Effect of walking distance on a queuing system of a totally asymmetric simple exclusion process equipped with functions of site assignments

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This paper proposes a totally asymmetric simple exclusion process on a traveling lane, which is equipped with a queueing system and functions of site assignments along the parking lane. In the proposed system, new particles arrive at the rear of the queue existing at the leftmost site of the system. A particle at the head of the queue selects one of the empty sites in the parking lane and reserves it for stopping at once during its travel. The arriving time and staying time in the parking sites follow half-normal distributions. The random selections of empty sites are controlled by the bias of the exponential function. Our simulation results show the unique shape of site usage distributions. In addition, the number of reserved sites is found to increase with an S-shape curve as the bias to the rightmost site increases. To describe this phenomena, we propose an approximation model, which is derived from the birth-death process and extreme order statistics. A queueing model that takes the effect of distance from the leftmost site of the traveling lane into consideration is further proposed. Our approximation model properly describes the distributions of site usage, and the proposed queueing model shows good agreement with the simulation results.

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I. INTRODUCTION

The queueing theory, which was started by Erlang [1] at the beginning of the 20th century, has attracted many scientists and researchers. Most of the theory is still in veil; nevertheless, a strong demand for this theory exists not only in academic studies of nonequilibrium statistical physics but also in many engineering fields such as traffic system [2], human dynamics [3,4], and molecular motor transport [5,6]. The study of queueing systems has been associated with the totally asymmetric simple exclusion process (TASEP) because of two main features: transportation in a one-way direction and the volume exclusion effect, which are suitable for the simulation of queueing systems [7–9].

This paper proposes a totally asymmetric simple exclusion process on a traveling lane, which is equipped with a queueing system and site assignments along the parking lane, under open boundary conditions. In the proposed system, new particles arrive at the rear of the queue existing at the leftmost site of the system. Thereafter, a particle at the head of the queue selects one of the empty sites in the parking lane and reserves it for stopping once during its travel. The arriving time and staying time in the parking sites follow half-normal distributions. The random selections of empty sites in the site assignments are controlled by the bias of the exponential function.

Similar mechanics of the proposed system can be observed in many real-world cases (e.g., parking problems in highways, airplane boarding, and airport ground transportation). Therefore, studying the proposed system is meaningful in many application fields. In particular, it is important to investigate the relationship between the occupancy of parking sites and the ways of site assignments because they are closely related to each other. The major scope of this research is to describe the relationship between the effect of site assignments in the proposed system and the occupancy of parking sites.

Because the parking site can be regarded as an absorption site in a wider sense, the proposed model is classified into the same category of the studies on multiple-lane systems with Langmuir Kinetics. Many previous studies have reported, exemplified by Refs. [10,11] for parallel-lane systems under periodic conditions, Ref. [12] for triple parallel-lane systems with Langmuir Kinetics, and Refs. [11–17] for two parallel-lane systems with Langmuir Kinetics. However, queueing problems are not taken into account in these studies. Regarding the function of site assignments, our previous research [18] was a pioneering study on site-assignment for parallel-lane systems; however, the problem of queueing was not discussed in that study.

In the modeling of queues, we consider the effect of walking distance from the entrance. Several previous studies have worked on this problem of walking-distance for specific cases, exemplified by the D-Fork system [19], D-Parallel system [19], and combinational queueing system of D-Fork with D-Parallel system [20]. A distinguishing factor of the proposed system compared to these previous systems is that the parking sites (service sites) push the particle back to the traveling lane, whereas in the previous studies, the particles pass through the service sites and exit from the opposite side of the traveling lane. Since the particle reentering the traveling lane causes delay in the traveling lane, the ways of site usage distributions affect the queueing system. Hence, the mechanics in the proposed system are different from those of the previous studies. In this paper, we propose an approximation model to

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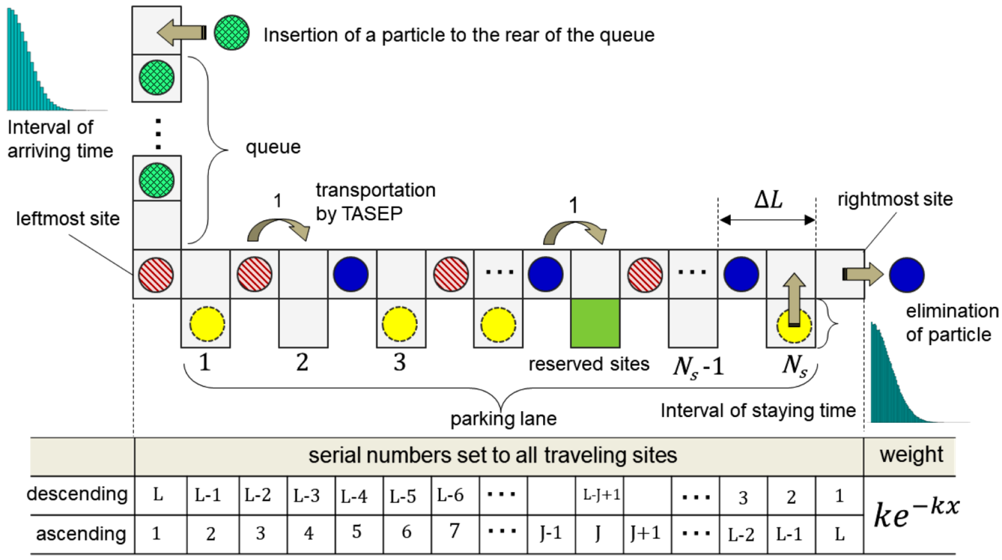

FIG. 1. Schematic view of the target system.

describe the site usage distribution of the proposed system on the basis of birth-death process for the spatial direction and extreme order statistics for the time direction.

The remainder of this paper is structured as follows. Section II provides a summary of the target system and that of the classical $M/M/c$ queueing model. Section III investigates the dependence of the utilization of parking sites on the distribution parameter of the exponential function through simulations. In Section IV, we propose an approximation model that describes the site usage distribution of the proposed system on the basis of the birth-death process and extreme statistics. Section V summarizes our results and concludes this paper.

II. MODELS

A. Target system

A schematic view of the target system is depicted in Fig. 1. The system consists of two parallel lanes: a traveling lane, which is composed of L sites, and a parking lane, which is composed of N_s sites ($N_s = L/2$). The distance between two parking sites Δl is set to be 2. In our system, a particle takes four different states. The cross-hatched green state indicates that the particle is in a state of queuing. The hatched red state indicates that the particle is in transport before stopping at the designated site of the parking lane. The dashed yellow state indicates that the particle is currently stopping at the site of the parking lane. The solid blue state indicates that the particle is in transport after exiting the parking lane. To sum up, the flow of a particle is described as follows. A particle arrives at the rear of the queue, which emerges at the leftmost site. After staying in the parking site, the particle goes back to the traveling lane, changing its state from the dashed yellow state to the solid blue state, and then moves towards the rightmost site. The particle in the traveling lane is eliminated from the system at the next step after moving to the rightmost site. Additionally, a parking site takes three kinds of states (reserved state, occupied state, and empty state). We call both the occupied state and reserved state simply as "a busy state" in this study. For the time integration, we adopt the parallel update method.

