Feynman-Kac equation revisited

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The functionals of particle paths have diverse applications in physics, mathematics, hydrology, economics, and other fields. Under the framework of a continuous-time random walk, the equations governing the probability density functions (PDFs) of the functionals, including those of the paths of stochastic processes of normal diffusion, anomalous diffusion, and even diffusion with reaction, have been derived. Sometimes the stochastic processes in physics and chemistry are naturally described by Langevin equations. The Langevin picture has the advantage of studying the dynamics with an external force field and analyzing the effect of noise resulting from a fluctuating environment. We derive the equations governing the PDFs of the functionals of paths of the Langevin system with both space- and time-dependent force fields and arbitrary multiplicative noise, and the backward version is proposed for a system with arbitrary additive noise or multiplicative Gaussian white noise together with a force field. For the newly built equations, their applications in solving the PDFs of the occupation time and area under the trajectory curve are provided, and the results are confirmed by simulations.

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I. INTRODUCTION

Stochastic processes are the basic mathematical tools to describe natural phenomena. Extracting statistical information on a stochastic process is one of the most important strategies in order to satisfy the demands of practical applications or to understand the microscopic mechanism. A functional, being a random variable, is an integral of a stochastic process. It has diverse applications across multiple disciplines, ranging from probability theory [1], mathematical finance [2], mesoscopic physics [3], and computer science [4], and it is used to understand the cooling and heating degree days relevant to weather derivatives [5]. This paper focuses on deriving the equations governing the probability density functions (PDFs) of the functionals of the paths of Langevin dynamics.

The popular microscopic models used to describe stochastic dynamics in the natural world include continuous-time random walks (CTRWs) and Langevin equations [6]. The Langevin picture is more convenient to apply if the effect of an external field and/or noises generated from a fluctuating environment [7] is considered; it builds a relationship between physically transparent and mathematically tractable descriptions for complex stochastic dynamics. The dynamical behaviors of the system depend fundamentally on the specific form of noise. The most common one is Lévy noise, generating the Lévy process [8], which is a stochastic process with stationary and independent increments and zero initial state. For Lévy noise, the solutions of the Langevin equation belong to the class of Markov processes [9–12]. As for the equations governing the PDFs of the displacement and/or velocity of particles described by the Langevin equation, there have been some developments. In particular, the Langevin equation with Gaussian white noise corresponds to the ordinary Fokker-Planck equation [10–12], and heavy-tailed stable noise corresponds to the spatial fractional Fokker-Planck equation [13–21]. In addition, the temporal fractional Fokker-Planck equation is obtained by the time-changed Langevin equations with an inverse α-stable subordinator [22].

There has also been some progress in deriving the equations governing the PDFs of a functional: $A = \int_0^t U[x(t')]dt'$, where $x(t)$ is the path of a stochastic process and $U(x)$ is some prescribed function. Influenced by Feynman's thesis about Schrödinger's equation, Kac derived the classical Feynman-Kac equation in 1949 for normal diffusion [1]. In recent years, Majumdar discussed the applications of Brownian functionals in [4] by the path-integral method. More and more Feynman-Kac equations for non-Brownian functionals have been established within the framework of CTRWs [23–29]. In particular, the ones in [27] are for the functionals of the reaction diffusion process. In some cases, by the method of subordination [15], a one-to-one correspondence of the Langevin picture and CTRWs can be achieved. But there are still a lot of cases in which the Langevin picture is a more natural choice than CTRWs, e.g., the Langevin equation with multiplicative noise, being effectively used to describe the motion of HaCaT cells [30], which are utilized for their high capacity to differentiate and proliferate *in vitro* (HaCaT is a spontaneously transformed aneuploid immortal keratinocyte cell line from adult human skin, widely used in scientific research). There has not been a lot of progress made in obtaining the Feynman-Kac equations governing the PDFs of the functionals of the paths of Langevin dynamics. Using the Itô formula, Cairoli and Baule [31,32] provided the derivation of the forward Feynman-Kac equation from the Langevin system with Gaussian white noise and arbitrary waiting time distribution. Along these lines, by adopting some different ideas, this paper presents the research of deriving the Feynman-Kac equations for more general Langevin pictures, for example the dynamical system with a fluctuating environment described by the overdamped Langevin equation:

$$\dot{x}(t) = f(x(t),t) + g(x(t),t)\xi(t), \tag{1}$$

where $x(t)$ is the particle coordinate, $f(x,t)$ is the force field, $\xi(t)$ is the noise resulting from a fluctuating environment, and $g(x,t)$ is the multiplicative noise term.

This paper extends the ideas in [14], which focus on a derivation of the generalized Fokker-Planck equation, to derive the generalized Feynman-Kac equation for the overdamped Langevin equation driven by an arbitrary Lévy noise together with a multiplicative noise term, and then we investigate applications for specific functionals of interest. To the best of our knowledge, all the existing backward Feynman-Kac equations are obtained from CTRWs, not a Langevin system, even with Gaussian white noise together with a force field. Here we derive the backward Feynman-Kac equation from the Langevin system with multiplicative Gaussian white noise or additive arbitrary Lévy noise. This paper is organized as follows. In Sec. II, we derive the forward and backward Feynman-Kac equations associated with the overdamped Langevin equation (1). In Sec. III, we use the derived equations to study two examples: the occupation time and the fraction of a particle moving in a box with reflecting boundary conditions, and the area under the curve of the particle trajectory. In addition, some numerical simulations are performed to verify the correctness of the theoretical results. Finally, summaries are provided in Sec. IV.

II. DERIVATION OF THE EQUATIONS

A. Forward equation

There are two parts in this subsection. We first explicitly derive the forward Feynman-Kac equation from the Langevin equation (1), then we introduce a time-changed Langevin equation with a Lévy subordinator and present its corresponding forward equation, with a detailed derivation given in Appendix A.

We use the Lévy noise $\xi(t)$, which is the formal time derivative of its corresponding Lévy process $\eta(t)$. That is to say, the increment $\delta\eta(t) = \eta(t+\tau) - \eta(t)$ of $\eta(t)$ could be defined as the time integral of $\xi(t)$, $\delta\eta(t) = \int_t^{t+\tau}\xi(t')dt'$. Similarly, the increment $\delta x(t) = x(t+\tau) - x(t)$ of the particle trajectory undergoing the Langevin system (1) during a time interval τ ($\tau \to 0$) satisfies

$$\delta x(t) = f(x(t),t)\tau + g(x(t),t)\delta\eta(t), \tag{2}$$

which defines the meaning of Eq. (1) in the Itô interpretation [12,33]. The particle location $x(t)$ only depends on the previous increments of $\eta(t)$ and thus it is independent of the increment $\delta\eta(t)$ since the increments of the Lévy process are independent between nonoverlapping intervals. Because of the stationary increment of the Lévy process, we know that $\delta\eta(t)$ has the same distribution as $\eta(\tau)$ with the characteristic function denoted by [8]

$$\langle e^{-ik\eta(\tau)}\rangle = e^{\tau\phi_0(k)}, \tag{3}$$

where the Lévy exponent $\phi_0(k)$ characterizes the jump structure of the Lévy noise $\xi(t)$. In the subsequent part, for a specific Lévy noise, it has the specific form that $\phi_0(k) = -k^2$ for Gaussian white noise and $\phi_0(k) = -|k|^\beta$ ($0 < \beta < 2$) for non-Gaussian β-stable Lévy noise.

Define the functional $A = \int_0^t U[x(t')]dt'$ and $G(x,A,t)$ as the joint PDF of position x and functional A at time t. To obtain the joint PDF $G(x,A,t)$, we define its Fourier transform $x \to k,\ A \to p$ as

$$G(k,p,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-ikx-ipA}G(x,A,t)dx\,dA,$$

and we write it in the usual way,

$$G(k,p,t) = \langle e^{-ikx(t)}e^{-ipA(t)}\rangle. \tag{4}$$

Throughout this work, we use the convention that the variables in parentheses indicate in what space we are working. Being similar to the increment $\delta x(t)$ in (2), one has the increment $\delta A(t) = A(t+\tau) - A(t) = U(x(t))\tau$ during the time interval τ ($\tau \to 0$). Then we consider the increment of $G(x,A,t)$ in Fourier space, $\delta G(k,p,t) := G(k,p,t+\tau) - G(k,p,t)$, which can be written as

$$\delta G(k,p,t) = \langle e^{-ikx(t+\tau)-ipA(t+\tau)}\rangle - \langle e^{-ikx(t)-ipA(t)}\rangle. \tag{5}$$

Substituting the increment $\delta x(t)$, $\delta A(t)$ into (5) and taking $\tau \to 0$, we obtain

$$\begin{aligned}\delta G(k,p,t) = &\ \langle e^{-ikx(t)-ipA(t)}(e^{-ikg(x(t),t)\delta\eta(t)} - 1)\rangle\\ &- ik\tau\langle e^{-ikx(t)-ipA(t)}f(x(t),t)\rangle\\ &- ip\tau\langle e^{-ikx(t)-ipA(t)}U(x(t))\rangle.\end{aligned} \tag{6}$$

Note that the angular brackets in the first term in (6) denote the average with the joint PDF $G(x,A,t)$ and the PDF of the noise increment $\delta\eta(t)$ since $\delta\eta(t)$ is independent of particle trajectory $x(t)$. The characteristic function of the noise increment $\delta\eta(t)$ in (3) gives

$$\lim_{\tau\to 0}\frac{1}{\tau}\langle(e^{-ikg(x(t),t)\delta\eta(t)} - 1)\rangle = \phi_0(kg(x(t),t)). \tag{7}$$

The second and third terms in (6) are just the Fourier transform of a compound function on $G(x,A,t)$, i.e.,

$$\begin{aligned}&ik\langle e^{-ikx(t)-ipA(t)}f(x(t),t)\rangle\\ &\quad = \mathcal{F}_x\mathcal{F}_A\left\{\frac{\partial}{\partial x}f(x,t)G(x,A,t)\right\},\end{aligned} \tag{8}$$

and

$$ip\langle e^{-ikx(t)-ipA(t)}U(x(t))\rangle = ip\mathcal{F}_x\mathcal{F}_A\{U(x)G(x,A,t)\}. \tag{9}$$

Basing on (7), (8), and (9), dividing (6) by τ, and taking the limit $\tau \to 0$, we obtain the forward Feynman-Kac equation in Fourier space:

$$\begin{aligned}&\frac{\partial G(k,p,t)}{\partial t}\\ &\quad = \mathcal{F}_x\{\phi_0(kg(x,t))G(x,p,t)\}\\ &\qquad - \mathcal{F}_x\left\{\frac{\partial}{\partial x}f(x,t)G(x,p,t) + ipU(x)G(x,p,t)\right\}.\end{aligned} \tag{10}$$

Once the form of $\phi_0(kg(x,t))$ is given for a specific noise, the forward Feynman-Kac equation in x space is obtained.

