Log-periodicity in piecewise ballistic superdiffusion: Exact results

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Small log-periodic oscillations have been observed in many systems and have previously been studied via renormalization-group approaches in the context of critical phenomena [Gluzman and Sornette, Phys. Rev. E **65**, 036142 (2002); Derrida and Giacomin, J. Stat. Phys. **154**, 286 (2014)]. Here we report their appearance in a random walk model with damaged memory, and we develop an exact discrete-time solution, free from adjustable parameters. Our results shed light on log-periodicity and how it arises. We also discuss continuous-time approaches to the solution along with their limitations and advantages. We show that, as a direct consequence of memory damage, the first moment for the model acquires piecewise ballistic behavior. The piecewise segments are separated by regularly placed singular points. Log-periodicity in this model is seen to be due to memory damage. Remarkably, piecewise ballistic behavior is only observed if one uses the discrete-time solution, because the continuous-time solution does not correctly account for the model's discrete-time dynamics.

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I. INTRODUCTION

Diffusion is a subject of intense research in the natural sciences, and it is one of the fundamental mechanisms of material transport. The rigorous physical and mathematical basis for the study of diffusion dates back to the beginning of the 20th century with the explanation of Brownian motion (BM) in terms of collisions between suspended particles and erratic molecules [1,2]. Brownian diffusion is characterized by a linear time dependence of the mean-squared displacement (MSD). In complex systems, however, transport dynamics are governed by anomalous diffusion (AD) [3–7], whose distinctive feature is a nonlinear time dependence of the MSD. The variance [8] typically scales according to $\langle x^2\rangle - \langle x\rangle^2 \sim t^{2H}$, where the exponent H is known as the Hurst exponent. Anomalous diffusion is termed subdiffusive for $H < 1/2$ and superdiffusive for $H > 1/2$. The value $H = 1/2$ is a necessary (but not sufficient) condition for normal diffusion. For the models we consider in this paper, the mean-squared displacement satisfies $\langle x^2\rangle \sim \langle x\rangle^2 \sim t^{2H}$. Anomalous diffusion and transport phenomena cannot be described in terms of Brownian diffusion models. However, they can be modeled by incorporating memory effects into the model's dynamics [9–15]. Stochastic differential equation models have been formulated to model anomalous diffusion phenomena, such as continuous-time random walks (CTRWs) [4,16–18], Langevin and generalized Langevin equations [19–23], fractional kinetic equations [4,24], and the fractional Fokker-Planck method [12,25].

Another important stochastic approach to the study of anomalous transport has been provided by the use of random walks (RWs) [26–33]. One successful example of memory implementation leading to anomalous behavior is the elephant random walk (ERW) model [34], where decisions made at the present time depend upon the entire history of previous decisions. The model's strong correlations give rise to superdiffusive behavior for long times, and, despite their nontrivial nature, the model can be solved analytically. The solution of this model inspired many further studies (e.g., see Refs. [35–39]). Other ways to include memory effects in the ERW dynamics have been devised, such as memory loss (memory damage), also known as the Alzheimer random walk [37,38], inclusion of intermittency or pauses to the walk [39] giving rise to one more regime (subdiffusion), and the minimal random walk model [40]. Newer solutions for the ERW model, besides the original solution, have also been proposed based on the CTRW [41], Fokker-Planck [38], and martingale [42] approaches.

Log-periodic (LP) corrections to scaling have been observed in many cases [43–46], and they are associated with a breakdown of a continuous scale invariance symmetry into a discrete symmetry [47]. Their presence has also been reported in highly correlated memory random walks with available analytical solutions [37,38,48,49], despite the strong non-Markovian character of these models. Small-amplitude log-periodic corrections have also been reported to appear in several systems, such as family names, ferromagnetic interactions on hierarchical lattices, diffusion-limited aggregation, rupture, earthquakes, and financial crashes, and their appearance in critical behavior has been studied via the renormalization group approach [50,51]. Random walks have already been used to account for log-periodic oscillations [52,53]. We recently found that Alzheimer random walk related memory correlated random walk models may also give rise to small log-periodic oscillations [54]. The special interest in these RW models comes from their analytical tractability, allowing for a deeper understanding of the general mechanisms behind the phenomenon of small log-periodicity, in particular the main ingredients necessary for their emergence.