The interesting subjects of this study are not only the problems with the random arrival, such as the parking in highways, but also the problems with the scheduled arrival with the random delay such as the airport ground transportation. Especially in the last case, the use of normal distribution is reasonable, however, it has a practical problem in that we have to cut the tail of the left side of the distribution in some cases. To avoid this problem, we take up to use the half-normal distribution in this study.

The interval of the arrival time is set to follow a half-normal distribution; the mean $\tau_{\rm in}$ and deviation $\sigma_{\rm in}$ of the interval of arrival time are given as follows:

$$\tau_{\rm in} = \bar{\tau}_{\rm in} + \bar{\sigma}_{\rm in}\sqrt{\frac{2}{\pi}}, \tag{1}$$

$$\sigma_{\rm in} = \bar{\sigma}_{\rm in}\sqrt{\left(1-\frac{2}{\pi}\right)}. \tag{2}$$

Here, $\bar{\tau}_{\rm in}$ and $\bar{\sigma}_{\rm in}$ are the mean and deviation of the original normal distribution, respectively. A schematic view of $\tau_{\rm in}$, $\sigma_{\rm in}$, $\bar{\tau}_{\rm in}$, and $\bar{\sigma}_{\rm in}$ is depicted in Fig. 2.

A particle at the head of the queue selects one of the empty sites in the parking lane and reserves it for stopping once during its travel. Similar to that of the arrival time, the interval of the staying time is also set to follow a half-normal distribution

$$\tau_s = \bar{\tau}_s + \bar{\sigma}_s\sqrt{\frac{2}{\pi}}, \tag{3}$$

$$\sigma_s = \bar{\sigma}_s\sqrt{\left(1-\frac{2}{\pi}\right)}. \tag{4}$$

Here τ_s and σ_s are the mean and deviation of the half-normal distribution, while $\bar{\tau}_s$ and $\bar{\sigma}_s$ are those of the original normal distribution, respectively. Unless otherwise noted, the mean

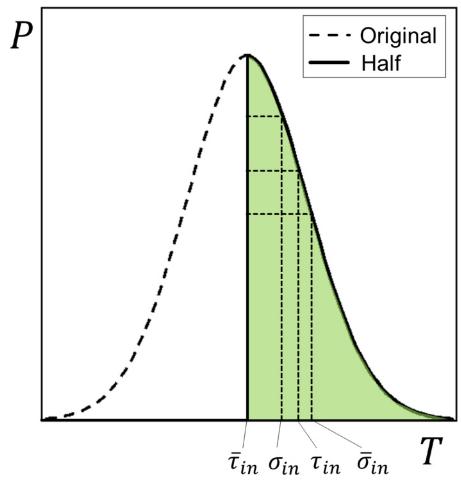

FIG. 2. Schematic view of the parameters of the interval of arrival time which follows half-normal distribution. The horizontal axis indicates the time step and the vertical axis indicates the probability.

and deviation of the two cases are indicated by $\tau_{\rm in}$, $\sigma_{\rm in}$, τ_s, and σ_s in this paper.

Two important rules are made to the system. The particle at the head of the queue is not permitted to reserve the parking site that has already been reserved by the other particle unless the site releases the particle. In addition, the dashed yellow particles in the parking sites have priority access to the upper site on the traveling lane compared to the hatched red or solid blue particle, which is to access the same traveling site.

The random selection of an empty parking site by the particle at the head of the queue is controlled by the exponential function ke^{-kx} ($x \geqslant 0, k > 0$). In the case of "ascending" in Fig. 1, the random selection of the empty sites becomes biased toward the leftmost site as the parameter increases. In the case of "descending" in Fig. 1, the random selection becomes biased toward the opposite rightmost site by setting the reversed sequential number to the number of sites L. Note that the inverse transform sampling (ITS) [21,22] is introduced to generate random variables that follow the exponential distribution.

We denote this last type of bias by multiplying the negative sign to parameter k for the sake of easy view. Namely, in the notation of ke^{-kx}, the random selection gets biased towards the leftmost site as parameter k increases while $k > 0$. In contrast, it gets biased towards the rightmost site as parameter k decreases while $k < 0$. In the case of setting parameter k to be zero, no external bias is given to the random selection.

B. Classical $M/M/c$ queue

In this section, we overview the classical queueing theory. A queueing system is characterized by six stochastic properties: the arrival process A, service process B, number of servers in the system C, maximum number of possible customers who will arrive at the system K, number of sources of customers N, and service discipline D. All these properties are summarized as $A/B/C/K/N/D$ by Kendall's notation [23]. The notation of K and N are abbreviated in case of infinity and that of D is abbreviated in the case of first come first served (FIFS); in this case, the system can be represented simply as $A/B/C$. Our system is categorized into $M/M/c$ queueing systems because the arriving time and staying time follows the Markov process and the system has a finite number of parking sites N_s. Note that the left and right M indicate the Markov process, and the notation c indicates the number of servers (the c corresponds to N_s in the system). In this section, several important formulas of the classical $M/M/c$ queueing system are enumerated. For more details on queueing theories, refer to Refs. [24,25].

The arrival rate λ and service rate μ are defined as the characteristic values of the queueing system. On the condition that the λ and μ are given as constant parameters, a distribution of probability P_n that the whole system (including queue) has n customers at a stationary state is obtained as a consequence of solving the transition equation of length of queue between the time step n and time step $n+1$ as follows:

$$P_n = \begin{cases} \frac{a^n}{n!} P_0, & (n = 1, 2, \dots, c), \\ \frac{a^n}{c^{n-c} c!} P_0, & (n = c+1, c+2, \dots), \end{cases} \tag{5}$$

$$P_0 = \left\{ \sum_{n=0}^{c-1} \frac{a^n}{n!} + \frac{a^c}{c!} \frac{1}{1-\rho} \right\}^{-1}, \tag{6}$$

$$a = \frac{\lambda}{\mu}, \tag{7}$$

$$\rho = \frac{\lambda}{c\mu}. \tag{8}$$

Additionally, the length of queue L_q, total number of customers in the whole system L_c, and the number of utilized servers U are obtained from Eq. (5) to Eq. (8), as follows:

$$L_q = \sum_{n=c+1}^{\infty} (n-c) P_n = C(c, a) \frac{a}{c-a}, \tag{9}$$

$$C(c, a) = \frac{c}{c-a} \frac{a^c}{c!} P_0, \tag{10}$$

$$L_c = \sum_{n=0}^{\infty} n P_n = L_q + a, \tag{11}$$

$$U = L_c - L_q = a. \tag{12}$$

Consequently, the utilization of servers corresponds to the parameter a, which is defined as the value of the arrival rate divided by the service rate, as shown in Eq. (7).

