Renormalization of viscosity in wavelet-based model of turbulence

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Statistical theory of turbulence in viscid incompressible fluid, described by the Navier-Stokes equation driven by random force, is reformulated in terms of scale-dependent fields $\mathbf{u}_a(x)$, defined as wavelet-coefficients of the velocity field $\mathbf{u}$ taken at point x with the resolution a. Applying quantum field theory approach of stochastic hydrodynamics to the generating functional of random fields $\mathbf{u}_a(x)$, we have shown the velocity field correlators $\langle \mathbf{u}_{a_1}(x_1) \dots \mathbf{u}_{a_n}(x_n) \rangle$ to be finite by construction for the random stirring force acting at prescribed large scale L. The study is performed in $d = 3$ dimension. Since there are no divergences, regularization is not required, and the renormalization group invariance becomes merely a symmetry that relates velocity fluctuations of different scales in terms of the Kolmogorov-Richardson picture of turbulence development. The integration over the scale arguments is performed from the external scale L down to the observation scale A, which lies in Kolmogorov range $l \ll A \ll L$. Our oversimplified model is full dissipative: interaction between scales is provided only locally by the gradient vertex $(\mathbf{u}\nabla)\mathbf{u}$, neglecting any effects or parity violation that might be responsible for energy backscatter. The corrections to viscosity and the pair velocity correlator are calculated in one-loop approximation. This gives the dependence of turbulent viscosity on observation scale and describes the scale dependence of the velocity field correlations.

I. INTRODUCTION

Statistical theory of hydrodynamic turbulence has a long history, which stems from the work of Osborne Reynolds [1], which suggested that turbulence should be described statistically rather than by the equations of laminar hydrodynamics. This practically implies statistical averaging over all possible trajectories, same as the Feynman's functional integral does in quantum field theory. There is a standard list of topics in turbulence theory: the stability of solutions of hydrodynamic equations, the instability of laminar flow, the origin of intermittency, and so on. This paper deals with only one of these topics—the description of fully developed homogeneous isotropic turbulence in an incompressible fluid. The detailed description of the problem can be found in classical monographs [2–4].

Turbulent flows occurring in various liquids and gasses at high Reynolds numbers reveal a number of general aspects: cascade of energy, scaling behavior of correlation functions, statistical correlation laws. However, the prediction of characteristic features of turbulent flow using basic equations of fluid dynamics still remains a challenge.

Starting from phenomenological theory of Kolmogorov-Obukhov [5–8], derived from simple dimensional consideration, it is well known that the probability distribution function of the velocity field fluctuations in fully developed turbulence is determined by two basic scales: the microscopic energy dissipation scale l and the macroscopic integral scale L, where the energy is injected into the fluid; and also by the energy dissipation rate per unit of mass ε. The experimental measurements show systematic deviations from Kolmogorov scaling for higher-order velocity correlation functions [3,4,9]. This should be explained from microscopic principles.

In this paper we consider the Navier-Stokes equation,

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \nu_0 \Delta \mathbf{u} - \nabla p + \mathbf{f}(t, \mathbf{x}), \qquad (1)$$

where $\mathbf{u}(t, \mathbf{x})$ is velocity field, ν_0 is the kinematic viscosity of the fluid, and p is the pressure, describing an incompressible fluid stirred by random force $f(t, \mathbf{x})$.

The observation scale a is not present in the micromodel Eq. (1) explicitly: the fields $\mathbf{u}$ and p are square-integrable functions formally defined in each point $(t, \mathbf{x})$. However, we know that the Eulerian velocity $\mathbf{u}(t, \mathbf{x})$ is meaningful only in case it is the velocity of fluid averaged over a volume

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δ^d, where δ is not less than a mean free path, to allow for hydrodynamic approximation. So, to introduce the *resolution* into mathematical consideration of a fluid dynamics problem as an extra coordinate $x \rightarrow (a, x)$ we extend the space of square-integrable functions $L^2(\mathbb{R}^{d+1})$ by means of continuous wavelet transform performed in a spatial variable $\mathbf{x}$. This trick is quite common in the numerical studies of turbulence [10–12].

The new feature of this paper is that we combine wavelet transform with the methods of quantum field theory, including the method of renormalization group, to study the statistical momenta of turbulent velocity fields at different scales.

The remainder of this paper is organized as follows. In Sec. II we review the quantum field theory approach to fully developed homogeneous isotropic turbulence. In Sec. III we reformulate the quantum field theory approach in terms of scale-dependent velocity fields $\mathbf{u}_a(t, \mathbf{x})$. The dependence of turbulent viscosity $\nu_a(k)$, which affects the scale components $\mathbf{u}_a(k)$, calculated in the developed framework, is presented in Sec. IV. Having calculated the second order statistical momenta $\langle u_a(\omega, \mathbf{k}) u_{a'}(\omega', \mathbf{k}') \rangle$ in one-loop approximation, in Sec. V we present the energy spectrum of the turbulent velocity field fluctuations as a function of dimensionless observation scale $\xi = A/L$. In Sec. VI we summarize the applicability of our theoretical results to the studies of turbulence. Technical details of calculations are presented in the Appendices.

II. QUANTUM FIELD THEORY APPROACH TO STATISTICAL HYDRODYNAMICS

Similarity between hydrodynamic turbulence and critical phenomena [13,14] made different authors cast the turbulent velocity field generating functional in a form of quantum field theory, and then apply renormalization group technique, see, e.g., Refs. [15–19]. This approach is valid for the stirred Navier-Stokes description of hydrodynamic turbulence Eq. (1), supplied by the equality condition between the energy injection by random force and the viscous energy dissipation.

The generating functional of the velocity field can be written in the form

$$G[A] = e^{W[A]} = \int \exp\left(S[\Phi] + \int d^d\mathbf{x}\, dt\, A\Phi \right) \mathcal{D}\Phi, \quad (2)$$

where the field $\Phi = (u, u')$ is the doublet of the Eulerian velocity field $u(t, \mathbf{x})$ and the Martin-Sigia-Rose auxiliary field $u'(t, \mathbf{x})$, introduced to exponentiate the delta-function of the equations of motion [20] (with functional Jacobian of the equations of motion with respect to velocity field being dropped due to appropriate redefinition of the Green functions [16], or using ghost fields [21]). The argument $A(t, \mathbf{x}) \equiv (A_u, A_{u'})$ is an arbitrary functional source. The "action" functional itself takes the form

$$S[\Phi] = \frac{1}{2} \int u' \hat{D}_0 u' + \int u'[-\partial_t u + \nu_0 \Delta u - (u \cdot \nabla)u], \tag{3}$$

where $\hat{D}_0(x - x') = \langle f(x) f(x') \rangle$ is the random force correlator. Integration over the space-time arguments $x \equiv (t, \mathbf{x})$ is tacitly assumed. The pressure term is eliminated from the

field theory Eqs. (2) and (3) by the imposed incompressibility conditions $\nabla \cdot \mathbf{u} = \nabla \cdot \mathbf{u}' = \nabla \cdot \mathbf{f} = 0$. The incompressibility is ensured by multiplication of all lines of the Feynman graphs of the field theory Eq. (2) by the orthogonal projector

$$P_{ij}(\mathbf{k}) = \delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2},$$

where $\mathbf{k}$ is the momentum of the line.

The perturbation expansion is performed by separating the action Eq. (3) into a free quadratic part $S_0[\Phi]$ and the cubic interaction term $V[\Phi]$:

$$S_0[\Phi] = \frac{\Phi K \Phi}{2}, \quad K = \begin{pmatrix} 0 & \partial_t + \hat{L} \\ -\partial_t + \hat{L} & \hat{D}_0 \end{pmatrix},$$

$$\hat{L} = \nu_0 \Delta, \quad V[\Phi] = -\frac{1}{2} u_i'[\delta_{ik}\nabla_j + \delta_{ij}\nabla_k] u_j u_k.$$

The inverse of the matrix K is the matrix of bare propagators. The potential $V[\Phi]$ gives the interaction vertex

$$v_{ijk} = -\frac{1}{2}[\delta_{ik}\nabla_j + \delta_{ij}\nabla_k].$$

Within the model Eq. (1) the pumping power is related to the spectral power of the stirring force $d_F(k)$. For the stationary isotropic turbulence, stirred by random force f, assumed to be δ-correlated in time,

$$\langle f_i(t, \mathbf{x}) f_j(t', \mathbf{x}') \rangle = \delta(t - t') \int \frac{d\mathbf{k}}{(2\pi)^d} P_{ij}(\mathbf{k}) d_F(\mathbf{k}) e^{\imath \mathbf{k}(\mathbf{x}-\mathbf{x}')},$$

$$(4)$$

the equality of energy injection by random force f to the viscous energy dissipation per unit of mass ε, according to Kolmogorov hypotheses [5,6,22], gives

$$\varepsilon = \frac{d-1}{2(2\pi)^d} \int d\mathbf{k}\, d_F(\mathbf{k}).$$

In this paper we are concerned with the dimension $d = 3$ for isotropic homogeneous hydrodynamic turbulence.

The field theory of fields $\Phi(x) \in L^2(\mathbb{R}^{d+1})$ determined by the action functional Eq. (3) is UV divergent [14,17]. To derive quantitative predictions for correlation functions the theory should be renormalized using the standard method of ϵ-expansion, used in quantum field theory, and the theory of phase transitions [13,23]. The difference from standard quantum field theory renormalization consists in the role of ϵ: in hydrodynamic theory it turns to be the spectral parameter of the stirring force rather than the deviation from the dimension of the space-time. The choice of the correlator $\hat{D}_0(x - x')$ for the stirring force f in the Navier-Stokes Eq. (1) is a long-standing problem, having been discussed at least since Ref. [15]. Most of the papers exploiting quantum field theory approach to turbulence use stirring force of IR-type, i.e., that concentrated on large scales. This corresponds to shaking the "container with turbulence" as a whole [15], although a UV-type noise can be also introduced by "statistical filtering" procedure of Eyink [24], which separates fluctuations into large-scale and small-scale parts in a way somehow similar to the discrete wavelet transform.

The main requirements for the stirring force f are the adequate description of the large-scale behavior of the turbulence and the compatibility with the renormalization procedure.

