*) = 1/2$.

In the limit $t \to \infty$, the random variable c does not have any fluctuations around the random variable $\bar{c}(v_0)$—i.e., $p_1(c,t|v_0) = \delta(c - \bar{c}(v_0))$ in Eq. (1). Therefore, using $d\bar{c}/dv_0 = -2\rho_+'(v_0) = 2\rho(v_0)$, we get

$$p(c) = \tfrac{1}{2} \quad \text{where} \quad -1 \leqslant c \leqslant 1. \qquad (2)$$

In words, in the $t \to \infty$ limit, m/t is uniformly distributed on $[-1, 1]$, for all continuous distributions $\rho(v)$, as claimed above.

A somewhat similar looking, but very different, result was obtained earlier for the so-called speed process [18,19]. In this, one considers TASEP with the step initial condition where the initial velocities satisfy $v_i > v_0$, for all $i < 0$, and $v_i < v_0$, for $i > 0$, and there is a single second-class particle at the origin. It was shown that in each realization, the second class particle shows a limiting speed for large times. The precise value of this limiting speed varies in different realizations. Some value gets selected in the early evolution, and in later evolution it does not change much. The probability distribution of this randomly selected value, averaged over all possible evolutions, is uniform on $[-1, 1]$.

For finite times, the fluctuations around $\bar{c}(v_0)$ are important. As $p_1(c,t|v_0)$ is expected to depend on v_0 only through $\rho_+(v_0)$, or equivalently $\bar{c}(v_0)$, making a change of variable from v_0 to $\bar{c}$ eliminates $\rho(v_0)$ completely from Eq. (1),

$$p(c,t) = \frac{1}{2}\int_{-1}^{1} p_2(c,t|\bar{c})\,d\bar{c}, \qquad (3)$$

where $p_2(c,t|\bar{c})$ is the conditional PDF for a given $\bar{c}$. Thus, not only the limiting distribution, but also the $p(c,t)$ at all time is independent of the velocity distribution $\rho(v)$.

Starting from an uncorrelated initial condition, we do not expect the correlations between the jumps of the tagged agent at different times to become important at early times. Therefore, a biased random walk (RW) description of motion of the tagged agent, that jumps to the left with the rate $\rho_+(v_0)$ and right with the rate $\rho_-(v_0)$, suggests that the typical fluctuations of c around $\bar{c}(v_0)$, at the scale of the standard deviation $\sigma_t = \sqrt{\langle[c - \bar{c}(v_0)]^2\rangle} = t^{-1/2}$, are Gaussian. However, correlations between jumps build up at later times, and eventually, it crosses over to $\sigma_t \propto \chi^{1/3} t^{-1/3}$ behavior [20–22] with $\chi = \rho_+(v_0)[1 - \rho_+(v_0)] = (1 - \bar{c}^2)/4$. The typical fluctuations, at large times, are described by [21,22]

$$p_2(c,t|\bar{c}) \simeq \tfrac{1}{4}(2\chi^{1/3}t^{-1/3})^{-1}\, G_{\text{scaling}}([c - \bar{c}]/[2\chi^{1/3}t^{-1/3}]), \qquad (4)$$

