Magnetorotational instability in spin quantum plasmas

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The magnetorotational instability (MRI) is analyzed using a two fluid model with the effect of spin magnetization in a differentially rotating degenerate electron-ion (e-i) quantum plasma. The electrons are taken to be degenerate, whereas ions are considered as classical owing to their large inertia. The general dispersion relation for spin quantum e-i plasma is derived and a local dispersion relation for MRI is obtained by applying MHD approximations. The obtained MRI criteria is discussed for both magnetized and unmagnetized plasma duly modified by spin correction terms. The instability criteria differ significantly from that reported for the case of classical plasma. Spin magnetization plays a vital role via coupling to the Alfvénic speed and can alter the instability criteria which leads to the transport phenomenon in compact astrophysical objects.

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I. INTRODUCTION

Currently there is a great deal of interest in studying and investigating physics of quantum plasmas. In plasma, quantum effects become prominent when the de Broglie wavelength of charged particles, i.e., $\lambda_{De}(=\hbar/m_e v_{te})$, becomes comparable to the system scale, e.g., interparticle distances $n^{-1/3}$, where n is the equilibrium particle density, $\hbar$ is the reduced Planck's constant, m_e is the mass of electron, and v_{te} is the thermal speed of electrons. Starting from Schrödinger's description of quantum electron, one can derive a set of equations for plasma either from N-body description, density matrix, or Madelung description of wave function [1,2]. Such quantum plasma has relevance in nanoscale electromechanical systems [3,4], dense laser plasma [5], lasers interaction with atomic systems [6,7], plasma echoes [8], and quantum plasma instabilities [9,10]. From an experimental perspective more interest is directed towards the relation of spin properties to the classical theory of motion. The effect of strong field on a single particle with spin has attracted experimental interest in the laser community [11–13]. In 2005, Haas [14] formulated a quantum magnetohydrodynamic (QMHD) model for degenerate dense quantum plasma and later this model was extended to spin quantum dense plasma [15] by taking the spin effect into account. The applicability of this sort of model to the dense astrophysical plasmas, dusty plasmas, and solid state plasmas were discussed and used to investigate the properties of hydrodynamic waves for the electron spin effects. Brodin and Marklund used spin MHD approach by using a nonrelativistic Pauli equation for spin-$1/2$ particles that gives the desired governing dynamics of spin plasmas [16]. It is to be noticed that spin effects are important for low temperature, high density, and strongly magnetized plasma. The effect of strong field has applications in astrophysical environments such as pulsars [17] and magnetars [18]. Therefore, much attention has been received in astrophysical surroundings, especially in strongly magnetized plasmas. In laboratory plasmas, many of the studies are motivated on the properties of single particles. It is expected that the collective spin effects can influence the propagation characteristics of waves in a strongly magnetized quantum plasma [19]. Andreev [20] investigated the separated spin up and spin down quantum hydrodynamics of degenerate electrons. He found the contribution of magnetic field in the wave dispersion via the difference of occupation of the two spin states. A lot of progress has been made to investigate electron spin magnetization [21], kinetic description of Fermi particles [22], and the spin dynamics of semirelativistic plasma [23]. Moreover, Groot and Suttorp [24] discussed the connection between microscopic and macroscopic spin dynamics. Maroof *et al.* [25] investigated the dispersive feature of magnetosonic waves in relativistic degenerate electron-positron-ion magnetoplasma with spin-$1/2$ effect. Mushtaq *et al.* [26] studied the oblique propagation of low frequency magnetosonic waves in spin-$1/2$ degenerate magnetoplasma consisting of mobile ions, electrons, and positrons. Asenjo [27] analyzed the propagation of low frequency magnetosonic waves with mobile electrons and ions by using nondegenerate temperature with the effects of Bohm potential and spin of electrons and discussed the effect of quantum corrections.

Recently, many theories of quantum plasma and hydrodynamic stability of magnetized plasma have been developed which have a great importance in various astronomical environments [28–30]. Astrophysical objects contain degenerate matter due to which quantum mechanical treatment is important in such regimes. Most of the work on magnetorotational instability (MRI) was made in classical based dynamics of plasma; however, the main stream of the MRI study belongs to the astrophysical trends. Beside its classical treatment one can analyze it by using a quantum mechanical approach. MRI is considered as an important candidate for the core collapse and many dynamical behavior of various stars. MRI is a type

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of MHD instability initially addressed by Velikov [31] in 1959 and then confirmed by Chandrasekhar [32] in 1960. It has much more attraction with the work performed by Balbus and Hawley [33]. They restudied the MRI and applied the concept to the accretion disk rotating around a compact astrophysical object and showed that the MRI growth rate does not depend on the magnetic field with even a very low magnetic field that can alter the instability criteria. However, in the presence of magnetic field the instability criteria shifts from the outwardly decreasing angular velocity to the radially increasing angular velocity. Hydrodynamical accretion disks are stable but they are unstable magnetohydrodynamically and lead to disk turbulence and angular momentum transport [34–36]. Many analytical explanations, numerical analysis, and experimental investigations of this MHD instability has certainly become a basic plasma phenomenon. MRI is also assumed to act as a dynamo in the accretion disks [37]. Hereinafter, there is an increasing interest in the MRI applications to the astrophysical problems concerning magnetized accretion disks like protoplanetry disks [38], stellar disks [39], differentially rotating protoneutron stars [40], etc. Using local linear analysis, Ren *et al.* [41] investigated MRI in differentially rotating dusty plasma with the dissipative and immobile dust effect. The two fluid hydromagnetic model [42] was used to investigate MRI by considering electron and ions having the same angular frequency which encompass both the electron and ion gyro effects. The instability criteria was presented in the case of magnetized and nonmagnetized plasma which is different from that reported in one fluid model. Mikhailovskii *et al.* [43] studied nonaxisymmetric MRI in rotating plasma by neglecting the gravitation and derived the dispersion relation for ideal plasma including the effect of viscosity.

