Work extraction using Gaussian operations in noninteracting fermionic systems

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We investigate work extraction from noninteracting fermions under arbitrary unitary operations and the more restricted class of Gaussian unitary operations that can be feasibly implemented. We characterize general quantum states in fermionic systems according to their ability to yield work (or not) under such transformations and study the limit for which multiple copies of passive states in fermionic systems can be activated for work extraction. We find that a sufficient number of copies of nonthermal passive states can achieve this, yielding an upper bound on the number of copies needed.

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I. INTRODUCTION

Since the inception of quantum thermodynamics, one of the important areas of research is the search for minimal resources and the least work-intensive protocols for the extraction of work out of thermal systems. This task entails finding quantum states that are freely available and quantum operations that can be feasibly implemented. In this context, the classification of passive states [1,2] as "freely available" [3,4], may have been overrated; this is because over a long period, "passive states" were generally believed to have no extractable work under cyclic unitary transformations. Surprisingly, recent studies [5–7] have shown that this situation holds if we have access to only a single copy of the state. However, if we collectively process many copies of the same system, extractable work can become available. If no work can be extracted unitarily, no matter how many copies are available then the state is said to be completely passive; thermal states are the only completely passive states [1,2]. The fact that passive states can be "activated" in such a way that work can be extracted from them is drawing increasing interest amongst researchers [5,8–10]. In this context, it seems that the underlying entanglement structure of the quantum system plays a crucial role. Indeed, recent results call into question the role of entanglement, free energy, correlations and coherence for such work extraction.

Ideally, work can be extracted from nonpassive states whose average energy can be lowered by acting on it with cyclic unitary operations. Generally in a cyclic Hamiltonian process [7], the maximal extractable work (called the ergotropy) between states ρ and σ is given as

$$W_{\max}(\rho) = \max_U \operatorname{tr}[H(\rho - \sigma)], \quad (1)$$

where H is the system's Hamiltonian, $\sigma = U\rho U^\dagger$, and U the unitary operator. For n copies of passive states, U would be a nonlocal (entangling) unitary acting on the total system with total Hamiltonian given by H.

Nonlocal (and thus entangling) unitary operations are capable of extracting more work than local operations from a set of quantum systems. However, the dynamics involving a nonlocal operation is slow, in the sense that it requires many different operations. Since such operations are difficult to implement, we are therefore left to consider which work extraction protocol is practically achievable when subjected to a restricted unitary operation. A large class of transformations that are easy to describe are Gaussian unitaries, which map Gaussian states into Gaussian states. The Gaussian unitary transformations being generated by quadratic Hamiltonians are in general more constrained than general unitary transformations.

A characterization of bosonic quantum states from which no (or maximal) work can be extracted using a Gaussian unitary transformation was recently established [11]. In this context, bosonic Gaussian passive states (and non-Gaussian passive states), from which no (or maximal) work can be extracted using a Gaussian unitary transformation, were defined. In this paper we investigate the corresponding situation for fermionic systems. Fermionic systems are similar to bosonic systems but differ in their statistics (Fermi-Dirac in the former and Bose-Einstein in the latter). There is a one-to-one map between n fermionic modes and the Hilbert space of n qubits. This allows for easy computations with fermions, providing an added advantage for quantum computational tasks [12]. It is the main aim of this paper to see how useful a fermionic system is for work extraction. We will show how Gaussian unitaries can yield a characterization of Gaussian passive and Gaussian nonpassive fermionic states, respectively.

Energy storage and its subsequent extraction has both fundamental and practical importance. The main goal of our study of work extraction from noninteracting fermion systems is to understand from which quantum states of fermionic systems energy can or cannot be minimized. We first consider general unitary transformations and then investigate the more restricted class of Gaussian unitary operations. Specifically, we consider a 1D noninteracting continuous variable fermionic systems composed of n modes.

In Sec. II, we describe the features of such systems and then discuss the characterization of fermionic Gaussian states.

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Identical thermal states do not allow for work activation no matter how many copies are available since a product of thermal states is a thermal state itself and hence passive. As a proof, we give an illustration of activation in product of thermal states using a general unitary transformation in Sec. III. Our main results are presented in Sec. IV where we characterize Gaussian and non-Gaussian passive states based on the availability of extractable work using Gaussian operations.

II. THE FERMIONIC SYSTEM

A system of N canonical fermionic modes κ is described by a Hilbert space (with dimension 2^N) $\mathcal{H} = \bigotimes_{\kappa=1}^N \mathcal{H}_\kappa$ spanned by the basis $|n_\kappa = 0\rangle$ and $|n_\kappa = 1\rangle$ known as the Fock or number state basis. The annihilation and creation operators a_k and $a_k^\dagger$ of a fermionic particle in mode k (with frequency ω_k) satisfy the canonical anticommutation relation (CAR) $\{a_k^\dagger, a_l\} = \delta_{kl}, \quad \{a_k^\dagger, a_l^\dagger\} = \{a_k, a_l\} = 0$, where δ is the Kronecker δ. Over the Fock state, the action of $\hat{a}$ and $\hat{a}^\dagger$ operators is given as $a|0\rangle = 0 = a^\dagger|1\rangle$, $a|1\rangle = |0\rangle$, and $a^\dagger|0\rangle = |1\rangle$. The number operator $\hat{n} = a^\dagger a$ is an eigenstate of the Fock state, i.e., $n|n\rangle = n|n\rangle$.

The fermionic system may be described by another set of field operators known as the Majorana operators. The kth mode Majorana operators c_{2k} and c_{2k-1} are related to the creation and annihilation operators through the relation

$$c_{2k-1} = \frac{1}{\sqrt{2}}(a_k + a_k^\dagger), \quad c_{2k} = \frac{\mathrm{i}}{\sqrt{2}}(a_k - a_k^\dagger), \tag{2}$$

where $k = 1, 2, \cdots, N$ labels the N modes of the system under study. The Majorana operators are Hermitian and satisfy the relation $\{c_i, c_j\} = \delta_{ij}$. They can be arranged into a vector

$$\mathbf{x} := (c_1, c_3, \cdots, c_{2N-1}; c_2, c_4, \cdots, c_{2N})^T,$$

so that in a compact form, the fermionic canonical anticommutation relations (CAR) become

$$\{x_i, x_j\} = \delta_{ij}. \tag{3}$$

Linear transformations on fermionic operators that preserve the CAR are of the form

$$c_k \to c_k' = \sum_i O_{kl} c_l,$$

where $O \in O(2N)$ is an element of the orthogonal group. These transformations can be implemented by unitary operations which are generated by quadratic Hamiltonians in the Majorana operators c_j.

A. Fermionic Gaussian states

Gaussian states for bosonic modes are easily accessible in laboratories and Gaussian unitaries for bosonic systems can be easily implemented [11]. The idea that the unitary operation necessary to extract work from passive states is rather general led to considerations of the set of more restricted set of Gaussian transformations as they are more practically implemented [11]. This brought the notion of bosonic Gaussian passive states as those states from which work cannot be extracted using Gaussian transformations in the bosonic regime [11].