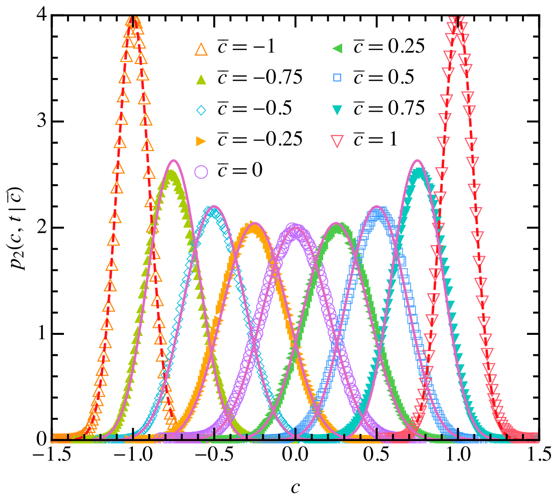

FIG. 2. The points are numerical simulation results for the conditional PDF of scaled net overtakings $c = m(t)/t$ for given $\bar{c}$, for our first choice of the overtaking rate $r = \theta(v_i - v_{i+1})$. The initial velocities are drawn from uniform distribution on $[-1, 1]$. The dashed lines plot two Gaussian distributions centered around $\bar{c} = \pm 1$ respectively, with a variance t^{-1}. The solid lines show the non-Gaussian distributions, given in Eq. (4), for $\bar{c} = 0,\ \pm 0.25,\ \pm 0.5,$ and ± 0.75. In all the cases, $t = 100$.

where $G_{\text{scaling}}(w)$ is the scaling function associated with the spatiotemporal two-point correlation function of the TASEP with the Bernoulli product measure initial condition [21–25]. The crossover time $t_* \propto \chi^{-2}$.

At large times, since $p_2(c, t|\bar{c})$ in Eq. (3) is peaked sharply around $\bar{c}$, the correction to Eq. (2) near the edges $c = \pm 1$ comes from $\bar{c} \to \pm 1$ respectively. In this case, t_* diverges. Hence, we can use the Gaussian form (see Fig. 2) $p_2(c, t|\bar{c}) \simeq \exp(-t[c - \bar{c}]^2/2)/\sqrt{2\pi t^{-1}}$ of the RW picture in Eq. (3). This yields

$$p(c,t) \simeq \frac{1}{4}\left[\operatorname{erf}\left(\frac{(c+1)\sqrt{t}}{\sqrt{2}}\right) - \operatorname{erf}\left(\frac{(c-1)\sqrt{t}}{\sqrt{2}}\right)\right], \qquad (5)$$

where $\operatorname{erf}(x) = (2/\sqrt{\pi}) \int_0^x e^{-y^2}\, dy$ is the error function. We compare this form with numerical results in Fig. 3 and find very good agreement. The finite large time correction for the central region around $c = 0$ can be computed by using Eq. (4) (see Fig. 2) in Eq. (3) [see Fig. 3(e)].

We next consider the case $\alpha = 1$. For this, our model, in fact, corresponds to the infinite-species Karimipour model [26]. As in the above $\alpha = 0$ case, here also, due to lack of strong correlations between jumps, we expect the motion of the tagged agent, with a given preferred velocity v_0, to be described by a RW at early times. The RW jumps to the right with the rate $\rho_R(v_0) = \int_{-\infty}^{v_0} (v_0 - v)\, \rho(v)\, dv$ and to the left with the rate $\rho_L(v_0) = \int_{v_0}^{\infty} (v - v_0)\, \rho(v)\, dv$. Therefore, the drift velocity $\bar{c}(v_0) = \rho_R(v_0) - \rho_L(v_0) = v_0 - \langle v \rangle$ and the standard deviation $\sigma_t = \sqrt{[\rho_L(v_0) + \rho_R(v_0)]/t}$.

In the limit $t \to \infty$, ignoring the fluctuations around $\bar{c}$—i.e., $p_1(c, t|v_0) = \delta[c - \bar{c}(v_0)]$ in Eq. (1), gives the limiting PDF $p(c) = \rho(c + \langle v \rangle)$, as claimed above. The shift of the