In this paper, we study a non-Markovian random walk model we proposed before [37], namely the Alzheimer random walk (ARW) model—a term coined by Kenkre [38]. In the model, long-range memory is introduced in the microscopic dynamics by including an explicit history dependence into the decision process. The evolution dynamics is the same as in the ERW [34]. These models exhibit non-Markovian behavior due to the strong memory correlation that is embedded in their dynamics. The ARW model, in particular, includes memory damage by restricting the working memory to the memories of the distant past. We argue below that memory damage is an essential ingredient in the rise of log-periodicities in these models [37,54]. Loss of memory of recent events is a common type of biological memory damage, mostly associated with Alzheimer's disease. Quantitatively, the memory damage in the ARW model means that only a fraction ft ($f < 1$) of the total time t is remembered, i.e., the events that took place after ft are not immediately available for use at decision time. Previous studies have shown that random walkers with a tendency to undo past decisions, which is known as the negative feedback regime, switch from nonpersistent to persistent behavior when inflicted with significant memory losses of the recent past. In this case, their power-law behavior can be decorated with log-periodic oscillations [37,38]. Log-periodic modulations small enough to be mistaken for numerical errors were also discovered in other memory-dependent RW models, now associated with positive feedback [54]. A discrete-time solution (DTS) was then provided to make sure the small oscillations were not merely numerical artifacts. It is worth mentioning that in order to prevent small log-periodicity signals from being averaged out, a proper canonical averaging is recommended [55,56]. Canonical averaging has found application in studying rupture in random media and in detecting log-periodicity in turbulence. It was also found that the usual approach of taking the continuous-time limit of a discrete-time equation could not be adequate to describe the microscopic details of the discrete-time problem [54,57]. On the other hand, we recently discovered evanescent small log-periodic oscillations in the ARW model. Their presence in the model went unnoticed until now, due to their small amplitude and also to the continuous-time solutions proposed, which were not adequate to deal with evanescent oscillations.

Motivated by such findings, we are carrying out studies aiming at examining the small log-periodicities in the ARW model and the question about the adequacy of the continuous-time approach to understand the walk. This model is very convenient for this study since its memory damage is clearly defined and its first moment is amenable to analytical treatments. We address questions regarding the characteristics of the oscillations and their origins by comparing existing continuous-time solutions with an exact discrete-time solution for the first moment. We show that all solutions of the model are decorated with piecewise ballistic sections that arise as a consequence of memory damage. Piecewise ballistic processes have been reported recently in the context of underdamped dynamics [58,59] and turbulent dispersion [60]. As far as we know, this is the first time that piecewise ballistic phenomena have been found in non-Markovian random walks, especially supported by first-principles exact calculations. We discuss the advantages and limitations of different solutions along with their adequacy to acknowledge the occurrence of the oscillations. An explanation is provided as to the origin of these phenomena in stochastic random walk models in terms of memory damage and breakdown of self-regulation mechanisms. We believe this study is important to finding the ingredients associated with the emergence of log-periodic oscillations and to understanding the limitations of continuous-time solutions in random walk memory models.

The remainder of this paper is organized as follows. In Sec. II we describe the models. Section III presents the results, and Sec. IV outlines the discussions. Final conclusions are drawn in Sec. V.

II. THE MODEL

In this section, we describe the non-Markovian Alzheimer random walk model we consider in this paper. It is a one-dimensional discrete-time RW that uses past decisions to make decisions at the present time. In the ARW model, the position X_{t+1} is obtained in terms of the previous position X_t using a recursive relation, namely