In this work we investigate MRI in electron-ion (e-i) quantum plasma by introducing a spin force term to the dynamic equation of motion. Solving the quantum hydrodynamic equations together with Maxwell's equations we obtain the generalized dispersion relation. It is shown that the instability criteria for MRI defined in the classical context using two fluid model and MHD model is duly modified by spin effect. We derived the instability criteria in both magnetized and unmagnetized cases for degenerate quantum plasma. The spin magnetization effect shows some important consequences on the instability criteria. Previously almost all of the studies about MRI are carried out in the classical based dynamics of plasma. We in this work are intending to make intensive analysis of MRI by looking into the quantum perspective of astrophysical objects and exploring the differences between the classical and quantum picture of the problem.

This manuscript is organized as follows. In Sec. II plasma quantum hydrodynamic equation of motion for electron-ion plasma in the presence of electron spin magnetization effect is presented. Based on this model, dispersion relation for the electron-ion plasma with axial magnetic field is derived in Sec. III. In Sec. IV a reduced dispersion relation is obtained using MHD approximations. In this section, different MRI characteristics (in the presence of spin magnetization) are discussed with and without magnetic field, for arbitrary magnetic field, and with ion Hall effect. Finally, in Sec. V the main results of the work are presented.

II. GOVERNING EQUATIONS

We consider a collisionless fully degenerate and quasineutral electron-ion plasma embedded in an external magnetic field $\mathbf{B} = B\hat{z}$. The strength of the magnetic field is assumed to be very high in a manner that it remarkably affects the dynamics of plasma. The charge neutrality condition at equilibrium is $n_{i0} = n_{e0} = n_0$, where n_{i0} and n_{e0} are the number densities of ions and electrons, respectively. The basic two fluid quantum hydrodynamic equation of motion in the presence of spin force is expressed as

$$\partial_t \mathbf{v}_j + (\mathbf{v}_j \cdot \nabla)\mathbf{v}_j = -\frac{1}{m_j n_j}\nabla P_j + \frac{q_j}{m_j}(\mathbf{E} + \mathbf{v}_j \times \mathbf{B}) - \nabla\Phi - \frac{2n_{j0}\mu_j}{\hbar}\nabla(\mathbf{S}\cdot\mathbf{B}_1). \quad (1)$$

Here $\mathbf{v}_j$ is the fluid velocity and P_j is the thermal pressure with j representing the species. For the massive ions, we used nondegenerate classical pressure, given by $P_i = \gamma_i n_i k_B T_i$, where γ is the polytropic index. For the degenerate electrons the Fermi pressure is defined as $P_{Fe} = \frac{(3\pi^2)^{\frac{2}{3}}\hbar^2}{5me} n_e^{\frac{5}{3}}$ and $\nabla P_{Fe} = \frac{1}{3}v_{Fe}^2 m_e \nabla n_e$ with $v_{Fe} = (3\pi^2 n_{e0})^{\frac{1}{3}}\frac{\hbar}{m_e}$ representing the Fermi velocity for degenerate electron with $\hbar$ being the reduced Planck's constant. The charge on particle is represented by q_j and the electric field by $\mathbf{E}$. The parameter $\Phi = -\frac{GM}{R}$ is the gravitational potential of the central object with mass M, gravitational constant G, and $R = (r^2 + z^2)^{\frac{1}{2}}$ with r being the distance from the rotation axis and $\hat{z}$ being the vertical coordinate. The last term in the above equation corresponds to spin correction in the equation of motion for electron and can be neglected in the case of ion dynamics. The parameter $\mu_j = \frac{q\hbar}{2m_j c}$ stands for the magnetic moment and B_1 represents the perturbed magnetic field. For electron the magnetic moment is defined by $\mu_e = -\mu_B$, with $\mu_B = \mid \frac{q\hbar}{2m_j c} \mid$ being the Bohr magneton. The spin evolution equation for the spin quantum plasma can be written as [16]

$$\frac{ds}{dt} = \frac{2\mu}{\hbar}(\mathbf{s}\times\mathbf{B}). \quad (2)$$

In MHD limit, under the assumption $\omega \leqslant \omega_{ci} \leqslant \omega_{ce}$, the spin inertia can be neglected well below the electron cyclotron frequencies and gives the spin equation of motion with solution [16]

$$\mathbf{S} = -\frac{\hbar}{2}\eta_j\left(\frac{\mu_j \mathbf{B}}{k_B T_{Fj}}\right)\hat{B}. \quad (3)$$