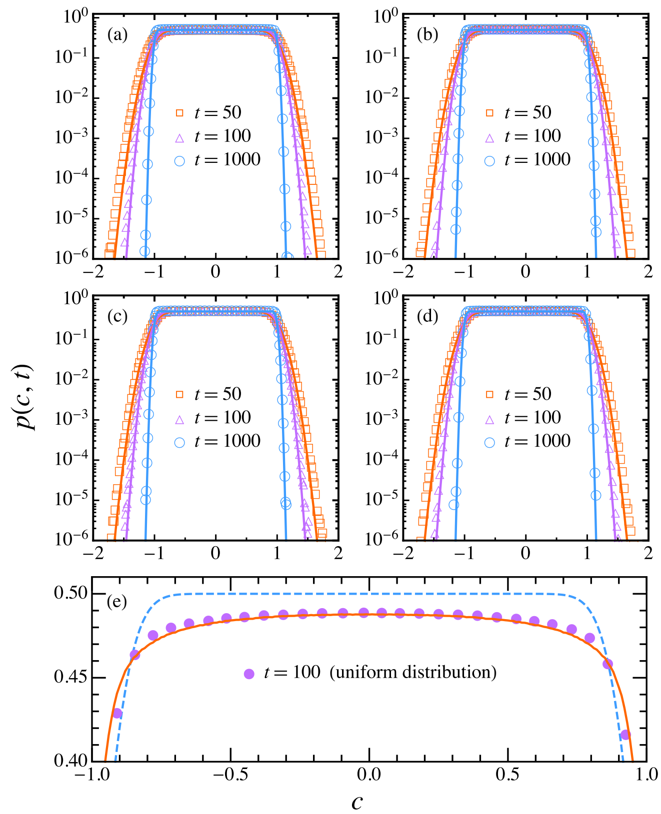

FIG. 3. (a)–(d) The points are numerical simulation results for the PDF of the scaled net overtakings $c = m(t)/t$ at different times, for our first choice of the overtaking rate $r = \theta(v_i - v_{i+1})$, where the initial velocities are chosen from (a) uniform, (b) Gaussian, (c) exponential, and (d) power-law distributions. The solid lines plot Eq. (5). (e) The points are from a numerical simulation where the initial velocities are chosen from a uniform distribution on $[-1, 1]$ and $t = 100$. The solid line plots Eq. (3) computed by using Eq. (4) and the dashed line plots Eq. (5) for $t = 100$.

PDF by the mean is easily understood, as the overtaking dynamics depends only on the velocity differences and the mean value of the overtaking rate, averaged over different agents, must be zero. For the approach to this limiting distribution, we note that as in the $\alpha = 0$ case, the contributions to the large $|c|$ tails of $p(c, t)$ come from large $|v_0 - \langle v \rangle|$ behavior of $p_1(c, t|v_0)$ in Eq. (1), for which we expect the crossover time $t_\#(v_0)$ to be large. Therefore, the tails of $p(c, t)$ can be computed by using a Gaussian distribution with mean $v_0 - \langle v \rangle$ and variance $[\rho_L(v_0) + \rho_R(v_0)]/t$ for $p_1(c, t|v_0)$ in Eq. (1),

$$p(c,t) \simeq \int_{-\infty}^{\infty} dv_0\, \rho(v_0)\, \frac{\sqrt{t}}{\sqrt{2\pi[\rho_L(v_0) + \rho_R(v_0)]}} \times \exp\left(-\frac{t[c + \langle v \rangle - v_0]^2}{2[\rho_L(v_0) + \rho_R(v_0)]}\right). \qquad (6)$$

Evidently, $p(c, t)$ now depends on the form of $\rho(v_0)$ and the integral has to be carried out separately for each case. Figure 4 shows very good agreement between Eq. (6) and numerical simulation results for four different choices of $\rho(v_0)$.

Finally, we comment on the relation of our model to the Jepsen gas, which has been studied in literature earlier [27]