A simple power-law choice,

$$d_F(\mathbf{k}) = D_0 |\mathbf{k}|^{4-d-2\epsilon},$$

in Ref. (4) will suffice these requirements at "realistic" value of $\epsilon = 2$, which makes the dimension of the constant D_0 equal to that of the mean energy dissipation per unit of mass ε. The correlator $d_F(\mathbf{k})$ can be generalized to $|\mathbf{k}|^{4-d-2\epsilon} h(m/k)$, where h is a certain fairly smooth function with $h(0) = 1$ [19]:

$$\langle \tilde{f}_i(t, \mathbf{k}) \tilde{f}_j(t', \mathbf{k}') \rangle = \delta(t - t') P_{ij}(\mathbf{k})(2\pi)^d \delta^d(\mathbf{k} + \mathbf{k}')$$
$$\times D_0 |\mathbf{k}|^{4-d-2\epsilon} h(m/k).$$

For convenience of ϵ-expansion the formal expansion parameter $g_0 = D_0/\nu_0^3$ is introduced. The renormalization procedure looks as follows. The original action Eq. (3), which depends on two parameters (g_0, ν_0), is declared a "bare" action, which yields divergences in the perturbation series for the velocity correlators. For realistic value of the space dimension $d = 3$, the new renormalized action,

$$S_R[\Phi] = \frac{1}{2} \int u' \hat{D} u' + \int u' [-\partial_t u + Z_\nu \nu \Delta u - (u \cdot \nabla) u],$$
(5)

is derived from the bare action by means of multiplicative renormalization,

$$\nu_0 = \nu Z_\nu, \quad D = D_0.$$

The renormalization constant Z_ν, which might be formally infinite, is chosen so that it adsorbs the divergences, emerging as poles in ϵ in the perturbation expansion of the velocity field correlator. The renormalized parameters (g, ν) are declared the *actual* parameters of the perturbation expansion, so that all poles in ϵ are subtracted from the perturbation expansion keeping its finite part intact.

The goal of renormalization procedure is to eliminate the divergences appearing in the velocity field correlators. The finite part of the renormalization constant Z_ν is not fixed by this procedure and therefore may be scheme-dependent. The most convenient is the minimal subtraction (MS) scheme [25]. To keep the renormalized coupling constant g dimensionless for an arbitrary value of ϵ an extra parameter μ of the dimension of inverse length is introduced:

$$g_0 = Z_g g \mu^{2\epsilon},$$

with the formal coupling constant renormalization defined as $Z_g := Z_\nu^{-3}$ to keep the force correlator invariant under renormalization:

$$\nu_0^3 g_0 = \nu^3 g \mu^{2\epsilon}.$$
(6)

In the space of point-defined functions $[L^2(\mathbb{R}^{d+1})]$ the only way to reveal the scale dependence of $\langle u(x) u(x') \rangle$ is to study the dependence of observed velocity correlators on $|\mathbf{x} - \mathbf{x}'|$, or alternatively on $|\mathbf{k}|$ in Fourier space. The dependence on the extra parameter of the dimension of length $(1/\mu)$ is an artifact of quantum field theoretic averaging procedure supplied with subtraction of divergences. The parameter $1/\mu$ can be qualitatively understood as the size of the domain over which the averaging is performed. But this is a qualitative consideration based on similarity of the roles played by the noise dispersion in chaotic systems and the Planck constant in quantum field theory models [14].

To get more physical insight into the problem, we need to use the Kolmogorov self-similarity ideas: the turbulence measured at different scales looks more or less similar. The space of square-integrable point-defined functions is too weak to encompass enough details required for more rigorous mathematical consideration of self-similarity properties.

At the assumptions on the stirring force mentioned above, the renormalized action Eq. (5) is constructed using a single counterterm, resulting in viscosity renormalization Z_ν. The hydrodynamic field theory S_R thus has two "charges," g and ν, the evolution of which with the normalization scale μ is determined by a single renormalization constant Z_ν. In one loop approximation its value is [16]

$$Z_\nu = 1 - \frac{ag}{2\epsilon} + O(g^2),$$
(7)

with $a = \frac{1}{20\pi^2}$ in $d = 3$ dimension. The β-function, which determines the evolution of the coupling g with the change of scale μ, is derived from the equality Eq. (6),

$$\beta(g) = \mu \frac{\partial}{\partial \mu} \Big|_{\nu_0, g_0} g = g(-2\epsilon + 3\gamma_\nu),$$
(8)

where $\gamma_\nu := \mu \frac{\partial}{\partial \mu} \ln Z_\nu$. In one-loop approximation Eq. (7) the β-function,

$$\beta(g) = -2\epsilon g + 3ag^2,$$
(9)

has a IR-stable fixed point $g_* = \frac{2\epsilon}{3a}$, which determines the properties of turbulence in large-scale asymptotics.

Since the fields (u, u') are not renormalized, any renormalized n-point correlator of velocity field W_R^n is invariant under RG transform:

$$\left[\mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} - \gamma_\nu(g) \nu \frac{\partial}{\partial \nu} \right] W_R^n = 0.$$
(10)

This means, the statistical momenta can depend only on invariant charges $\bar{g}, \bar{\nu}$ – the first integrals of the RG Eq. (10) normalized so that $\bar{g}(s = 1, g) = g$ at the normalization scale, $s = k/\mu$.

The dependence of the invariant charge $\bar{g}$ on the invariant scale $s = k/\mu$ is implicitly given by the integral of the inverse β-function:

$$\ln s = \int_g^{\bar{g}} \frac{dx}{\beta(x)}.$$
(11)

The invariant viscosity $\bar{\nu} = \bar{\nu}(s, g)$ is the second invariant of RG Eq. (10):

$$\bar{\nu} = \nu \exp \left[\int_{\bar{g}}^g \gamma_\nu(x) \frac{dx}{\beta(x)} \right] = \left(\frac{g \nu^3}{\bar{g} s^{2\epsilon}} \right)^{1/3} = \left(\frac{g_0 \nu_0^3}{\bar{g} k^{2\epsilon}} \right)^{1/3}.$$

At the presence of the IR-stable fixed point $\beta(g_*) = 0$ in Eq. (9), the turbulence behavior at large scales is determined by the value of invariant viscosity at fixed point g_*:

$$\bar{\nu}_*(k) = \nu_0 \left(\frac{g_0}{g_*} \right)^{1/3} k^{-\frac{2\epsilon}{3}}.$$
(12)

The pair correlator of velocity field $C = \langle uu \rangle$ that satisfies RG Eq. (10) has the form,

$$C = \bar{v}^2 k^{-d} R(\bar{g}, \bar{z}),$$

where $R(\cdot)$ is some function of the invariant coupling constant $\bar{g}$ and the invariant frequency $\bar{z} = \frac{\omega}{\bar{v}k^2}$. (More details and the incorporation of IR scale parameter m into consideration can be found in Ref. [19].) The equal-time pair correlator is obtained by integrating C over the frequency argument:

$$C_{st} = \int C \frac{d\omega}{2\pi} = \bar{v}^2 k^{2-d} R(\bar{g}). \tag{13}$$

The substitution of the viscosity Eq. (12) into Eq. (13) at $\epsilon = 2, d = 3$ yields the IR asymptotics of the Kolmogorov type: $C_{st} \propto k^{-\frac{11}{3}}$. Further discussion on anomalous scaling, different from the Kolmogorov regime, the effects of anisotropy, compressibility, and finite correlation time effects can be found, e.g., in Ref. [26].

III. MULTISCALE THEORY OF TURBULENCE IN WAVELET BASIS

Kolmogorov (K41) hypotheses [5] assume statistically homogeneous and isotropic turbulence. This justifies the evaluation of velocity field correlations in wave-number space, but does not provide any rigorous mathematical definition of the "fluctuation of scale a." They are tacitly assumed in the literature as Fourier components of velocity field with wave numbers equal to the inverse scale: $|\mathbf{k}| \approx \frac{2\pi}{a}$. Such nonlocal definition meets global characteristics of the homogeneous isotropic turbulence, but is hardly applicable to nonlinear phenomena such as coherent structure formation.

To analyze the *local* properties of turbulent velocity field at a given scale a, same as in quantum field theory [27,28], the wavelet decomposition $\mathbf{u}(t, \mathbf{x}, \cdot) \to \mathbf{u}_a(t, \mathbf{x}, \cdot)$ was performed by many authors, see, e.g., Refs. [11,29–33]. Among those, the wavelet transform was applied to the iterative solution of the stochastic Navier-Stokes equation [34].

To perform wavelet decomposition of the velocity field we need some aperture function $g(\mathbf{x}) \in \mathrm{L}^2(\mathbb{R}^d)$, called a *basic wavelet*, which satisfies an admissibility condition,

$$C_g = \int_0^\infty |\tilde{g}(a)|^2 \frac{da}{a} < \infty, \tag{14}$$

so that the original ("no-scale") field $u(t, \mathbf{x}, \cdot)$ can be reconstructed from the set of its wavelet coefficients $u_a(\mathbf{b}, \cdot)$:

$$u(\mathbf{x}, \cdot) = \frac{1}{C_g} \int_0^\infty \frac{da}{a} \int_{\mathbb{R}^d} \frac{1}{a^d} g\left(\frac{\mathbf{x} - \mathbf{b}}{a}\right) u_a(\mathbf{b}, \cdot) d^d\mathbf{b},$$

$$u_a(\mathbf{b}, \cdot) = \int_{\mathbb{R}^d} \frac{1}{a^d} \bar{g}\left(\frac{\mathbf{x} - \mathbf{b}}{a}\right) u(\mathbf{x}, \cdot) d^d\mathbf{x}. \tag{15}$$

The wavelet coefficients $u_a(\mathbf{b}, \cdot)$ can be considered as the scale components of the velocity field u measured with the aperture function g. To keep the fields u and u_a the same physical dimension the L^1 norm is used in wavelet transform Eq. (15) instead of the traditional L^2 norm [27,35].