If the deterministic time variable in the Langevin equation (1) is replaced by a positive nondecreasing one-dimensional Lévy process, called a subordinator [8], then the subordinated stochastic process could be described by the following coupled Langevin equation:

$$\begin{aligned}\dot{x}(s) &= f(x(s), T(s)) + g(x(s), T(s))\xi(s),\\ \dot{T}(s) &= \theta(s).\end{aligned} \tag{11}$$

Here we adopt the fully skewed α-stable Lévy noise $\theta(s)$ with $0 < \alpha < 1$, which is independent of the arbitrary Lévy noise $\xi(s)$. Then the combined process is defined as $y(t) = x(S(t))$ with the inverse α-stable subordinator $S(t)$, which is the first-passage time of the α-stable subordinator $\{T(s), s \geqslant 0\}$ and is defined [34,35] as $S(t) = \inf_{s>0}\{s : T(s) > t\}$. Note that the time-dependent force f and multiplicative noise term g should depend on the physical time $T(s)$, rather than the operation time s, due to a physical interpretation [36,37]. Denote the corresponding functional of process $y(t)$ as $W(t) = \int_0^t U(y(t'))dt'$. Then the forward Feynman-Kac equation of the joint PDF $G(y, W, t)$ in Fourier space $(y \to k, W \to p)$ is

$$\begin{aligned}\frac{\partial G(k, p, t)}{\partial t} &= \mathcal{F}_y\{\phi_0(kg(y, t))\mathcal{D}_t^{1-\alpha}G(y, p, t)\}\\ &\quad - \mathcal{F}_y\left\{\frac{\partial}{\partial y}f(y, t)\mathcal{D}_t^{1-\alpha}G(y, p, t)\right.\\ &\quad \left. + ipU(y)G(y, p, t)\right\},\end{aligned} \tag{12}$$

which recovers (10) when $\alpha = 1$; the detailed derivation is presented in Appendix A. The symbol $\mathcal{D}_t^{1-\alpha}$ is the fractional substantial derivative operator [38,39] defined as

$$\begin{aligned}\mathcal{D}_t^{1-\alpha}G(y, p, t) &= \frac{1}{\Gamma(\alpha)}\left[\frac{\partial}{\partial t} + ipU(y)\right]\\ &\quad \times \int_0^t \frac{e^{-(t-t')ipU(y)}}{(t-t')^{1-\alpha}}G(y, p, t')dt'.\end{aligned}$$

B. Special cases

This subsection provides some special cases of the derived equations in the above subsection:

(i) *Generalized Fokker-Planck equation.* Let $p = 0$ in (10). In this case, $G(x, p = 0, t) = \int_0^\infty G(x, A, t)dA$ reduces to $G(x, t)$, the marginal PDF of finding the particle at position x at time t. Correspondingly, the forward Feynman-Kac equation (10) reduces to the generalized Fokker-Planck equation [14], where three kinds of noises (Gaussian white noise, Poisson white noise, and Lévy stable noise) are considered for the specific forms of this equation.

(ii) *Gaussian white noise.* If the noise $\xi(t)$ is the Gaussian white noise in (12) for arbitrary $f(x, t)$ and $g(x, t)$, we get the forward Feynman-Kac equation:

$$\begin{aligned}&\frac{\partial G(y, p, t)}{\partial t}\\ &= \left[-\frac{\partial}{\partial y}f(y, t) + \frac{\partial^2}{\partial y^2}g^2(y, t)\right]\\ &\quad \times \mathcal{D}_t^{1-\alpha}G(y, p, t) - ipU(y)G(y, p, t).\end{aligned} \tag{13}$$

This equation is consistent with the forward Feynman-Kac equation with the inverse α-stable subordinator proposed in [32] by the Langevin-type approach. Especially when $g(x, t) \equiv 1$, one recovers the equation in [24] derived from CTRWs.

(iii) *Non-Gaussian β-stable noise.* If the noise $\xi(t)$ is the non-Gaussian β-stable noise in (12) for arbitrary $f(x, t)$ and $g(x, t)$, the forward Feynman-Kac equation becomes

$$\begin{aligned}\frac{\partial G(y, p, t)}{\partial t} &= \left[-\frac{\partial}{\partial y}f(y, t) + \nabla_y^\beta|g(y, t)|^\beta\right]\\ &\quad \times \mathcal{D}_t^{1-\alpha}G(y, p, t) - ipU(y)G(y, p, t),\end{aligned} \tag{14}$$

where ∇_y^β is the Riesz space fractional derivative operator with Lévy exponent $-|k|^\beta$ [25,26]; and in y space,

$$\nabla_y^\beta h(y) = -\frac{{}_{-\infty}D_y^\beta h(y) + {}_yD_\infty^\beta h(y)}{2\cos(\beta\pi/2)},$$

where for $n - 1 < \beta < n$,

$${}_{-\infty}D_y^\beta h(y) = \frac{1}{\Gamma(n-\beta)}\frac{d^n}{dy^n}\int_{-\infty}^y \frac{h(y')}{(y-y')^{\beta+1-n}}dy',$$

$${}_yD_\infty^\beta h(y) = \frac{(-1)^n}{\Gamma(n-\beta)}\frac{d^n}{dy^n}\int_y^\infty \frac{h(y')}{(y'-y)^{\beta+1-n}}dy'.$$

This equation extends (13) to the case corresponding to Lévy stable noise, denoting the heavy-tailed jump length in CTRWs, which will be further studied by an application in the next section.

(iv) *A positive functional.* If the functional A is positive at any time t, the Fourier transform $A \to p$ will be replaced by the Laplace transform $G(x, p, t) = \int_0^\infty e^{-pA}G(x, A, t)dA$. Eventually, the forward Feynman-Kac equation corresponding to (12) is obtained by replacing ip with p.

C. Backward equation

The forward Feynman-Kac equation (12) describes the joint PDF $G(x, A, t)$ of position x and functional A. But sometimes, especially in practical applications [4,24,25], what we are interested in may be only the distribution of functional A, which prompts us to develop the backward Feynman-Kac equation governing $G_{x_0}(A, t)$—the PDF of functional A at time t, given that the process has started at x_0. In this subsection, the stochastic process we consider is

$$\dot{x}(t) = f(x(t)) + g(x(t))\xi(t), \tag{15}$$

where $\xi(t)$ is also a Lévy noise. Compared with the model (1), f and g do not explicitly depend on t. This assumption is necessary and can also be found in [24]. If not, the time-dependent force field (or the multiplicative term) induces a different displacement for a particle located at the same position but different time. In this case, it is difficult to let the functional A only depend on the initial position x_0 without using the information of the whole path $x(t)$.

Noting that x_0 here is a deterministic variable instead of a random one, we should explore how functional A depends on the initial position x_0. Different from the increment δA

considered in the forward Feynman-Kac equation, here we should build the relation between A and x_0 as, during the time interval τ ($\tau \to 0$),

$$A(t+\tau)|_{x_0} = \int_0^\tau U(x(t'))dt' + \int_\tau^{t+\tau} U(x(t'))dt' = U(x_0)\tau + A(t)|_{x(\tau)}, \tag{16}$$

where $A(t+\tau)|_{x_0}$ denotes the functional A at time $t+\tau$ with the initial position x_0. Letting $t = 0$ in (2), $x(\tau)$ can be written as

$$x(\tau) = x_0 + f(x_0)\tau + g(x_0)\eta(\tau). \tag{17}$$

Expressing $G_{x_0}(A,t)$ in Fourier space,

$$G_{x_0}(p,t) = \langle e^{-ipA(t)|_{x_0}}\rangle,$$

we could get the form of $G_{x_0}(p,t+\tau)$ from (16) as

$$G_{x_0}(p,t+\tau) = \langle\langle e^{-ipA(t)|_{x(\tau)}}\rangle\rangle e^{-ipU(x_0)\tau}. \tag{18}$$

Since $A(t)|_{x(\tau)}$ denotes the functional A at time t with the initial position $x(\tau)$, it is independent of the event before $x(\tau)$, e.g., $\eta(\tau)$. So the internal angular brackets in (18) denote the average of $A(t)|_{x(\tau)}$ while the external ones denote the average of $\eta(\tau)$. Then the increment $\delta G_{x_0}(p,t)$ can be expressed as

$$\begin{aligned}\delta G_{x_0}(p,t) &:= G_{x_0}(p,t+\tau) - G_{x_0}(p,t)\\ &= \langle\langle e^{-ipA(t)|_{x(\tau)}}\rangle\rangle e^{-ipU(x_0)\tau} - \langle e^{-ipA(t)|_{x_0}}\rangle.\end{aligned}$$

Taking $\tau \to 0$, omitting the higher-order terms of τ, we get

$$\begin{aligned}\delta G_{x_0}(p,t) = &\ \langle\langle e^{-ipA(t)|_{x(\tau)}}\rangle\rangle - \langle e^{-ipA(t)|_{x_0}}\rangle\\ &- ipU(x_0)\tau\langle e^{-ipA(t)|_{x_0}}\rangle,\end{aligned} \tag{19}$$

where the last term is equal to $-ipU(x_0)\tau G_{x_0}(p,t)$. Next, we will deal with the first two terms on the right-hand side of (19) carefully by keeping the terms of $O(\tau)$ but removing the terms $o(\tau)$.