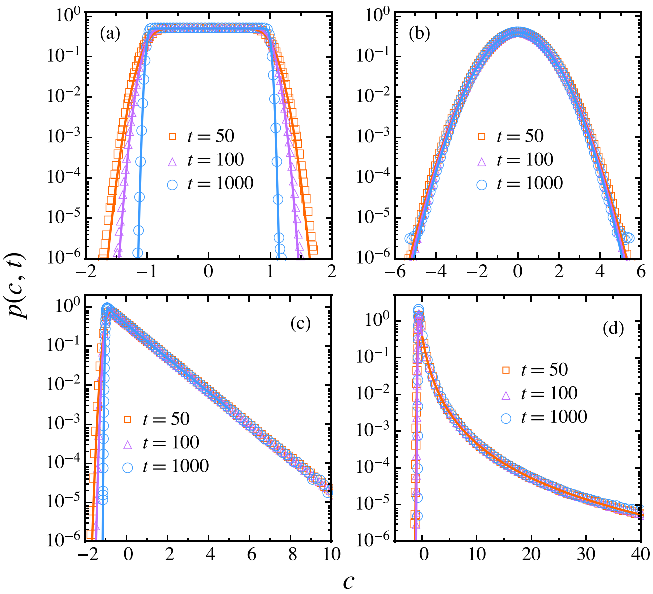

FIG. 4. The points are numerical simulation results for the PDF of the scaled net overtakings $c = m(t)/t$ at different times, for the second choice of the overtaking rate $r = \theta(v_i - v_{i+1})(v_i - v_{i+1})$, where the initial velocities are chosen from (a) uniform, (b) Gaussian, (c) exponential, and (d) power-law distributions. The solid lines plot the theoretical results obtained from Eq. (6).

for different quantities. In our language, the Jepsen gas corresponds to cars on multilane highways (in the continuum space, not on a lattice), each moving with a constant velocity (in cruise control) that is drawn independently from $\rho(v)$. The cars can pass each other freely without any obstacles. We consider the density of the cars along the highway to be uniform, which is set to unity. The space-time trajectories are given by slanted straight lines where the slopes with respect to the time axis represent the velocities. Evidently, any two lines (cars) can cross (pass) each other at the most once. The net number of overtake events by a tagged car, in a given duration, is the number of lines crossing the tagged line (corresponds to the trajectory of the car) from its right minus the number of lines crossing the tagged line from its left, in that duration. It is easy to see that, for a tagged car with a velocity v_0, the crossings are mutually independent and the rates of crossing from the left and right are $\rho_L(v_0)$ and $\rho_R(v_0)$ respectively—the same rates that occur in the RW description of the motion of the tagged agent in the $\alpha = 1$ case above. Therefore, the RW description of the $\alpha = 1$ case at early times is exact for this Jepsen gas model for all times.

In conclusion, we have found two categories of overtaking behavior. In these cases, the net number of overtake events $m(t)$ grows as t. We have also obtained the limiting distribution of the time-averaged overtaking rate, defined by the total number of net overtakings in a given time period divided by the total time, as well as the approach to the limiting distributions. We contrast this behavior with what happens when the exchange rates r is independent of the velocities and without consideration of which is faster. In this case, evidently, a tagged agent performs a symmetric RW, and therefore, the net overtakings $m(t)$, in a given time t, has the diffusive scaling $m(t) \sim \sqrt{t}$ and PDF of the scaled variable $m/\sqrt{t}$, in the limit $t \to \infty$, is Gaussian. In the overtaking dynamics, the behavior is very different, as seen above. Also, in the case where the random velocity v takes only two distinct values, clearly the value of α does not matter anymore, and the model corresponds to TASEP with identifying agents carrying one type of velocity as particles and the agents carrying the other type of velocity as holes. In this case, the tagged particle displacement $m(t)$ depends on the initial condition [28–31]. In particular, for independent and identically drawn initial velocities, a tagged particle performs a totally asymmetric RW in continuous time, where the fluctuations about the mean displacement are Gaussian and grows diffusively in time [32–35].

There are several interesting open directions for future research. The first and foremost one is of course to analyze real data. Another question is whether there are other classes, and if any, how to identify them. Third, here we have studied only a single time property. However, one can study correlations between overtake events at different times or the overtaking dynamics itself as a process. Here, it is somewhat assumed that the density of agents is homogeneous in real space, so that velocity is the only relevant variable for overtaking. One can explore the effect of inhomogeneity by considering a dilute case, where a finite number of sites, chosen randomly with a given density, are not occupied by agents (equivalently, occupied by agents having zero velocity). The simple picture presented in this paper can serve as a stepping stone for future studies. The model on a finite line is also of interest. In this case, there are important end effects, and there are shock waves that start at the ends and travel inwards, and determine the qualitative behavior in the region deep inside for times of order of the system size. These will be discussed in a future publication.

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