$$X_{t+1} = X_t + \sigma_{t+1}. \tag{1}$$

The set of discrete time step directions $\{\sigma_t = \pm 1\}$ constitutes a binary random sequence that contains two-point correlations carrying all the information on the walk history. The walk starts at time $t = 0$ at $X_{t=0} = 0$ and the first time step is set to $\sigma_{t=1} = 1$. As in the ERW model, at time $t + 1$ a previous time $1 \leqslant t' < t + 1$ is randomly chosen from a uniform distribution with equal *a priori* probabilities. The current step direction σ_{t+1} is then chosen either as $\sigma_{t+1} = +\sigma_{t'}$ (with probability p) or $\sigma_{t+1} = -\sigma_{t'}$ (with probability $1 - p$). According to this discrete-time dynamics, the decision process acts to compensate for past behavior (reformer) if $p < 1/2$, mimicking a negative feedback mechanism. It can also act to reproduce past behavior (traditional) if $p > 1/2$, which we call a positive feedback mechanism. We now cause a disruption in the memory by restricting the memory usage to only a fraction ft ($0 < f \leqslant 1$) of the total time steps, and we refer to such disrupted memory simply as memory damage. We can define the working memory as the portion of the total memory that can be used in the decision process. The (integer) size $L = L(t)$ of the working memory at time t can be written as $L = \lfloor f(t-1) \rfloor + 1$. Here $y = \lfloor y \rfloor$ represents the greatest integer less than or equal to y, and the $+1$ is here to avoid a working memory of a zero length when $ft < 1$. For example, for $f = 0.01$ we have $L = 1$ for $0 < t \leqslant 100$, $L = 2$ for $100 < t \leqslant 200$, and so on. We see for $f < 1$ there is a time delay, inversely proportional to f, for new memories to be introduced in the working memory. This drastically changes the behavior of the walk, as will be seen below. The choice of the current step direction σ_{t+1} is done as in the ERW model. The non-Markovian character is retained for $f < 1$, while for $f = 1$ we recover the full memory model. The (f, p) phase diagram of the ARW at long times reveals superdiffusion for $p \neq 1/2$, according to the value of f. Log-periodic corrections to scaling were found in superdiffusive regimes for $p < 1/2$.

Log-periodic corrections in the ARW model are clearly due to memory damage, because no such modulations exist in the original ERW model. At first, they seemed to be restricted to negative feedback regimes [37,48]. However, small-amplitude LP oscillations in the positive feedback region ($p > 1/2$) were recently reported for binomial and δ-memory profile models [54]. Motivated by such new results, after some careful investigation we were able to show that small oscillations also take place in the simple rectangular memory model associated with the ARW model. In fact, we believe that small oscillations are directly associated with disruptions in the memory pattern in general. As shown below, besides the "regular" (nonsmall) log-periodic oscillations for $p < 1/2$ (negative feedback regime), the ARW also exhibits small-amplitude log-periodic oscillations, which only occur for $p > 1/2$. Moreover, all solutions of the model are decorated with piecewise ballistic branches that are carried through all finite values of t. Such small oscillations are evanescent, i.e., they disappear at long times. We also discuss possible analytic solutions for Eq. (1), namely two continuous-time solutions derived by taking the continuous-time limit of (1) and one discrete-time solution. We have carefully examined all three solutions and compared their advantages, limitations, and conveniences to understand the long-time behavior of the ARW model. We show that only the discrete-time solution can predict piecewise ballistic behavior. Moreover it provides the best fit to numerical data.

Damaged recent memory seems to be an essential ingredient to the appearance of LP oscillations. In nondamaged memory, as in the ERW model, the working memory is readily updated, meaning that the decisions at time $t-1$ are immediately available for use in the decision process at time t. In contrast, in the ARW's damaged memory model, some of the latest decisions only enter into the working memory after a time delay. This leads to alternate persistence windows [54] in $\langle x(t)\rangle$, which consist of steps predominantly in a given direction. The step sizes grow exponentially with time leading to log-periodic behavior.

In the next sections, we develop an exact discrete-time solution to the model, which is then compared with existing continuous-time solutions. The advantages, conveniences, and limitations of each approach are discussed.

III. RESULTS AND DISCUSSION

In this section, we derive an exact discrete-time solution to describe the walk introduced above. Continuous-time solutions are also described below.

We start by writing the first moment $x(t) \equiv \langle x(t)\rangle$ in terms of a recursive equation by taking average of (1), namely

$$\begin{aligned} x(t+1) &= x(t) + \frac{\alpha}{L}x(L) \\ &= x(t) + \frac{\alpha}{\lfloor f(t-1)\rfloor + 1}x(\lfloor f(t-1)\rfloor + 1), \end{aligned} \tag{2}$$

where $x(t) = \langle X_t\rangle$ and $\alpha = 2p-1$. Here $(\alpha/L)x(L)$ represents the walker's average speed [61].