In contrast to "statistical filtering" procedure of Ref. [24], which is also given by convolution with a filtering function $G_l(x) := l^{-d} G(x/l)$, the continuous wavelet transform

Eq. (15) is *invertible*. Usage of basic wavelet that satisfies the admissibility condition Eq. (14) makes our theory significantly different from "statistical filtering." The difference is briefly as follows. The statistical filtering operator G_l projects the velocity field onto the space of functions $\mathcal{F}_k$ with wave vectors less or equal to a given value $k \sim 1/l$. Thus, G_l is a low-pass filter:

$$v_l(x) := G_l * v(x).$$

The projection $\mathrm{L}^2(\mathbb{R}^d) \xrightarrow{G_l} \mathcal{F}_k$ is a homomorphism. The details lost by this projection are given by the high-pass filter H_l, so that

$$\hat{G}_l(k) + \hat{H}_l(k) = 1.$$

Statistical filtering applies the low-pass and high-pass filters only once, and then treat large-scale and small-scale modes [24]. The Kadanoff blocking procedure [36] applies it sequentially to coarser and coarser scale, each time increasing the size of the block by an integer factor and loosing some details on each step. The renormalization group can do it gradually, integrating over the difference space,

$$D_{k,\Delta k} := \mathcal{F}_k \setminus \mathcal{F}_{k-\Delta k}, \tag{16}$$

on each step. Since $\ldots \mathcal{F}_{k-2\Delta k} \subset \mathcal{F}_{k-\Delta k} \subset \mathcal{F}_k$, the spaces Eq. (16) allow for an evident decomposition,

$$\mathcal{F}_k = D_{k,\Delta k} \oplus D_{k-\Delta k,\Delta k} \oplus D_{k-2\Delta k,\Delta k} \oplus \ldots . \tag{17}$$

To study the behavior of a function on a ladder of scales $k, k - \Delta k, k - 2\Delta k, \ldots$ it is sufficient to project it onto a set of the difference subspaces Eq. (17), with no need to keep the whole set $\{\mathcal{F}_k\}_k$. The decomposition Eq. (17) is exactly what wavelet transform does, if discretized in an orthogonal basis; see, e.g, Ref. [37]. So, in our approach we separately treat the fluctuations concentrated *near* each given scale $\{u_l(x)\}_l$, rather than all fluctuations concentrated above the given scale l, as G.Eyink does. Thus integrating from external size of the system L down to the observation scale A we can reconstruct velocity field in the sense of Eq. (17). No need to say that the Gaussian filtering cannot be used for wavelet decomposition for it does not satisfy the admissibility condition Eq. (14), and therefore the original function $v(x)$ cannot be uniquely reconstructed from the set of its coefficients $v_l(x)$.

Referring the reader to general textbooks in wavelet analysis [37,38] for more details on the continuous wavelet transform Eq. (15), we assume for simplicity the basic wavelet $g(\mathbf{x})$ to be isotropic function of $\mathbf{x}$, having fairly good localization properties; it may be a derivative of Gaussian, for instance, Refs. [4,11]. In Fourier space the convolution becomes a product: $\tilde{u}_a(\mathbf{k}) = \bar{\tilde{g}}(a\mathbf{k})\tilde{u}(\mathbf{k})$.

The stirring force can be represented by its scale components f_a, Gaussian random functions with zero mean, concentrated at a fixed large scale L. The correlator of the stirring force scale components can be taken in the form

$$\langle \tilde{f}_{ai}(t, \mathbf{k}) \tilde{f}_{a'j}(t', \mathbf{k}') \rangle = \delta(t - t') P_{ij}(\mathbf{k}) g_0 v_0^3 C_g (2\pi)^d$$
$$\times \delta^d(\mathbf{k} + \mathbf{k}') a\delta(a - a')\delta(a - L), \tag{18}$$

where g_0 is formal dimensionless strength of forcing. The random force Eq. (18) in our approach simulates random initial and boundary conditions, and also the effect of external forces injecting energy from large scales comparable to the size of the turbulent domain. The delta-function-type correlator is of course an approximation that enables analytical calculations of the diagram expansion. A random force concentrated on a narrow range of gross scales would be more realistic, but is hard for analytical calculations. We follow the Kolmogorov-Richardson scenario of turbulence development: the kinetic energy injected at large scale by external forcing is transferred to smaller and smaller scales until it reaches the scale l, where it is dissipated by viscosity. We do not consider the effect of small scales close to Kolmogorov dissipative length l on the dynamics of larger eddies. In this respect there is a difference from the approach [24], where the random force is split into a large-scale part $\bar{f}$ and the high-frequency noise acting on molecular scales.

In our model the molecular noise and the subgrid effects below the observation scale A are neglected for they do not seriously affect the large-scale motion. The reason is that our consideration is concerned with a homogeneous isotropic stationary turbulence with no parity violation, i.e., with the mean helicity assumed to be zero $\langle \mathbf{u} \cdot \text{curl } \mathbf{u} \rangle = 0$. We have only two inviscid invariants: the kinetic energy and the helicity. The inverse energy cascade can be induced by the presence of extra topological invariant—the conservation of enstrophy the $Z = \frac{1}{2} \int \text{curl}^2 \mathbf{u}$ [2,4]. In a fully three-dimensional turbulence the enstrophy is not conserved and the inverse energy cascade is not significant in the inertial range of scales. In our simple model we consider isotropic basic wavelets. The quasi-two-dimensional turbulence, with one dimension being much less than the others, causing inverse energy cascade [39], can be hardly handled analytically in our framework. Our consideration may be not true for helical turbulence, but this is not the subject of this paper, keeping it for future research.

The basic objects of our model are the correlation functions of the of the velocity field scale components $\langle \Phi_{a_1}(x_1) \dots \Phi_{a_n}(x_n) \rangle$. They can be evaluated from the generating functional,

$$G[A] = e^{W[A]} = \int \mathcal{D} \Phi_a(x) e^{S[\Phi_a] + \int \frac{dxda}{a} A_a(x) \Phi_a(x)}, \quad (19)$$

which is different from its classical counterpart Eq. (2) only by making the integration measure $dx \equiv dt d^d \mathbf{x}$ into $dt \frac{d^d \mathbf{x} da}{a}$ for each space-time argument, and substituting the interaction vertex v_{ijk} by its wavelet transform. The substitution of wavelet transform Eq. (15) into the action functional Eq. (3) yields the action functional of the scale-dependent fields,

$$S[\Phi_a] = \frac{1}{2} \int \frac{dxda}{a} \frac{dx'da'}{a'} u'_a(x) D_{aa'}(x - x') u'_{a'}(x')$$
$$+ \int \frac{dxda}{a} u'_a(x)[-\partial_t u_a(x) + v_0 \Delta u_a(x) + V_a[u]], \quad (20)$$

where $V_a[u]$ is an integral nonlinear operator, obtained by wavelet transform of the cubic interaction term $u'u\nabla u$.

We use the first derivative of the Gaussian as a basic wavelet g. The equality between the energy injection and the energy dissipation then defines the bare coupling constant g_0:

$$\varepsilon = \frac{g_0 v_0^3}{L^4} \frac{3}{8\pi^{3/2}}. \quad (21)$$

See Appendix A for details.

The functional derivatives are taken with respect to the formal source $A_a(t, \mathbf{x})$:

$$\langle \Phi_{a_1}(x_1) \dots \Phi_{a_n}(x_n) \rangle_c = \left. \frac{\delta^n W[A]}{\delta A_{a_1}(x_1) \dots \delta A_{a_n}(x_n)} \right|_{A=0}.$$

Integration over all scale arguments $\int_0^\infty \frac{da_i}{a_i}$ would certainly drive us back to the known divergent description of fully developed turbulence, in case we substitute the force correlator Eq. (18) by wavelet image of a wide-band correlator power-law correlator [40].

Taking into account that statistical properties of fully developed turbulence are determined by the energy flux from large scales to small scales, we apply the following rule for the calculation of any Feynman graph for the correlation functions $\langle \Phi_{a_1}(x_1) \dots \Phi_{a_n}(x_n) \rangle$. Let $A = \min(a_1, \dots, a_n)$, then the integration in all internal lines is to be performed within the range $\int_A^\infty \frac{da_i}{a_i}$. The theory defined in this way is finite by construction [27,28,41]. In contrast to standard means of regularization, such as introduction of cutoff momenta, our method provides an exact conservation of momentum in each vertex.

IV. ONE-LOOP CORRECTIONS TO VISCOSITY

Bare Green functions of the field theory Eq. (2) are determined by the linear part of the Navier-Stokes operator. In multiscale theory the bare response function between the scales α and β is obtained by multiplication of ordinary one by the wavelet factors:

$$G_{\alpha,i,\beta,j}^{(0)}(k) = \frac{\tilde{g}(\alpha \mathbf{k}) \overline{\tilde{g}}(\beta \mathbf{k})}{-\iota k_0 + v_0 \mathbf{k}^2} P_{ij}(\mathbf{k}).$$

One loop contribution to this Green function is graphically shown in Fig. 1. For the simplicity of calculations, same as in Ref. [27], we have chosen the first derivative of the Gaussian as the basic wavelet. Its shape naturally resembles a localized wave. Its Fourier transform is $\tilde{g}_1(k) = -\iota k e^{-\frac{k^2}{2}}$, with $C_{g_1} = \frac{1}{2}$.
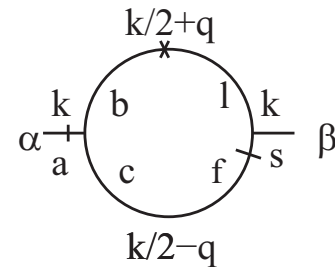

FIG. 1. One-loop contribution to the Green function. Primed lines denote the u' fields incident to the vertices. Crossed line denotes the stirring force correlators. Latin letters indicate vector indices, Greek letters stand for the scale arguments of external lines.

Due to the limited range of integration over the scale variables $\int_A^\infty \frac{da_i}{a_i}$, bounded from below, all internal lines will contribute a cutoff factor $f_g^2(Ak)$, where k is the momentum of the line, and $f_g(x) = \frac{1}{C_g} \int_x^\infty \frac{|\tilde{g}(a)|^2}{a} da$; for g_1 wavelet cutoff factor is $f_{g_1}(x) = e^{-x^2}$; see Refs. [27,28] for details.