Taking Fourier transform $x_0 \to k_0$ in (19), $\langle e^{-ipA(t)|_{x_0}}\rangle$ then becomes $G_{k_0}(p,t)$. But for $\langle\langle e^{-ipA(t)|_{x(\tau)}}\rangle\rangle$, it is not easy to get the form in Fourier space. Hence, we first take $g(x) \equiv 1$, i.e., the noise in this system is additive noise. Then for the convenience of the reader, we put the detailed derivations for nonconstant $g(x)$ in Appendix B and directly present the results for general $g(x)$ here.

Denote $T_\eta = \langle e^{-ipA(t)|_{x(\tau)}}\rangle$. Since $g(x) \equiv 1$, (17) becomes $x(\tau) = x_0 + f(x_0)\tau + \eta(\tau)$, where $f(x_0)$ depends on the initial position x_0. Therefore, $x(\tau)$ is not a simple shift of x_0 and we write the Fourier transform ($x_0 \to k_0$) of $\langle T_\eta\rangle$ as

$$\mathcal{F}_{x_0}\{\langle T_\eta\rangle\} = \left\langle \int_{-\infty}^{\infty} e^{-ik_0x(\tau)}T_\eta e^{ik_0[f(x_0)\tau+\eta(\tau)]}dx_0\right\rangle.$$

Then we turn dx_0 into $dx(\tau)$ and get

$$\begin{aligned}\mathcal{F}_{x_0}\{\langle T_\eta\rangle\} = &\left\langle \int_{-\infty}^{\infty} e^{-ik_0x(\tau)}T_\eta e^{ik_0[f(x_0)\tau+\eta(\tau)]}dx(\tau)\right\rangle\\ &-\left\langle \int_{-\infty}^{\infty} e^{-ik_0x(\tau)}T_\eta e^{ik_0[f(x_0)\tau+\eta(\tau)]}\frac{df(x_0)}{dx_0}\tau dx_0\right\rangle.\end{aligned} \tag{20}$$

Since all x_0 and $f(x_0)$ are multiplied by τ in (20), replacing all x_0 by $x(\tau)$ in (20) yields higher-order terms of τ, which can be omitted. Then writing $e^{ik_0f(x_0)\tau} \simeq 1 + ik_0f(x_0)\tau$, the first term on the right-hand side of (20) reduces to

$$\left\langle \int_{-\infty}^{\infty} e^{-ik_0x(\tau)}T_\eta e^{ik_0\eta(\tau)}dx(\tau)\right\rangle + ik_0\tau\left\langle \int_{-\infty}^{\infty} e^{-ik_0x(\tau)}T_\eta f(x(\tau))dx(\tau)\right\rangle,$$

where the latter term of the above is equal to

$$\tau\mathcal{F}_{x_0}\left\{\frac{\partial}{\partial x_0}f(x_0)G_{x_0}(p,t)\right\}. \tag{21}$$

The second term on the right-hand side of (20) gives

$$\begin{aligned}&-\tau\left\langle \int_{-\infty}^{\infty} e^{-ik_0x(\tau)}T_\eta\frac{df(x(\tau))}{dx(\tau)}dx(\tau)\right\rangle\\ &= -\tau\mathcal{F}_{x_0}\left\{\frac{df(x_0)}{dx_0}G_{x_0}(p,t)\right\}.\end{aligned}$$

Therefore, the Fourier transform of $\langle\langle e^{-ipA(t)|_{x(\tau)}}\rangle\rangle - \langle e^{-ipA(t)|_{x_0}}\rangle$ in (19), replacing $x(\tau)$ by y, reduces to

$$\left\langle \int_{-\infty}^{\infty} e^{-ik_0y}T_\eta(e^{ik_0\eta(\tau)}-1)dy\right\rangle + \tau\mathcal{F}\left\{f(x_0)\frac{\partial G_{x_0}(p,t)}{\partial x_0}\right\},$$

i.e.,

$$\tau\phi_0(-k_0)G_{k_0}(p,t) + \tau\mathcal{F}_{x_0}\left\{f(x_0)\frac{\partial G_{x_0}(p,t)}{\partial x_0}\right\}$$

on account of (7). Dividing (19) by τ and taking the limit $\tau \to 0$, we obtain the backward Feynman-Kac equation in Fourier space:

$$\begin{aligned}&\frac{\partial G_{k_0}(p,t)}{\partial t}\\ &= \phi_0(-k_0)G_{k_0}(p,t) + \mathcal{F}_{x_0}\left\{f(x_0)\frac{\partial G_{x_0}(p,t)}{\partial x_0}\right.\\ &\left.-ipU(x_0)G_{x_0}(p,t)\right\}.\end{aligned} \tag{22}$$

If the noise $\xi(t)$ is Gaussian white noise, then $\phi_0(-k_0) = -k_0^2$ and we get the backward Feynman-Kac equation:

$$\begin{aligned}\frac{\partial G_{x_0}(p,t)}{\partial t} = &\frac{\partial^2}{\partial x_0^2}G_{x_0}(p,t) + f(x_0)\frac{\partial}{\partial x_0}G_{x_0}(p,t)\\ &- ipU(x_0)G_{x_0}(p,t),\end{aligned} \tag{23}$$

which is the same as the backward Feynman-Kac equation proposed in [24] with $\alpha = 1$ in the CTRW framework. Here, α is the exponent characterizing the waiting time PDF in CTRWs or the subordinator PDF in the Langevin system.

If the noise $\xi(t)$ is non-Gaussian β-stable noise, i.e., $\phi_0(-k_0) = -|k_0|^\beta$, then the backward Feynman-Kac equation becomes

$$\frac{\partial G_{x_0}(p,t)}{\partial t} = \nabla_{x_0}^{\beta} G_{x_0}(p,t) + f(x_0)\frac{\partial}{\partial x_0} G_{x_0}(p,t) - ipU(x_0)G_{x_0}(p,t), \tag{24}$$

which is an extension for the backward Feynman-Kac equation derived in the CTRW framework [25], in which jump length obeys heavy-tailed distribution but without a force field $f(x)$. In the case that $g(x)$ is not a constant, we assume $\xi(t)$ to be Gaussian white noise and derive the backward Feynman-Kac equation as

$$\frac{\partial G_{x_0}(p,t)}{\partial t} = g^2(x_0)\frac{\partial^2}{\partial x_0^2} G_{x_0}(p,t) + f(x_0)\frac{\partial}{\partial x_0} G_{x_0}(p,t) - ipU(x_0)G_{x_0}(p,t), \tag{25}$$

which goes back to (23) when $g(x_0) \equiv 1$. See the detailed derivation in Appendix B.

III. APPLICATIONS

For the stochastic dynamics driven by additive white noise (or Gaussian jump length in CTRWs), there have been many applications for their corresponding Feynman-Kac equations [24,25]. Here we provide the applications for Feynman-Kac equations of more general stochastic processes discussed above. More concretely, two applications of the generalized Feynman-Kac equations are given, including the occupation time in the positive half-space of a particle moving in a box with multiplicative Gaussian white noise and the area under the curve of trajectory of the stochastic process with a harmonic potential driven by additive Lévy noise.

A. Occupation time in the positive half of a box

The distribution of the occupation time of Brownian motion was first computed by Lévy with probabilistic methods [40]. Later, Kac derived it using the Feynman-Kac formalism [1]. More recently, Majumdar derived it based on the backward Fokker-Planck approach [4], and Carmi and Barkai derived it for the non-Brownian case from the Feynman-Kac equation [24]. Here, we extend it to the case with multiplicative Gaussian white noise. We first discuss the occupation time in $x > 0$ for a particle moving freely but with a multiplicative Gaussian white noise in a box $[-L, L]$, $L > 0$, and then we give its direct application—the first-passage time. As a special occupation time, the first-passage time has also attracted a lot of attentions, particularly in relation to persistence. Persistence and the first-passage time for a Lévy flight and fractional Brownian motion have been discussed in [41]; one can refer to the review [42] for more properties of first-passage time in nonequilibrium systems.