We now derive an exact discrete-time recursive solution for Eq. (2), free from adjustable parameters, that describes correctly the details of the walk for any t, big or small.

A. Discrete-time solution

The initial conditions for (2) are set as $x(0) = 0$ and $x(1) = 1$, and we start looking for an exact iterative process to get a solution. We then write

$$\begin{aligned} x(2) &= 1+\alpha, \\ x(3) &= 1+2\alpha, \\ &\vdots \quad \vdots \\ x(t+1) &= 1+\alpha t, \end{aligned} \tag{3}$$

while $t \leqslant 1/f$. For $t > 1/f$, we write

$$\begin{aligned} x(1/f+2) &= 1 + (1/f)\alpha + \frac{\alpha}{2}(1+\alpha), \\ x(1/f+3) &= 1 + (1/f)\alpha + 2\frac{\alpha}{2}(1+\alpha), \\ &\vdots \quad \vdots \\ x(t+1) &= 1 + (1/f)\alpha + \frac{\alpha}{2}(1+\alpha)t \end{aligned} \tag{4}$$

for $1/f < t \leqslant 2/f$. For $t > 2/f$, we write

$$\begin{aligned} x(2/f+2) &= 1 + (1/f)\alpha + (1/f)\frac{\alpha}{2}(1+\alpha) + \frac{\alpha}{3}(1+2\alpha), \\ x(2/f+3) &= 1 + (1/f)\alpha + (1/f)\frac{\alpha}{2}(1+\alpha) + 2\frac{\alpha}{3}(1+2\alpha), \\ &\vdots \quad \vdots \\ x(t+1) &= 1 + (1/f)\alpha + (1/f)\frac{\alpha}{2}(1+\alpha) + \frac{\alpha}{3}(1+2\alpha)t \end{aligned} \tag{5}$$

for $2/f < t \leqslant 3/f$, and we proceed likewise for $t > 3/f$. We see that within the intervals $[n/f, (n+1)/f]$, the behavior of $x(t)$ is linear with t, characterizing what we refer to as discrete-time ballistic behavior. This discrete-time iterative process can be represented by a single recursive equation, namely

$$x(t) = 1 + \sum_{j=1}^{t-1} \frac{\alpha}{\lfloor f(j-1)\rfloor + 1} x(\lfloor f(j-1)\rfloor + 1). \tag{6}$$

Notice the absence of any adjustable parameters in this equation. This discrete-time approach thus predicts a piecewise ballistic decoration to $\langle x(t)\rangle$ with period $1/f$, which is not contemplated by the previous solutions. It is worth noting that these decorations to the first moment curve could actually be predicted directly from the basic equation of motion, namely Eq. (2), if it is written as

$$\Delta x = \frac{\alpha}{\lfloor f(t-1)\rfloor + 1} x(\lfloor f(t-1)\rfloor + 1).$$

We see that the speed of the walker $dx/dt \approx \Delta x$ depends on $L = \lfloor f(t-1)\rfloor + 1$, which remains constant for $t \in [n/f, (n+1)/f]$, with $n \in \{0, 1, 2, \dots\}$. Within these time intervals $\langle x(t)\rangle$ is a straight line, i.e., $x(t) = at + b$. This *piecewise* ballistic behavior is caused by the memory damage. The decisions taken while t runs within one of these intervals are not within the range of the working memory. Thus the time intervals $t_n = n/f$ up to $t_{n+1} = (n+1)/f$ define memory chunks within which the working memory does not change. The effect of memory damage in the model affects the walker's speed in such a way that speed changes depend exclusively on a fixed position point until the next update in the working memory occurs, characterizing the *piecewise* ballistic behavior.

Equation (6) represents a first-principles solution for $\langle x(t)\rangle$, free from adjustable parameters, which reproduces all the microscopic details of the walk, even for small times. It correctly accounts for all the log-periodic modulations that appear in the ARW model, both "regular" and small, along with the *piecewise* decorations to the first moment, as shown in the discussions below. An added convenience is the possibility of avoiding using numerical simulations to analyze the walk behavior. In fact, this solution provides data faster and more accurately than computing simulations.

