Enstrophy spectrum in freely decaying two-dimensional self-similar turbulent flow

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We consider freely decaying two-dimensional homogeneous and isotropic turbulent motion in a self-similar limit that is achieved at large Reynolds numbers based on time and the mean kinetic energy of the flow provided that initial average enstrophy tends to infinity as fluid viscosity tens to zero. In this case, the enstrophy dissipation rate has a nonzero finite limit. The vorticity correlation function and the spectral enstrophy density are investigated in an inertial range of distances and wave numbers where these functions are free from the influence of viscosity and large-scale flow parameters. It turns out that in freely decaying two-dimensional self-similar turbulence, the inertial range exists in real space but is absent in the space of wave numbers. This means that turbulent eddies of the appropriate size do not contribute to the spectral density and the known k^{-1} law does not hold. The spectral enstrophy density at large wave numbers behaves nonmonotonically: it first decreases faster than in accordance with the k^{-1} law and then, in the dissipation region, has a growth portion and a second maximum. The enstrophy spectral flux at the boundary of the dissipation region is only a fraction of the enstrophy dissipation rate.

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I. INTRODUCTION

Freely decaying two-dimensional isotropic turbulence is described by the two-dimensional Navier-Stokes equations in which external forces are absent. Its main properties, according to Batchelor [1], follow from the fact that for plane motion the kinetic energy dissipation tends to zero when the fluid viscosity $\nu \to 0$. In this limit, the average kinetic energy per unit mass E is conserved and is an invariant of the motion under consideration. Batchelor assumed that the statistical characteristics of turbulence depend on only two governing parameters ν and E and, therefore, obey self-similar laws. The enstrophy, in particular, reads [1]

$$\frac{1}{2}\langle\omega^2\rangle = \frac{A}{t^2}, \tag{1}$$

where A is a dimensionless constant. The law of conservation of energy

$$\frac{dE}{dt} = -\nu\langle\omega^2\rangle = -\frac{2A\nu}{t^2}$$

after integration gives the equality

$$E = V^2 + \frac{2A\nu}{t}, \quad V^2 = E(\infty),$$

which implies that for $\tau = tV^2/\nu \sim 1$, when both terms on the right-hand side have the same order of magnitude, average kinetic energy changes by an amount of order V^2. This happens because for $\tau \sim 1$ according to (1), enstrophy has the order $V^4\nu^{-2}$, and the relative energy dissipation rate is of order unity. Therefore, at the initial instant of time, energy differs significantly from V^2. Hence we can conclude that the self-similar flow arises for $\tau \gg 1$ provided that the initial velocity field is weakly correlated, it has a small spatial scale of inhomogeneity proportional to viscosity, and the Reynolds number $R_0 = V^2/\omega_0\nu$, where ω_0 is the root of initial average enstrophy, is of order unity.

According to (1), the self-similar representation of the enstrophy dissipation rate

$$\chi = -\frac{d\langle\omega^2\rangle}{2dt} \tag{2}$$

is $\chi = -At^{-3}$ and does not depend on viscosity. This does not contradict Eyink's rigorous mathematical result [2], according to which in two-dimensional flow in the limit of vanishing viscosity, $\chi \to 0$ for a finite value of initial enstrophy, and the limit $\chi \neq 0$ is possible if initial enstrophy tends to infinity, which is just achieved under the condition $R_0 \sim 1$. Thus, the self-similar flow with the initial enstrophy $\omega_0^2 \sim V^4\nu^{-2}$ is exactly the case when the well-known Batchelor conjecture [1] about a finite nonzero value of the enstrophy dissipation rate is fulfilled.