In the case of $c = 1$ ($M/M/1$ queue), the arrival rate λ and service rate μ are defined as the inverse values of arrival time and service time, as follows:

$$\lambda = {\tau_{\rm in}}^{-1}, \tag{13}$$

$$\mu = {\tau_s}^{-1}. \tag{14}$$

Even in general cases of $c > 1$, all the servers are assumed to have the same values of arrival rate λ and service rate μ defined by Eqs. (13) and (14), respectively, similar to the $M/M/1$ queue in the classical $M/M/c$ queueing model. However, this assumption causes a nonnegligible deviation from real-world systems because the effect of walking distance to each server is not considered in the classical queueing theory. In particular, this assumption becomes a serious problem in our system because the reentering customer causes a delay in transportation on the traveling lane. To solve this

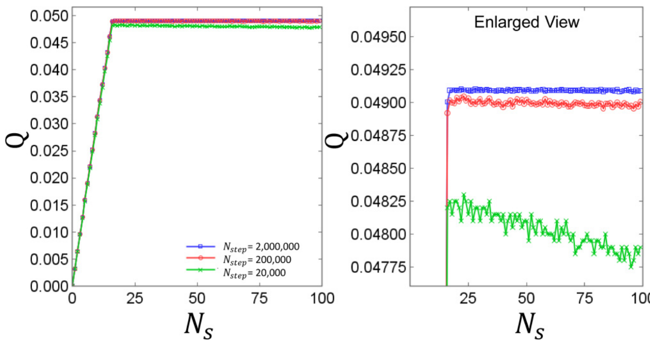

FIG. 3. Dependence of particle flux Q at the rightmost site on the number of parking sites N_s for different values of the total time steps N_{step} from 20 000 to 2 000 000.

problem, in this paper, approximation models that consider the effect of walking distance are proposed in Sec. IV.

III. SIMULATIONS

The input parameters of our system are given as the set of (N_s, $\bar{\tau}_{\mathrm{in}}$, $\bar{\sigma}_{\mathrm{in}}$, $\bar{\tau}_s$, $\bar{\sigma}_s$). From Eqs. (1) and (4), we immediately obtain the set of (N_s, τ_{in}, σ_{in}, τ_s, σ_s), which our system follows in practice during the simulation. In the parameter settings, the coefficient of variation (CV), defined as the ratio of the deviation to the mean value must be considered since the CV determines the degree of the stochastic dispersion of simulations; to complete the simulations in acceptable computational time, we set the input parameters ($\bar{\tau}_{\mathrm{in}}$, $\bar{\sigma}_{\mathrm{in}}$, $\bar{\tau}_s$, $\bar{\sigma}_s$) to (20, 1, 300, 10) to obtain resulting CVs $\sigma_{\mathrm{in}}/\tau_{\mathrm{in}}$ and σ_s/τ_s smaller than 0.03.

We define a particle flux Q as an average number of particles that exit from the rightmost site per time step. To determine the condition of reaching stationary state, the dependence of particle flux Q at the rightmost site on the number of parking sites N_s was measured for different values of the total time steps N_{step} between 20 000 and 2 000 000, as shown in Fig. 3. The break in line occurs at approximately $N_s = 17$ as all the sites are in use due to the lack of capacity of sites for $N_s < 17$. The particle flux Q was observed to become almost stable after N_{step} reaches 2 000 000. Through this primary investigation, we set N_s and N_{step} to be 48 and 2 000 000, respectively.

Figure 4 shows the dependence of the number of reserved or occupied sites and the number of busy sites (the sum of reserved sites and occupied sites) on the different values of distribution parameter k between -10 and 10. It was observed that the number of busy sites N_b increases with a gentle S-shape curve as the parameter k decreases. Substantially, the increase in the number of busy sites N_b is found to be determined only by the increase in reserved sites N_r. On the other hand, the number of occupied sites N_o remained constant during the simulations. The result in Fig. 4 indicates that the utilization of U servers in the classical $M/M/c$ queueing theory corresponds not to the number of occupied sites N_o, but to the number of busy sites N_b in our system. This is because a parking site becomes accessible every time the previous busy time ends. In the next section, we investigate the relationship between N_b and the distribution parameter k of the exponential function.

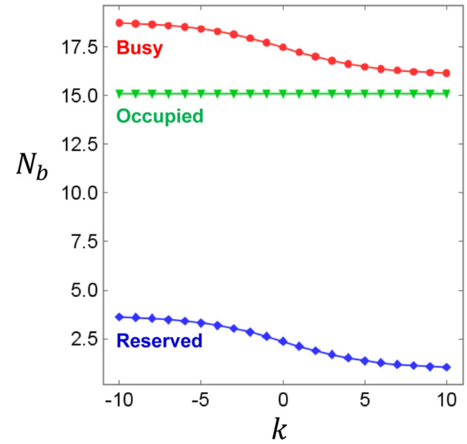

FIG. 4. Dependence of the number of reserved or occupied sites and the number of busy sites (the sum of reserved sites and occupied sites) on the different values of distribution parameter k between -10 and 10.

IV. ANALYSIS

A. First approximation level

At the beginning of this study, we proposed a fundamental model, which is simulated by the concept of D-Fork system. By considering the effect of walking distance from the left-most site, the occupied time T_i of the ith parking site can be modeled as follows:

$$T_i = \tau_s + i\Delta l + \alpha. \tag{15}$$

Here α is a constant parameter and Δl is the distance between two parking sites.

In the right-hand side of Eq. (15), the first term indicates the staying time in a parking site. The second term indicates the traveling time to the parking site. In this approximation level, we ignore the volume exclusion effect in the second term; a particle is assumed to hop to the neighboring cell per step. Thus the velocity of the particle becomes 1 since the length of a cell is set to 1. That is why the notation of velocity does not emerge in the second term. Instead, we introduce the α in the third term, assuming that the volume exclusion effect can be approximated as constant values in the target system.