B. Series expansion continuous-time solutions

Another approach to study this problem consists in writing a continuous-time limit for Eq. (2) by taking into account the first derivative, i.e.,

$$\frac{dx(t)}{dt} = \frac{\alpha}{ft}x(ft), \tag{7}$$

from which continuous-time solutions can be obtained. We actually used this approach before [37,62]. Notice, however, that while (7) provides a good representation of the walk for very large times, it definitely fails to do so for small values of t. Therefore, one should not expect continuous-time solutions to be able to describe the microscopic details of the walk. In particular, the damaged memory pattern of the model leads to microscopic details in the walk that can only be seen in finite times. Indeed, as we shall see below, only the full discrete-time solution can fully describe all details of the walk for finite times and small value parameter f.

In a previous work [61], we employed a continuous-time solution for Eq. (7) given in terms of a time-series expansion, namely

$$x(t) = \sum_{r=0}^{\infty} a_i \sin\left[B_i \ln(t) + C_i\right] t^{\delta_i}. \tag{8}$$

The choice of coefficients in this time series was motivated by numerical results, which indicated the presence of log-periodic oscillations. Only the dominant term of the series was kept since we were interested mainly in the model's diffusive behavior in the long-time limit. The simplicity of (8) offers great help in the overall analysis of the problem, a major advantage being that it can be used straightforwardly to determine the Hurst exponent H in superdiffusive regimes. In fact, H can be obtained by replacing the dominant term of (8) into (7) and solving the resulting transcendental equations. This approach was successfully used to draw the phase diagram $H = H(f, p)$ for the ARW model for large t. As expected, solution (8) is also able to account for the nonsmall nonevanescent log-periodicities for $p < 1/2$ (the negative feedback regime) providing excellent agreement with numerical calculations (details can be found in [49,61]). This solution represents an exact solution to Eq. (7), valid in the asymptotic limit. It also provides a transcendental equation $H = \alpha f^{H-1}$ for $p > 1/2$ ($\alpha > 0$), which can be easily linked to the Lambert W function $W(u)$,

$$H = \frac{1}{\ln(1/f)} W(u), \tag{9}$$

with $u = \alpha/f$. This function can be expanded in the form [63]

$$W(u) = \ln(u) + 2\pi\imath k - \ln[\ln(u) + 2\pi\imath k] + \sum_{k=0}^{\infty}\sum_{m=1}^{\infty} c_{km} \ln^m[\ln(u)+2\pi\imath k][\ln(u) + 2\pi\imath k]^{-k-m}.$$

This complex exponent could represent a log-periodic behavior on $x(t)$. However, the exact solution (6) indicates an evanescent log-periodic behavior that disappears in the asymptotic limit. Therefore, the unique solution for the Hurst exponent for $\alpha > 0$ is given by (9) with $k = 0$. Then the imaginary part of H does not contribute to the leading term of the series for $x(t)$. In other words, the log-periodicity appears only in the correction to scaling terms, and it is non-negligible for finite (even large) t (for some values of parameters f and p).

Another point worth mentioning concerns the piecewise ballistic decorations that appear in the solutions of the ARW model for all $p \neq 1/2$. These decorations are due to the microscopic consequences of the memory damage and are incorporated in Eq. (2). Solutions to the continuous-time equation (7) smooth the discrete results and cannot display these microscopic details.