In the sequel we shall consider the analogous problem for fermions. Although there have been relatively few studies of fermionic Gaussian systems, we note that they can in principle be rendered useful for quantum computation purposes because of their analogy with two qubit systems [13,14]. To this end, we recapitulate some basic formalism on fermionic Gaussian states, which may be defined based on either covariance matrix approach or a Grassmann approach [15].

1. Covariance matrix approach

Here, arbitrary fermionic Gaussian states are operators that are exponentials of quadratic form in the Majorana operators

$$\rho = \mathcal{Z}^{-1} \exp\left[\frac{-\mathrm{i}}{4}\mathbf{x}^T G \mathbf{x}\right], \tag{4}$$

where $\mathcal{Z}$ is a normalization constant that can be obtained from the condition $\mathrm{tr}(\rho) = 1$; we provide its derivation in the Appendix. G is a real antisymmetric $2N \times 2N$ matrix, which can be brought to a $2N \times 2N$ block diagonal form by a special orthogonal matrix $O \in SO(2N)$, that is

$$\tilde{G} = OGO^T = \bigoplus_{j=1}^{N} \begin{pmatrix} 0 & \beta_j \\ -\beta_j & 0 \end{pmatrix}, \tag{5}$$

where the β_j are real numbers that characterize G. With an inverse transformation $G = O^T \tilde{G} O$, the density matrix Eq. (4) can be written as

$$\rho = \mathcal{Z}^{-1} \exp\left[-\frac{\mathrm{i}}{4}\mathbf{x}^T O^T \tilde{G} O \mathbf{x}\right]$$

and upon defining a new set of transformed Majorana operators $\tilde{\mathbf{x}} = \mathbf{O}\mathbf{x}$, the density matrix becomes

$$\rho = \mathcal{Z}^{-1} \exp\left[-\frac{\mathrm{i}}{4}\tilde{\mathbf{x}}^T \tilde{G} \tilde{\mathbf{x}}\right]. \tag{6}$$

Substituting $\tilde{\mathbf{x}} = (\tilde{c}_{2j-1}, \tilde{c}_{2j})^T$ and Eq. (5), we get the fermionic Gaussian state in standard form [16]

$$\rho = \frac{1}{2^n} \Pi_{j=1}^n \left(\mathbb{1} - \mathrm{i} \tanh\left(\frac{\beta_j}{2}\right) \tilde{c}_{2j-1} \tilde{c}_{2j}\right), \tag{7}$$

using $(\tilde{c}_{2j-1}\tilde{c}_{2j})^2 = -1$ for any j (see the Appendix).

Let us define a real and antisymmetric matrix Γ with elements

$$\Gamma_{kl} = \frac{\mathrm{i}}{2}\langle[c_k, c_l]\rangle = \begin{cases} \mathrm{i}\langle c_k c_l\rangle, & \text{for } k \neq l \\ 0, & \text{for } k = l \end{cases}, \tag{8}$$

where for a given state ρ and an observable A, we define $\langle A\rangle = \mathrm{Tr}[\rho A]$. In terms of the transformed Majorana operators $\tilde{c} = Oc$, the antisymmetric matrix Γ transforms as

$$\tilde{\Gamma}_{kl} = \frac{\mathrm{i}}{2}\mathrm{Tr}(\rho[\tilde{c}_k, \tilde{c}_l]) = \frac{\mathrm{i}}{2}\langle[O_{km}c_m, O_{ln}c_n]\rangle$$
$$= \sum_{kl} O_{km}\frac{\mathrm{i}}{2}\mathrm{Tr}(\rho[c_m, c_n])O_{nl}^T = \mathbf{O}\mathbf{\Gamma}\mathbf{O}^T.$$

Using the density matrix from Eq. (7), we can calculate $\tilde{\Gamma}_{kl} = \frac{\mathrm{i}}{2}\mathrm{Tr}(\rho[\tilde{c}_k, \tilde{c}_l])$, obtaining

$$\lambda_j = \tilde{\Gamma}_{2j-1,2j} = \mathrm{i}\,\mathrm{Tr}(\rho[\tilde{c}_{2j-1}, \tilde{c}_{2j}]) = \tanh\left(\frac{\beta_j}{2}\right),$$

with all other $\tilde{\Gamma}_{kl}$ zero, yielding

$$\tilde{\Gamma} = O\Gamma O^T = \bigoplus_{j=1}^{M} \begin{pmatrix} 0 & \lambda_j \\ -\lambda_j & 0 \end{pmatrix} \tag{9}$$

and demonstrating that Γ and G can both be brought to block diagonal form by the same orthogonal matrix $\mathbf{O}$.

By definition, Γ is the covariance matrix of the fermionic Gaussian state. The direct link between G and Γ indicates that a fermionic Gaussian state can be fully characterized by either its density matrix or its covariance matrix. Hence, every Γ corresponding to a physical state has to fulfill $\mathrm{i}\Gamma \leqslant \mathbb{1}$ or equivalently $\Gamma\Gamma^T \leqslant \mathbb{1}$ with equality if and only if the state is pure.

Thermal states with inverse temperature (β),

$$\tau(\beta) = \frac{1}{\mathcal{Z}} e^{-\beta H},$$

are examples of a more general class of Gaussian state. Here, $\mathcal{Z} = \mathrm{tr}(\tau H)$ is the partition function. If we define the Hamiltonian for a single fermionic mode with frequency ω as $H = \omega a^\dagger a$, then in the Fock basis, a thermal state can be expressed as

$$\tau(\beta) = \frac{1}{(1 + e^{-\beta\omega})} \sum_{n=0}^{1} e^{-n\beta\omega} |n\rangle\langle n|,$$

with covariance matrix

$$\Gamma = \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix}, \quad \Gamma^2 < -\lambda^2 \mathbb{1},$$

where $\lambda = \tanh\left(\frac{\beta\omega}{2}\right)$. For n noninteracting fermionic modes, the Hamiltonian is defined as $H = \sum_{i=1}^{n} \omega_i a_i^\dagger a_i$ and the covariance matrix for the product of n fermionic thermal states is

$$\Gamma_n = \bigoplus_{i}^{n} \begin{pmatrix} 0 & \lambda_i \\ -\lambda_i & 0 \end{pmatrix}, \tag{10}$$

$\lambda_i = \tanh\left(\frac{\beta_i \omega_i}{2}\right)$. We will make reference to this later in the paper.