The maximum service rate $\mu_{\max}$ and the minimum service rate $\mu_{\min}$ are obtained by substituting N_s and 1 to Eq. (15), as follows:

$$\mu_{\max} = T_{N_s}{}^{-1}, \tag{16}$$

$$\mu_{\min} = T_1{}^{-1}. \tag{17}$$

The averaged value of the service rate of the system μ_{avr} is obtained by calculating the arithmetic mean of Eq. (15), as follows:

$$\mu_{\mathrm{avr}} = \left\{\tau_s + \frac{1}{N_s}\sum_{i=1}^{N_s}(i\Delta l + \alpha)\right\}^{-1} \tag{18}$$

$$= \left\{\tau_s + \frac{\Delta l}{2}N_s + \frac{\Delta l}{2} + \alpha\right\}^{-1}. \tag{19}$$

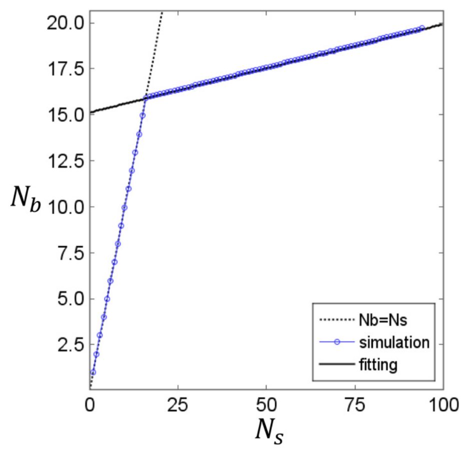

FIG. 5. Dependence of the number of busy sites N_b on the total number of sites N_s in the case of $k = 0$.

Finally, the averaged number of busy sites N_b is obtained by dividing Eq. (13) by Eq. (19), as follows:

$$N_b = \frac{\lambda}{\mu_{\text{avr}}} \tag{20}$$

$$= \left(\frac{\Delta l}{2\tau_{\text{in}}}\right) N_s + \frac{1}{\tau_{\text{in}}}\left(\tau_s + \frac{\Delta l}{2} + \alpha\right). \tag{21}$$

Equation (21) shows that the number of busy sites N_b is a linear function of the number of sites N_s. Figure 5 shows the simulation result of the dependence of the number of busy sites N_b on the total number of sites N_s, in the case of $k = 0$. It was observed that the break in the blue circle line occurs at around $N_s = 17$, which is because all the sites are in use due to the lack of capacity of sites when $N_s < 17$. In comparison to Fig. 4, the y-intercept value of the fitting line in Fig. 5 corresponds to the value of the number of occupied sites N_o. This is because the increment of the number of busy sites N_b depends only on the increase in the number of reserved sites N_r. By fitting the line at $N_s > 17$ according to Eq. (21), the parameter α is obtained to be 6.215 as a fitting result. It was confirmed from the red circle line in Fig. 6 that the simulation results are bounded between the maximum case (the dotted line) and the minimum case (the dashed line). In addition, the averaged case of classical $M/M/c$ with the parameter α is found to semi-experimentally correspond to the case of $k = 0$.

B. Second approximation level

On the basis of an assumption that the site usage distributions obey the exponential function ke^{-kx}, we correct Eq. (19) by replacing the arithmetic mean by the weighted average using the exponential function ke^{-kx}. The number of busy sites N_b is calculated as follows:

$$N_b = \frac{\tau_s}{\tau_{\text{in}}} + \frac{1}{\tau_{\text{in}}} \frac{\sum_{i=1}^{N_s}(i\Delta l + \alpha)E_i}{\sum_{i=1}^{N_s} E_i}, \tag{22}$$

$$E_i = k \times \exp\left(-k\frac{i}{N_s}\right). \tag{23}$$

The red circle line and blue square line in Fig. 6 show the comparison of simulation results and the estimated values

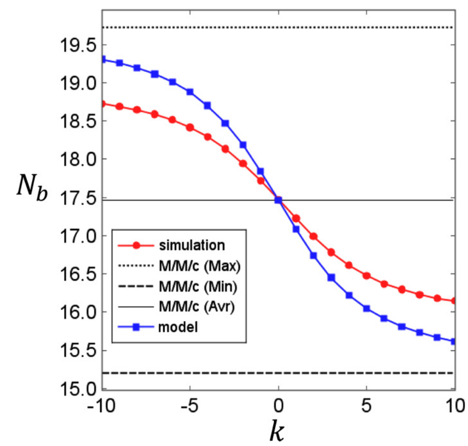

FIG. 6. Comparison of the simulation results, the estimated values obtained using Eq. (22), the maximum values by dividing Eq. (13) by $\mu_{\max}$, the minimum values by dividing Eq. (13) by $\mu_{\min}$, the averaged values obtained by using Eq. (19), for different values of distribution parameter k between -10 and 10.

obtained using Eq. (22), respectively. It was confirmed that the feature of S-shape curve is observed in both simulations and approximations. However, the number of busy sites N_b calculated by Eq. (22) becomes overestimated or underestimated at both sides of $k < 0$ and $k > 0$.

To clarify the reason for the deviation, the site usage distributions were investigated. Figure 7 shows all the distributions for different values of parameter k between -10 and 10. Obviously, the shape of each distribution is different from that of the exponential function. It should be noted that the reason why the distribution gets slightly biased to the leftmost site in the case of $k = 0$ is that the parking site, which is closer to the leftmost site, has a higher turnover rate because of the shorter walking distance.

C. Third approximation level

1. Birth-death process for walking direction

In this section, we describe the site usage distributions by introducing the birth-death process for walking direction. Namely, the particles in transport before stopping at a site on the parking lane (hatched red particles in Fig. 1) are regarded

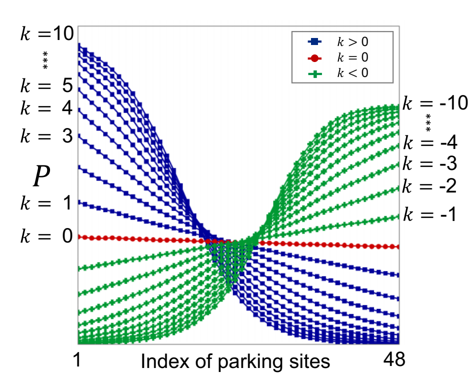

FIG. 7. All the distributions of the site usage for different values of parameter k between -10 and 10.

as "surviving." On the contrary, an event in which a particle stops at a site indicates the "death" of the particle.