1. Distribution of occupation time

We take $U(x_0)$ in (25) to be $\Theta(x_0)$ [$\Theta(x) = 1$ for $x \geqslant 0$ and $\Theta(x) = 0$ otherwise], and then get the occupation time of a particle in the positive half-space as $T_+(t) = \int_0^t \Theta[x(t')]dt'$. In this case, $T_+(t)$ is always positive. We replace the Fourier transform by the Laplace transform in (25) and remove i in it. To find the distribution of $T_+(t)$, we take the Laplace transform of the backward Feynman-Kac equation (25) $(t \to s)$:

$$sG_{x_0}(p,s) - 1 = g^2(x_0)\frac{\partial^2}{\partial x_0^2} G_{x_0}(p,s) + f(x_0)\frac{\partial}{\partial x_0} G_{x_0}(p,s) - pU(x_0)G_{x_0}(p,s). \tag{26}$$

To consider the effect of multiplicative noise, we specify $f(x_0) = 0$ and $g(x_0) = aL - x_0$ with $a > 1$ to keep $g(x_0)$ positive. The constant aL in $g(x_0)$ measures the intensity of the additive component of the random force. Systems described by Langevin equations involving both multiplicative and additive components of the random force are common in nature [43]. Examples include polymers in turbulent flow [44,45], motion of a passive scalar in a random velocity field [46,47], and noise in dye lasers [48,49]. The linear term $g(x_0)$ is also discussed in [30]. Based on these potentially physical applications and its relatively simple form, we take $g(x_0) = aL - x_0$ to analytically obtain the solution of the backward Feynman-Kac equation (26).

To further examine the effect of a different sign of multiplicative noise, $g(x_0) = aL + x_0$ is also discussed. Interestingly, the theoretical results are quite close, just replacing $a + 1$ by $a - 1$ in (32) and (34). The simulation results for the cases $g(x_0) = aL \pm x_0$ are shown together in Figs. 1 and 2.

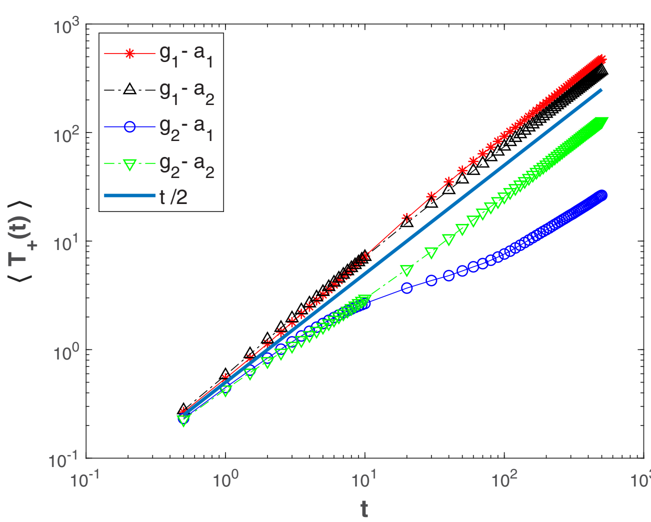

FIG. 1. Mean value of the occupation time T_+ in positive half-space for a particle moving in the box $[-1, 1]$. Here g_1 represents the case $g(x) = aL - x$ and g_2 the case $g(x) = aL + x$. The other parameters are $a_1 = 1.1$ and $a_2 = 2$. Four kinds of different cases ($g_1 - a_1$ denoted with stars; $g_1 - a_2$ denoted with triangles; $g_2 - a_2$ denoted with inverted triangles; $g_2 - a_1$ denoted with circles) are simulated with 1000 trajectories and the total time $T = 500$.

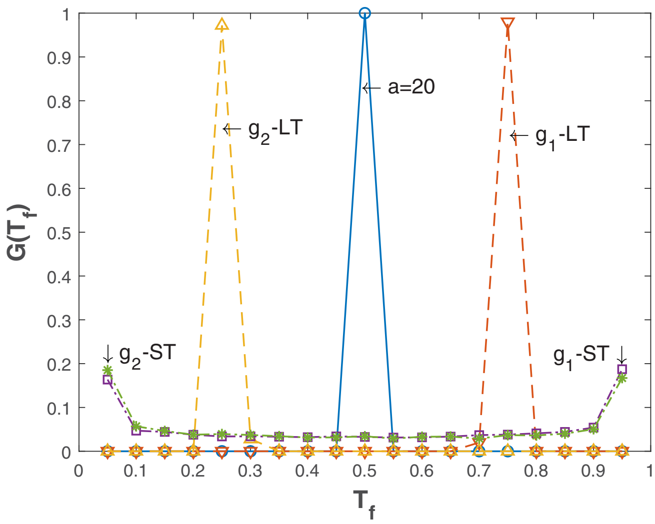

FIG. 2. PDF of the occupation fraction T_f in positive half-space for a particle moving in the box $[-1, 1]$. Here the PDFs of T_f for long times ($T = 500$) and short times ($T = 0.01$) are shown together, recorded as "LT" and "ST," respectively. 1000 trajectories are used. In this figure, g_1 and g_2 represent the cases $g(x) = aL - x$ and $g(x) = aL + x$, respectively. The solid line denotes $G(T_f)$ in long times with $a = 20$ as well as $g = aL \pm x$ (the lines coincide for the cases $g = g_1$ and $g = g_2$). The other lines represent the case $a = 2$, but g or times are different.

For the case of $g(x_0) = aL - x_0$, (26) becomes

$$(aL - x_0)^2 \frac{\partial^2 G_{x_0}(p,s)}{\partial x_0^2} - [s + pU(x_0)]G_{x_0}(p,s) = -1.$$

With a variable substitution $y = aL - x_0 > 0$, the celebrated Euler equation is obtained:

$$y^2 \frac{\partial^2 \tilde{G}_y(p,s)}{\partial y^2} - [s + p\tilde{U}(y)]\tilde{G}_y(p,s) = -1.$$

It can be solved by a new variable substitution $y = e^t$. Finally, we get the solutions of (26) in two half-spaces, respectively,

$$G_{x_0}(p,s) = \begin{cases} C_1(aL - x_0)^{\lambda_1} + C_2(aL - x_0)^{\lambda_2} + \frac{1}{s+p}, & x_0 > 0, \\ C_3(aL - x_0)^{\lambda_3} + C_4(aL - x_0)^{\lambda_4} + \frac{1}{s}, & x_0 < 0, \end{cases} \tag{27}$$

where

$$\lambda_{1,2} = \frac{1 \mp \sqrt{1 + 4(s+p)}}{2}, \quad \lambda_{3,4} = \frac{1 \mp \sqrt{1 + 4s}}{2}. \tag{28}$$

Specify the reflecting boundary condition to (27), i.e.,

$$\left.\frac{\partial G_{x_0}(p,s)}{\partial x_0}\right|_{x_0 = \pm L} = 0. \tag{29}$$

The two conditions in (29) together with another two conditions [$G_{x_0}(p,s)$ and its derivative are continuous at $x_0 = 0$] can solve the four coefficients C_{1-4} in (27). Then we get the final solution $G_{x_0}(p,s)$ at $x_0 = 0$:

$$G_0(p,s) = \frac{p}{s(p+s)} \frac{F_1 F_2}{F_3 F_4 - F_1 F_2} + \frac{1}{s}, \tag{30}$$

where

$$F_1 = a^{\lambda_4} - \frac{\lambda_4}{\lambda_3}(a+1)^{\lambda_4 - \lambda_3} a^{\lambda_3},$$
$$F_2 = \lambda_2[a^{\lambda_2} - (a-1)^{\lambda_2 - \lambda_1} a^{\lambda_1}],$$
$$F_3 = \lambda_4[a^{\lambda_4} - (a+1)^{\lambda_4 - \lambda_3} a^{\lambda_3}],$$
$$F_4 = a^{\lambda_2} - \frac{\lambda_2}{\lambda_1}(a-1)^{\lambda_2 - \lambda_1} a^{\lambda_1}.$$

Equation (30) is the PDF of T_+ in Laplace space, but it cannot be inverted easily. Nevertheless, the first moment of the occupation time $T_+(t)$ can be computed by taking the inverse Laplace transform [50] of

$$\langle T_+(s)\rangle = -\left.\frac{\partial G_0(p,s)}{\partial p}\right|_{p=0}.$$

Using this formula, from (30) one can get

$$\langle T_+(s)\rangle = -\frac{1}{s^2} \left.\frac{F_1 F_2}{F_3 F_4 - F_1 F_2}\right|_{p=0}. \tag{31}$$

For long times, i.e., $s \ll 1$ ($\lambda_1 = \lambda_3 \sim -s$, $\lambda_2 = \lambda_4 \sim 1$),

$$\langle T_+(t)\rangle \simeq \frac{a+1}{2a} t. \tag{32}$$

For short times, i.e., $s \gg 1$ ($\lambda_1 = \lambda_3 \sim -\sqrt{s}$, $\lambda_2 = \lambda_4 \sim \sqrt{s}$),

$$\langle T_+(t)\rangle \simeq \tfrac{1}{2} t. \tag{33}$$

It can be seen that for both long times and short times, $\langle T_+(t)\rangle$ scales asymptotically as t, which is also verified in Fig. 1. Four curves begin as $t/2$ and finally turn to $\frac{a+1}{2a}t$ for the case $g(x) = aL - x$ or $\frac{a-1}{2a}t$ for the case $g(x) = aL + x$. Therefore, it is natural to consider the PDF of the occupation fraction $T_f \equiv T_+/t$.