2. Grassmann approach

The connection between the covariance matrix approach and Grassmann approach is the map assigning the Grassmann variables to each Majorana operator,

$$\omega(c_{2M-1}, c_{2M}, \gamma) = \gamma_{2M-1}\gamma_{2M}, \quad \omega(\mathbb{1}, \gamma) = 1, \tag{11}$$

where $\gamma_k \in \mathcal{G}_{2n}$ is the algebra of Grassmann variables. Then we define a state ρ of n fermionic modes to be Gaussian if its Grassmann representation $\omega(\rho, \gamma)$ is Gaussian,

$$\omega(\rho, \gamma) = \frac{1}{2^n} \exp\left(\frac{\mathrm{i}}{2} \gamma^* \Gamma \gamma\right), \tag{12}$$

where Γ is a $2n \times 2n$ real antisymmetric matrix also known as the covariance matrix of the state as defined in (8) [16].

3. Coherent states

Under the Grassmann representation, one can define a fermionic coherent state [15]. For any set of variables $\{\gamma_i\}$ of Grassmann numbers, a normalized coherent state $|\gamma\rangle$ is defined as the displaced vacuum state $|\gamma\rangle = D(\gamma)|0\rangle$, where $D(\gamma)$ is the displacement operator, which acts on fermionic $\hat{a}$ and $\hat{a}^\dagger$ operators as $D(\gamma)\hat{a}D^\dagger(\gamma) = a + \gamma$ and $D(\gamma)\hat{a}^\dagger D^\dagger(\gamma) = a^\dagger + \gamma^*$ respectively [15]. This operation preserves the anticommutation relations. In this paper we restrict ourselves to Gaussian operators that contain no Grassmann variables [17].

B. Gaussian unitaries

Gaussian unitaries are generated by Hamiltonians quadratic in Majorana operators and transform Gaussian states to Gaussian states. This definition applies to the specific case of unitary transformations that preserve the Gaussian character of a quantum state. Gaussian transformations in Hilbert space are special orthogonal transformations on phase space. In terms of the statistical moment $\hat{\mathbf{x}}$ and Γ, the special orthogonal transformation is defined by the action

$$\hat{\mathbf{x}} = O\hat{\mathbf{x}}, \quad \Gamma = O\Gamma O^T, \quad OO^T = \mathbb{1}. \tag{13}$$

Unlike boson field operators, whose algebraic properties are preserved by symplectic transformations, fermion anticommutation relations are invariant under rotations. Examples of such Gaussian transformations that preserve the canonical anticommutation relations of fermionic modes (thus transforming fermionic Gaussian states to fermionic Gaussian states) are as follows.

(1) Phase rotation operator:

$$R(\theta) = e^{-\mathrm{i}\theta a^\dagger a},$$

$$\hat{\mathbf{x}} \to O(\theta)\hat{\mathbf{x}}, \quad O(\theta) = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}. \tag{14}$$

(2) Two-mode squeezing operator [18]:

$$S(r) = \exp[r(ab - b^\dagger a^\dagger)],$$

$$\hat{\mathbf{x}} \to S(r)\hat{\mathbf{x}}, \quad S(r) = \begin{pmatrix} \cos(r)\mathbb{1} & -\sin(r)\sigma_z \\ \sin(r)\sigma_z & \cos(r)\mathbb{1} \end{pmatrix}, \tag{15}$$

where $\sigma_z = \mathrm{diag}(1, -1)$ is the usual Pauli matrix.

(3) Beam splitting operation:

$$B(\phi) = \exp[\phi(ab^\dagger + a^\dagger b)]$$

$$\hat{\mathbf{x}} \to B(\phi)\hat{\mathbf{x}}, \quad B(\phi) = \begin{pmatrix} \cos(\phi)\mathbb{1} & -\sin(\phi)\mathbb{1} \\ \sin(\phi)\mathbb{1} & \cos(\phi)\mathbb{1} \end{pmatrix}. \tag{16}$$

III. PASSIVITY AND ACTIVATION

A state ρ is passive if its average energy cannot be lowered when a unitary operation acts on it, that is

$$\mathrm{Tr}[H\rho] \leqslant \mathrm{Tr}[HU\rho U^\dagger], \tag{17}$$

where $H = \sum_{i=0}^{d-1} E_i |i\rangle\langle i|$ is the Hamiltonian of the finite dimensional quantum system associated with the Hilbert space $\mathcal{H} \equiv \mathbb{C}^d$, with energy eigenstates $|i\rangle$ and eigenvalues E_i. A state may be passive given only a single copy but can become active for n copies. Completely passive states remain passive no matter how many copies of the system are available, while those states that become active for some $k \geqslant n$ copies of the system is termed k-activable [7]. This naturally leads to the question of what the class of states is that remains passive, even given an infinite number of copies. Thermal states defined by $\rho = \frac{1}{\mathcal{Z}} e^{-\beta H}$ with $\mathcal{Z} = \mathrm{Tr}[e^{-\beta H}]$ are the only completely passive states [1,2].

Given that some passive states can be activated for some $k \geqslant n$ copies of the system to yield work, the aim of this section is to find the value of k for which a passive but not thermal state of fermionic modes can be activated to yield work.

A. Passive states

Passivity of a quantum state is often expressed as a property of the state and its Hamiltonian. Consider a state ρ and a reference Hamiltonian H, both written in their respective eigenbasis,

$$H := \sum E_n |n\rangle\langle n|, \quad \text{with} \quad E_{n+1} \geqslant E_n \ \forall n,$$
$$\rho := \sum p_n |\rho_n\rangle\langle \rho_n|, \quad \text{with} \quad p_{n+1} \leqslant p_n \ \forall n,$$

where $0 \leqslant p_n \leqslant 1$ and $\sum_n p_n = 1$. ρ is passive if and only if it is diagonal in the same basis as the Hamiltonian H of the system, that is $[\rho, H] = 0$. This can be interpreted as $\{|\rho_n\rangle\}$ coinciding with $\{|n\rangle\}$, with no population inversion, that is with decreasing population $p_j < p_k$ and increasing energy $E_j > E_k$. Otherwise, we say ρ is nonpassive.

In a two-dimensional continuous variable system spanned by the states $|m\rangle$ and $|n\rangle$, it can be shown[11] that a product of two thermal states of two bosonic modes at the same inverse temperature β and frequency ω, form an example of a passive state, whereas given the modes with the same frequency and at different inverse temperature, the state is non passive. We ask if this is true for fermionic systems.