To calculate one-loop contribution to viscosity coming from large-scale turbulent pulsations we consider 1PI diagrams for the two-point vertex function $\Gamma^{(2)}$:

$$\Gamma^{(2)} = \Gamma^{(2)}_{\text{bare}} + \Sigma_{\alpha\beta}, \tag{22}$$

where $-\Sigma_{\alpha\beta}$ is the value of the diagram shown in Fig. 1. The explicit equation for the "self-energy" $\Sigma_{\alpha\beta}(k)$ is (the details are presented in the Appendix C):

$$\Sigma_{as} = -\,v_0 g_0 k L \int_0^\infty \frac{y^2 dy}{(2\pi)^2} e^{-(kL)^2(1+4\xi^2)(\frac{1}{4}+y^2)}$$

$$\times \int_0^\pi d\theta \sin\theta L_{as}(k, p^+, p^-) \frac{e^{-(kL)^2 y \cos\theta}}{\frac{1}{4} + y^2 - \iota \frac{\omega}{2v_0 \mathbf{k}^2}}, \tag{23}$$

where we have introduced a dimensionless scale $\xi = \frac{A}{L}$ and the dimensionless momentum $\mathbf{y} = \frac{\mathbf{q}}{|\mathbf{k}|}$ for integration in $\mathbb{R}^3$. The one-loop tensor structure of the diagram shown in Fig. 1 is

$$L_{as}(k, p^+, p^-) = \frac{\delta_{as}}{4} \left[\frac{(p^+ k)(p^+ p^-)}{p^{+2}} - (kp^-) \right]$$

$$+ \frac{p_a^+ p_s^-}{2} \left[\frac{(p^+ k)}{p^{+2}} - \frac{(p^- k)(p^+ p^-)}{p^{-2} p^{+2}} \right]$$

$$+ p_a^- p_s^- \left[\frac{(kp^-)}{p^{-2}} - \frac{(p^+ k)(p^+ p^-)}{2p^{-2} p^{+2}} \right]$$

$$- \frac{k_a p_s^-}{2} + p_a^+ k_s \frac{p^+ p^-}{4p^{+2}} - \frac{p_a^- k_s}{4}, \tag{24}$$

with $p^\pm = \frac{k}{2} \pm q$, and all scalar products taken in $\mathbb{R}^3$. For the isotropic turbulence, the tensor structure of the "self-energy" diagram Eq. (23) may depend only on the direction of vector $\mathbf{k}$. It can be written in the form

$$\Sigma_{as} = v_0 g_0 \Sigma^\delta k^2 \left(\delta_{as} - \frac{k_a k_s}{k^2} \right) + v_0 g_0 \Sigma^\lambda k_a k_s. \tag{25}$$

After standard algebraic manipulations this gives

$$\Sigma^\delta = \frac{kL}{128 C_g} \int_0^\infty \frac{y^2 dy}{(2\pi)^2} \frac{e^{-(kL)^2(1+4\xi^2)(\frac{1}{4}+y^2)}}{\frac{1}{4} + y^2 - \iota \frac{\omega}{2v_0 \mathbf{k}^2}} \int_{-1}^1 d\mu$$

$$\times \frac{(1-\mu^2)(8\mu^2 y^2 + \mu(8y^3 - 10y) + 4y^2 + 1)}{(\frac{1}{4y} + y - \mu)(\frac{1}{4y} + y + \mu)}$$

$$\times e^{-(kL)^2 y \mu},$$

where $\mu = \cos\theta$, with θ being the polar angle between $\mathbf{k}$ and $\mathbf{q}$.

In our model the "self-energy" contribution to viscosity is finite by construction. It determines the relation between the viscosity measured at certain reference scale, and the viscosity at the observation scale A. Following the Kolmogorov-Richardson scenario of turbulence development [42], we sum up all fluctuations from the large stirring scale L up to the measuring scale A, where $\xi = A/L \ll 1$, but still much above the Kolmogorov scale $A \gg l = (v^3/\varepsilon)^{1/4}$. Similar settings were used for renormalization group studies of the general case of helical turbulence, where the dependence of energy spectra on observation scale was shown to be significant only on the edges of of the energy-containing range by means of affecting the stability of RG fixed points [43,44].

Measuring all lengths in units of L we rewrite renormalization of viscosity in the form

$$\nu(\xi) = \nu_L Z_\nu, \quad Z_\nu = 1 - g_L \Sigma^\delta(\xi). \tag{26}$$

Equation (26) works fairly well if the difference between the observation scale A and the stirring scale L is not too big, otherwise we need to solve RG equations to determine g_L and ν_L as functions of the microscopic parameters g_0 and ν_0.

The RG equations may be obtained by iterating Eq. (26) over the set of scales,

$$1 = \xi_L > \xi_{L-1} > \xi_{L-2} > \ldots > \xi_0 = \frac{l}{L},$$

$$\xi_k = \xi_0 \delta^k, \quad \delta > 1.$$

In continuous limit this leads to the RG equation,

$$\frac{d \ln \nu}{d \ln \xi} = g(\xi) \frac{\Sigma(\xi)}{\ln \delta}, \tag{27}$$

see Appendix E 1 for the derivation.

Evolution of the formal coupling constant $g(\xi)$ is determined by the scale corrections to the stirring force correlator,

$$D(\xi) = \frac{g(\xi)\nu^3(\xi)}{L}. $$

In one-loop approximation, the renormalization of stirring force correlator is given by

$$\frac{d \ln D}{d \ln \xi} = -\frac{K(\xi)}{\ln \delta}, \tag{28}$$

where $K(\xi)$ is one-loop contribution to the stirring force correlator, see Appendix D for details. Making use of RG Eqs. (27) and (28), and since $\ln g(\xi) = \ln D(\xi) - \ln L - 3 \ln \nu(\xi)$, we get the RG equation for the formal coupling constant $g(\xi)$:

$$\frac{d \ln g}{d \ln \xi} = -\frac{K(\xi)}{\ln \delta} - 3g(\xi) \frac{\Sigma(\xi)}{\ln \delta}, \tag{29}$$

which has the solution

$$g(\xi) = \frac{g_L e^{\int_\xi^1 \frac{d\eta}{\eta} K(\eta)}}{1 - 3g_L \int_\xi^1 \frac{d\xi'}{\xi'} \Sigma(\xi') e^{\int_{\xi'}^1 \frac{d\eta}{\eta} K(\eta)}}, \tag{30}$$

where we have set $\ln \delta = 1$.

In view of $K(\xi) \ll 1$, the value of $D(\xi)$ is utmost scale-invariant and we can use the equality,

$$g(\xi)\nu^3(\xi) = g_0 \nu_0^3 = g_L \nu_L^3,$$

to evaluate $\nu(\xi)$ for the known values of $g(\xi)$.

The solution of the RG Eq. (27) in the IR region, calculated for a fixed normalization momentum $x_* = 4\pi$, is presented in Fig. 2. The renormalized viscosity $\nu_\xi(k)$ is a counterpart of renormalized viscosity in the action of ordinary theory (5). The difference is that $\nu_\xi(k)$ is taken not in IR-stable
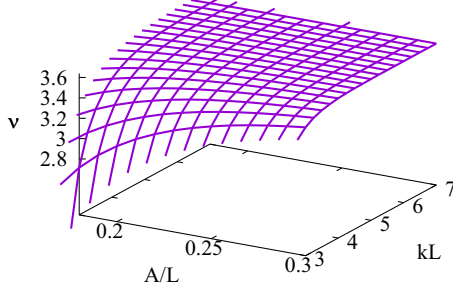
FIG. 2. The dependence of turbulent viscosity $\nu_A(k)$ on the observation scale $\xi = A/L$ and the dimensionless momentum $x = kL$. Calculations were performed in the IR region using the equation (30) at the assumption of invariant $g_A\nu_A^3 = g_L\nu_L^3$, i.e., small $K(\xi)$, and normalization momentum $x_* = 4\pi$ used to evaluate g_L, in accordance to $k = 2$ energy injection limit in simulated turbulence from John Hopkins Turbulence Database.

fixed point, and therefore describes the asymptotic behavior of large-scale eddies, but is taken at a fixed observation scale $A = \xi L$. In terms of the scale-dependent action $S[u_a, u'_a]$ $\nu_\xi(k)$ can be understood as a viscosity acting on the wavelet-type pulsations of the velocity field $u_a(x)$ measured at scale A. The wave vector k of such perturbations can take arbitrary values.

In this paper we neither aim to construct a turbulent stress tensor, as is presented in statistical filtering theory [24] for the wave vectors less or equal to $1/a$, nor we construct statistical closures for scale-dependent fields $\mathbf{u}_a(x)$ for it would result in integral equations. Instead, since what is really measured in turbulence are the n-point correlation functions, we consider the Fourier transform of such functions $\langle \tilde{\mathbf{u}}_{a_1}(k_1) \ldots \tilde{\mathbf{u}}_{a_n}(k_n) \rangle$,
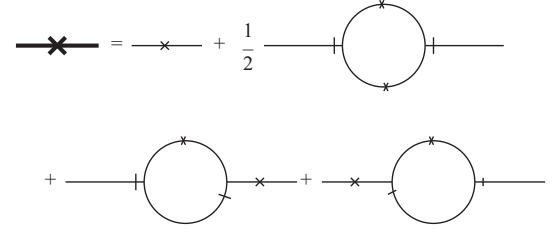

FIG. 3. Feynman diagrams used for evaluation of the pair correlator of velocity field.

where the wave vectors k_i are responsible for the *separation* between the observation points, while the scale arguments a_i are responsible for the observation scales.