For long times, i.e., $s \ll 1$, together with $p \ll 1$ due to the scale of $T_+(t)$, we have $\lambda_1 \sim -(s+p)$, $\lambda_2 \sim 1$, $\lambda_3 \sim -s$, $\lambda_4 \sim 1$ from (28) and $F_1 \sim (a+1)/s$, $F_2 \sim 1$, $F_3 \sim -1$, $F_4 \sim (a-1)/(s+p)$, which give the asymptotic expression of (30):

$$G_0(p,s) \simeq \frac{2a}{2as + (a+1)p}.$$

By inverting the scaling form of a double Laplace transform in [51], after some calculations, using the nascent δ function:

$$\lim_{\epsilon \to 0} \frac{\epsilon}{\pi(x^2 + \epsilon^2)} = \delta(x),$$

we obtain the PDF of T_f:

$$G(T_f) \simeq \frac{r}{T_f}\delta(T_f - r) \overset{d}{=} \delta(T_f - r), \tag{34}$$

where $r = \frac{a+1}{2a}$ and $\overset{d}{=}$ denotes identical distribution. Note that the PDF of T_f in (34) is normalized. Especially, T_f reduces to a deterministic event for large t, occurring at r with probability 1. But the value r depends on a. When a is sufficiently large, this value will approach $\frac{1}{2}$ (see the curve for $a = 20$, which has a peak at $\frac{1}{2}$ in Fig. 2). This phenomenon has an intuitive explanation that in this case the multiplicative noise term approximates an additive noise term aL and thus it is consistent with the case of $\alpha = 1$ in [24]. On the other hand, when a is small and close to 1, the value r is near 1, which means that the particle stays in a positive half-plane all the time. This phenomenon results from the multiplicative noise term. We simulate $G(T_f)$ with $a = 2$ and it has a peak at $\frac{a+1}{2a}$ for $g(x) = aL - x$ (see g_1- LT in Fig. 2) and a peak at $\frac{a-1}{2a}$ for $g(x) = aL + x$ (see g_2- LT in Fig. 2).

For short times, i.e., $s \gg 1$, we have $\lambda_1 \sim -\sqrt{s+p}$, $\lambda_2 \sim \sqrt{s+p}$, $\lambda_3 \sim -\sqrt{s}$, $\lambda_4 \sim \sqrt{s}$ from (28) and $F_1 \sim (a+1)^{2\sqrt{s}}a^{-\sqrt{s}}$, $F_2 \sim \sqrt{s+p}a^{\sqrt{s+p}}$, $F_3 \sim -\sqrt{s}(a+1)^{2\sqrt{s}}a^{-\sqrt{s}}$, $F_4 \sim a^{\sqrt{s+p}}$, which result in the asymptotic expression of (30):

$$G_0(p,s) \simeq -\frac{p}{s(p+s)}\frac{\sqrt{s+p}}{\sqrt{s}+\sqrt{s+p}} + \frac{1}{s}.$$

Then we obtain the PDF of T_f:

$$G(T_f) \simeq \frac{1}{\pi}\frac{1}{\sqrt{x}\sqrt{1-x}}, \tag{35}$$

which coincides exactly with the first Lévy arcsine law as the classical Brownian functional [4] (generalized arcsine laws for fractional Brownian motion have been discussed in [52]). This result is as expected since for short times the particle does not interact with the boundaries and behaves like a free particle. Furthermore, if the time t is sufficiently small, such that $x \ll aL$, then the multiplicative noise term approximates an additive noise term aL, so the PDFs of occupation fractions T_f in cases $g(x) = aL \pm x$ all become the Lamperti PDF and present a symmetric curve with two peaks at $T_f = 0$ and $T_f = 1$ (see g_2- ST and g_1- ST in Fig. 2). Though $x \ll aL$, there is still a slight difference between two kinds of multiplicative noises $g(x) = aL \pm x$. Therefore, the two curves in Fig. 2 look a little skewed to one side (0 or 1).

For both long times and short times, from another perspective, the particle driven by the multiplicative noise term $g(x) = aL - x$ is more likely to move to the positive half-space since the distribution of T_f has a larger proportion on the right side of 0.5 in Fig. 2. On the contrary, for $g(x) = aL + x$, the distribution of T_f concentrates on the left side of 0.5. This phenomenon can be explained by the corresponding Fokker-Plank equation of (13). Taking $\alpha = 1$, $p = 0$, $f(x,t) = 0$ in (13) and using the notation $g'(x) = dg(x)/dx$, one obtains

$$\begin{aligned}\frac{\partial G(x,t)}{\partial t} &= \frac{\partial^2}{\partial x^2}g^2(x)G(x,t)\\ &= g^2(x)\frac{\partial^2 G(x,t)}{\partial x^2} + 4g(x)g'(x)\frac{\partial G(x,t)}{\partial x}\\ &\quad + 2(g'(x)^2 + g(x)g''(x))G(x,t),\end{aligned} \tag{36}$$

where the coefficient $E := 4g(x)g'(x)$ in front of the first-order derivative of $G(x,t)$ is called noise-induced drift [7]. The cases of $g(x) = aL - x$ and $g(x) = aL + x$ lead to $E = 4(x - aL) < 0$ and $E = 4(x + aL) > 0$, respectively, which means that the multiplicative noise term $g(x) = aL - x$ induces a positive drift while $g(x) = aL + x$ induces a negative drift. Furthermore, as the time goes on, the curve of the distribution of occupation fraction T_f in Fig. 2 changes from concave upwards to convex upwards, which is similar to the case of Brownian motion. The reason is that by a variable substitution, the Langevin equation for $x(t)$ with multiplicative noise can be turned into another Langevin equation for a new process $y(t)$ with an additive noise [i.e., $y(t)$ is the Brownian motion with a drift]; see [53,54] for more details.

2. Distribution of first-passage time

The application of occupation time in the previous part is a good beginning to consider a problem of the first-passage time t_f. Still assume a particle moves freely in the box $[-L, L]$, with t_f denoting the time it takes a particle starting at $x_0 = -bL$, $0 < b < 1$ to reach $x = 0$ for the first time [55]. The distribution of t_f can be obtained from the occupation time functional by using an identity due to Kac [56]:

$$\mathbb{P}(t_f > t) = \mathbb{P}\left(\max_{0\leqslant\tau\leqslant t} x(\tau) < 0\right) = \lim_{p\to\infty} G_{x_0}(p,t),$$

where $G_{x_0}(p,t)$ is the Laplace transform of the PDF of functional T_+ in the previous subsection. If the particle has crossed $x = 0$ at time t, we have $T_+ > 0$ and $e^{-pT_+} = 0$ for $p \to \infty$. Then the two sides of the last equation are equal to 0. Otherwise $T_+ = 0$ and $e^{-pT_+} = 1$ lead two sides equal to 1. So now, taking $x_0 = -bL$ in (27) in the previous subsection, we get

$$G_{-bL}(p,s) = \frac{p}{s(p+s)}\frac{F_{1b}F_2}{F_3F_4 - F_1F_2} + \frac{1}{s}, \tag{37}$$

where F_{1-4} are the same as the ones in (30) and

$$F_{1b} = (a+b)^{\lambda_4} - \frac{\lambda_4}{\lambda_3}(a+1)^{\lambda_4-\lambda_3}(a+b)^{\lambda_3}.$$

When $p \to \infty$, we consider the long-time behavior (i.e., $s \to 0$) and have $\lambda_1 \sim -\sqrt{p}$, $\lambda_2 \sim \sqrt{p}$, $\lambda_3 \sim -s$, $\lambda_4 \sim 1$. Substituting λ_{1-4} into (37) yields

$$\lim_{p\to\infty} G_{-bL}(p,s) \simeq \ln\left(1 + \frac{b}{a}\right) - \frac{b}{1+a} =: C_{ab},$$

which is a constant only depending on a and b. Considering the first-passage time PDF satisfying $f(t) = \frac{\partial}{\partial t}[1 - \mathbb{P}(t_f > t)]$, we have the PDF of t_f in Laplace s space,

$$f(s) \simeq 1 - C_{ab}s \simeq e^{-C_{ab}s},$$

and thus

$$f(t_f) \simeq \delta(t_f - C_{ab}).$$

This means that the first-passage time is a deterministic event, occurring at C_{ab} with probability 1; see the distribution of first-passage time t_f in Fig. 3. Furthermore, for $0 < b < 1 < a$, C_{ab} is monotonously increasing with the increase of b but decreasing with the increase of a, being the same as physical intuition.

B. Area under the random-walk curve

Now we turn to another application of the Langevin system containing a force field and non-Gaussian β-stable noise. In this case, we take $U(x) = x$ and get the functional $A_x = \int_0^t x(t')dt'$, denoting the total area under the curve of trajectory $x(t)$ [57,58]. This functional A_x is also related to the

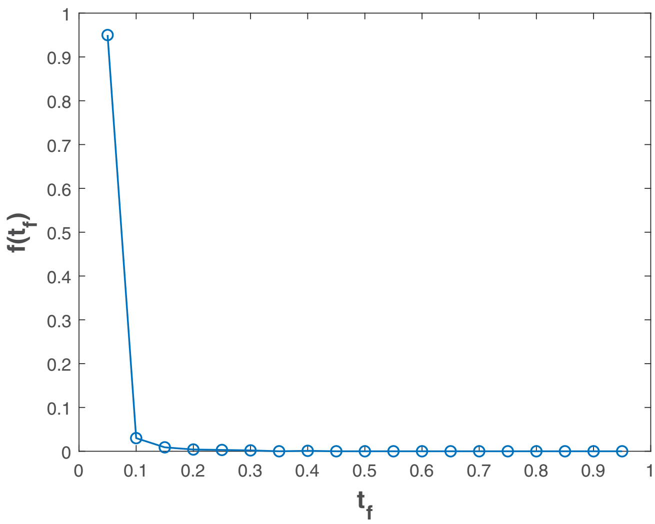

FIG. 3. PDF of the first passage time of a particle in the box $[-1, 1]$ starting at $-b$ and reaching $x = 0$ for the first time. We simulate it with 1000 trajectories and the total time $T = 10$. The parameters are $b = 1/2$, $a = 2$, and then $C_{ab} = 0.0565$, which is consistent with the curve that has a peak near 0.0565 in the figure.

phase accumulated by spins in an NMR experiment [58]. If we further assume that the particle starts and ends at the origin $x(0) = x(t) = 0$ but stays positive in between, this motion is called excursion [4]. The question about the area under Brownian excursion has been studied quite extensively by mathematicians [59–64]. Recently, the applications of Brownian excursion were further studied [65,66] and even extended to the Bessel excursion for anomalous diffusion [67].