B. Activation of passive states to generate work

In the Fock basis, a thermal state for a fermionic mode with inverse temperature β is given as

$$\tau(\beta) = (1 + e^{-\beta\omega})^{-1} \sum_{n=0}^{1} e^{-n\beta\omega} |n\rangle\langle n|$$
$$= (1 + e^{-\beta\omega})^{-1} (|0\rangle\langle 0| + e^{-\beta\omega} |1\rangle\langle 1|) \qquad (18)$$

Consider a noninteracting two-mode fermionic system of equal frequency ω each with local Hamiltonian $h_i = \omega a_i^\dagger a_i$. The total Hamiltonian H of the system is simply the sum of the individual local Hamiltonians: $H_s = \omega(a_1^\dagger a_1 + a_2^\dagger a_2)$. The fermionic two-mode thermal state in the Fock basis may then be expressed as

$$\tau(\beta_1, \beta_2) = \frac{1}{\mathcal{Z}_1 \mathcal{Z}_2} \sum_{m,n=0}^{1} e^{-\omega(n\beta_1 + m\beta_2)} |m\rangle\langle m| \otimes |n\rangle\langle n|,$$

where $\mathcal{Z}_1 \mathcal{Z}_2 = (1 + e^{-\beta_1\omega})(1 + e^{-\beta_2\omega})$ and up to a common factor, the matrix elements are

$$\epsilon = e^{-\omega(\beta_1 n + \beta_2 m)} = e^{-\frac{\omega}{T_1 T_2}(m T_1 + n T_2)}.$$

We see that H_s commutes with the product state $\tau(\beta_1, \beta_2)$ composed of states of the form in Eq. (18). The occupational numbers $n, m \in \{0, 1\}$. The sum of the occupational numbers in the state is $N_i = m + n$. Consider a unitary transformation from the state $\tau(\beta_1, \beta_2)$ to $\tau'(\beta_1, \beta_2)$ such that

$$\tau'(\beta_1, \beta_2) = \frac{1}{\mathcal{Z}_1 \mathcal{Z}_2} \sum_{m',n'=0}^{1} e^{-\omega(n'\beta_1 + m'\beta_2)} |m'\rangle\langle m'| \otimes |n'\rangle\langle n'|,$$

with new occupational number given as $N_i' = m' + n'$ and matrix element proportional to

$$\epsilon' = e^{-\omega(\beta_1 n' + \beta_2 m')} = e^{-\frac{\omega}{T_1 T_2}(m' T_1 + n' T_2)}.$$

The state $\tau(\beta_1, \beta_2)$ is nonpassive if there exist pairs of non-negative integers m, n, m', n' such that

$$\epsilon' > \epsilon, \quad \text{while} \quad m' + n' > m + n, \qquad (19)$$

which up to a common factor yields the condition

$$m T_1 + n T_2 > m' T_1 + n' T_2, \quad \text{while} \quad m' + n' > m + n, \qquad (20)$$

by making use of the fact that $e^{-AX} > e^{-AY} \Rightarrow X < Y$. Given that $m, n \in \{0, 1\}$, Eq. (20) cannot be satisfied. We then conclude that for two-mode fermionic states, regardless of frequencies of the modes and its temperature, the product of two thermal states is always passive, this is in contrast to the bosonic case [11].

However, for a product $\tau(\beta_1, \beta_2, \beta_3)$ of three fermionic thermal states,

$$\tau(\beta_1, \beta_2, \beta_3) = \frac{1}{Z_1 Z_2 Z_3} \sum_{m,n,l=0}^{1} e^{-\omega(n\beta_1 + m\beta_2 + l\beta_3)} \times |m\rangle\langle m| \otimes |n\rangle\langle n| \otimes |l\rangle\langle l|, \qquad (21)$$

the situation changes. The nonpassivity condition becomes

$$n\beta_1 + m\beta_2 + l\beta_3 > n'\beta_1' + m'\beta_2' + l'\beta_3', \qquad (22)$$
$$\text{while} \quad m' + n' + l' > m + n + l.$$

The matrix element is now proportional to $e^{-\omega(n\beta_1 + m\beta_2 + l\beta_3)}$ and $m, n, l \in \{0, 1\}$. One can now find a three-dimensional subspace in which a unitary can reduce the average energy, proving that the state $\tau(\beta_1, \beta_2, \beta_3)$ is not always passive. For example, let $m' = n' = 1, l' = 0$ and $m = n = 0, l = 1$, it is obvious that $m' + n' + l' > m + n + l$. Also,

$$\beta_3 > \beta_1 + \beta_2, \qquad (23)$$

which can hold for sufficiently large β_3. In general, the condition Eq. (22) can be satisfied provided $\beta_i \ll \beta_j, \beta_k$ for distinct i, j, k. Hence, a product of thermal states $\rho = \prod_j^n \tau(\beta_j) = \tau(\beta_1) \otimes \cdots \otimes \tau(\beta_n)$ for fermionic modes can be activated to become nonpassive for $n \geqslant 3$. In other words, the state is 3-activable [6].

From the above we can construct the following.

Protocol: Consider the three-mode fermionic system described by the state Eq. (21). From the nonpassivity condition Eq. (22), we note that for the above transformation to be possible, the action of the unitary operation must be such that

(1) The initial state with a composition of the three modes should consist at least of an unpopulated mode and a populated mode. That is, initial states of the system of the form $|111\rangle$ and $|000\rangle$ are not allowed.

(2) The action of the unitary should take the initially populated (unpopulated) mode to an unpopulated (populated) mode of the final state.

(3) One can always guess the temperature relationship of the different modes: The sum of the inverse temperature of the initially unpopulated modes must be less than the inverse temperature of the populated mode.

(4) If a transformation leaves a mode unaffected, then the temperature of such mode does not matter during the transformation process.

We now turn to a practical example of such transformation. The three mode state can be written as

$$\begin{aligned}\rho_{nml} = \frac{1}{Z_1Z_2Z_3}&[e^{-\omega\beta_1}|100\rangle\langle 001| + e^{\omega\beta_3}|001\rangle\langle 100|\\ &+ e^{-\omega\beta_2}|010\rangle\langle 010| + e^{-\omega(\beta_1+\beta_3)}|101\rangle\langle 101|\\ &+ e^{-\omega(\beta_1+\beta_2)}|110\rangle\langle 011| + e^{-\omega(\beta_2+\beta_3)}|011\rangle\langle 110|\\ &+ |000\rangle\langle 000| + e^{-\omega(\beta_1+\beta_2+\beta_3)}|111\rangle\langle 111|],\end{aligned}$$

upon expanding the sum in Eq. (21). Consider a unitary of the form

$$\begin{aligned}U = |101\rangle\langle 010| + |010\rangle\langle 101| - |101\rangle\langle 101|\\ - |010\rangle\langle 010| + \mathbb{1},\end{aligned} \tag{24}$$

where U induces a transition between the two degenerate states,

$$|010\rangle \leftrightarrow |101\rangle. \tag{25}$$

We note that $U = U^{\dagger}$. This type of unitary has been applied to generate a mixed state of the Werner- type thermal state [19] necessary for quantum information processing. The amount of work extracted from the system (the change in its average energy) is given by [7]

$$\begin{aligned}W &= \mathrm{Tr}[H(\rho_{nml} - U\rho_{nml}U^{\dagger})]\\ &= \hbar\omega e^{-\omega\beta_2}(1 - e^{-\omega((\beta_1+\beta_3)-\beta_2)}),\end{aligned}$$

which must be positive for the state to be nonpassive. Clearly this will hold whenever $(\beta_1 + \beta_3) - \beta_2 < 0$, or in other words,

$$\beta_2 > \beta_1 + \beta_3, \tag{26}$$

which agrees with the nonpassivity condition in Eq. (22). Alternatively one could employ a unitary that interchanges the $|001\rangle$ and $|110\rangle$ states and one would obtain Eq. (23).