V. ENERGY SPECTRA

The full kinetic energy of homogeneous isotropic turbulence can be expressed in terms of scale components of the velocity field: $E = \frac{1}{2} \int \langle |\tilde{\mathbf{u}}_a(\mathbf{k})|^2 \rangle \frac{d^d k}{(2\pi)^d} \frac{1}{C_g} \frac{da}{a}$. We evaluate the equal-time pair correlator $C(k, \xi) = \langle \tilde{u}_A(t, \mathbf{k}) \tilde{u}_A(t, -\mathbf{k}) \rangle$ of velocity field wavelet coefficients according to the diagram rule shown in Fig. 3. The first term, the bare correlator integrated over the frequency and two internal scales, is equal to

$$C^{(0)} = \frac{1}{\nu_0 k^2} \frac{g_0 \nu_0^3}{L} (Lk)^2 e^{-(Lk)^2} |\tilde{g}(Ak)|^2 P_{ik}(\mathbf{k}). \quad (31)$$

The next contribution comes from the symmetric diagram in first line of Fig. 3, integrated over the frequency arguments. This gives

$$C^{(2)} = \frac{1}{2} \left(\frac{g_0 \nu_0^3}{2C_g} \right)^2 \frac{L^2 k^3}{16 \nu_0^3} |\tilde{g}(Ak)|^2 f^2(Ak) P_{ik}(\mathbf{k}) \int \frac{y^2 dy}{(2\pi)^2} \frac{e^{-2(Lk)^2(1+2\xi^2)(\frac{1}{4}+y^2)}(\frac{1}{4}+y^2)}{4(\frac{1}{4}+y^2)\nu_0 k^2(1+\frac{1}{2}(\frac{1}{4}+y^2))} \int d\mu \frac{(1-\mu^2)(8\mu^2 y^2 + 4y^2 + 1)}{(\frac{1}{4}+y^2-y\mu)(\frac{1}{4}+y^2+y\mu)}, \quad (32)$$

where $f(Ak) = e^{-(Ak)^2}$ are wavelet filters in the legs of diagram. $1/2$ before the whole equation is a topological factor.

Two last diagrams in Fig. 3 contribute equally to the correlation function. Their joint contribution is

$$2C^{(1)} = -2P_{ik}(\mathbf{k})|\tilde{g}(Ak)|^2 f^2(Ak) \frac{g_0 \nu_0^3}{C_g L} |\tilde{g}(Lk)|^2 \frac{g_0 kL}{128 C_g} \int \frac{y^4 dy}{(2\pi)^2} \frac{e^{-(Lk)^2(1+4\xi^2)(\frac{1}{4}+y^2)}}{2(\nu_0 k^2)^2[1+2(\frac{1}{4}+y^2)]}$$

$$\times \int d\mu \frac{(1-\mu^2)(8\mu^2 y^2 + 2\mu y(4y^2 - 5) + 4y^2 + 1)}{(\frac{1}{4}+y^2-y\mu)(\frac{1}{4}+y^2+y\mu)} e^{-(kL)^2 y\mu}. \quad (33)$$

The common sign minus stands for the fact Σ^δ is equal to minus diagram.

The final equation, without common wavelet factor $\tilde{g}(kA)\bar{\tilde{g}}(kA)$ on the legs of each diagram, calculated with g_1 wavelet is given by

$$C(k, \xi) = \frac{g_0 \nu_0^3}{\nu_A(k)} L e^{-(Lk)^2} + \frac{(g_0 \nu_0^3)^2}{128} \frac{(Lk)L}{\nu_A^4(k)} e^{-2\xi^2(Lk)^2} \int_0^\infty \frac{y^2 dy}{(2\pi)^2} \frac{e^{-2(kL)^2(1+2\xi^2)(\frac{1}{4}+y^2)}}{1+\frac{1}{2}(\frac{1}{4}+y^2)} \int_{-1}^1 d\mu \frac{(1-\mu^2)(8\mu^2 y^2 + 4y^2 + 1)}{(\frac{1}{4}+y^2-y\mu)(\frac{1}{4}+y^2+y\mu)}$$

$$+ \frac{(g_0 \nu_0^3)^2}{32} \frac{(Lk)L}{\nu_A^4(k)} e^{-(Lk)^2(1+2\xi^2)} \int_0^\infty \frac{y^4 dy}{(2\pi)^2} \frac{e^{-(kL)^2(1+4\xi^2)(\frac{1}{4}+y^2)}}{1+2(\frac{1}{4}+y^2)}$$

$$\times \int_{-1}^1 d\mu \frac{(\mu^2 - 1)(8\mu^2 y^2 + 2\mu y(4y^2 - 5) + 4y^2 + 1)}{(\frac{1}{4}+y^2-y\mu)(\frac{1}{4}+y^2+y\mu)} e^{-(kL)^2 y\mu}. \quad (34)$$

The coupling g_0 is related to the mean energy dissipation rate per unit of mass ε by the energy balance equation $\varepsilon L^4 = g_0 v_0^3 \chi[g]$, where $\chi[g]$ is numeric factor, which depends on the shape of basic wavelet Eq. (21). For the g_1 wavelet used in this paper $\chi[g_1] = \frac{3}{8\pi^{3/2}}$.

The obtained function Eq. (34) can be used to study the dependence of the turbulent pulsations energy spectrum $E(x, \xi) = 4\pi k^2 C(k, \xi)$, which is assumed to be Kolmogorov spectrum, if there is no dependence on the observation scale $\xi = A/L$.

We have compared our results with the 1024^3 grid simulations of homogeneous isotropic turbulence presented in John Hopkins Turbulence Database (JHTDB) and described, e.g., in Ref. [45], with the following parameters of simulation: cubic domain of size $L = 2\pi$, Kolmogorov length $l = 0.0028$, dissipation rate $\varepsilon = 0.103$, viscosity $v_0 = 0.000185$.

In traditional theory of turbulence the dependence of results of measurements, *viz.* the fields $v_\delta := \frac{1}{\delta^d} \int v(x) d^d x$ and their statistical momenta, on the averaging scale δ is usually considered as an artifact of inappropriate choice: if δ is too small, say is close to the mean free path, we may get out of applicability of hydrodynamic approximation; alternatively, if it is too large, that is of the same order as the system size L, we get out of the limits of the Kolmogorov theory. Thus, the "legitimate" choice of observation scale lies deeply inside the Kolmogorov range of scales: $l \ll \delta \ll L$. Experimental processing of turbulence data stepped a little further when the wavelet transform was used to study turbulence behavior in (k, a) plane [32]. The study of distributions in (k, a) plane gives more information than that in k only: the window width (a) for each mode (k) may tell whether this mode originates from the small or from the large-scale dynamics.

In the limit of large observation scales $a \lesssim L$ there is no need for pulsations $u_a(k)$ to obey Kolmogorov's laws. The energy of such pulsations decreases with the increase of resolution $a \to L$. The analytic tools, based on continuous wavelet transform, we propose in this paper may be useful in analytical computations of correlations of velocity pulsations measured at different spatial resolution.

As we can see from Fig. 4, the slope of the curves $E(k, \xi)$ depends on the observation scale ξ. The curves $A = 0.1, 0.2$, corresponding to small observation scales, i.e., those more than an order of magnitude less than external scale L, have the slopes close to the Kolmogorov $k^{-\frac{5}{3}}$ regime. In contrast, the larger observation window $A = 0.5, 1.0$, i.e., only one order less than L, results in a steeper falloff of the energy curves. Same thing happens with the dependence of energy on the dimensionless wave vector $x = kL$. Our analyzing wavelet $\tilde{g}(ak) \sim (ak)e^{-\frac{(ak)^2}{2}}$ is mostly sensitive to the wave numbers $k \sim \frac{1}{a}$, hence the observation scale $A \sim 0.1$ results in dimensionless wavevectors of the order $x \sim \frac{2\pi}{0.1} \approx 63$, and similar for other curves, which qualitatively agrees with that observed in Fig. 4.

The spectral index is close to the Kolmogorov value $-\frac{5}{3}$ for the values of observation scale in the middle of the inertial range $\xi_0 \ll \xi \ll 1$, but becomes steeper when approaching the dissipation scale ξ_0. Since our correlation functions Eq. (34) represent only partial energies of the given scale $\xi = A/L$ fluctuations, only the slopes can be compared.
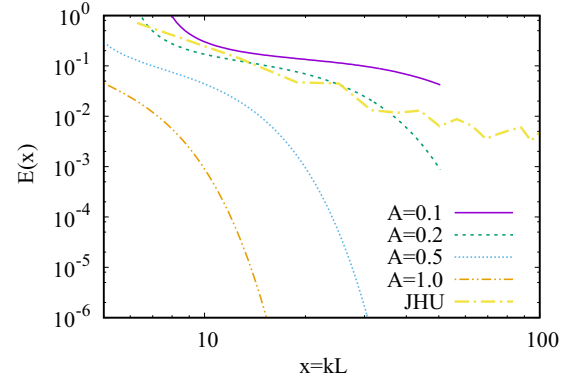

FIG. 4. One-dimensional energy spectra. Calculated for the set of parameters of the John Hopkins Turbulence Database for the isotropic turbulence grid simulations. The curves show the spectra for observation scales $A = 0.1, 0.2, 0.5, 1.0$. The $E(k)$ numerical energy spectrum, obtained for the 128^3 decimation of the original JHTDB 1024^3 data, is shown by dashed line. The normalization scale $x_* = 4\pi$ corresponds to the $k = 2$ energy injection limit. All quantities are normalized to $(2\pi)^3$ volume.

The integral over all scales should give a "no-scale" energy spectrum. Since we are interested in the dependence of the energy spectra on the observation scale ξ in Fig. 4 we present the graphs of such spectra and the standard "no-scale" spectrum (shown in dashed line) obtained from numerical simulations [46].

VI. CONCLUSION

We conclude, that using the multiscale representation of fields in field-theoretic calculations of turbulent velocity correlations we can obtain more information on the statistics of turbulent pulsations, than by standard spectral methods. Presented formalism is more relevant to experimental studies: any measured statistics of turbulent pulsations is always obtained at certain observation scale by averaging velocity fluctuations over the measuring volume. This volume should be somehow taken into account by theoretical description of turbulence.

It should be noted that our method of renormalization, acting from larger scales to smaller scales, ignores any effects of inverse energy cascade. This is significant simplification. The authors think, however, that a rigorous approach to the effect of small-scale fluctuations on large-scale fluctuations, which remains a chalenging problem, would involve more complicated description than just the forced Navier-Stokes equation itself. First of all the parity breaking effects and the inviscid topological invariants should be taken into account [44]. The complexity of the problem can be seen for instance from the recent paper [47]. We can hardly foresee such theory of an incompressible fluid flow capable of analytic calculations in the nearest future. The incompressibility itself is also a simplification. Wavelets might be of some use here as well, but up to the best authors knowledge their use in studying inverse energy cascade is limited to numerical simulation [48].