Since the analytical solutions of $G_{x_0}(p,t)$ in (24) cannot be easily obtained due to the Riesz space fractional derivative operator ∇_x^β, we resort to the forward Feynman-Kac equation (14) by integrating the solution $G(x, p, t)$ over x with initial position x_0 to get the marginal PDF of $G_{x_0}(p,t)$.

In the case of a harmonic potential, where $V(x,t) = bx^2/2\,(b > 0)$ $[f(x,t) = -\partial V(x,t)/\partial x = -bx]$ and $g(x,t) \equiv 1$, $U(x) = x$, $\alpha = 1$, the forward Feynman-Kac equation (10) takes the form

$$\frac{\partial G(k,p,t)}{\partial t} + (bk - p)\frac{\partial}{\partial k}G(k,p,t) = \phi_0(k)G(k,p,t).$$

Its general solution is given by the following [68]:

$$G(k,p,t) = \exp\left[\int_0^k \frac{\phi_0(z)}{bz - p}dz + c_1\right] \times \Psi\left[\frac{1}{b}\ln|bk - p| - t + c_2\right], \tag{38}$$

where c_1, c_2 are constants and $\Psi(x)$ is an arbitrary function. Using the initial condition $G(k, p, 0) = 1$ (the particle starts at $x_0 = 0$), we get

$$\Psi\left[\frac{1}{b}\ln|bk - p| + c_2\right] = \exp\left[-\int_0^k \frac{\phi_0(z)}{bz - p}dz - c_1\right]. \tag{39}$$

Then replacing k by $l(k) := \frac{bk-p}{be^{bt}} + \frac{p}{b}$, (39) becomes

$$\Psi\left[\frac{1}{b}\ln|bk - p| - t + c_2\right] = \exp\left[-\int_0^{l(k)} \frac{\phi_0(z)}{bz - p}dz - c_1\right].$$

Substituting this result into (38) gives

$$G(k,p,t) = \exp\left[\int_{l(k)}^k \frac{\phi_0(z)}{bz - p}dz\right].$$

Letting $k = 0$, we get the PDF of functional A_x in Fourier space ($A_x \to p$):

$$G(p,t) := G(k,p,t)|_{k=0} = \exp\left[\int_{\frac{p}{b}(1-e^{-bt})}^0 \frac{\phi_0(z)}{bz - p}dz\right]. \tag{40}$$

Now we discuss the specific dynamical behavior of the functional A_x with Lévy β-stable noise $[\phi_0(k) = -|k|^\beta]$. Considering a variable substitution $z = \frac{p}{b}(1 - e^{-bt})y$, (40) can be represented in the form

$$\ln G(p,t) = -C_b(t)\left(\frac{1 - e^{-bt}}{b}\right)^{\beta+1}|p|^\beta, \tag{41}$$

where $C_b(t)$ is independent of p [69]:

$$C_b(t) = \int_0^1 \frac{y^\beta}{1 - (1 - e^{-bt})y}dy = B(\beta + 1, 1)\,{}_2F_1(1, \beta + 1; \beta + 2; 1 - e^{-bt}).$$

It can be seen from (41) that the functional A_x also obeys Lévy β-stable distribution. Next what we need to pay attention to is the coefficient in front of $|p|^\beta$ in (41).

For long times $t \to \infty$, we find that

$$C_b(t) = \int_0^1 \frac{y^\beta}{1 - (1 - e^{-bt})y}dy \simeq bt,$$

since this integral scales as bt in two extreme cases ($\beta = 0$ and $\beta = 2$). Substituting it into (41) makes

$$G(p,t) \simeq \exp(-b^{-\beta}t|p|^\beta) \quad \text{as} \quad t \to \infty. \tag{42}$$

For short times $t \to 0$, ${}_2F_1(1, \beta + 1; \beta + 2; 1 - e^{-bt}) \sim 1$, and thus

$$G(p,t) \simeq \exp\left(-\frac{t^{\beta+1}}{\beta + 1}|p|^\beta\right) \quad \text{as} \quad t \to 0. \tag{43}$$

For the special case $\beta = 2$, i.e., Gaussian white noise, by the formula

$$\langle A_x^2\rangle = \frac{\partial^2}{\partial p^2}G(p,t)\bigg|_{p=0},$$

we get

$$\langle A_x^2\rangle \simeq 2b^{-2}t \quad \text{as} \quad t \to \infty \tag{44}$$

and

$$\langle A_x^2\rangle \simeq \tfrac{2}{3}t^3 \quad \text{as} \quad t \to 0, \tag{45}$$

which are verified by numerical simulations. In Fig. 4, the functional A_x exhibits a crossover between different scaling regimes (from t^3 to t). When the particle begins its movement from the origin, i.e., $x \ll 1$, the effect of force ($f = -bx$) can

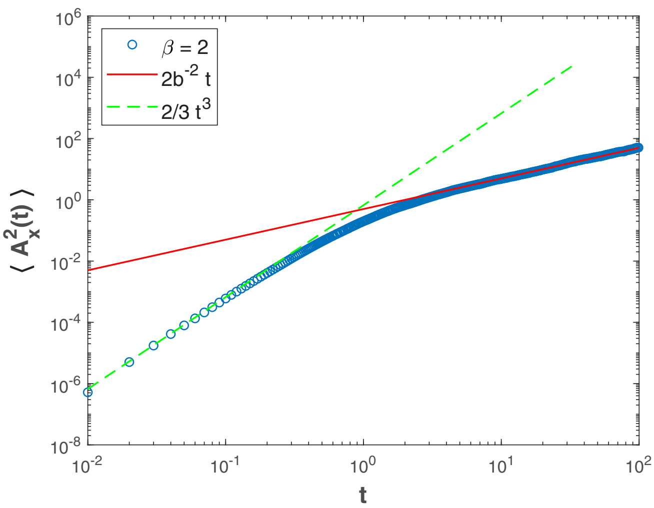

FIG. 4. Second moment $\langle A_x^2\rangle$ of the area under the trajectory curve with $\beta = 2$ and $b = 2$, generated with 1000 trajectories and the total time $T = 100$. The circles denote the simulation results. The dotted line denotes the theoretical result $\langle A_x^2\rangle \simeq \frac{2}{3}t^3$ for short time while the solid line represents $\langle A_x^2\rangle \simeq \frac{2}{b^2}t$ for long time. This figure shows a crossover of scaling regimes from t^3 to t.

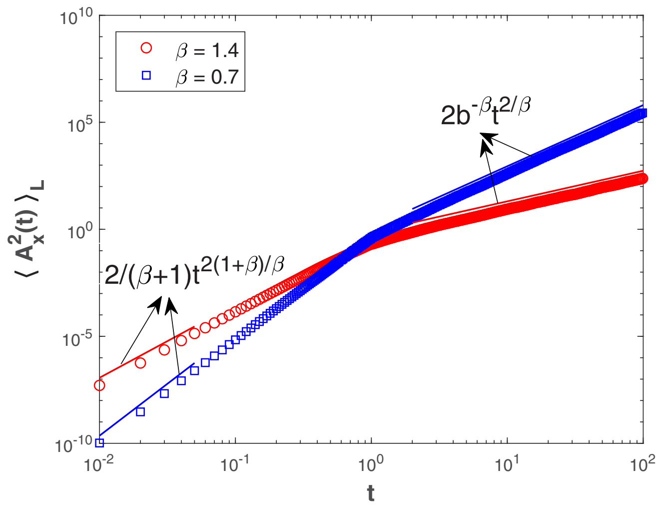

FIG. 5. Pseudo second moment $\langle A_x^2\rangle_L$ by a cutoff approach, generated with 1000 trajectories and the total time $T = 100$ with $\beta =$ 1.4, 0.7, and $b = 2$. The circles and squares denote the simulation results of $\beta = 1.4$ and 0.7, respectively. The solid lines are the theoretical results with slope $2(\beta+1)/\beta$ for short time and $2/\beta$ for long time. It shows that for different Lévy β-stable noise ($0 < \beta < 2$), there is a crossover of scaling regimes from $t^{2(\beta+1)/\beta}$ to $t^{2/\beta}$.

be omitted. As time goes on, this effect is getting bigger and bigger, and eventually it produces the multiscale phenomenon. On the contrary, for the case without the force field f, i.e., $b = 0$, it is equivalent to $b \to 0$ for any t from (41). Then only the single-scale phenomenon $\langle A_x^2\rangle \simeq \frac{2}{3}t^3$ can be observed, which is consistent with [25] by taking $\alpha = 1$ there.