We define a random variable X, which donates a position on the traveling lane, $f(x)$ is defined as the probability density function of X. From these definitions, the cumulative density function $F(x)$ of $f(x)$ can be expressed as follows:

$$F(x) := P(X \leqslant x) = \int_0^x f(t)dt. \tag{24}$$

$F(x)$ represents the probability of "death" at position x. Conversely, we define $S(x)$ as the probability of surviving at position x, as follows:

$$S(x) := 1 - F(x) = P(X > x). \tag{25}$$

In addition, a hazard function $h(x)$ is defined as follows:

$$h(x) := \lim_{\Delta\to 0} \frac{1}{\Delta} P(x < X < x + \Delta | X > x). \tag{26}$$

$h(x)$ is the probability density that a particle stops at a site at the position between x and $x + \Delta$. Equation (26) can be transformed as follows:

$$= \lim_{\Delta\to 0} \frac{1}{\Delta} \times \frac{P\{(x < X < x + \Delta) \cap (X > x)\}}{P(X > x)} \tag{27}$$

$$= \lim_{\Delta\to 0} \frac{1}{\Delta} \times \frac{P(x < X < x + \Delta)}{P(X > x)}. \tag{28}$$

From Eqs. (24) and (25),

$$= \frac{1}{P(X > x)} \lim_{\Delta\to 0} \frac{F(x + \Delta) - F(x)}{\Delta} \tag{29}$$

$$= \frac{F'(x)}{S(x)} \tag{30}$$

$$= -\frac{S'(x)}{S(x)} = -\frac{d}{dx}\{\log[S(x)]\}. \tag{31}$$

Here we impose an initial condition $S(0) = 1$ to Eq. (31) since the probability that a particle survives at the position of zero always becomes 1. Then the differential equation Eq. (31) can be solved as follows:

$$S(x) = \exp[-H(x)]. \tag{32}$$

Here we introduce the $H(x)$, as follows:

$$H(x) := \int_0^x h(t)dt. \tag{33}$$

We obtain $F(x)$ and $f(x)$ from Eq. (32) as follows:

$$F(x) = 1 - \exp[-H(x)], \tag{34}$$

$$f(x) = h(x) \times \exp[-H(x)], \tag{35}$$

$f(x)$ corresponds to the probability distribution of site usage since $F(x)$ is the probability of death at position x. We successfully obtained a general formula for site usage distribution in our system. We introduce extreme statistics in Sec. IV C 3 to determine the specific formula of $H(x)$.

2. *Introduction of order statistics*

The derivation of Eq. (34) lacks the information of order statistics of the random variable X. We introduce the concept of order statistics to our system in this section, as a preliminary work for the approximation by extreme statistics in Sec. IV C 3.

Let us consider the situation that a single particle is inserted from the queue to the leftmost site at certain intervals of arrival time during the total n time steps. We name the ith inserted particle to the leftmost site simply as "ith particle". We define the random variables X_i, which indicates the position that the ith particle stops at during its travel. As similarly in the previous section, the ith particle is judged as "death" when $X_i \leqslant x$. On the contrary, the particle is judged as "surviving" when $X_i > x$.

An identical cumulative density function $A(x)$ of $X_1, X_2, \ldots, X_n$ is expressed, as follows:

$$A(x) := P(X_i \leqslant x), \quad (i = 1, 2, \ldots, n). \tag{36}$$

The order statistics of $X_1, X_2, \ldots, X_n$, which is obtained by rearranging the $X_1, X_2, \ldots, X_n$ in an ascending order, is represented as follows:

$$X_{(1:n)} \leqslant X_{(2:n)} \leqslant \cdots \leqslant X_{(n:n)}. \tag{37}$$

From the definition of the order statistics, it is obvious that the ith order statistic $X_{(i:n)}$ corresponds to the ith largest original independent variables. The relationship between the original independent variables and the order statistics in case of $n = 5$ is depicted in Fig. 8.

A cumulative density function $A_{X_{(m:n)}}(x)$ of the mth statistic $X_{(m:n)}$ is defined as follows:

$$A_{X_{(m:n)}}(x) := P(X_{(m:n)} \leqslant x). \tag{38}$$

Considering the fact that $X_{(i:n)}$ corresponds to the ith largest original independent variables, it can be said that Eq. (38) indicates the probability that at least m variables of $X_1, X_2, \ldots, X_n$ become equal or less than the position x (= the state of "death").

By using the probability P_j that exactly j variables of $X_1, X_2, \ldots, X_n$ becomes equal or less than the position x, the right-hand side of Eq. (38) is represented as follows:

$$P(X_{(m:n)} \leqslant x) = \sum_{j=m}^{n} P_j. \tag{39}$$

The probability P_i is further decomposed as follows; there are ${}_nC_j$ different combinations of j variables from $X_1, X_2, \ldots, X_n$. In each case, j variables become "death" with the probability $A(x)$ and that of $n - j$ variables become "survival" with the probability $1 - A(x)$; the cumulative density function $A_{X_{(m:n)}}(x)$ is represented by using Eqs. (38) and (39), as follows:

$$A_{X_{(m:n)}}(x) = \sum_{j=m}^{n} \binom{n}{j} A(x)^j [1 - A(x)]^{n-j}. \tag{40}$$

Now we obtain the precise expression of Eq. (38). Equation (40) includes all the possible patterns of particle arrivals for the time direction during the total n time steps.

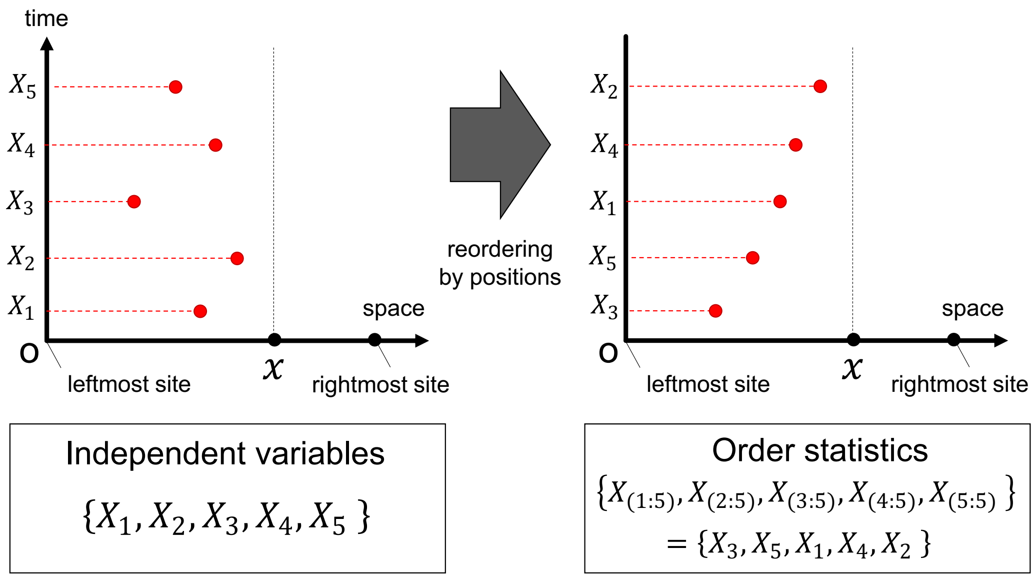

FIG. 8. Relationship between the independent variables and the order statistics in case of $n = 5$.

Unfortunately, it is difficult to derive the probability distribution of site usage directly from Eq. (40) because the cumulative density function $A(x)$ at each time step is unknown. To solve this problem, we propose to approximate the site usage distribution by the asymptotic distribution of the distribution of extreme order statistics in the next section.