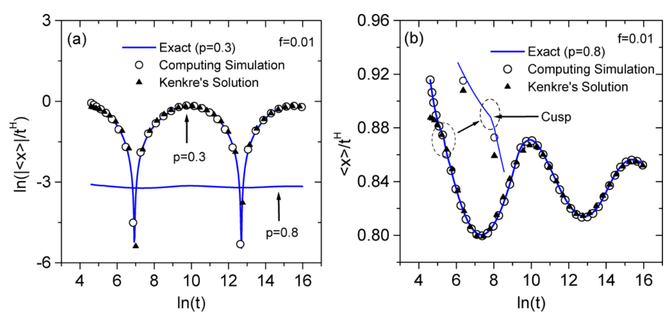

FIG. 1. Computing simulations (black open circles) and analytical results (Kenkre's solution as full black triangles and exact discrete-time exact results as blue solid lines) for the first moment $\langle x(t)\rangle$ vs time for $f = 0.01$. The period of the oscillations is equal to $\ln(1/f) \approx 4.6$ (see the text) and the Hurst exponents ($H = 0.806\,11$ for $p = 0.3$ and $H = 0.909\,64$ for $p = 0.8$). The main figure in (a) shows $\ln[|\langle x\rangle|/t^H]$ for $p = 0.3$ with numerical data (10^3 runs and $t_{\max} \approx 9 \times 10^6$ time units each) along with discrete-time exact results and Kenkre's solution. Note that the simulation results adjust perfectly well to the exact curve. The log-periodic curve in this case ($p < 1/2$) represents a nonsmall log-periodic oscillation. A small log-periodic oscillation corresponding to $p = 0.8$ is also drawn for comparison (solid blue curve in the middle). Note that the amplitude is so small that the oscillation is barely noticed in this scale. In (b) numerical data (2×10^5 runs and $t_{\max} \approx 9 \times 10^6$ time units each) and exact results are shown in a $\langle x\rangle/t^H$ vs $\ln(t)$ plot with $p = 0.8$. This small log-periodic oscillation is evanescent. The zoom emphasizes a cusp indicating a singularity in the exact curve. Note that the oscillatory aspect of the curve is clearly seen now, along with the presence of the singularity.

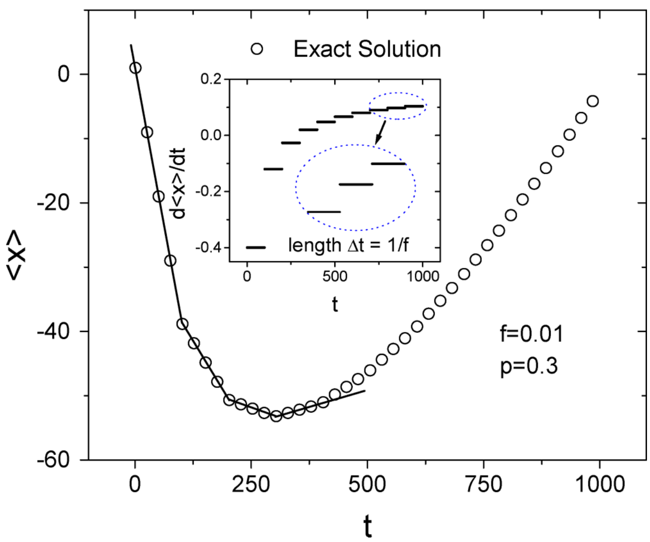

FIG. 2. Discrete-time exact results for $f = 0.01$ and $p = 0.3$. The main curve equally sized straight line time segments are repeated periodically every $\Delta t = 1/f = 100$ time units. The diffusion is ballistic $[\langle x(t)\rangle \sim t]$ within each straight line section of the curve characterizing *piecewise* ballistic behavior repeated with period $\Delta t = 1/f$. The derivative of the main curve is shown in the inset for an easy view of the size of the time sections. Note that the whole curve consists of *piecewise* sections that persist for long times.

Another continuous-time solution for Eq. (7) was provided by Kenkre [38], which is written as

$$x(t) = x_0 \sum_{r=0}^{\infty} \frac{\left[(\alpha/f)\ln(1/f)\left(\frac{\ln(t/t_0)}{\ln(1/f)} - r\right)\right]^r}{r!} \times \mathcal{H}(t - t_0(1/f)^r), \tag{10}$$

where $\mathcal{H}$ stands for the Heaviside step function, i.e., $\mathcal{H}(t) = 1$ for $t \geqslant 0$ or $\mathcal{H}(t) = 0$ for $t < 0$. This equation is also an exact solution to Eq. (7), valid for all t. For this reason, it accounts for the model's log-periodicities, both small and nonsmall. However, it is not adequate for determining the Hurst coefficient, for example. Moreover, like Eq. (8), it does not explain the ballistic branches of the walk.