Here the Langevin parameter $\eta_j(\alpha) = \tanh(\alpha)$ appears due to the magnetization of spin distribution in thermodynamic equilibrium with $\alpha = \frac{\mu_B B_0}{k_B T_{Fj}}$ and $T_{Fj} = \frac{(3\pi^2 n_j)^{2/3}\hbar^2}{2k_B m_j}$ represents the Fermi temperature of the degenerate j_{th} particle. The singly charged ions are considered to be nondegenerate due to the fact that ions are massive and have large inertia in comparison to electrons, so we neglect the quantum spin effect in the case of ions. The first order continuity equation for the species is given as

$$\partial_t n_j + \nabla\cdot(n_j \mathbf{v}_j) = 0. \quad (4)$$

The above equations are coupled to Maxwell equations in the following form:

$$\nabla\cdot\mathbf{E}=\frac{e}{\epsilon_0}(n_i-n_e), \tag{5}$$

$$\nabla\cdot\mathbf{B}=0, \tag{6}$$

$$\partial_t\mathbf{B}=-\nabla\times\mathbf{E}, \tag{7}$$

$$\nabla\times\mathbf{B}=\mu_\circ\mathbf{J}+\frac{1}{c^2}\partial_t\mathbf{E}, \tag{8}$$

where $\mathbf{J}=\sum_{j=e,i}q_jn_jv_j+cJ_{M_e}$ is the current density with $J_{M_e}=\nabla\times\mathbf{M_e}$ being the spin magnetization current density of electron for degenerate Fermi plasma with magnetization density vector $\mathbf{M_e}=\mu_Bn_e\tanh(\alpha)\hat{B}$ and $c=(\varepsilon_0\mu_0)^{-\frac{1}{2}}$ being the speed of light in vacuum. We considered an axisymmetric plasma rotating in the azimuthal θ direction having angular frequency $\Omega=\Omega(r)$ in a standard cylindrical coordinates system (r,θ,z). The equilibrium quantities are $\mathbf{v}_{j0}=(0,\,r\Omega,0)$, $\mathbf{B}_0=(0,0,B_0)$, $\mathbf{E}_0=(E_0,0,0)$, and $P_{j0}=P_{j0}(r)$. Electrons and ions are supposed to have the same angular frequency in order to eliminate the unperturbed current, i.e., $\mathbf{J}_0=0$; hence the equilibrium magnetic field is assumed to be homogeneous.

III. DISPERSION RELATION

We consider the plasma in the cylindrical coordinates and the perturbed magnetic field, electric field, and velocity are represented by $\mathbf{B}_1=(\widetilde{B}_r,\widetilde{B}_\theta,\widetilde{B}_z)$, $\mathbf{E}_1=(\widetilde{E}_r,\tilde{E}_\theta,\widetilde{E}_z)$, and $\mathbf{v}_{j1}=(\widetilde{v}_{jr},\widetilde{v}_{j\theta},\widetilde{v}_{jz})$, respectively, while the perturbed pressure is given by $\widetilde{P}_j$ and the perturbed number density by $\widetilde{n}_j$. Each perturbed profile is considered to be proportional to $e^{-i\omega t+ik_zz}$, where ω and k_z are the wave frequency and wave number directed along z, respectively. The plasma is considered to be incompressible so that the mass conservation is reduced to $\nabla\cdot\mathbf{v}_j=0$. To ignore the density fluctuations the equilibrium state is assumed to be homogeneous, such that $n_0=\text{const}$, yielding $\widetilde{n}_j=0$. The perturbed Poisson equation $\nabla\cdot\mathbf{E}_1=0$, resulting in $\hat{E}_z=i\frac{\hat{L}}{k_z}\tilde{E}_r$, and the incompressible condition gives rise to $\widetilde{v}_{jz}=i\frac{\hat{L}}{k_z}\widetilde{v}_{jr}$. The divergence free property of the magnetic field $\nabla\cdot\mathbf{B}_1=0$ provides that $\mathbf{B}_z=i\frac{\hat{L}}{k_z}\widetilde{B}_r$ and from the perturbed Faraday's law we obtain $\hat{E}_\theta=\frac{\omega}{k_z}\widetilde{B}_r$ and $\mathbf{B}_\theta=\dfrac{k_z^2-\partial_r\hat{L}}{\omega k_z}\tilde{E}_r$, where we define the operator $\hat{L}=\frac{1}{r}+\partial_r$. Now linearizing Eq. (1) for the electron up to the first order results in the following relation:

$$\begin{aligned}&\partial_t(\tilde{v}_{er}\hat{r}+\tilde{v}_{e\theta}\hat{\theta}+\tilde{v}_{ez}\hat{z})+\frac{\kappa^2}{2\Omega}\tilde{v}_{er}\hat{\theta}-2\Omega\tilde{v}_{e\theta}\hat{r}\\&=-\frac{\nabla_{r,\theta,z}}{m_en_e}\tilde{P}_e+\frac{q_e}{m_e}[(\widetilde{E}_r\hat{r}+\tilde{E}_\theta\hat{\theta}+\widetilde{E}_z\hat{z})\\&\quad+(\tilde{v}_0\tilde{B}_z+\tilde{v}_\theta\tilde{B}_0)\hat{r}-\tilde{v}_0\tilde{B}_r\hat{z}-\tilde{v}_r\tilde{B}_0\hat{\theta}]-\eta_e(\alpha)\mu_en_e\nabla_zB_z.\end{aligned} \tag{9}$$