Another known Batchelor's result [1] is the k^{-1} law for the spectral enstrophy density

$$\Omega(k,\, t) = C\chi^{2/3}k^{-1} \tag{3}$$

(C is a universal constant), which was obtained under the assumption of the existence of an inertial range of wave numbers where this quantity depends on the wave number k and χ only. It must be said that DNS data on this law do not agree well with each other. One part [3–6] confirms Eq. (3), but the other one [7–11] demonstrates a stronger decrease of $\Omega \sim k^{-\alpha}$ with the exponent $\alpha > 1$. We will show that for self-similar flow, the k^{-1} law cannot be satisfied, since it contradicts Eq. (2), but there is a more rapid decrease in the spectral enstrophy density at large wave numbers.

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Thus, the aim of the present study is to find out what a consistent application of Batchelor's conjecture about the self-similarity of freely decaying two-dimensional turbulence yields.

II. PROBLEM FORMULATION

Let us assume that the limiting at $\tau \to \infty$ self-similar flow depends on three governing parameters ν, V, and ω_0. Then for the average enstrophy and the rate of its dissipation from dimensional considerations, we obtain the expressions

$$\frac{1}{2}\langle\omega^2\rangle = \frac{f(R_0)}{t^2}, \quad \chi = -\frac{2f(R_0)}{t^3}, \tag{4}$$

in which f is a universal function. Relations (4), like all subsequent ones, are obtained under the condition $\tau \to \infty$ with $R_0 = O(1)$.

Since V and Vt are the characteristic velocity and length scale of large-scale motion, the spectral density characterizing the flow on large scales can be written

$$\Omega(k, t) = \frac{V}{t} g_e(K, R_0), \quad K = Vtk, \tag{5}$$

where g_e is a universal function.

Following Batchelor, we assume that there exists a direct enstrophy cascade, and the parameter χ determines the properties of freely decaying two-dimensional turbulence on the dissipation scale. Then the inner length scale is calculated in terms of the quantities ν and χ and equals $\nu^{1/2}\chi^{-1/6}$. The fluctuation velocity on this scale is of order $\nu^{1/2}\chi^{1/6}$. Since χ characterizes an enstrophy flux from large eddies to small ones, the characteristic time of motion for all scales (up to the enstrophy dissipation scale inclusive) has the same order of magnitude $\chi^{-1/3}$. In accordance with these estimates and the second formula in Eq. (4), the representation of the spectral density on the enstrophy dissipation scale is

$$\Omega(k, t) = \sqrt{\frac{\nu}{t^3}}\, g_i(\varkappa, R_0), \quad \varkappa = k\sqrt{\nu t} = \frac{K}{\sqrt{\tau}}, \tag{6}$$

where g_i is a universal function.

Thus, on large and small scales, the spectral density is described by self-similar representations (5) and (6).

III. INERTIAL RANGE OF WAVE NUMBERS: THE k^{-1} LAW

Suppose that there exist an inertial range of wave numbers where the spectral enstrophy density does not depend on the outer parameter V and viscosity. In this case, Eqs. (5) and (6) give the k^{-1} law

$$g_e(K, R_0) \to \frac{G(R_0)}{K}, \quad K \to \infty, \tag{7}$$

$$g_i(\varkappa, R_0) \to \frac{G(R_0)}{\varkappa}, \quad \varkappa \to 0, \tag{8}$$

where G is a certain function. Relations (7) and (8) are matching conditions [12] that relations (5) and (6) must satisfy in the inertial range.

Substituting (5) and (6) into the integrals

$$\frac{1}{2}\langle\omega^2\rangle = \int_0^\infty \Omega(k, t)\, dk, \quad \chi = 2\nu\int_0^\infty k^2\Omega(k, t)\, dk$$

yields

$$\frac{1}{2}\langle\omega^2\rangle = \frac{1}{t^2}\left[\int_0^{\tau^{1/4}} g_e(K, R_0)\, dK + \int_{\tau^{-1/4}}^\infty g_i(\varkappa, R_0)\, d\varkappa\right], \tag{9}$$

$$\chi = \frac{2}{t^3}\left[\frac{1}{\tau}\int_0^{\tau^{1/4}} K^2 g_e(K, R_0)\, dK + \int_{\tau^{-1/4}}^\infty \varkappa^2 g_i(\varkappa, R_0)\, d\varkappa\right]. \tag{10}$$