From experimental point of view the dependence of the energy spectra on the observation scale A becomes important when that scale approaches the Kolmogorov dissipative

length l. In this limit the really observed steepness of the energy spectra should significantly exceed the Kolmogorov value of $-5/3$. In the opposite case, if the observation scale A approaches the system size L, the steepness also increases for a large-scale aperture can hardly resolve the energy-containing range fluctuations. If the observation scale belongs to the inertial range, the steepness is utmost equal to the Kolmogorov value, and does not significantly depend on scale [49].

ACKNOWLEDGMENTS

APPENDIX A: LARGE-SCALE STIRRING FORCE

According to the Kolmogorov hypotheses, the energy injection should be equal to viscous energy dissipation per unit of mass. For stationary turbulence, to justify that the rate of energy dissipation per unit of mass,

$$\varepsilon = \frac{d}{dt} \frac{\langle \mathbf{u}^2(t, \mathbf{x}) \rangle}{2} = \langle \dot{u}_i(t, \mathbf{x}) u_i(t, \mathbf{x}) \rangle, \qquad \text{(A1)}$$

is compensated by the stirring force f, we chose the random force correlator by defining the correlation function of its scale components:

$$\langle \tilde{f}_{ai}(t, \mathbf{k}) \tilde{f}_{a'j}(t', \mathbf{k}') \rangle$$
$$= \delta(t - t') P_{ij}(\mathbf{k}) g_0 v_0^3 C_g (2\pi)^d \delta^d(\mathbf{k} + \mathbf{k}') a \delta(a - a') \delta(a - L),$$

where g_0 is formal dimensionless strength of forcing, to be used for the perturbation expansion [19]. The δ functions in scale arguments ensure that fluctuations of different scales a and a' are uncorrelated, and the work is exerted over the fluid only at large scale $a = L$.

Reconstruction of Eulerian velocities u_i from their scale components u_{ai} by means of inverse wavelet transform gives

$$\langle \dot{u}_i(t, \mathbf{x}) u_i(t, \mathbf{x}) \rangle$$
$$= \frac{1}{C_g^2} \int \exp[\iota \mathbf{x}(\mathbf{k}_1 + \mathbf{k}_2)] \tilde{g}(a_1 \mathbf{k}_1) \tilde{g}(a_2 \mathbf{k}_2) \langle \tilde{f}_{a_1 i}(t, \mathbf{k}_1)$$
$$\times \int^t \tilde{f}_{a_2 i}(\tau, \mathbf{k}_2) d\tau \rangle \frac{d^d \mathbf{k}_1}{(2\pi)^d} \frac{da_1}{a_1} \frac{d^d \mathbf{k}_2}{(2\pi)^d} \frac{da_2}{a_2}$$
$$= \frac{g_0 v_0^3}{L^{d+1}} \frac{(d-1)}{2 C_g} \int_0^\infty |\tilde{g}(\mathbf{y})|^2 \frac{d^d \mathbf{y}}{(2\pi)^d}, \qquad \text{(A2)}$$

where factor $(d - 1)$ comes from the trace of orthogonal projector, and the factor $\frac{1}{2}$ is the value of the θ function at the discontinuity. For particular case of

$$\tilde{g}_1(k) = -\iota \mathbf{k} e^{-\mathbf{k}^2/2}$$

wavelet in $d = 3$ dimensions, we get

$$C_g = \frac{1}{2}, \quad \int_0^\infty y^2 e^{-y^2} \frac{4\pi y^2 dy}{8\pi^3} = \frac{3}{16\pi^{3/2}} \equiv \frac{\chi[g_1]}{2},$$

Substituting this integral into Eq. (A2), one gets

$$\varepsilon = \frac{g_0 v_0^3}{L^4} \frac{3}{8\pi^{3/2}}. \qquad \text{(A3)}$$

APPENDIX B: FEYNMAN DIAGRAM TECHNIQUE

Using the generating functional Eq. (19) with the action Eq. (20) we can easily derive the Feynman diagram technique for the scale-dependent fields Φ_a.

The correlation functions are given by functional derivatives

$$\langle \Phi_{a_1}(x_1) \dots \Phi_{a_n}(x_n) \rangle_c = \frac{\delta^n W[A]}{\delta A_{a_1}(x_1) \dots \delta A_{a_n}(x_n)} \bigg|_{A=0}.$$

The difference is that each spatial integration measure dx is substituted by the integration measure over affine group $\frac{dx da}{a}$, with functional derivatives taken with respect to this measure.

Using the Fourier transform,

$$u(x) = \int e^{\iota(\mathbf{kx} - \omega t)} u(k) \frac{d^d \mathbf{k} d\omega}{(2\pi)^{d+1}},$$

of the fields in Eq. (20) we make each convolution with basic wavelet g into a multiplicative factor $\tilde{g}(a\mathbf{k})$. In this way we obtain the following diagram technique:

(1) Each external line is labeled by a pair (a, k) (scale,momentum) and a vector index (i).

(2) The integration in each internal line is performed over the measure $\frac{d\omega}{2\pi} \frac{d^d \mathbf{k}}{(2\pi)^d} \frac{da}{a} \frac{1}{C_g}$.

(3) There are two type of lines: (a) Green functions $\langle uu' \rangle$ and (b) correlation functions $\langle uu \rangle$; the auxiliary field u' has zero moments $\langle u'u' \rangle = 0$. These Green functions are given by propagator matrix K^{-1} multiplied by wavelet factors $\tilde{g}(a\mathbf{k})$ on each leg.

(4) Each line carrying momentum k is proportional to orthogonal projector $P_{ij}(\mathbf{k})$, where i and j are vector indices of the line, i.e.,

$$G_{i\alpha, j\beta}^{(0)}(k) = \frac{\tilde{g}(\alpha \mathbf{k}) P_{ij}(\mathbf{k}) \bar{\tilde{g}}(\beta \mathbf{k})}{-\iota \omega + v_0 \mathbf{k}^2}$$

for the Green function, and

$$D_{i\alpha, j\beta}^{(0)}(k) = P_{ij}(\mathbf{k}) \frac{g_0 v_0^3}{C_g L} \frac{\tilde{g}(\alpha \mathbf{k}) |\tilde{g}(kL)|^2 \bar{\tilde{g}}(\beta \mathbf{k})}{|-\iota \omega + v_0 \mathbf{k}^2|^2}$$

for the bare velocity pair correlation function.

(5) Each vertex of the diagram is given by $m_{abc}(k) = \frac{\iota}{2}(k_b \delta_{ac} + k_c \delta_{ab})$, multiplied by 3 wavelet factors of adjusted lines.

(6) We consider the observation scale A to be much bigger than the viscous dissipation scale l and to belong the Kolmogorov range: $\xi = A/L \ll 1$. For this reason we assume the statistical momenta of the turbulent velocity field are determined by direct energy cascade. *At the language of Feynman diagrams this means it should be no scales a_i in internal lines less than minimal scale of all external lines.*

APPENDIX C: ONE-LOOP CONTRIBUTIONS TO THE GREEN FUNCTIONS

The scale dependence of the viscosity is given by renormalization of the Green function $\langle \Phi_\alpha(x)\Phi_\beta(x')\rangle$ by means of loop corrections. In Fourier space the value of the Green function $G^{(2)}_{\alpha a, \beta s}(k)$, shown in diagram Fig. 1, can be evaluated from the above mentioned expressions for the vertices and the Green functions after the integration over internal line scale arguments. The upper line of the diagram Fig. 1, (p^+), contains the random stirring force correlator, the bottom line (p^-) is the Green function.

The frequency integration in the loop integral can be done explicitly

$$\int_{-\infty}^{\infty} \frac{dq_0}{2\pi} \left| G_0\left(\frac{k}{2}+q\right)\right|^2 G_0\left(\frac{k}{2}-q\right)$$
$$= \frac{1}{2\nu_0^2\left(\frac{\mathbf{k}}{2}+\mathbf{q}\right)^2} \frac{1}{-\frac{\iota k_0}{\nu_0^2}+\frac{\mathbf{k}^2}{2}+2\mathbf{q}^2},$$

where

$$G_0(k) = \frac{1}{-\iota k_0 + \nu_0 \mathbf{k}^2}.$$

The loop tensor structure $L_{as}(k, p^+, p^-)$ Eq. (24) is given by the convolution of

$$L(k, p^+, p^-) = m_{abc}(k)m_{fls}(p^-)P_{bl}(p^+)P_{cf}(p^-), \quad \text{(C1)}$$

where

$$m_{abc}(k) = \frac{\iota}{2}(k_b\delta_{ac}+k_c\delta_{ab}), \quad p^\pm = \frac{k}{2}\pm q,$$

over repeated indices. The explicit equation for tensor structure is Eq. (24).

Let $A = \min(\alpha, \beta)$ be the minimal scale of two external lines of the diagram Fig. 1. The lower line (p^-) contributes two identical g_1 wavelet factors

$$e^{-(A\mathbf{p}^-)^2} = \frac{1}{C_g}\int_A^\infty |\tilde{g}(a\mathbf{p}^-)|^2 \frac{da}{a},$$

so the factor $e^{-2(A\mathbf{p}^-)^2}$ will be prescribed to the bottom line, and similar factor $e^{-2(A\mathbf{p}^+)^2}$ will be prescribed to the upper line.

Introducing the dimensionless momentum $\mathbf{q} = |\mathbf{k}|\mathbf{y}$, with the polar angle between $\mathbf{q}$ and $\mathbf{k}$ measured from the $\mathbf{k}$ direction, we can write the whole one-loop integral in $d=3$ dimension in the form

$$-\Sigma_{as}(k) = \frac{g_0\nu_0^3}{C_g L}\int k^3 \frac{y^2 dy}{(2\pi)^3}\sin\theta d\theta d\varphi L_{as}(k, p^+, p^-)\frac{1}{\nu_0^2 k^4}\frac{1}{\frac{1}{2}+2y^2+2y\cos\theta}\frac{1}{\frac{1}{2}+2y^2-\frac{\iota k_0}{\nu_0 k^2}}$$

$$\times (Lk)^2\left[\frac{1}{4}+y^2+y\cos\theta\right]e^{-(Lk)^2[\frac{1}{4}+y^2+y\cos\theta]}e^{-2(Lk)^2\xi^2[\frac{1}{4}+y^2-y\cos\theta]}e^{-2(Lk)^2\xi^2[\frac{1}{4}+y^2+y\cos\theta]}.$$

This can be simplified to

$$-\Sigma_{as}(k) = \frac{g_0\nu_0^3 C_g^{-1}Lk}{4\nu_0^2}\int \frac{y^2 dy}{(2\pi)^3}\sin\theta d\theta d\varphi L_{as}(k, p_+, p_-)\frac{1}{\frac{1}{4}+y^2-\frac{\iota k_0}{2\nu k^2}}.$$

The sign $-\Sigma$ is introduced for the value of one-loop integral is equal to minus "self-energy" contribution.