The problem of generating a unitary analogous to Eq. (24) for more copies of fermion states is rather challenging. In the next section, we discuss a more restricted class of unitary transformations.

IV. GAUSSIAN PASSIVITY

In the previous section we saw that, unlike the situation for bosonic modes, for fermionic modes a product of two thermal states at different temperatures is passive. Given that constructing a heat engine requires access to two thermal baths at different temperatures, can one construct a heat engine out of a product of thermal states in fermionic modes?

To answer this question, we note that passivity of quantum states requires a cyclic unitary transformation. In our work, we consider a Gaussian unitary transformation to characterize fermionic states according to their abilities to generate work or not.

Suppose we have access to a Gaussian unitary. We are interested in the effect of the Gaussian transformation induced by this unitary on an arbitrary state via the effect on the corresponding covariance matrix. We ask for which (not necessarily Gaussian) states of two noninteracting fermionic modes with frequencies ω_a and ω_b ($\omega_a \leqslant \omega_b$) can energy can be extracted using only Gaussian operations. States from which energy cannot be extracted using Gaussian operations are called Gaussian passive [11].

V. ENERGY AS A FUNCTION OF STATE COVARIANCE MATRIX

Before we proceed, we define the average energy of a state in terms of its covariance matrix.

Definition 1: The average energy of a quantum state ρ of a fermionic mode with frequency ω is given in terms of its covariance matrix Γ by the relation

$$E(\Gamma) = \frac{\omega}{2}[1 - \mathrm{Tr}(\Omega\Gamma)], \tag{27}$$

for some real symplectic matrix Ω

To demonstrate this, the covariance matrix Γ of a quantum state ρ for a fermionic mode with frequency ω is

$$\Gamma = \frac{\mathrm{i}}{2}\begin{pmatrix} 0 & \langle[c_1, c_2]\rangle \\ \langle[c_2, c_1]\rangle & 0 \end{pmatrix}, \tag{28}$$

in terms of Majorana operators Eq. (2) for this mode. Defining the 2×2 symplectic matrix

$$\Omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and taking the product $\Omega\Gamma$, we find

$$\mathrm{Tr}(\Omega\Gamma) = \mathrm{i}\langle[c_1, c_2]\rangle = \mathrm{Tr}(\mathrm{i}[c_1, c_2]\rho). \tag{29}$$

The average energy $E(\rho) = \omega\,\mathrm{Tr}[\rho a^{\dagger} a]$, which becomes

$$E(\rho) = \frac{\omega}{2}\,\mathrm{Tr}\left[\rho\left(c_1^2 - \mathrm{i}[c_1, c_2] + c_2^2\right)\right]. \tag{30}$$

Substituting Eq. (29) into Eq. (30) and taking note that $c_1^2 = 1/2 = c_2^2$, we obtain $E(\Gamma) = \frac{\omega}{2}(1 - \mathrm{Tr}(\Omega\Gamma))$ as expected. This is the average energy for a single mode of the state with frequency ω.

As we consider noninteracting fermionic modes, the average energy of an n-mode state is defined as the sum of the average energy of each of the individual modes. In terms of covariance matrix this is given as

$$E(\Gamma_n) = \frac{\omega_1}{2}[1 - \mathrm{Tr}(\Omega_1\Gamma_1)] + \cdots + \frac{\omega_n}{2}[1 - \mathrm{Tr}(\Omega_n\Gamma_n)], \tag{31}$$

where the symplectic matrix for the entire system is $\Omega = \bigoplus_{j=1}^{n}\Omega_j$.

VI. CHARACTERIZING A GAUSSIAN PASSIVE AND NONPASSIVE FERMIONIC STATE

We are now ready to characterize quantum states with a covariance matrix Γ for which the average energy Eq. (31) can be minimized by a Gaussian unitary transformation.

A. Standard form of a covariance matrix

Let ρ be the state of a two-mode system each with frequencies ω_a and $\omega_b \geqslant \omega_a$, and define Γ as the covariance matrix of the two-mode system. Any two-mode covariance matrix can be brought to the form

$$\Gamma_{sf} = \begin{pmatrix} 0 & a & 0 & -e_1 \\ -a & 0 & -e_2 & 0 \\ 0 & e_2 & 0 & b \\ e_1 & 0 & -b & 0 \end{pmatrix}, \tag{32}$$

by a local orthogonal operation (LOO) $O_{\mathrm{loc}} = O_{\mathrm{loc,a}} \oplus O_{\mathrm{loc,b}}$, that is $\Gamma_{sf} = O_{\mathrm{loc}}\Gamma O_{\mathrm{loc}}^T$ [14].

The more restrictive set of pure Gaussian states are characterized by $\Gamma_{sf}^2 = -\mathbb{1}$. This implies that the covariance matrix of the two-mode pure fermionic Gaussian state can be brought to the form

$$\Gamma_{sf}^p = \begin{pmatrix} 0 & a & 0 & -e \\ -a & 0 & -e & 0 \\ 0 & e & 0 & a \\ e & 0 & -a & 0 \end{pmatrix}, \tag{33}$$

with $e = (1 - a^2)^{1/2}$ [13,14] so that the fermionic system depends only on one parameter a.

Now suppose we have a product of two fermionic modes with the covariance matrix in the standard form Eq. (32). Its average energy according to Eq. (31) is given as

$$E(\Gamma_{sf}) = \frac{\omega_a}{2}(1 - 2a) + \frac{\omega_b}{2}(1 - 2b), \tag{34}$$

where ω_a and ω_b are the frequencies of the modes. We shall now prove:

Theorem 1: Any (not necessarily Gaussian) state of two noninteracting fermionic modes with frequencies $\omega_b \geqslant \omega_a$ is Gaussian-passive if and only if its covariance matrix Γ is

(i) in Williamson standard form [13]

$$\Gamma = \begin{pmatrix} 0 & a & 0 & 0 \\ -a & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & -b & 0 \end{pmatrix}, \tag{35}$$

with $\lambda_a > \lambda_b$ for $\omega_b \neq \omega_a$, or

(ii) in the form

$$\Gamma = \begin{pmatrix} 0 & a & 0 & -e \\ -a & 0 & e & 0 \\ 0 & -e & 0 & b \\ e & 0 & -b & 0 \end{pmatrix}, \tag{36}$$

for equal frequencies $\omega_b = \omega_a$.

To prove this theorem, we start with the most general covariance matrix that any state ρ may have and apply Gaussian operations to reduce its average energy until minimal. At this point we obtain a state ρ' with minimal energy. We compare the energy of ρ' with that of ρ and identify under which conditions the energy of ρ has been lowered. We thus can identify the characteristics of Gaussian-passive states from these conditions. We consider here even fermionic systems for which $\mathrm{Tr}(X) = 0$. As noted in Sec. II A, these have no Grassmann variables and so have vanishing first moment.