3. Approximation by extreme statistics

The maximum order statistics Z_n and the minimum order statistics Y_n are defined, respectively, as follows:

$$Z_n := X_{(n:n)} = \max\{X_1, X_2, \dots, X_n\} = \max_{1\leqslant i\leqslant n} X_i, \quad (41)$$

$$Y_n := X_{(1:n)} = \min\{X_1, X_2, \dots, X_n\} = \min_{1\leqslant i\leqslant n} X_i. \quad (42)$$

In this paper, we propose to approximate the cumulative probability distribution of site usage in Eq. (24) by the distribution of extreme order statistics. Here we have two candidates of $P(Z_n \leqslant x)$ and $P(Y_n \leqslant x)$.

We are able to say that the selection of $P(Y_n \leqslant x)$ is appropriate for the approximation of Eq. (24) considering the physical meaning of these extreme order statistics. $P(Z_n \leqslant x)$ describes the probability that not a single X_i becomes larger than the position x during the time step n, as shown in Fig. 9(a). Because the situation depicted in Fig. 9(a) seldom occurs in our system, replacing the random variables X in Eq. (24) by the maximum order statistics of Z_n is not appropriate. On the contrary, as shown in Fig. 9(b), $P(Y_n \leqslant x)$ describes the probability that at least one of X_i becomes smaller than the position x during time step n; therefore, the asymptotic distribution of minimum order statistics of Y_n is suitable for describing the behaviors of our system, compared to the former case.

We approximate the cumulative density distribution of site usage in Eq. (24) by the distribution of minimum order statistics Y_n, as follows:

$$F(x) \approx P(Y_n \leqslant x). \quad (43)$$

For adequate large n, it is known that the distribution of minimum order statistics Y_n asymptotic to the following extreme value distribution $M(x)$ in case that the random variables of $X_1, X_2, \dots, X_n$ follow exponential distributions [26,27]:

$$P(Y_n \leqslant x) \to M\left(\frac{x-\tilde{c}}{\tilde{b}}\right) \quad (44)$$

$$= 1 - \exp\left[-\exp\left(\frac{kx-c}{b}\right)\right]. \quad (45)$$

Here $(\tilde{c}, \tilde{b})$ are normalizing constants, which are selected to convert the location and scale so that the extreme value distribution M does not diverge and degenerate. The (c, b) is $(k\tilde{c}, k\tilde{b})$, respectively. For a detailed description on the derivation of Eq. (45), see the Appendix. Now we obtain the distribution $F(x)$ of our system, as follows:

$$F(x) = 1 - \exp\left[-\exp\left(\frac{kx-c}{b}\right)\right]. \quad (46)$$

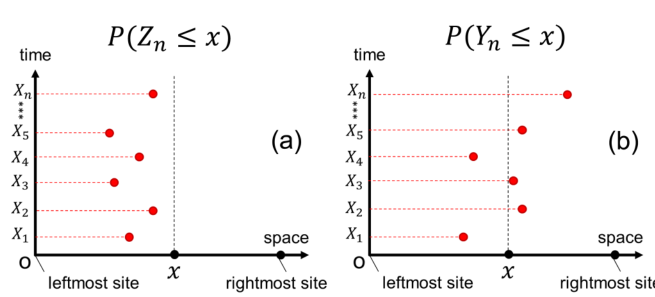

FIG. 9. Schematic view of the distributions of two different extreme order statistics.

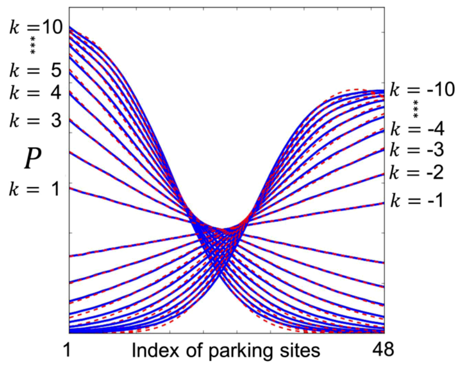

FIG. 10. All the distributions of the site usage fitted by Eq. (49) for different values of parameter k between -10 and 10.

The probability density function $f(x)$ is obtained as follows:

$$f(x) = \frac{k}{b} \times \exp\left(\frac{kx-c}{b}\right) \times \exp\left[-\exp\left(\frac{kx-c}{b}\right)\right]. \quad (47)$$

From Eq. (47), the cumulative hazard function $H(x)$ is found to become an exponential function:

$$H(x) = \exp\left(\frac{kx-c}{b}\right). \quad (48)$$

The selection of Y_n is validated from the point of mathematical derivation. If we select Z_n, the right-hand side of Eq. (46) becomes $\exp[-e^{-(kx-f)/g}]$. This description contradicts with the formula obtained in Eq. (34). In this case, the relationship between Eq. (34) and Eq. (35) is not satisfied.

It is not easy to mathematically derive the constant parameters of (c, b) of minimum order statistics Y_n, we determine these parameters by fitting the simulation results in the next section.

4. Corrections of the $M/M/c$ queueing model

Let us get back to the subject of queueing theory. We attempt to correct the weighted calculation in Eq. (22) by replacing the exponential function by the fitting function of the simulation results. We adopt Eq. (47) as the fitting function, admitting the transformation of the scale of Eq. (47) by using constant parameter a, as follows:

$$f_{\mathrm{FIT}}(x) = a\frac{k}{b} \times \exp\left(\frac{kx-c}{b}\right) \times \exp\left[-\exp\left(\frac{kx-c}{b}\right)\right]. \quad (49)$$

Figure 10 shows all the cases of exponential distributions fitted by Eq. (49) for different values of parameter k between -10 and 10. The dashed red colored lines indicate fitting results by the least-squares method. Figure 11 shows the dependence of the chi-square of fitting results in Fig. 10 on the different values of parameter k. Obviously, in Fig. 11, it was observed that the accuracy of curve fitting deteriorates as the bias to the right or left side increases. The reason for this is interpreted as follows. As the bias to the right or left side increases, congestion occurs in the neighboring area of the rightmost or leftmost site. Because the effect of congestion is not considered in the deviation of Eq. (49), the difference at both sides of the edges emerges.

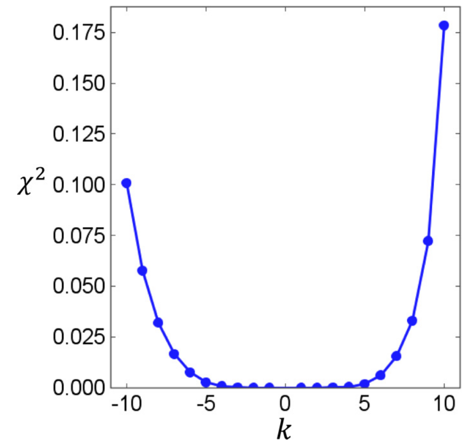

FIG. 11. Dependence of the chi-square of fitting results in Fig. 10 on the different values of parameter k.