Figures 1(a) and 1(b) show numerical data and discrete-time results for $\langle x(t)\rangle$ versus $\ln t$ for $p = 0.3$ ("normal" amplitude) and $p = 0.8$ (small amplitude), respectively, with $f = 0.01$. The Hurst exponents in these cases, obtained from (9), characterize superdiffusion with $H = 0.806\,11$ for $p = 0.3$ and $H = 0.909\,64$ for $p = 0.8$. The Hurst exponents in both cases were determined using the dominant term of Eq. (8). Notice that the small log-periodic oscillations are evanescent. The relative sizes of the two oscillations can be noticed by comparing the solid curve for $p = 0.8$ drawn in the middle in Fig. 1(a), with the main curve drawn for $p = 0.3$. Kenkre's solution provides a good fit for $p = 0.3$ and a fairly good fit to the small oscillations occurring for $p = 0.8$. The main point here is that solution (10) is able to acknowledge the presence of the small oscillations occurring for $p > 1/2$. We see that the exact discrete-time solution adjusts perfectly to the numerical data in both cases, as expected. Notice the small cusp by the zoom indicating a singularity in the exact solution. The presence of the singularity can also be noticed from the numerical data, but it can only be definitely settled by the exact results.

In Fig. 2, a section of the log-periodic curve for $(f, p) = (0.01, 0.3)$ is shown in detail using only discrete-time solution points. As shown, the log-periodic curve is composed of straight line time segments of equal size $\Delta t = 1/f$. Such *piecewise* ballistic behavior is repeated periodically in time every $\Delta t = 1/f$ and is sustained for large t and disappears for $t \to \infty$.

IV. CONCLUSIONS

We have studied possible solutions for a non-Markovian random walk model with long-range memory correlations and damaged memory. We note that by memory damage in the model, we simply mean that only the ancient part of the memory is available to the decision process, or, more generally, two different parts of the memory are treated differently. It is shown that memory damage and discrete time lead to the truncation term in Eq. (2), which ultimately causes the emergence of the piecewise ballistic behavior for small f. Specifically, it is shown that, besides the already reported asymptotic nonsmall log-periodic oscillations in the negative feedback region ($p < 1/2$), the model also presents small amplitude evanescent LP corrections to scaling in the positive feedback regime ($p > 1/2$). It is also shown that, as a direct consequence of memory damage, the first moment curve is characterized by an assembly of straight line time segments regularly separated by singular points. The singular points are placed evenly over the $\langle x(t)\rangle$ versus t curve and constitute an infinite discrete set. The straight line sections decorate all solutions for the first moment except for $p = 1/2$, representing *piecewise* ballistic branches over the entire curve for all finite values of t. The model is thus far richer than we first thought.

The small log-periodic behavior along with the straight line branches and singularities are studied via two existing continuous-time solutions for the model. The conveniences and limitations of the solutions are discussed. They are shown to represent suitable choices to study the large time macroscopic aspects of the walk, like the period of the oscillations and the Hurst exponent. It is shown that neither of the continuous-time solutions resolves the decorations characterized by the *piecewise* ballistic behavior. Despite its non-Markovian character, the ARW model offers the great advantage of analytical tractability. An exact discrete-time recursive solution was then developed, free from adjustable parameters, that is able to account for all the microscopic details of the walk, particularly the piecewise ballistic sections.

Our results leave open some questions and suggest possible directions for future work. A primary and challenging task would be to obtain a series expansion solution for the discrete-time approach to replace Eq. (6). Another interesting question involves the implications of the small log-periodic oscillations to the underlying symmetry of the model. In fact, small log-periodic oscillations can be nonevanescent [54]. It might be worthwhile to investigate their consequences to the model's symmetry. We also have found indications that other singularities, besides the ones leading to piecewise ballistic behavior, may exist. This array of singular points in the first moment indicates that the probability density function for these non-Markovian processes can be more complex than we first thought [35]. The results show that the description of inherently discrete problems using continuous-time approaches must be done with care. The findings in this work can provide guidance on investigations involving transitions from discrete-time equations to continuous-time equations. They can also lead to new insights on questions regarding the ingredients leading to the appearance of log-periodic corrections to scaling, both normal and small.

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