Here $\kappa^2=\frac{d\Omega^2}{d\,\ln r}+4\Omega^2$ is the square of the epicyclic frequency. The above linearized equation (9) in component form (r,θ,z) can be expressed as

$$\partial_t\tilde{v}_{er}\hat{r}-2\Omega\tilde{v}_{e\theta}\hat{r}=-\frac{\nabla_r}{m_en_e}\tilde{P}_e+\frac{q_e}{m_e}[\widetilde{E}_r+\tilde{v}_0\tilde{B}_z+\tilde{v}_\theta\tilde{B}_0]\hat{r}, \tag{10}$$

$$\partial_t\tilde{v}_{e\theta}\hat{\theta}+\frac{\kappa^2}{2\Omega}\tilde{v}_{er}\hat{\theta}=-\frac{\nabla_\theta}{m_en_e}\tilde{P}_e+\frac{q_e}{m_e}[\tilde{E}_\theta-\tilde{v}_r\tilde{B}_0]\hat{\theta}, \tag{11}$$

$$\partial_t\tilde{v}_{ez}\hat{z}=-\frac{\nabla_z}{m_en_e}\tilde{P}_e+\frac{q_e}{m_e}[\tilde{E}_z-\tilde{v}_0\tilde{B}_r]\hat{z}-\eta_e(\alpha)\mu_en_e\nabla_zB_z. \tag{12}$$

Using the plane wave solution Eq. (12) can be rewritten in the following form:

$$\frac{\tilde{P}_e}{m_en_e}=\frac{\omega}{k_z}\widetilde{v}_{ez}-\frac{i}{k_z}\frac{q_e}{m_e}[\tilde{E}_z-\tilde{v}_0\tilde{B}_r]-\eta_e(\alpha)\mu_en_eB_z. \tag{13}$$

Substituting Eq. (13) into Eqs. (10) and (11) we obtain

$$-i\omega\widetilde{v}_{e\theta}+\left(\frac{\kappa^2}{2\Omega}+\omega_c\right)\tilde{v}_{er}=-\frac{\omega\omega_c}{k_zB_0}B_r \tag{14}$$

and

$$\begin{aligned}&-i\omega\widetilde{v}_{er}-(2\Omega+\omega_c)\tilde{v}_{e\theta}\\&=\frac{-i\omega}{k_z^2}\partial_r\hat{L}\tilde{v}_{er}+\frac{\omega\omega_c}{k_zB_0}B_\theta-m\frac{i\omega_c}{k_zB_0}\frac{d\Omega}{d\,\ln r}\tilde{B}_r\\&\quad+\frac{i}{k_z}\eta_e(\alpha)\mu_en_{e\circ}\partial_r\hat{L}\tilde{B}_r.\end{aligned} \tag{15}$$

Here $\omega_c=\frac{eB_0}{m_e}$ is the electron cyclotron frequency associated with external magnetic field and ω is the wave frequency. The local approximation is adopted by assuming $\partial_r\simeq ik_r$ and $k_rr\gg1$, where k_r is the radial wave number. Thus $\partial_r\hat{L}\simeq-k_r^2$ and $k=(k_r^2+k_z^2)^{1/2}$ is the total wave number. The perturbed magnetic field is determined by

$$\nabla\times\mathbf{B}_1=en_0\mu_0(\mathbf{v}_{i1}-\mathbf{v}_{e1})+\eta_e(\alpha)\mu_en_e(\nabla\times\mathbf{B}_1). \tag{16}$$

From (16) the ion perturbed velocities in component form can be obtained by using the problem geometry as

$$\tilde{v}_{ir}=-\frac{ik_z}{en_0\mu_0}\tilde{B}_\theta-\frac{ik_z}{en_\circ\mu_\circ}\eta_e(\alpha)\mu_en_{e0}\tilde{B}_\theta+\tilde{v}_{er}, \tag{17}$$

$$\tilde{v}_{i\theta}=\frac{ik^2}{k_zen_0\mu_0}\tilde{B}_r+\frac{ik^2}{en_0\mu_0}\eta_e(\alpha)\mu_en_{e0}\tilde{B}_r+\tilde{v}_{e\theta}. \tag{18}$$

Equations (14) and (15) engender $\tilde{B}_r$ and $\tilde{B}_\theta$ in the terms of $\tilde{v}_{er}$ and $\tilde{v}_{e\theta}$ as

$$\tilde{B}_r=\frac{-ik_zB_0}{\omega_c}\widetilde{v}_{e\theta}+\frac{k_zB_0}{\omega\omega_c}\left(\frac{\kappa^2}{2\Omega}-\omega_c\right)\tilde{v}_{er} \tag{19}$$

and

$$\begin{aligned}\tilde{B}_\theta=&\frac{ik^2B_0}{k_z\omega_c}\tilde{v}_{er}+\frac{k_zB_0}{\omega\omega_c}\left(\frac{\kappa^2}{2\Omega}-\omega_c\right)\tilde{v}_{e\theta}\\&+\frac{ik_zB_0}{\omega^2\omega_c}\frac{d\Omega}{d\,\ln r}\left(\frac{\kappa^2}{2\Omega}-\omega_c\right)\tilde{v}_{er}-\frac{k^2B_0^2}{\omega\omega_c^2}\eta_e(\alpha)\mu_en_e\tilde{v}_{e\theta}\\&-\frac{ik^2B_0^2k_z}{\omega^2\omega_c^2}\left(\frac{\kappa^2}{2\Omega}-\omega_c\right)\eta_e(\alpha)\mu_en_e\tilde{v}_{er}.\end{aligned} \tag{20}$$