We correct the weighted calculations of our queueing model in Eq. (22) by replacing the weighted function with the function in Eq. (49), as follows:

$$N_b = \frac{\tau_s}{\tau_{\mathrm{in}}} + \frac{1}{\tau_{\mathrm{in}}}\frac{\sum_{i=1}^{N_s}(i\Delta l + \alpha)E_i}{\sum_{i=1}^{N_s} E_i}, \quad (50)$$

$$E_i := f_{\mathrm{FIT}}\left(\frac{i}{N_s}\right). \quad (51)$$

Figure 12 shows a comparison of (a) the simulation results in Fig. 6 to (b) the estimated values obtained using Eq. (22) and (c) the estimated values obtained using Eq. (50). It was confirmed that our proposed model shows a good agreement with the simulation results compared to the model exhibited in Eq. (22). This result indicates that our method, which estimates the service rate μ_s by using weighted calculations of the site usage distributions, is an effective approach under certain conditions.

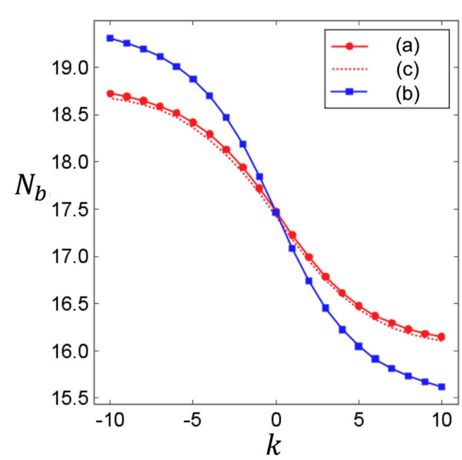

FIG. 12. (a) Simulation results in Fig. 6, (b) second-order proposed model exhibited in Eq. (22), and (c) third-order proposed model exhibited in Eq. (50).

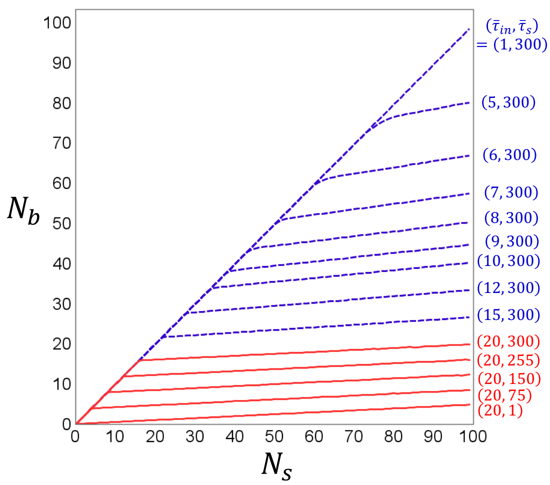

FIG. 13. Dependence of the number of busy sites N_b compared to the number of sites N_s for each case of the parametric investigation in Fig. 14.

To investigate the effective range of our model, we performed the same comparison as the one performed for Fig. 12 with different values of input parameters. We set the pairs of $(\bar{\tau}_{in}, \bar{\tau}_s)$ and $(\bar{\sigma}_{in}, \bar{\sigma}_s)$ to obtain resulting CVs σ_{in}/τ_{in} and σ_s/τ_s having a constant value; in this case, the system can be identified as the function of $(N_s, \bar{\tau}_{in}, \bar{\tau}_s)$. Figure 13 shows the results of the dependence of the number of busy sites N_b compared to the number of sites N_s, for each case of this investigation. The brackets represent a pair of $(\bar{\tau}_{\rm in}, \bar{\tau}_s)$. The result in the case of (20,300) corresponds to the blue circle line in Fig. 5 . The solid red lines in Fig. 13 represent the results when setting the parameter $\bar{\tau}_{\rm in}$, whereas the dashed blue lines illustrate the results when setting the parameter $\bar{\tau}_s$.

We performed three kinds of simulations, as shown in Fig. 14: (a) the simulations with the settings $(N_s, \bar{\tau}_{\rm in})$ being (48, 20) with different values of $\bar{\tau}_s$ between 1 and 300, (b) the simulations with the settings $(N_s, \bar{\tau}_s)$ being (48, 300) with different values of $\bar{\tau}_{\rm in}$ between 1 and 20, and (c) the simulations with the settings $(\bar{\tau}_{\rm in}, \bar{\tau}_s)$ being (20, 300) with different values of N_s between 48 and 1536. For (a), the simulations and models showed good agreement for all the cases. In the case of (b), a slight deviation between simulations and models was observed, increasing as the parameter $\bar{\tau}_{\rm in}$ decreases. As we simply approximate the volume exclusion effect using a constant parameter, as mentioned in Sec. IV A, the accuracy of our model deteriorates when the traveling lane becomes denser and the nonlinearity of the congestion phenomena becomes nonnegligible; this is the main cause of the deviation of (b). Additionally, a very slight deviation was observed as N_s increases when performing (c). This is due to the influence of the increasing number of reentering particles coming from the parking lane since the traveling time until they stop at the parking site increases, especially for a distribution parameter $k < 0$. Another important feature is that the dependence of N_b on the parameter k was not observed in the first place when $\bar{\tau}_{\rm in} < 8$ as all the parking sites are in use or reserved.

Our model was further validated within the scope of this investigation. In this paper, we focused on the case of CV values kept constant to control the stochastic dispersion of the system. There is still room for further investigation for CVs comprised in a wider range, which is expected to be studied in the future.

V. CONCLUSION

We introduce a totally asymmetric simple exclusion process on a traveling lane equipped with a queueing system and functions of site assignments along the parking lane. In this study, we investigate the relationship between the utilization of parking sites and the effect of site assignments in the proposed system. The contributions of this study are as follows.

We propose an approximation model to describe the site usage distributions of the proposed system on the basis of birth-death process for the spatial direction and extreme statistics for the time direction. The specific formula in the case where the random variables follow exponential distributions are described. In addition, our proposed $M/M/c$ queueing model, whose service rate is determined by the weighted calculation of site usage distributions, shows good agreement with the simulation results.