As for the general case $0 < \beta < 2$, the mean-squared displacement of A_x diverges [70]: $\langle A_x^2\rangle \to \infty$. But for a particle with nondiverging mass, a finite velocity of propagation exists, making long instantaneous jumps impossible. Their fractional moments can be written as

$$\langle |A_x|^\delta\rangle \propto \tilde{t}^{\delta/\beta}, \tag{46}$$

where $0 < \delta < \beta < 2$. From (42) and (43), one can get that in (46) $\tilde{t}$ should be $t^{\beta+1}$ for short times and t for long times. So we rescale the fractional moments and get the pseudo second moment $[A_x^2] \propto \tilde{t}^{2/\beta}$. An alternative method is to consider the $(A_x - t)$ scaling relations, or to measure the width of the PDF $G(A_x, t)$ rather than its variance [70]. More precisely, enclose the particle in an imaginary growing box [19] and define

$$\langle A_x^2\rangle_L := \int_{L_1 t^{1/\beta}}^{L_2 t^{1/\beta}} A_x^2 G(A_x, t)\, dA_x \simeq \tilde{t}^{2/\beta},$$

where L_1 and L_2 are chosen to adapt the scaling regimes in (42) and (43), i.e., for long time $-L_1 = L_2 = \sqrt{2b^{-\beta}}$ while for short time $-L_1 = L_2 = \sqrt{2/(1+\beta)}$. This has been implemented numerically and can be seen in Fig. 5. We take β to be 1.4 or 0.7 and $b = 2$. The markers denote simulation results while the solid lines are the theoretical ones,

$$\langle A_x^2\rangle \simeq 2b^{-\beta} t^{2/\beta} \quad \text{as} \quad t \to \infty,$$

and

$$\langle A_x^2\rangle \simeq \frac{2}{\beta+1} t^{2(\beta+1)/\beta} \quad \text{as} \quad t \to 0,$$

which go back to (44) and (45), respectively, when $\beta = 2$.

IV. SUMMARY AND DISCUSSION

The Feynman-Kac equations have striking advantages in characterizing the PDFs of various general statistical quantities. Under the CTRW framework, there have been systematic derivations of the equations. But the CTRWs cannot well describe the multiplicative noise and the arbitrary additive noise together with a force field, being more conveniently modeled by the Langevin system.

The contributions of this paper are twofold: deriving the forward Feynman-Kac equation from the overdamped Langevin equation driven by an arbitrary Lévy noise together with a time-dependent multiplicative noise term and an arbitrary time-dependent external force field, and deriving the backward Feynman-Kac equation with an arbitrary additive Lévy noise or a multiplicative Gaussian white noise, together with an arbitrary force field. For some special noises and force fields, the obtained equations are consistent with the existing works. Two applications of the derived equations to solve PDFs of the occupation time T_+ and the total area A_x under the curve of the particle trajectory are carefully provided. In the first application, we take a multiplicative Gaussian white noise and restrict the particle in a box $[-L, L]$ with reflecting boundary conditions. Then we find the phenomenon that for long times the PDF of the occupation fraction is a δ-function, which is similar to the case of Brownian motion [3,71]. In the second application, we take an additive Lévy β-stable noise and find that the functional A_x also obeys Lévy stable distribution but experiences a crossover between different scaling regimes. Using the techniques of a subordinator in deriving Feynman-Kac equations presented in [32], we also derive the forward Feynman-Kac equations from the coupled Langevin equation with an α-stable subordinator and arbitrary Lévy noise based on the Langevin framework.

In view of the complexity of deriving these equations, it is quite important to verify the correctness of the equations. So we choose the specific multiplicative term $g(x) = aL \pm x$ in the first application and the harmonic potential in the second application. In these cases, the analytic solutions can be obtained, and many simulations are presented to implicitly show the correctness of the forward and backward Feynman-Kac equations while providing the applications of the equations. In turn, based on the correctness of the models, the cases with more general multiplicative noises or potentials can be investigated; without analytic solutions, some results can also be obtained through simulations or numerical approximations, which will be considered in our future works and will not be presented here due to space limitations.

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APPENDIX A: FORWARD FEYNMAN-KAC EQUATION WITH A SUBORDINATOR

Since the forward Feynman-Kac equation in the case of Gaussian white noise $\xi(s)$ has been derived in [32], we can make the best of some techniques of the subordinator in that paper and extend its result to arbitrary Lévy noise $\xi(s)$ in the Langevin framework. Some of the calculations about the subordinator may be omitted for simplicity; see [32] for details.

Since $y(t) = x(S(t))$, we can build the Langevin equation of $y(t)$ from model (11) as

$$\dot{y}(t) = f(y(t), t)\dot{S}(t) + g(y(t), t)\xi(S(t))\dot{S}(t).$$

Being similar to (2), with the Itô interpretation, the increment of $y(t)$ reads

$$\delta y(t) = f(y(t), t)\delta S(t) + g(y(t), t)\delta\eta(S(t)), \tag{A1}$$

where $\delta S(t) = S(t+\tau) - S(t)$ and $\delta\eta(S(t)) = \eta(S(t+\tau)) - \eta(S(t))$. Next, similar to (6), we obtain the increment of $G(y, W, t)$ in Fourier space ($y \to k$, $W \to p$):

$$\begin{aligned}\delta G(k, p, t) = &\langle e^{-iky(t)-ipW(t)}(e^{-ikg(y(t),t)\delta\eta(S(t))} - 1)\rangle \\ &- ik\langle e^{-iky(t)-ipW(t)} f(y(t), t)\delta S(t)\rangle \\ &- ip\tau\langle e^{-iky(t)-ipW(t)} U(y(t))\rangle,\end{aligned} \tag{A2}$$

where the first term on the right-hand side can be reduced to

$$\langle e^{-iky(t)-ipW(t)}\phi_0(kg(y(t), t))\,\delta S(t)\rangle$$

as usual due to the characteristic function of $\delta\eta(t)$ in (3). So dividing (A2) by τ and taking the limit $\tau \to 0$, we obtain

$$\begin{aligned}\frac{\partial}{\partial t}G(k, p, t) = &\langle e^{-iky(t)-ipW(t)}\phi_0(kg(y(t), t))\dot{S}(t)\rangle \\ &- ik\langle e^{-iky(t)-ipW(t)} f(y(t), t)\dot{S}(t)\rangle \\ &- ip\langle e^{-iky(t)-ipW(t)} U(y(t))\rangle \\ =: &\; Q_1 + Q_2 + Q_3.\end{aligned} \tag{A3}$$

It is obvious that the inverse Fourier transform ($k \to y$) of Q_3 is $-ipU(y)G(y, p, t)$. But for Q_1 and Q_2, they look a little bit difficult due to the new term $\dot{S}(t)$ compared with (6). Note that the angular brackets in Q_1 denote the average of two kinds of independent stochastic processes with the joint PDF $G(y(t), W(t), t)$ and Lévy α-stable noise $\theta(t)$ on which $S(t)$ depends. To deal with the term Q_1, we first add a technical delta function $\delta(y - y(t))$ in it and get

$$Q_1 = \int_{-\infty}^{\infty} e^{-iky}\phi_0(kg(y, t))\langle e^{-ipW(t)}\delta(y - y(t))\dot{S}(t)\rangle dy.$$

Then applying the technique in [32] of rewriting the functional $W(t)$ as a subordinated process,

$$W(t) = V(S(t)), \qquad V(s) = \int_0^s U(x(s'))\theta(s')ds'.$$

Substituting $y(t) = x(S(t))$ and $W(t) = V(S(t))$ into Q_1 gives the middle term of Q_1 as

$$\begin{aligned}&\langle e^{-ipV(S(t))}\delta(y - x(S(t)))\dot{S}(t)\rangle \\ &\quad = \int_0^{\infty}\langle e^{-ipV(s)}\delta(y - x(s))\delta(t - T(s))\rangle ds.\end{aligned} \tag{A4}$$

Taking the Laplace transform ($t \to u$) of (A4), we obtain

$$\begin{aligned}Q_1(u) = &\int_{-\infty}^{\infty} e^{-iky}\phi_0(kg(y, t)) \\ &\times \int_0^{\infty}\langle e^{-ipV(s)-uT(s)}\delta(y - x(s))\rangle ds\, dy.\end{aligned} \tag{A5}$$

On the other hand, $G(y, p, t)$ can be rewritten as

$$\begin{aligned}G(y, p, t) &= \langle e^{-ipV(S(t))}\delta(y - x(S(t)))\rangle \\ &= \int_0^{\infty}\langle e^{-ipV(s)}\delta(s - S(t))\delta(y - x(s))\rangle ds.\end{aligned}$$

So its Laplace transform ($t \to u$) is

$$G(y, p, u) = \int_0^{\infty}\langle e^{-ipV(s)-uT(s)}\theta(s)\delta(y - x(s))\rangle ds. \tag{A6}$$

The characteristic function of the Lévy process $T(s)$ in (11) is

$$\langle e^{-uT(s)}\rangle = e^{-su^{\alpha}},$$

which yields an important equality in [32] from (A6):

$$\begin{aligned}G(y, p, u) = &[u + ipU(y)]^{\alpha-1} \\ &\times \int_0^{\infty}\langle e^{-ipV(s)-uT(s)}\delta(y - x(s))\rangle ds.\end{aligned} \tag{A7}$$

Comparing (A5) with (A7), we find that

$$\begin{aligned}Q_1(u) = &\int_{-\infty}^{\infty} e^{-iky}\phi_0(kg(y, t)) \\ &\times [u + ipU(y)]^{1-\alpha}G(y, p, u)dy.\end{aligned}$$

Taking the inverse Laplace transform ($u \to t$), we obtain

$$Q_1 = \int_{-\infty}^{\infty} e^{-iky}\phi_0(kg(y, t))\mathcal{D}_t^{1-\alpha}G(y, p, t)dy. \tag{A8}$$

As for Q_2, it can be obtained by the procedure similar to Q_1, i.e.,

$$Q_2 = -ik\int_{-\infty}^{\infty} e^{-iky} f(y,t)\mathcal{D}_t^{1-\alpha}G(y,p,t)dy. \tag{A9}$$