In view of the existence of only one preferred direction, that is the direction of $\mathbf{k}$, the integrals of L_{as} should depend upon two scalars only:

$$T_1 \equiv \text{Tr}L_{as}, \quad T_2 = \frac{k_a k_s}{k^2}L_{as}, \quad \text{(C2)}$$

$$T_1 = -\frac{(p^-k)(p^+p^-)^2}{2(p^+)^2(p^-)^2}+\frac{(p^+k)(p^+p^-)}{(p^+)^2}-\frac{(p^-k)}{2},$$

$$T_2 = -\frac{(p^+k)(p^+p^-)(p^-k)^2}{(p^+)^2(p^-)^2k^2}+\frac{(p^-k)(p^+k)^2}{2(p^+)^2k^2}+\frac{(p^+k)(p^+p^-)k^2}{2(p^+)^2k^2}+\frac{(p^-k)^3}{(p^-)^2k^2}-(p^-k).$$

Assuming

$$\Sigma_{as} = \nu_0 g_0 \Sigma^\delta \delta_{as}k^2 + \nu_0 g_0 \Sigma^T k_a k_s = -\int L_{as}d\mu(y, \theta), \quad \text{(C3)}$$

we get

$$(d\Sigma^\delta + \Sigma^T)k^2 = -\int T_1 d\mu(y, \theta) = -k^2 I_1, \quad (\Sigma^\delta + \Sigma^T)k^2 = -\int T_2 d\mu(y, \theta) = -k^2 I_2,$$

where $d=3$ is the space dimension. So, we need to calculate I_1 and I_2. From where

$$\Sigma^\delta = -\frac{I_1 - I_2}{2}, \quad \Sigma^T = -I_2 - \Sigma^\delta = -\frac{3I_2 - I_1}{2}.$$

The substitution $\mathbf{q} = k\mathbf{y}$, $(kq) = ky\mu$, with $\mu \equiv \cos\theta$, gives:

$$T_1 = \mathrm{Tr} L_{as} = k^2 \frac{-\mu^3 y^3 + 2\mu^2 y^4 + \mu y^3 - 2y^4}{2\left[\left(\frac{1}{4} + y^2\right)^2 - y^2\mu^2\right]} = \frac{8k^2(\mu - 1)(\mu + 1)(2y - \mu)y^3}{(4y^2 - 4y\mu + 1)(4y^2 + 4y\mu + 1)},$$

$$T_2 = \frac{k_a k_s}{k^2} L_{as} = k^2 \frac{-\mu^3 y^5 + \mu y^5 - \mu^4 y^4 + \frac{5}{2}\mu^2 y^4 - \frac{3}{2}y^4 + \frac{1}{4}\mu^3 y^3 - \frac{1}{4}\mu y^3 - \frac{1}{8}\mu^2 y^2 + \frac{1}{8}y^2}{2(1/4 + y\mu + y^2)(1/4 - y\mu + y^2)}$$

$$= -k^2 \frac{(\mu - 1)(\mu + 1)(8\mu y^3 + 8\mu^2 y^2 - 12y^2 - 2\mu y + 1)}{(4y^2 - 4y\mu + 1)(4y^2 + 4y\mu + 1)} y^2,$$

$$\frac{T_1 - T_2}{2} = k^2 \frac{(\mu^2 - 1)y^2(8\mu^2 y^2 + \mu(8y^3 - 10y) + 4y^2 + 1)}{2(4y^2 - 4y\mu + 1)(4y^2 + 4y\mu + 1)},$$

$$\frac{3T_2 - T_1}{2} = -k^2 \frac{(\mu^2 - 1)y^2(24\mu^2 y^2 + \mu(24y^3 - 14y) - 20y^2 + 3)}{2(4y^2 - 4y\mu + 1)(4y^2 + 4y\mu + 1)}.$$

After standard algebraic manipulations, this gives

$$\Sigma^\delta = \frac{kL}{128 C_g} \int_0^\infty \frac{y^2 dy}{(2\pi)^2} \frac{e^{-(kL)^2(1+4\xi^2)(\frac{1}{4}+y^2)}}{\frac{1}{4} + y^2 - \iota\frac{2k_0}{v_0 \mathbf{k}^2}} \int_{-1}^1 d\mu \frac{(1 - \mu^2)(8\mu^2 y^2 + \mu(8y^3 - 10y) + 4y^2 + 1)}{\left(\frac{1}{4y} + y - \mu\right)\left(\frac{1}{4y} + y + \mu\right)} e^{-(kL)^2 y\mu},$$

$$\Sigma^T = -\frac{kL}{128 C_g} \int_0^\infty \frac{y^2 dy}{(2\pi)^2} \frac{e^{-(kL)^2(1+4\xi^2)(\frac{1}{4}+y^2)}}{\frac{1}{4} + y^2 - \iota\frac{2k_0}{v_0 \mathbf{k}^2}} \int_{-1}^1 d\mu \frac{(1 - \mu^2)(24\mu^2 y^2 + \mu(24y^3 - 14y) - 20y^2 + 3)}{\left(\frac{1}{4y} + y - \mu\right)\left(\frac{1}{4y} + y + \mu\right)} e^{-(kL)^2 y\mu},$$

$$\text{(C4)}$$

where $\mu = \cos\theta$ determines the polar angle between $\mathbf{k}$ and $\mathbf{q}$. Since the velocity field is transversal, it is natural to present the self-energy as a sum of transversal and longitudinal terms:

$$\Sigma_{as} = \nu g \Sigma^\delta \left(\delta_{as} - \frac{k_a k_s}{\mathbf{k}^2}\right)\mathbf{k}^2 + \nu g \Sigma^l k_a k_s, \quad \text{(C5)}$$

where $\Sigma^l = \Sigma^\delta + \Sigma^T$.

APPENDIX D: ONE-LOOP CONTRIBUTION TO STIRRING FORCE CORRELATOR

The one-loop contribution to stirring force correlator is shown in Fig. 5 below. The tensor structure of this diagram has the form

$$C_{as}(k, p^+, p^-) = -m_{abc}(k)m_{sfl}(k)P^{bl}(\mathbf{p}^+)P^{cf}(\mathbf{p}^-), \quad \text{(D1)}$$

or explicitly:

$$C_{as}(k, p^+, p^-) = \delta_{as}\left[k^2 - \frac{(p^+k)^2}{4(p^+)^2} - \frac{(p^-k)^2}{4(p^-)^2}\right] + \frac{k_a k_s}{2}$$

$$+ \frac{p_a^+ p_s^+}{4}\left[\frac{(p^-k)^2}{(p^-)^2(p^+)^2} - \frac{k^2}{(p^+)^2}\right]$$

$$+ \frac{p_a^- p_s^-}{4}\left[\frac{(p^+k)^2}{(p^-)^2(p^+)^2} - \frac{k^2}{(p^-)^2}\right]$$

$$= (p_a^+ p_s^- + p_a^- p_s^+)\frac{(p^+k)(p^-k)}{4(p^+)^2(p^-)^2}$$

$$- (k_a p_s^+ + p_a^+ k_s)\frac{(p^+k)}{4(p^+)^2}$$

$$- (k_a p^- s + p_a^- k_s)\frac{(p^-k)}{4(p^-)^2}.$$

The corresponding invariants the tensor structure can depend on are

$$C_1 = \mathrm{Tr} C_{as} = \frac{3}{2}k^2 - \frac{(p^-k)^2}{(p^-)^2} - \frac{(p^+k)^2}{(p^+)^2}$$

$$+ \frac{(p^+k)(p^-k)(p^+p^-)}{2(p^+)^2(p^-)^2},$$

$$C_2 = \frac{k^a k^s}{k^2} C_{as} = k^2 - \frac{(p^-k)^2}{(p^-)^2} - \frac{(p^+k)^2}{(p^+)^2} + \frac{(p^-k)^2(p^+k)^2}{(p^-)^2(p^+)^2}.$$

In dimensionless variables

$$C_1 = -\frac{2k^2(\mu^2 - 1)y^2(12y^2 + 1)}{(4y^2 - 4y\mu + 1)(4y^2 + 4y\mu + 1)},$$

$$C_2 = \frac{16k^2(\mu^2 - 1)^2 y^4}{(4y^2 - 4y\mu + 1)(4y^2 + 4y\mu + 1)},$$

$$\frac{C_1 - C_2}{2} = -\frac{k^2(\mu^2 - 1)y^2(8y^2\mu^2 + 4y^2 + 1)}{(4y^2 - 4y\mu + 1)(4y^2 + 4y\mu + 1)},$$

$$\frac{3C_2 - C_1}{2} = \frac{k^2(\mu^2 - 1)y^2(24y^2\mu^2 - 12y^2 + 1)}{(4y^2 - 4y\mu + 1)(4y^2 + 4y\mu + 1)}. \quad \text{(D2)}$$
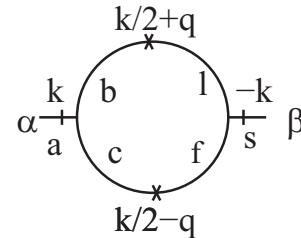

FIG. 5. One-loop contribution to the correlation function. Greek letters α and β denote scale arguments.