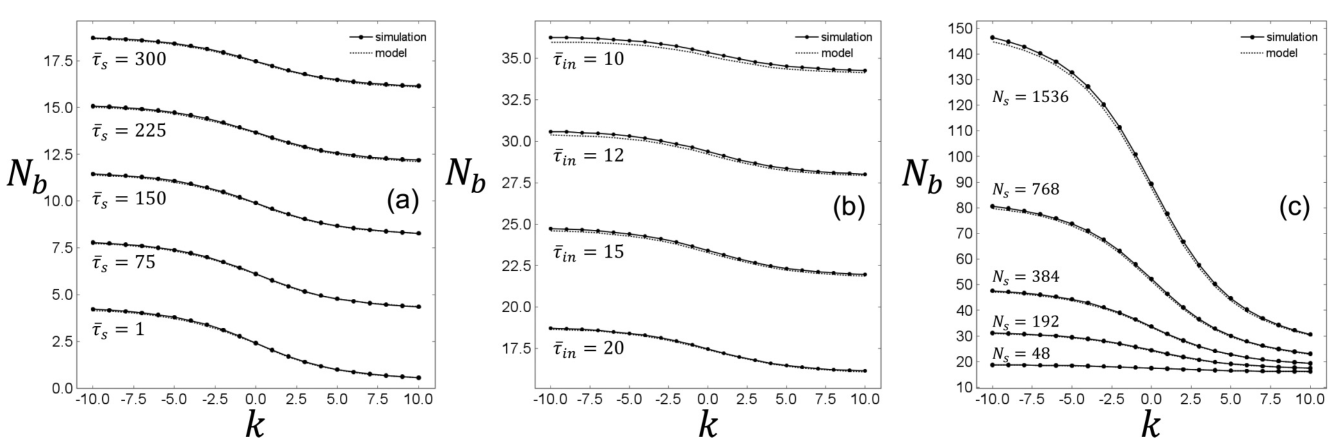

FIG. 14. (a) Simulations with the settings $(N_s, \bar{\tau}_{\rm in})$ being (48, 20) with different values of $\bar{\tau}_s$ between 1 and 300, (b) the simulations with the settings $(N_s, \bar{\tau}_s)$ being (48, 300) with different values of $\bar{\tau}_{\rm in}$ between 1 and 20, and (c) the simulations with the settings $(\bar{\tau}_{\rm in}, \bar{\tau}_s)$ being (20, 300) with different values of N_s between 48 and 1536.

As mentioned in the Introduction, the major scope of the current research is to describe the relationship between the utilization of parking sites and the effect of site assignments in the proposed system. Accordingly, we obtain insightful results from the findings of this study.

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APPENDIX A: ASYMPTOTIC DISTRIBUTIONS OF THE DISTRIBUTIONS OF EXTREME ORDER STATISTICS

The distributions of maximum order statistics Z_n and minimum order statistics Y_n are represented as follows:

$$P(Z_n \leqslant x) = P(X_{(n:n)} \leqslant x) = A_{X_{(n:n)}}(x), \tag{A1}$$

$$P(Y_n \leqslant x) = P(X_{(1:n)} \leqslant x) = A_{X_{(1:n)}}(x). \tag{A2}$$

For adequate large n, the distributions of these two extreme order statistics are assumed to asymptotic to the extreme value distributions, respectively, as follows:

$$P(Z_n \leqslant x) \to G\left(\frac{x - a_n}{b_n}\right), \tag{A3}$$

$$P(Y_n \leqslant x) \to M\left(\frac{x - c_n}{d_n}\right). \tag{A4}$$

Here (a_n, b_n) are normalizing constants, which are selected to convert the location and scale of G so that the extreme value distribution G does not diverge and degenerate. The same is true of (c_n, d_n). The assumption of the existence of G and Eq. (A3) are validated on the condition that Eq. (B1) is satisfied [26,27]. If they are validated, the asymptotic distribution $M(x)$ of minimum order statistics Y_n is obtained from the following relationship:

$$M(x) = 1 - G(-x). \tag{A5}$$

APPENDIX B: TRINITY THEOREM

A population distribution F is assumed to belong to a domain of attraction of an extreme value distribution G; this assumption is denoted as $F \in \mathcal{D}(G)$. Fisher and Tippett [28] mathematically proved the following relationship for maximum order statistics Z_n:

$$F \in \mathcal{D}(G) \Leftrightarrow \lim_{n\to\infty} F^n(a_n x + b_n) = G(x), \quad a_n > 0, b_n \in \mathbb{R}. \tag{B1}$$

After considerable efforts, mathematicians Fréchet [29], Fisher and Tippett [28], and Gnedenko [30] proved a notable fact that only three types of extreme distributions exist, which are as follows:

$$\text{Gumbel}: G(x) = \exp[-\exp(-x)], \quad x \in \mathbb{R}, \tag{B2}$$

$$\text{Fréchet}: G(x) = \exp(-x^{-\alpha}), \quad x \geqslant 0, \ \alpha > 0, \tag{B3}$$

$$\text{Weibull}: G(x) = \exp[-(-x)^{\alpha}], \quad x \leqslant 0, \alpha \geqslant 0. \tag{B4}$$

The series of equations from Eq. (B2) to Eq. (B4) is called the Trinity Theorem, which indicates that any population distribution F is asymptotic to one of the three kinds of extreme distributions listed from Eq. (B2) to Eq. (B4), on the condition that the relation $F \in \mathcal{D}(G)$ is satisfied.

APPENDIX C: EXTREME VALUE DISTRIBUTIONS OF AN EXPONENTIAL DISTRIBUTION

The asymptotic distribution for the case in which the random variables of $X_1, X_2, \ldots, X_n$ follow exponential distributions is obtained, as follows. A cumulative exponential function is written as follows:

$$F(x) := 1 - \exp(-kx). \tag{C1}$$

Here we use the following identity equation:

$$F^n(a_n x + b_n) = \left\{1 + \frac{-n[1 - F(a_n x + b_n)]}{n}\right\}^n. \tag{C2}$$

By selecting $a_n = 1$ and $b_n = k^{-1}\log(n)$,

$$-n[1 - F(a_n x + b_n)] = -\exp(-kx), \quad x \geqslant 0. \tag{C3}$$

By substituting Eq. (C3) into Eq. (C2),

$$\lim_{n\to\infty} F^n(a_n x + b_n) = \lim_{n\to\infty}\left(1 + \frac{-e^{-kx}}{n}\right)^n \tag{C4}$$

$$= \exp[-\exp(-kx)]. \tag{C5}$$

From Eq. (B1), we obtain the expression of $G(x)$, as follows:

$$G(x) = \exp[-\exp(-kx)]. \tag{C6}$$

Equation (C5) indicates that the asymptotic distribution $G(x)$ of maximum order statistics Z_n, when the random variables $X_1, X_2, \ldots, X_n$ follow an exponential distribution, belongs to the family of Eq. (B2) in the Trinity Theorem.

From the relationship in Eq. (A5), we obtain the expression of $M(x)$, as follows:

$$M(x) = 1 - \exp[-\exp(kx)]. \tag{C7}$$

By substituting Eq. (C7) into Eq. (A4), we obtain the following:

$$P(Y_n \leqslant x) \to M\left(\frac{x - c_n}{d_n}\right) \tag{C8}$$

$$= 1 - \exp\left[-\exp\left(\frac{kx - \tilde{c}_n}{\tilde{d}_n}\right)\right]. \tag{C9}$$

Here $\tilde{c}_n$ and $\tilde{d}_n$ are kc_n and kd_n, respectively.

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