B. Local orthogonal transformations

We note that the covariance matrix Γ of a two-mode fermionic system can be brought to its standard form through a local orthogonal transformation $O_{\mathrm{loc}} = O_{\mathrm{loc}_a} \oplus O_{\mathrm{loc}_b}$, that is

$$O_{\mathrm{loc}}\Gamma O_{\mathrm{loc}}^T = \Gamma_{sf} = \begin{pmatrix} A & E \\ -E^T & B \end{pmatrix}, \tag{37}$$

with

$$O_{\mathrm{loc}_a} = \begin{pmatrix} \cos(\phi_a) & \sin(\phi_a) \\ -\sin(\phi_a) & \cos(\phi_a) \end{pmatrix},$$

and where each element of Γ_{sf} is a 2×2 matrix

$$A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & e_1 \\ e_2 & 0 \end{pmatrix}.$$

A and B describe the local covariance matrix of each mode and E describes the correlation between the two modes. By inverting Eq. (37), we can write the local covariance matrix of a two-mode system as

$$O_{\mathrm{loc}}^T\Gamma_{sf}O_{\mathrm{loc}} = \Gamma, \tag{38}$$

and we note that the inverse operations are also local orthogonal transformations.

We ask here: Given a state with a two-mode covariance matrix Γ, can work be extracted from the system? That is, can the average energy corresponding to Γ be reduced? To answer this question we compute the average energy $E(\Gamma)$ corresponding to the covariance matrix Γ and find using Eq. (37) that $E(\Gamma) = E(\Gamma_{sf})$ as given in Eq. (34). Since the energies are the same, it becomes clear that states with covariance matrix $\Gamma = \Gamma_{sf}$ are Gaussian passive under a local orthogonal transformation.

However, the energy of such states may be reduced by global orthogonal transformations, as we will show in the next section.

C. Two mode Squeezing

Now suppose a state has a covariance matrix in the standard form Eq. (37). We have seen in the previous subsection that such a state is Gaussian passive under a local orthogonal transformation. In this section, we will apply the global orthogonal transformation Eq. (15) to the system and see if its average energy can be reduced. Computing the corresponding two-mode squeezed covariance matrix $\hat{\Gamma}_{\rm TM} = S(r)\Gamma_{sf}S(r)^T$, we find

$$\hat{\Gamma}_{\rm TM} = \begin{pmatrix} 0 & a' & 0 & -e_1' \\ -a' & 0 & -e_2' & 0 \\ 0 & e_2' & 0 & b' \\ e_1' & 0 & -b' & 0 \end{pmatrix}, \tag{39}$$

where

$$a' = ac_r^2 - bs_r^2 - \tfrac{1}{2}(e_1+e_2)s_{2r}, \tag{40a}$$
$$b' = -as_r^2 + bc_r^2 - \tfrac{1}{2}(e_1+e_2)s_{2r}, \tag{40b}$$
$$e_1' = \tfrac{1}{2}(a+b)s_{2r} + e_1c_r^2 - e_2s_r^2, \tag{40c}$$
$$e_2' = \tfrac{1}{2}(a+b)s_{2r} + e_2c_r^2 - e_1s_r^2, \tag{40d}$$

with $c_r = \cos(r)$ and $s_r = \sin(r)$, respectively. To see if this transformation can reduce the average energy, we compute $E(\hat{\Gamma}_{\rm TM})$ using Eq. (27), obtaining

$$E(\hat{\Gamma}_{\rm TM}) = \frac{\omega_a}{2}(1-2a') + \frac{\omega_b}{2}(1-2b'), \tag{41}$$

and substituting Eq. (40) into Eq. (41), we get

$$\begin{aligned} E(\hat{\Gamma}_{\rm TM}) = {} & \omega_a[b\sin^2(r) - a\cos^2(r)] \\ & + \omega_b[a\sin^2(r) - b\cos^2(r)] \\ & + \frac{(\omega_a+\omega_b)}{2}[1 + (e_1+e_2)\sin(2r)], \end{aligned} \tag{42}$$

and minimizing this with respect to the squeezing parameter r, we find the condition

$$\begin{aligned} \frac{\partial}{\partial r}E(\hat{\Gamma}_{\rm TM}) = 0, \\ \Rightarrow (a+b)\sin(2r) + (e_1+e_2)\cos(2r) = 0, \end{aligned} \tag{43}$$

whose solution is

$$r = -\frac{1}{2}\tan^{-1}\left[\frac{e_1+e_2}{(a+b)}\right] = -\frac{1}{2}\tan^{-1}(\lambda), \tag{44}$$

where $\lambda = (e_1+e_2)/(a+b)$. The minimized energy is

$$\begin{aligned} E_{\rm min}(\hat{\Gamma}_{\rm TM}) = {} & \frac{(\omega_a+\omega_b)}{2}(1-(a+b)\sqrt{1+\lambda^2}) \\ & + \frac{1}{2}(\omega_b-\omega_a)(a-b). \end{aligned} \tag{45}$$

Defining $e = (e_1 - e_2)/2$, the elements of the covariance matrix Eq. (40) are now

$$\tilde{a}' = \frac{(a+b)}{2}\sqrt{1+\lambda^2} + \frac{(a-b)}{2}, \tag{46a}$$
$$\tilde{b}' = \frac{(a+b)}{2}\sqrt{1+\lambda^2} - \frac{(a-b)}{2}, \tag{46b}$$
$$\tilde{e}_1' = e, \quad \tilde{e}_2' = -e. \tag{46c}$$

We pause to comment on the interpretation of these matrix elements. In addition to minimizing the system's average energy, the squeezing parameter Eq. (44) reduces the off-diagonal elements in Eq. (39) to a single parameter e so that the resulting covariance matrix is of the form

$$\Gamma_{\rm GP} = \begin{pmatrix} 0 & \tilde{a}' & 0 & -e \\ -\tilde{a}' & 0 & e & 0 \\ 0 & -e & 0 & \tilde{b}' \\ e & 0 & -\tilde{b}' & 0 \end{pmatrix}. \tag{47}$$

If the state is a two-mode pure fermionic Gaussian state whose covariance matrix is of the form of Eq. (33), the two-mode squeezing operation takes the state's covariance matrix to the form

$$\Gamma_{\rm GP}^p = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \tag{48}$$

with property $(\Gamma_{\rm GP}^p)^2 = -\mathbb{1}$. This corresponds to the covariance matrix of a pure fermionic Gaussian state in the Williamson normal form [13]. To achieve a Williamson normal form covariance matrix for the general two-mode fermionic system, we consider further Gaussian unitary transformations on the system.