Using Eqs. (14) and (15) for ions and substituting the values of $\tilde{v}_{ir}$ and $\tilde{v}_{i\theta}$ from Eqs. (17) and (18) leads to the following expressions:

$$\frac{\omega}{k_z B_0}\left(\frac{k^2 V_A^2}{\Omega_c}+\Omega_c\right)\tilde{B}_r+\frac{\omega k^2 V_A^2}{k_z B_0 \Omega_c}\eta_e(\alpha)\mu_e n_e \tilde{B}_r - i\omega\tilde{v}_{e\theta} - \frac{i k_z V_A^2}{B_0\Omega_c}\left(\frac{\kappa^2}{2\Omega}+\Omega_c\right)\tilde{B}_\theta - \frac{i k_z V_A^2}{B_0\Omega_c}\left(\frac{\kappa^2}{2\Omega}+\Omega_c\right) \times \eta_e(\alpha)\mu_e n_e \tilde{B}_\theta + \left(\frac{\kappa^2}{2\Omega}+\Omega_c\right)\tilde{v}_{er}=0 \tag{21}$$

and

$$-\frac{\omega k^2 V_A^2}{k_z B_0\Omega_c}\tilde{B}_\theta+\frac{\omega k^2 V_A^2}{k_z B_0\Omega_c}\eta_e(\alpha)\mu_e n_e\tilde{B}_\theta - i\omega\frac{k^2}{k_z^2}\tilde{v}_{er} - \frac{i k^2 V_A^2}{k_z B_0 \Omega_c}(2\Omega+\Omega_c)\tilde{B}_r - \frac{i k^2 V_A^2}{k_z B_0 \Omega_c}(2\Omega+\Omega_c)\eta_e(\alpha)\mu_e n_e \tilde{B}_r - (2\Omega+\Omega_c)\tilde{v}_{e\theta} - \frac{\omega\Omega_c}{k_z B_0}\tilde{B}_\theta + \frac{i\Omega_c}{k_z B_0}\frac{d\Omega}{d\ \ln r}\tilde{B}_r = 0. \tag{22}$$

In the above equations (21) and (22) $\Omega_c=\frac{eB_0}{m_i}$ stands for ion gyrofrequency, while $V_A=\sqrt{\frac{B_\circ^2}{m_i n_i \mu_\circ}}$ and $\omega_A=k_z V_A$ are the Alfvén speed and frequency, respectively. Substituting the value of the perturbed magnetic field from Eqs. (19) and (20) into Eqs. (21) and (22) and eliminating the perturbations we arrive at the following dispersion relation:

$$\begin{aligned}&(\omega^2\alpha_o+\omega_A^2\beta)^2-\omega^2\frac{k_z^2}{k^2}\left[2\Omega\alpha_o+\frac{\kappa^2}{2\Omega}\left(\alpha_o-1-\frac{\Omega_c}{\omega_c}\right)-\frac{k^2V_A^2}{\Omega_c}\left(1-\frac{\Omega_c}{\omega_c}\right)\right]\left[\left(2\alpha_o-1-\frac{\Omega_c}{\omega_c}\right)\right.\\&\times\left(\frac{\kappa^2}{2\Omega}+\omega^2\eta_e(\alpha)\mu_e n_e\right)-\alpha_o\omega^2\eta_e(\alpha)\mu_e n_e-\frac{\omega_A^2}{\Omega_c}\left(1-\frac{\Omega_c}{\omega_c}\right)[1+\eta_e(\alpha)\mu_e n_e]+\left.\frac{\omega_A^2}{\omega^2}\beta\left(\frac{d\Omega}{d\ \ln r}-\frac{k_zB_0\omega}{\omega_c^2}\eta_e(\alpha)\mu_e n_e\right)\right]\\&+\frac{\omega^2}{\omega_c^2}(\omega^2\alpha_o+\omega_A^2\beta)\left[\left(\alpha_o-1-\frac{\Omega_c}{\omega_c}\right)\omega^2+\omega_A^2\beta-\frac{\omega^2k_z}{\omega_c\Omega_c}\left(\frac{\kappa^2}{2\Omega}+\Omega_c\right)\left(\frac{\eta_e(\alpha)\mu_e n_e}{\omega_c}-1\right)\eta_e(\alpha)\mu_e n_e\right.\\&-\left.\frac{\omega^2}{\omega_c^2}\omega_A^2\left(\frac{k^4V_A^2}{\omega_c\Omega_c}[1-\eta_e(\alpha)\mu_e n_e]+\frac{\Omega_ck^2}{\omega_c^2}\right)\right]\left[\left(2\alpha_o-1-\frac{\Omega_c}{\omega_c}\right)\left(\frac{\kappa^2}{2\Omega}+\omega^2\eta_e(\alpha)\mu_e n_e\right)+\alpha_o\omega^2\eta_e(\alpha)\mu_e n_e\right.\\&+\left.\frac{\omega_A^2}{\Omega_c}\left(1-\frac{\Omega_c}{\omega_c}\right)-\frac{\omega_A^2}{\omega^2}\beta\left(\frac{d\Omega}{d\ \ln r}-\frac{k_zB_0\omega}{\omega_c^2}\eta_e(\alpha)\mu_e n_e\right)\right]\eta_e(\alpha)\mu_e n_e=0,\end{aligned}\tag{23}$$

and here we denote

$$\alpha_o = 1+\frac{k^2V_A^2}{\Omega_c\omega_c}+\frac{\Omega_c}{\omega_c},$$

$$\beta=\frac{1}{\Omega_c\omega_c}\left(\frac{\kappa^2}{2\Omega}-\omega_c\right)\left(\frac{\kappa^2}{2\Omega}+\Omega_c\right).$$

This formula refers to the general dispersion relation determining the dynamics of axisymmetric MRI in the quantum e-i plasma. By letting $\eta_e(\alpha)=0$, the result is reduced to the relation obtained in Ref. [42]. The classical dispersion relation for plasma mode is modified by including the spin quantum correction showing the complete physical picture about the MRI in both low and high frequency regimes.