The total symmetric contribution to the 1PI correlation function has the form

$$\Delta_{as}^{(2)} = \left(\frac{g_0 v_0^3}{LC_g}\right)^2 \int \frac{d^4q}{(2\pi)^4} \frac{|\tilde{g}(Lp^+)|^2 f^2(Ap^+)}{|-\iota\omega^+ + v_0(p^+)^2|^2} \frac{|\tilde{g}(Lp^-)|^2 f^2(Ap^-)}{|-\iota\omega^- + v_0(p^-)^2|^2} C_{as}(k, p^+, p^-).$$

In case the nonzero frequency $k_0 \neq 0$,

$$\int_{-\infty}^{\infty} \frac{dq_0}{2\pi} \left|G_0\left(\frac{k}{2}+q\right)\right|^2 \left|G_0\left(\frac{k}{2}-q\right)\right|^2 = \frac{1}{4v_0^3} \frac{\frac{\mathbf{k}^2}{4}+\mathbf{q}^2}{\left(\frac{\mathbf{k}^2}{4}+\mathbf{q}^2+\iota\frac{2k_0}{v_0}\right)\left(\frac{\mathbf{k}^2}{4}+\mathbf{q}^2-\iota\frac{2k_0}{v_0}\right)} \cdot \frac{1}{\frac{\mathbf{k}^2}{4}+\mathbf{q}^2+\mathbf{kq}} \cdot \frac{1}{\frac{\mathbf{k}^2}{4}+\mathbf{q}^2-\mathbf{kq}}. \tag{D3}$$

In the case of the zero frequency $k_0 = 0$, this turns out to be

$$\int_{-\infty}^{\infty} \frac{dq_0}{2\pi} \left|G_0\left(\frac{k}{2}+q\right)\right|^2 \left|G_0\left(\frac{k}{2}-q\right)\right|^2 = \frac{1}{4v_0^3} \frac{1}{\frac{\mathbf{k}^2}{4}+\mathbf{q}^2} \cdot \frac{1}{\frac{\mathbf{k}^2}{4}+\mathbf{q}^2+\mathbf{kq}} \cdot \frac{1}{\frac{\mathbf{k}^2}{4}+\mathbf{q}^2-\mathbf{kq}}. \tag{D4}$$

The δ function in scale variables results in square of wavelets in both upper and lower parts of the loop

$$|\tilde{g}(Lp^+)|^2|\tilde{g}(Lp^-)|^2 = L^2\left(\frac{\mathbf{k}}{2}+\mathbf{q}\right)^2 e^{-L^2(\frac{\mathbf{k}}{2}+\mathbf{q})^2} L^2\left(\frac{\mathbf{k}}{2}-\mathbf{q}\right)^2 e^{-L^2(\frac{\mathbf{k}}{2}-\mathbf{q})^2}.$$

The angle arguments in the exponents are canceled due to symmetry. The filter factors for g_1 wavelet are

$$f(Ap^{\pm}) = e^{-(Ak)^2(1/4+y^2\pm y\cos\theta)}.$$

After integration over the loop frequency variable $\int_{-\infty}^{\infty} \frac{dq_0}{2\pi} \ldots$ we get

$$\Delta_{as}^{(2)} = \left(\frac{g_0 v_0^3}{LC_g}\right)^2 (kL)^4 \int \left[\left(\frac{1}{4}+y^2\right)^2 - y^2\cos^2\theta\right] e^{-2(kL)^2(\frac{1}{4}+y^2)} \frac{1}{4v_0^3} \frac{\frac{\mathbf{k}^2}{4}+\mathbf{q}^2}{\left(\frac{\mathbf{k}^2}{4}+\mathbf{q}^2+\iota\frac{2k_0}{v_0}\right)\left(\frac{\mathbf{k}^2}{4}+\mathbf{q}^2-\iota\frac{2k_0}{v_0}\right)} \cdot$$

$$\times \frac{1}{\frac{\mathbf{k}^2}{4}+\mathbf{q}^2+\mathbf{kq}} \cdot \frac{1}{\frac{\mathbf{k}^2}{4}+\mathbf{q}^2-\mathbf{kq}} e^{-4\xi^2(Lk)^2(\frac{1}{4}+y^2)} C_{as}(k, p^+, p^-)\frac{q^2 dq}{(2\pi)^2} \sin\theta d\theta.$$

After algebraic simplification we get

$$\Delta_{as}^{(2)}(L, k) = \left(\frac{g_0 v_0^3}{2C_g}\right)^2 \frac{L^2}{v_0^3} \int \frac{q^2 dq}{(2\pi)^2} \sin\theta d\theta \frac{e^{-2(Lk)^2(1+2\xi^2)(\frac{1}{4}+y^2)} C_{as}(k, p^+, p^-)\left(\frac{\mathbf{k}^2}{4}+\mathbf{q}^2\right)}{\left(\frac{\mathbf{k}^2}{4}+\mathbf{q}^2+\iota\frac{2k_0}{v_0}\right)\left(\frac{\mathbf{k}^2}{4}+\mathbf{q}^2-\iota\frac{2k_0}{v_0}\right)}.$$

In view of isotropy, using Eq. (D2), we get

$$C_{as} = C^{\delta}\delta_{as} + C^T \frac{k_a k_s}{k^2}$$

with

$$C^{\delta}(L, k) = \left(\frac{g_0 v_0^3}{2C_g}\right)^2 \frac{L^2 k^3}{16 v_0^3} \int \frac{y^4 dy}{(2\pi)^2} d\mu \frac{e^{-2(Lk)^2(1+2\xi^2)(\frac{1}{4}+y^2)}\left(\frac{1}{4}+y^2\right)}{\left(\frac{1}{4}+y^2+\iota\frac{2k_0}{v_0\mathbf{k}^2}\right)\left(\frac{1}{4}+y^2-\iota\frac{2k_0}{v_0\mathbf{k}^2}\right)} \frac{(1-\mu^2)(8y^2\mu^2+4y^2+1)}{\left(\frac{1}{4}+y^2-y\mu\right)\left(\frac{1}{4}+y^2+y\mu\right)}. \tag{D5}$$

APPENDIX E: RENORMALIZATION GROUP EQUATIONS

1. Renormalization of viscosity

We use the following set of scales $l = A_0 < A_1 < A_2 < \ldots < A_L = L$. In terms of dimensionless variable $\xi = A/L$ this corresponds to $\xi_0 = R^{-1}, \xi_1, \xi_2, \ldots, \xi_L = 1$, where $R = L/l \gg 1$. If we use the iteration from the Kolmogorov dissipation scale l to the macro scales, we would get the following chain of equations:

$$v_1 = v_0[1 + g_0\Sigma(\xi_0)], \quad v_2 = v_1[1 + g_1\Sigma(\xi_1)], \quad v_{k+1} = v_k[1 + g_k\Sigma(\xi_k)].$$

If we go from large scales to smaller scales the proposed inversion formulas are

$$v_0 \approx v_1[1 - g_1\Sigma(\xi_0)], \quad v_1 \approx v_2[1 - g_2\Sigma(\xi_1)], \quad v_{k-1} \approx v_k[1 - g_k\Sigma(\xi_{k-1})], \tag{E1}$$

where $g_k \equiv g(\xi_k)$. The iteration scheme above can be written in a form of difference equation

$$\frac{v_{k-1} - v_k}{v_k} = -g(\xi_k)\Sigma(\xi_{k-1}). \tag{E2}$$

For the equal scale steps $A_k = A_0\delta^k, \xi_k = \xi_0\delta^k, \Delta\ln\xi = \ln\delta\Delta k$, we get

$$\frac{\Delta\ln v}{\Delta k} = g(\xi_k)\Sigma(\xi_k), \quad \text{or} \quad \frac{d\ln v}{d\ln\xi} = g(\xi)\frac{\Sigma(\xi)}{\ln\delta}. \tag{27}$$

2. Renormalization of stirring force

The one-loop contribution to the stirring force correlation function is shown in Fig. 5.

Since the stirring force acts on large scales only, its yield on smaller scales will be the sum of correlator itself and the one loop correction

$$D_{L-1} = D_L + D_L^2 * \text{OneLoop}K(\xi_{L-1}) \tag{E3}$$

with $D(\xi) = g(\xi)v^3(\xi)/L$, hence

$$\frac{D_{L-1} - D_L}{D_L} = D_L * \text{OneLoop}K(\xi_{L-1}).$$

In differential form the latter difference equation yields

$$\frac{d \ln D}{d \ln \xi} = -\frac{K(\xi)}{\ln \delta}, \tag{28}$$

where

$$K(\xi) = \frac{(kL)^3}{16} \int \frac{y^4 dy}{(2\pi)^2} d\mu \frac{e^{-2(Lk)^2(1+2\xi^2)(\frac{1}{4}+y^2)}(\frac{1}{4}+y^2)}{(\frac{1}{4}+y^2+\iota\frac{2k_0}{v_0\mathbf{k}^2})(\frac{1}{4}+y^2-\iota\frac{2k_0}{v_0\mathbf{k}^2})} \frac{(1-\mu^2)(8y^2\mu^2+4y^2+1)}{(\frac{1}{4}+y^2-y\mu)(\frac{1}{4}+y^2+y\mu)}. \tag{E4}$$

The value of $K(\xi)$ is typically a few orders of magnitude less than $\Sigma(\xi)$.

APPENDIX F: FREQUENCY INTEGRALS THAT CONTRIBUTE TO VELOCITY PAIR CORRELATOR

The bare correlation function, being integrated over the frequency, contains the integral

$$I_{w0} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{1}{|-\iota\omega + A|^2} = \frac{1}{2A}, \quad A = v\mathbf{k}^2. \tag{F1}$$

The integral coming from 1PI correlation function multiplied by two conjugated correlation functions from the legs of the diagram has the form

$$I_{w2} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{1}{|-\iota\omega + A|^2} \frac{1}{|B + \iota\frac{2\omega}{A}|^2}, \quad B = \frac{1}{4} + y^2. \tag{F2}$$

Substituting $C = \frac{AB}{2}$, we get

$$I_{w2} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{A^2}{4} \frac{1}{\omega^2 + A^2} \frac{1}{\omega^2 + C^2} = \frac{1}{8} \frac{A}{C(A+C)} = \frac{1}{4(\frac{1}{4}+y^2)v\mathbf{k}^2[1+\frac{1}{2}(\frac{1}{4}+y^2)]}.$$

The third integral comes from two conjugated diagrams with self-energy multiplied by correlator and propagator

$$I_{1w} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{1}{\omega^2 + A^2} \cdot \frac{1}{-\iota\omega + A} \cdot \frac{2A}{4C - \iota\omega} = \frac{1}{2A(A+4C)} = \frac{1}{2(v\mathbf{k}^2)^2[1+2(\frac{1}{4}+y^2)]}.$$

This integral contributes twice for there are two conjugated diagrams, shown in the lower part of Fig. 3.

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