Finally, substituting (A8) and (A9) into (A3), we obtain the forward Feynman-Kac equation in Fourier space:

$$\frac{\partial G(k,p,t)}{\partial t} = \mathcal{F}_y\{\phi_0(kg(y,t))\mathcal{D}_t^{1-\alpha}G(y,p,t)\} - \mathcal{F}_y\left\{\frac{\partial}{\partial y}f(y,t)\mathcal{D}_t^{1-\alpha}G(y,p,t) + ipU(y)G(y,p,t)\right\}.$$

APPENDIX B: BACKWARD FEYNMAN-KAC EQUATION WITH MULTIPLICATIVE NOISE

If $g(x)$ is not a constant in (15), then the Fourier transform of $\langle T_\eta\rangle$ becomes

$$\mathcal{F}_y\{\langle T_\eta\rangle\} = \left\langle\int_{-\infty}^{\infty} e^{-ik_0x(\tau)}T_\eta e^{ik_0[f(x_0)\tau+g(x_0)\eta(\tau)]}dx_0\right\rangle.$$

We turn dx_0 into $dx(\tau)$ and get

$$\begin{aligned}\mathcal{F}_{x_0}\{\langle T_\eta\rangle\} = &\left\langle\int_{-\infty}^{\infty} e^{-ik_0x(\tau)}T_\eta e^{ik_0[f(x_0)\tau+g(x_0)\eta(\tau)]}dx(\tau)\right\rangle - \left\langle\int_{-\infty}^{\infty} e^{-ik_0x(\tau)}T_\eta e^{ik_0[f(x_0)\tau+g(x_0)\eta(\tau)]}\frac{df(x_0)}{dx_0}\tau\, dx_0\right\rangle\\ &-\left\langle\int_{-\infty}^{\infty} e^{-ik_0x(\tau)}T_\eta e^{ik_0[f(x_0)\tau+g(x_0)\eta(\tau)]}\frac{dg(x_0)}{dx_0}\eta(\tau)dx_0\right\rangle.\end{aligned} \tag{B1}$$

Letting $\tau\to 0$, the second term of $\mathcal{F}_{x_0}\{\langle T_\eta\rangle\}$ is the same as (21), i.e.,

$$-\tau\mathcal{F}_{x_0}\left\{\frac{\partial f(x_0)}{\partial x_0}G_{x_0}(p,t)\right\}. \tag{B2}$$

Though $\eta(\tau)\to 0$ as $\tau\to 0$, how fast it tends to 0 is not specific, which brings about the challenge of dealing with the first and third terms in (B1). To make this point clear, we should define

$$M_n(k,\tau) = \langle e^{-ik\eta(\tau)}\eta^n(\tau)\rangle.$$

When $n=0$, M_0 is the characteristic function of $\eta(\tau)$, given in (3). When $n\geqslant 1$, $M_n\to 0$ as $\tau\to 0$. For the case of Gaussian white noise: $M_0 = e^{-\tau k^2}$, by some calculations we have, as $\tau\to 0$,

$$\begin{aligned} &M_1 \sim -2ik\tau, && M_2\sim 2\tau,\\ &M_n\sim \tau^2\sim 0, && \forall\, n\geqslant 3;\end{aligned} \tag{B3}$$

since M_n $(n\geqslant 3)$ are all higher-order terms of τ. But for Lévy β-stable noise, $M_0 = e^{-\tau|k|^\beta}$ and all M_n $(n\geqslant 1)$ contain the first-order term of τ. Here we focus on the case that $\eta(\tau)$ is Gaussian white noise, and we use the property (B3) to deal with the first and third terms in (B1).

Denoting the first term as T_1 for convenience and using $e^{ik_0f(x_0)\tau}\simeq 1+ik_0f(x_0)\tau$ as before, we get

$$T_1 = \left\langle\int_{-\infty}^{\infty} e^{-ik_0x(\tau)}T_\eta e^{ik_0g(x_0)\eta(\tau)}dx(\tau)\right\rangle + ik_0\tau\left\langle\int_{-\infty}^{\infty} e^{-ik_0x(\tau)}T_\eta f(x(\tau))dx(\tau)\right\rangle, \tag{B4}$$

where the latter term is equal to

$$\tau\mathcal{F}_{x_0}\left\{\frac{\partial}{\partial x_0}f(x_0)G_{x_0}(p,t)\right\}.$$

Turn $g(x_0)$ into $g(x(\tau))$ in (B4) by Taylor expansion $g(x_0) = g(x(\tau)) + R_g$, where

$$R_g = -[f(x_0)\tau + g(x_0)\eta(\tau)]g'(x(\tau)) + \frac{1}{2}[f(x_0)\tau+g(x_0)\eta(\tau)]^2g''(x(\tau)) + \cdots,$$

where for convenience we use a prime to denote the first-order derivative. Then the former term of (B4), denoted as T_{11}, becomes

$$T_{11} = \left\langle\int_{-\infty}^{\infty} e^{-ik_0x(\tau)}T_\eta e^{ik_0g[x(\tau)]\eta(\tau)}e^{ik_0R_g\eta(\tau)}dx(\tau)\right\rangle, \tag{B5}$$

where $e^{ik_0R_g\eta(\tau)} = 1 + ik_0R_g\eta(\tau) + \cdots$. Considering $M_n \sim 0\,(n \geqslant 3)$ in (B3), the second term of R_g and all the latter terms can be omitted since these terms contribute to $\eta^2(\tau)$ and then yield $M_n\ (n \geqslant 3)$ when substituted into (B5). Therefore, we have

$$\begin{aligned} T_{11} &= \left\langle \int_{-\infty}^{\infty} e^{-ik_0x(\tau)}T_\eta e^{ik_0g(x(\tau))\eta(\tau)}e^{-ik_0g(x_0)g'(x(\tau))\eta^2(\tau)}dx(\tau)\right\rangle \\ &= \left\langle \int_{-\infty}^{\infty} e^{-ik_0x(\tau)}T_\eta e^{ik_0g(x(\tau))\eta(\tau)}dx(\tau)\right\rangle - ik_0\left\langle \int_{-\infty}^{\infty} e^{-ik_0x(\tau)}T_\eta e^{ik_0g(x(\tau))\eta(\tau)}g(x_0)g'(x(\tau))\eta^2(\tau)dx(\tau)\right\rangle \\ &= \mathcal{F}_{x_0}\{\langle e^{ik_0g(x_0)\eta(\tau)}\rangle G_{x_0}(p,t)\} - ik_0\mathcal{F}_{x_0}\{\langle e^{ik_0g(x_0)\eta(\tau)}\eta^2(\tau)\rangle g(x_0)g'(x_0)G_{x_0}(p,t)\}, \end{aligned}$$

where we replace $g(x_0)$ by $g(x(\tau))$ in the latter term and omit the high-order term M_3. Substituting M_2 in (B3) and T_{11} into (B4) gives

$$T_1 = \mathcal{F}_{x_0}\{\langle e^{ik_0g(x_0)\eta(\tau)}\rangle G_{x_0}(p,t)\} - 2ik_0\tau\mathcal{F}_{x_0}\{g(x_0)g'(x_0)G_{x_0}(p,t)\} + \tau\mathcal{F}_{x_0}\left\{\frac{\partial}{\partial x_0}f(x_0)G_{x_0}(p,t)\right\}. \tag{B6}$$

Similarly, still using the property $M_n \sim 0,\ n \geqslant 3$, we get the third term in (B1),

$$T_3 = -2\tau\mathcal{F}_{x_0}\left\{g(x_0)g'(x_0)\frac{\partial G_{x_0}(p,t)}{\partial x_0}\right\}. \tag{B7}$$

Combining (B6), (B7), and (B2), we finally get

$$\begin{aligned} \mathcal{F}_{x_0}\{\langle T_\eta\rangle\} =\ & \mathcal{F}_{x_0}\{\langle e^{ik_0g(x_0)\eta(\tau)}\rangle G_{x_0}(p,t)\} - 2\tau\mathcal{F}_{x_0}\left\{\frac{\partial}{\partial x_0}g(x_0)g'(x_0)G_{x_0}(p,t)\right\} \\ & + \tau\mathcal{F}_{x_0}\left\{f(x_0)\frac{\partial G_{x_0}(p,t)}{\partial x_0}\right\} - 2\tau\mathcal{F}_{x_0}\left\{g(x_0)g'(x_0)\frac{\partial G_{x_0}(p,t)}{\partial x_0}\right\}. \end{aligned} \tag{B8}$$

The characteristic function of Lévy noise (3) leads to

$$\langle e^{ik_0g(x_0)\eta(\tau)}\rangle - 1 \simeq \tau\phi_0(-k_0g(x_0)) = -\tau k_0^2g^2(x_0) \qquad \text{as } \tau \to 0. \tag{B9}$$

Combining (B8) and (B9), by some calculations we obtain

$$\mathcal{F}_{x_0}\{\langle T_\eta\rangle\} - \mathcal{F}_{x_0}\{G_{x_0}(p,t)\} = \tau\mathcal{F}_{x_0}\left\{g^2(x_0)\frac{\partial^2G_{x_0}(p,t)}{\partial x_0^2}\right\} + \tau\mathcal{F}_{x_0}\left\{f(x_0)\frac{\partial G_{x_0}(p,t)}{\partial x_0}\right\}.$$

Substituting this formula into (19), dividing (19) by τ, taking the limit $\tau \to 0$, and making the inverse Fourier transform ($k_0 \to x_0$), we obtain the backward Feynman-Kac equation:

$$\frac{\partial G_{x_0}(p,t)}{\partial t} = g^2(x_0)\frac{\partial^2G_{x_0}(p,t)}{\partial x_0^2} + f(x_0)\frac{\partial G_{x_0}(p,t)}{\partial x_0} - ipU(x_0)G_{x_0}(p,t). \tag{B10}$$

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