IV. DISCUSSION

Equation (23) reveals new contributions due to the spin magnetization effect to the wave dispersion at the quantum scale depending on the strength of the magnetic field. This contribution is useful in understanding the features of long wavelength or low frequency MHD waves in quantum plasmas that exist in astrophysical environments like neutron stars and white dwarfs. In the low frequency limits, i.e., $kV_A \ll \Omega_c$, $\omega \ll \Omega_c$, and $\Omega \ll \Omega_c$ assuming $\frac{\Omega_c}{\omega_c}=\frac{m_e}{m_i}\simeq 0$, $\alpha_0\simeq 1$, and $\beta\simeq -1$, the dispersion relation (23) can be expressed as

$$\omega^2-k_z^2V_A^2-\frac{k_z^2}{k^2}\left[\frac{d\Omega^2}{d\ \ln r}+\frac{4\Omega^2\omega^2}{\omega^2-k_z^2V_A^2}-\frac{k_z^2V_A^2}{\omega^2-k_z^2V_A^2}\frac{d\Omega^2}{d\ \ln r}\eta_e(\alpha)\mu_e n_e\right]=0. \tag{24}$$

Equation (24) represents the modified dispersion relation for the MRI in which the two fluid quantum hydrodynamic model is reduced to the ideal MHD model with the inclusion of axial magnetic field and spin magnetization effect. If the spin magnetization effect is considered to be zero $[\eta_e(\alpha)=0]$ in Eq. (24), then we recover the classical relation reported in Ref. [33]. In many magnetized plasmas, the spin magnetization has a small contribution to the total magnetic field. For the case when the factor $\mu_B B_0/k_B T_{Fe} \ll 1$, the spin quantum effects are negligible and are more significant in the case when $\mu_B B_0/k_B T_{Fe} \gg 1$. In high density and low temperature plasmas like in the vicinity of pulsars and magnetars, the spin contributions arise due to the fact that the component of spin force is parallel to the ambient magnetic field. For the higher values of magnetic field the magnetization energy shows some important effects on the dynamics of the system.

A. Magnetized plasma

To illustrate the instability criteria, the dispersion relation (23) can be reduced to the following equation under the assumption $\omega^2/\omega_c^2 \ll 1$:

$$\omega^4\alpha_0^2 + \omega^2\frac{k_z^2}{k^2}(D_1 - D_3) + \frac{k_z^4}{k^4}D_2 = 0, \tag{25}$$

where

$$D_1 = \left[\frac{\kappa^2}{2\Omega}\left(2\alpha_0 - 1 - \frac{\Omega_c}{\omega_c}\right) - \frac{k^2V_A^2}{\Omega_c}\left(1 - \frac{\Omega_c}{\omega_c}\right)\right]\left[2\Omega\alpha_0 + \frac{\kappa^2}{2\Omega}\left(\alpha_0 - 1 - \frac{\Omega_c}{\omega_c}\right) - \frac{k^2V_A^2}{\Omega_c}\left(1 - \frac{\Omega_c}{\omega_c}\right)\right] - 2\alpha_0 k^2V_A^2\beta, \tag{26}$$

$$D_2 = k^2V_A^2\beta\left[k^2V_A^2\beta - \frac{d\Omega}{d\,\ln r}[1 + \eta_e(\alpha)\mu_e n_e]\left\{2\Omega\alpha_0 + \frac{\kappa^2}{2\Omega}\left(\alpha_0 - 1 - \frac{\Omega_c}{\omega_c}\right) - \frac{k^2V_A^2}{\Omega_c}\left(1 - \frac{\Omega_c}{\omega_c}\right)\right\}\right], \tag{27}$$

and

$$D_3 = \left[2\Omega\alpha_0 + \frac{\kappa^2}{2\Omega}\left(\alpha_0 - 1 - \frac{\Omega_c}{\omega_c}\right) - \frac{k^2V_A^2}{\Omega_c}\left(1 - \frac{\Omega_c}{\omega_c}\right)\right] \times \left[\frac{\kappa^2}{2\Omega}\left(\alpha_0 - 1 - \frac{\Omega_c}{\omega_c}\right) + \frac{k^2V_A^2}{\Omega_c}\left(1 - \frac{\Omega_c}{\omega_c}\right) - \frac{\omega_A^2}{\omega_c\omega^2}k_zB_0\eta_e(\alpha)\mu_e n_e\right]\eta_e(\alpha)\mu_e n_e. \tag{28}$$

Here we look into the unstable condition for which $D_2 < 0$, i.e.,

$$k^2V_A^2\beta\left[k^2V_A^2\beta - \frac{d\Omega}{d\ln r}(1 + \eta_e(\alpha)\mu_e n_e)\left\{2\Omega\alpha_0 + \frac{\kappa^2}{2\Omega}\left(\alpha_0 - 1 - \frac{\Omega_c}{\omega_c}\right) - \frac{k^2V_A^2}{\Omega_c}\left(1 - \frac{\Omega_c}{\omega_c}\right)\right\}\right] < 0. \tag{29}$$