D. Beam splitting

The last Gaussian operation we have to consider is the beam-splitting operation. This transformation on fermionic phase space is represented by the transformation matrix Eq. (16). We find

$$\Gamma_{\rm BS} = B(\theta)\hat{\Gamma}_{\rm GP}B^\dagger(\theta) = \begin{pmatrix} 0 & A & 0 & D \\ -A & 0 & -D & 0 \\ 0 & D & 0 & B \\ -D & 0 & -B & 0 \end{pmatrix}, \tag{49}$$

where

$$A = \tilde{a}'\cos^2\theta + \tilde{b}'\sin^2(\theta) + e\sin(2\theta), \tag{50a}$$
$$B = \tilde{b}'\cos^2\theta + \tilde{a}'\sin^2(\theta) - e\sin(2\theta), \tag{50b}$$
$$D = \tfrac{1}{2}(\tilde{a}' - \tilde{b}')\sin 2\theta - e\cos(2\theta). \tag{50c}$$

The average energy corresponding to $\Gamma_{\rm BS}$ is

$$E(\Gamma_{\rm BS}) = -\omega_a[\tilde{b}'\sin^2(\theta) + \tilde{a}'\cos^2(\theta)]$$
$$- \omega_b[\tilde{a}'\sin^2(\theta) + \tilde{b}'\cos^2(\theta)] \tag{51}$$
$$+ \frac{(\omega_a+\omega_b)}{2} + (\omega_b-\omega_a)e\sin(2\theta). \tag{52}$$

Again, energy is minimized for the value of θ satisfying the equation

$$(\omega_b-\omega_a)[(\tilde{b}'-\tilde{a}')\sin(2\theta) + 2e\cos(2\theta)] = 0, \tag{53}$$

implying

$$\theta = -\frac{1}{2}\tan^{-1}\left(\frac{2e}{\tilde{b}'-\tilde{a}'}\right) = -\frac{1}{2}\tan^{-1}\mu,$$

where $\mu = 2e/(\tilde{b}' - \tilde{a}')$. The minimized energy under the beam splitting operation is then

$$E_{\min}(\hat{\Gamma}_{\rm BS}) = \frac{(\omega_b - \omega_a)}{2}[(\tilde{a}' - \tilde{b}')\sqrt{1+\mu^2}] + \frac{1}{2}(\omega_b + \omega_a)[1 - (\tilde{a}' + \tilde{b}')], \tag{54}$$

and the corresponding minimized matrix element is

$$A = \frac{(\tilde{a}' + \tilde{b}')}{2} + \frac{(\tilde{a}' - \tilde{b}')}{2}\sqrt{1+\mu^2}, \tag{55a}$$

$$B = \frac{(\tilde{a}' + \tilde{b}')}{2} - \frac{(\tilde{a}' - \tilde{b}')}{2}\sqrt{1+\mu^2}, \tag{55b}$$

$$D = 0. \tag{55c}$$

For equal frequencies $\omega_a = \omega_b$, the average energy is unchanged, that is $E_{\min}(\hat{\Gamma}_{\rm TM}) = E_{\min}(\hat{\Gamma}_{\rm BS})$ and we conclude that the state with covariance matrix Eq. (47) is Gaussian passive. However, for different frequencies assume w.l.o.g. that $\omega_b > \omega_a$, the covariance matrix for the minimized state under beam splitting operation is in the Williamson normal form [13]

$$\Gamma^1_{\rm GP} = \begin{pmatrix} 0 & A & 0 & 0 \\ -A & 0 & 0 & 0 \\ 0 & 0 & 0 & B \\ 0 & 0 & -B & 0 \end{pmatrix}, \tag{56}$$

with eigenvalues given as $\lambda_a = \pm iA$ and $\lambda_b = \pm iB$. If $a > b$, we find that $\lambda_a > \lambda_b$ and so the lower frequency mode has the higher population.

We see that the effect of the orthogonal transformation on the fermionic two-mode covariance matrix is to decompose the modes and bring them into a product of single-mode locally thermal states diagonal in the Fock basis. An example of Gaussian passive state of two modes with different frequencies is that of a product of single mode thermal states, in which each mode has different temperature. In this case, the Williamson eigenvalues are $\lambda_i = \tanh(\frac{\omega_i}{2T_i})$. For $T_b \neq 0$ the condition $\lambda_a > \lambda_b$ for Gaussian passivity can be expressed as

$$\frac{\omega_a}{\omega_b} > \frac{T_a}{T_b}. \tag{57}$$

As shown in Sec. III B, within the framework of general operations, the product of two thermal states at different temperature is passive, regardless of the frequencies of the modes involved. And from above, we see that such a state is also Gaussian passive showing that all passive states are obviously Gaussian passive, but the converse may not be true [11] as we will show in the next section.

E. More general operations

So far we have focused on characterizing a general fermionic state according to whether work can be extracted or not using Gaussian unitary transformations. We started with the covariance matrix of a general two-mode fermionic system, applied Gaussian unitary operations to extract the energy from the system and then we arrived at the Gaussian passive state Eq. (56), where no further energy could be extracted by an additional Gaussian unitary transformation. A reasonable question then arises: in the process of characterizing a (not necessarily Gaussian) state, how much extractable work is sacrificed by using Gaussian unitary transformations instead of general unitary transformations? To address this question we will follow a procedure similar to that in the bosonic case [11].

In the characterization process we fixed the second moment of the fermionic state, which only uniquely identifies a state if it is Gaussian. Two steps therefore lead us to answering the above question. (1) First, we must find a non-Gaussian state that is compatible with a given Gaussian passive state, or in other words we must find a non-Gaussian state with the same second moment as that of the Gaussian passive state. (2) We must show that a general unitary transformation on the resulting non-Gaussian state can lower its energy to the minimal value.

To proceed, we first note that the covariance matrix of a general two-mode Gaussian-passive state Eq. (56) is identical to the covariance matrix of a product of locally thermal states of two different modes each with different effective temperatures. One could then consider a single fermionic mode in a thermal state with arbitrary temperature and then find a pure state whose second moment is that of this single mode thermal state. Then one could certainly find pairs of states of this kind whose tensor product is compatible with a Gaussian-passive locally thermal two-mode state. For example, in the Fock basis, the fermionic state

$$|\psi\rangle = \sqrt{1-p}|0\rangle + \sqrt{p}|1\rangle, \quad 0 \leqslant p \leqslant 1 \tag{58}$$

has a covariance matrix of the form

$$\begin{pmatrix} 0 & 2p-1 \\ 1-2p & 0 \end{pmatrix}, \tag{59}$$

and so by carefully choosing the continuous parameter p, we can bring the covariance matrix to look like that of a single-mode fermionic thermal state with inverse temperature β,

$$\Gamma_{th} = \begin{pmatrix} 0 & \tanh\left(\frac{\beta\omega}{2}\right) \\ -\tanh\left(\frac{\beta\omega}{2}\right) & 0 \end{pmatrix}, \tag{60}$$

where ω is the mode frequency. Unfortunately, the state Eq. (58) is prohibited by a super-selection rule [20] and so does not exist.