In MHD limits the relation (29) reduces to

$$\frac{d\Omega^2}{d\,\ln r} + \frac{k^2V_A^2}{[1 + \eta_e(\alpha)\mu_e n_e]} < 0. \tag{30}$$

Equation (30) is the basic criteria for MRI which is duly modified in the presence of spin correction. By ignoring the spin effect of electron the criteria is reduced to the classical one in a magnetized plasma previously studied in Ref. [33]. The first term in Eq. (30) represents the Velikhov effect driving the rotational instability. On the other hand, the second term in the inequality describes the spin modified Alfvén wave, where the spin effect plays a vital role via coupling to the Alfvénic speed. In MHD the magnetic field perturbations travel along the direction of the ambient magnetic field with the Alfvén velocity. Due to the contribution of spin effect within the single fluid model, the Alfvén velocity is decreased. The factor $\tanh(\mu_B B_0/k_\beta T_{Fe})$ that appears in the spin contribution term gives the net effect of spin; however, it is limited to certain conditions. In the thermodynamic equilibrium $\mu_B B_0/k_\beta T_{Fe} \ll 1$, the correction factor is close to unity and, under the approximation $\mu_0 H_0 \approx B_0$ (which is the magnetic field due to the external sources only), the spin correction may be omitted. Moreover, the envelope of weakly modulated Alfvén waves travel for $\omega \ll \Omega_c$, with group velocity close to the Alfvénic speed [44]. For the case of strong magnetic field the factor $\mu_B B_0/k_\beta T_{Fe} \gg 1$, that shows the strong magnetization effect and the spin contribution in this case, is important, e.g., in compact astrophysical objects like white dwarfs and magnetars (a type of neutron star with an extremely powerful magnetic field).

B. Unmagnetized plasma

In the case of unmagnetized plasma, we have $B_0 = 0$ and define $\xi = \frac{k^2V_A^2}{\Omega_c\omega_c} = \frac{k^2 m}{(e^2 n_\circ \mu_\circ)}$ along with $\alpha_0 \simeq 1 + \xi$ and $\beta k^2V_A^2 = (\frac{\kappa^2}{2\Omega})^2\xi$. Neglecting the terms proportional to $\frac{\Omega_c}{\omega_c}$, the criteria in Eq. (29) can then be written as

$$\left(\frac{\kappa^2}{2\Omega}\right)^2\xi\left[\left(\frac{\kappa^2}{2\Omega}\right)^2\xi - \frac{d\Omega^2}{d\,\ln r}[1 + \eta_e(\alpha)\mu_e n_e] \times \left\{2\Omega(1+\xi) + \left(\frac{\kappa^2}{2\Omega}\right)\xi\right\}\right] < 0. \tag{31}$$

In MHD limits, the inequality (31) reduces to the following form:

$$\frac{4\Omega^2\xi}{[1 + \eta_e(\alpha)\mu_e n_e]} - \frac{d\Omega^2}{d\,\ln r} < 0. \tag{32}$$

Equation (32) gives the critical condition for instability to occur in quantum electron-ion plasma with electron spin magnetization correction and is identical to the criteria for classical regimes described in [42] for the two fluid model. In the case of electron spin contribution, when the external applied magnetic field is zero ($B_0 = 0$), the spin effect vanishes due to coupling to the field and has no contribution in the dynamics of the system. However, it still exists due to induced magnetization (bulk magnetization) due to the spin. This is possible for oblique J_S coupling. For completely aligned the spin term will also vanished with $B_0 = 0$. It is therefore concluded that in the case of strongly magnetized plasma, the instability criteria may alter remarkably due to the presence of spin effect. In the case of weakly magnetized plasma the spin of the electrons is equally distributed in spin up and spin down populations. However, in the presence of electromagnetic perturbations, the pondermotive force separates the two populations locally that enhance the original magnetic field with a fixed ion background (magnetic dynamos) resulting in a nonlinear phenomena. The criteria defined in Eq. (32) can be reduced to the MHD one under the assumption $\xi = 0$ in the low frequency limits and can be expressed as

$$\frac{d\Omega^2}{d\,\ln r}[1 + \eta_e(\alpha)\mu_e n_e] > \frac{2c^2k^2}{\omega_{pe}^2}. \tag{33}$$

Ignoring the spin effects $[\eta_e(\alpha) = 0]$ we exactly retrieve the results obtained for classical plasma reported in Ref. [41].