However, another example would be the fermionic vacuum state $|0\rangle$ and a single fermion state $|1\rangle$ each having covariance matrices

$$\Gamma_{\rho_0} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \Gamma_{\rho_1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

respectively. Given that the $(|1\rangle, |0\rangle)$ states are pure, their covariance matrices satisfy the condition $\Gamma^2_{|i\rangle} = -\mathbb{1}$. We define the free energy of these states as

$$F(\rho) = E(\rho) - TS(\rho), \tag{61}$$

where $S(\rho) = -\mathrm{Tr}[\rho \ln(\rho)]$ is the von Neumann entropy, which is vanishing for pure states, and $E(\rho)$ is the average (internal) energy.

Now to achieve our first task, consider pairs of the single fermionic systems encoded into a bipartite Hilbert space $\mathcal{H}_{ab} = \mathcal{H}_a \otimes \mathcal{H}_b$ of subsystems a and b, respectively. The state is defined by a density operator $\rho^1_{ab} = |00\rangle_{ab}\langle 00|$ and $\rho^2_{ab} = |11\rangle_{ab}\langle 11|$, respectively, the resulting states correspond to direct sum of locally pure fermionic Gaussian states. Their covariance matrices are, respectively,

$$\Gamma_{\rho^1_{ab}} = \Gamma^a_{\rho_1} \oplus \Gamma^b_{\rho_1}, \quad \Gamma_{\rho^2_{ab}} = \Gamma^a_{\rho_2} \oplus \Gamma^b_{\rho_2},$$

which is the same as the CM of pure fermionic Gaussian passive state Eq. (48). For our second task, given that the constructed states are pure, their free energy is thus identical to the average energy. Interestingly, there is no way to lower the average energy of the constructed state ρ^1_{ab}; however, the energy of the state ρ^2_{ab} can be lowered by applying a (non Gaussian) unitary transformation that takes the pure state to vacuum state. This shows that ρ^2_{ab} is Gaussian passive but not passive while ρ^1_{ab} is both passive and Gaussian passive, as expected.

VII. CONCLUSION

We have investigated the problem of work extraction from fermionic systems, finding a number of similarities and differences with their bosonic counterparts.

Thermal states at positive temperatures are the only completely passive states from which work cannot be extracted no matter the number of available copies [1,6,7].

Any quantum state out-of-equilibrium is a potential resource for work extraction. However, for fermions the situation is somewhat subtle. We have shown that under arbitrary unitary transformations there is no way to process a product of two fermionic modes in different thermal states to extract work, independent of mode temperatures and frequency. This is quite unlike the situation for the bosonic counterpart [11] and suggests that fermionic systems are not as useful for quantum thermodynamic applications such as construction of quantum heat engines [21]. However, we found that a product of more than two fermionic modes in different thermal states was nonpassive (under a certain temperature constraint), implying work extraction is possible in this system. The challenge of generating the necessary unitary operation for this work extraction could be a limitation.

We extended the notion of Gaussian passivity to fermionic systems and presented criteria for identifying fermionic states according to their Gaussian (non-Gaussian) passivity; that is, according to our ability (inability) to extract work from them using Gaussian unitary transformations. This characterization is based on the second statistical moment of the two-mode fermionic system, which is known to have complete information about the system. This implies that our characterization provides information about the Gaussian ergotropy of the system (that is the maximum extractable energy in a Gaussian unitary process). Our result showed that under non-Gaussian (general) unitaries, we showed that work can be extracted from a general two-mode fermionic system.

There is still much that can be done with Fermionic Gaussian systems. A classification of their dynamics for open systems (analogous to the bosonic case [22]) remains to be carried out, along with their time evolution under rapid bombardment. Work on these topics is in progress.

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APPENDIX: STANDARD FORM OF FERMIONIC GAUSSIAN STATE

In this Appendix, we give a brief calculation on how the standard form of fermionic Gaussian state Eq. (7) is obtained and the explicit expression of the normalization constant [23,24]. To begin, when we substitute $\tilde{\mathbf{x}} = (\tilde{c}_{2j-1}, \tilde{c}_{2j})^T$ and Eq. (5), the density matrix Eq. (7) becomes

$$\rho = \mathcal{Z}^{-1} \exp\left[-\frac{\mathrm{i}}{4}\left(\sum_{j=1}^{N} \tilde{c}_{2j}\beta_j\tilde{c}_{2j-1} - \tilde{c}_{2j-1}\beta_j\tilde{c}_{2j}\right)\right].$$

The relation $\tilde{c}_{2j-1}\beta_j\tilde{c}_{2j} = -\tilde{c}_{2j}\beta_j\tilde{c}_{2j-1}$ between the majorana operators holds so that

$$\rho = \mathcal{Z}^{-1} \exp\left[\frac{\mathrm{i}}{2}\left(\sum_{j=1}^{N} \beta_j\tilde{c}_{2j-1}\tilde{c}_{2j}\right)\right]$$
$$= \frac{1}{\mathcal{Z}}\prod_{j=1}^{N} \exp\left[\frac{\mathrm{i}}{2}\beta_j\tilde{c}_{2j-1}\tilde{c}_{2j}\right] = \frac{1}{\mathcal{Z}}\prod_{j=1}^{N}\sum_{n=0}^{\infty}\frac{\left(\frac{\mathrm{i}}{2}\beta_j\tilde{c}_{2j-1}\tilde{c}_{2j}\right)^n}{n!}.$$

We have

$$(\tilde{c}_{2j-1}\tilde{c}_{2j})^2 = \tilde{c}_{2j-1}\tilde{c}_{2j}\tilde{c}_{2j-1}\tilde{c}_{2j} = -\tilde{c}_{2j-1}\tilde{c}_{2j-1}\tilde{c}_{2j}\tilde{c}_{2j} = -\mathbb{1}.$$

This gives

$$\rho = \frac{1}{\mathcal{Z}}\prod_{j=1}^{N}\left[\cosh\left(\frac{\beta_j}{2}\right)\mathbb{1} + \mathrm{i}\sinh\left(\frac{\beta_j}{2}\right)\tilde{c}_{2j-1}\tilde{c}_j\right]. \quad \text{(A1)}$$

Renormalization entails that $\mathrm{tr}\rho = 1$. To take the trace of Eq. (A1), we can transform to the Fock basis $|N\rangle = (\tilde{a}^\dagger)^N|0\rangle$ with the Majorana $\tilde{c}_k$ operators expressed in terms of the creation $\tilde{a}^\dagger_k$ and annihilation $\tilde{a}_k$ operators. Hence, the normalization constant becomes

$$\mathcal{Z} = \prod_{j=1}^{N} 2^N \cosh\left(\frac{\beta_j}{2}\right). \quad \text{(A2)}$$

Substituting Eq. (A2) into Eq. (A1) gives the fermionic Gaussian state in standard form

$$\rho = \frac{1}{2^N}\prod_{j=1}^{N}\left[\mathbb{1} + \mathrm{i}\tanh\left(\frac{\beta_j}{2}\right)\tilde{c}_{2j-1}\tilde{c}_j\right], \quad \text{(A3)}$$

where we have taken note that $\tanh(A) = \sinh(A)/\cosh(A)$.

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