C. Arbitrary magnetic field

For the magnetic field with an arbitrary strength we can derive the instability criteria by rewriting Eq. (29) in the following form:

$$\left(\frac{d\Omega}{d\ \ln r} + 2\Omega + \Omega_c\right)\left(\frac{d\Omega}{d\ \ln r} + 2\Omega + \omega_c\right) \times \left[\frac{d\Omega}{d\ \ln r} + \frac{k^2 V_A^2}{2\Omega\left(1+\frac{\Omega_c}{\omega_c}\right)[1+\eta_e(\alpha)\mu_e n_e]} \times \left(1+\frac{2\Omega}{\Omega_c}\right)\left(1-\frac{2\Omega}{\omega_c}\right)\right] > 0. \tag{34}$$

By letting $\frac{d\Omega}{d\ \ln r} = X$, Eq. (34) has three roots, $X_1 = -2\Omega - \Omega_c$, $X_2 = -2\Omega + \Omega_c$, and $X_3 = -\vartheta X_2$, with

$$\vartheta = \frac{k^2 V_A^2}{2\Omega\left(1+\frac{\Omega_c}{\omega_c}\right)[1+\eta_e(\alpha)\mu_e n_e]\omega_c}\left(1+\frac{2\Omega}{\Omega_c}\right).$$

X_1 is always negative, X_2 may be negative or positive, while the sign of X_3 depends on the value of X_2. The instability takes place in the region when $X > X_2$, resulting in $X < X_3$, and Eq. (34) gives

$$\frac{d\Omega}{d\ \ln r} < -\frac{k^2 V_A^2}{2\Omega\left(1+\frac{\Omega_c}{\omega_c}\right)[1+\eta_e(\alpha)\mu_e n_e]} \times \left(1+\frac{2\Omega}{\Omega_c}\right)\left(1-\frac{2\Omega}{\omega_c}\right). \tag{35}$$

In the long wavelength limits, Eq. (35) reduces to the following inequality:

$$\frac{d\Omega}{d\ \ln r}[1+\eta_e(\alpha)\mu_e n_e] < 0. \tag{36}$$

The system is still unstable to the magnetic perturbations. The spin of electrons are aligned that develops a magnetic field and pushes the electrons away from its locality. This local displacement of electrons increases the share rate and is responsible for the particle transport leading to the accretion process in the interiors of compact degenerate astrophysical objects. When $X_2 < 0$, this leads to $X_3 > 0$, resulting in $X > X_3$, and Eq. (34) can be expressed as

$$\frac{d\Omega}{d\ \ln r} > -\frac{k^2 V_A^2}{2\Omega\left(1+\frac{\Omega_c}{\omega_c}\right)[1+\eta_e(\alpha)\mu_e n_e]} \times \left(1+\frac{2\Omega}{\Omega_c}\right)\left(1-\frac{2\Omega}{\omega_c}\right), \tag{37}$$

$$\frac{d\Omega}{d\ \ln r}[1+\eta_e(\alpha)\mu_e n_e] > 0. \tag{38}$$

The criteria shifts from the outwardly decreasing angular velocity to the radially increasing angular velocity. Many regions are developed corresponding to the instability that exists in multicomponent plasmas like $X > X_2$, $X < X_1$, and $X_3 < X$. The region discussed above in Eq. (37) gives the criteria for the MRI in the presence of electron and ion gyro effects duly modified by the spin force term.

D. Hall-MHD

The MHD frequency has restrictions to the unstable region mentioned in Eq. (34) and the instability criterion is reduced to MHD under the assumption $\Omega \ll \Omega_c$. For the same region if we ignore the electron gyro effect ($\Omega \ll \omega_c$), while taking into account the ion gyro effect, i.e., assuming that Ω is comparable to or greater than Ω_c, then the criteria for MRI in the presence of spin effect can be deduced from Eq. (34) as

$$\frac{d\Omega}{d\ \ln r} + \frac{k^2 V_A^2}{2\Omega[1+\eta_e(\alpha)\mu_e n_e]}\left(1+\frac{2\Omega}{\Omega_c}\right) < 0. \tag{39}$$

The term proportional to $\frac{\Omega}{\Omega_c}$ is called the Hall effect. For the case when $\Omega \ll \Omega_c$, we can write Eq. (39) in the following form:

$$\frac{d\Omega}{d\ \ln r} + \frac{k^2 V_A^2}{2\Omega[1+\eta_e(\alpha)\mu_e n_e]} < 0. \tag{40}$$

By ignoring the spin term in Eq. (40), the result is identical to that of Ref. [42].

V. CONCLUSIONS

In this work, we have analytically studied the MRI by using QHD equations including spin magnetization effect and obtained the dispersion relation for axisymmetric MRI in a rotating degenerate quantum e-i plasma. The dispersion relation for e-i quantum plasma has been presented in Eq. (23), which describes the waves in both longer and shorter wavelength limits. Our main focus was on low frequency of the mode with the effect of spin magnetization. The electron spin magnetization can introduce some new aspects to the MRI due to its low inertia and quantum signature. The local dispersion relation was obtained by using MHD approximations and the criteria for instability has been defined in magnetized plasma along with the modified form of the spin term. The spin magnetization effect can be minimized through the condition $\mu_B B_0 \ll k_\beta T_{Fe}$; however, it can even influence the criteria for instability and the properties of MHD waves in the degenerate plasma. Spin contributions are important in high density, low temperature, and strongly magnetized plasmas. The latter can be found in astrophysical surroundings, e.g., pulsars and magnetars. The instability criteria for MRI was obtained for various cases, e.g., magnetized plasma, unmagnetized plasma, arbitrary magnetic field, and for Hall regime, duly modified by the spin correction term. The comparison of both classical and quantum description has been shown with the same qualitative behavior. A general case was presented for an arbitrary magnetic field generating many regions which are unstable to the MRI in both the MHD and quantum two fluid model. The instability criteria shifts from inwardly increasing angular velocity to the outwardly increasing angular momentum. The displacement of electrons due to the aligned spin may give rise to the accretion process in heavenly compact astrophysical objects.

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