In view of the relation between the two dimensionless wave numbers $K = \varkappa\sqrt{\tau}$, the boundary of the dissipation range is characterized by the conditions $K = O(\sqrt{\tau})$ or $\varkappa = O(\tau^{-1/2})$. Therefore, the first integrals (9) and (10) encompass the region of large-scale motion lying outside the dissipation range, where representation (5) and asymptotics (7) are true, and the second ones encompass the region of viscous dissipation, where representation (6) and asymptotics (8) are true. In other words, asymptotic representations (7) and (8) are satisfied up to a conditional boundary of the dissipation region, where the dimensionless wave numbers reach the values $K = O(\sqrt{\tau})$ and $\varkappa = O(\tau^{-1/2})$, while the limits of integration have the values $K = \tau^{1/4}$ and $\varkappa = \tau^{-1/4}$, which guarantees the validity of asymptotic representations (7) and (8) within the integration intervals. In the limit $\tau \to \infty$, Eqs. (9) and (10) give

$$\frac{1}{2}\langle\omega^2\rangle = t^{-2}\left[\frac{1}{2}G(R_0)\ln\tau + I(R_0)\right], \tag{11}$$

$$I = \int_0^1 g_e\, dK + \int_1^\infty [g_e - GK^{-1}]\, dK + \int_0^1 [g_i - G\varkappa^{-1}]\, d\varkappa + \int_1^\infty g_i\, d\varkappa, \tag{12}$$

$$\chi = \frac{2}{t^3}\int_0^\infty \varkappa^2 g_i\, d\varkappa. \tag{13}$$

The convergence of the second and third integrals (12) is ensured by asymptotic representations (7) and (8). Expression (11) does not agree with (4). The time derivative of (11) contains $\ln\tau$ and cannot be equal to (13). It implies that the assumption of the existence of the inertial range of wave numbers [the overlap region for representations (5) and (6)] and the k^{-1} law provide such a form of enstrophy spectrum that does not ensure Eq. (2).

IV. INERTIAL RANGE IN REAL SPACE

In the case of homogeneous and isotropic two-dimensional turbulence, the spectral density is calculated in terms of the vorticity correlation function

$$\Phi(r, t) = \langle\omega(\mathbf{x}, t)\omega(\mathbf{x}+\mathbf{r}, t)\rangle \tag{14}$$

by the formula [13]

$$\Omega(k, t) = \frac{k}{2}\int_0^\infty \Phi(r, t)J_0(kr)r\, dr, \tag{15}$$

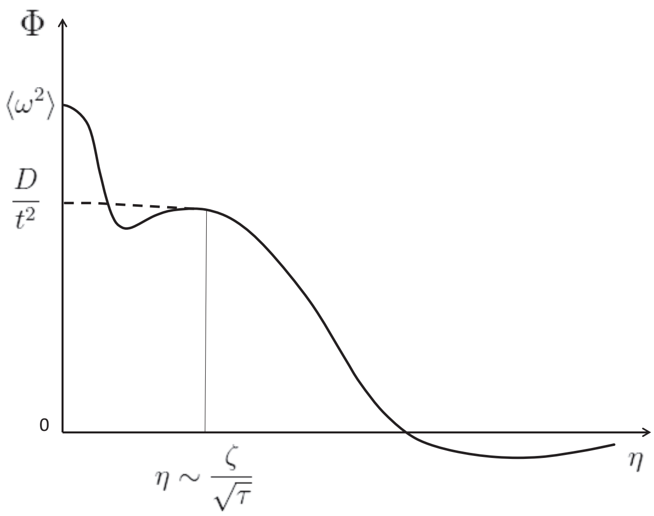

FIG. 1. Qualitative form of the vorticity correlation function.

where J_0 is the zero-order Bessel function. In accordance with the estimates given above, we introduce the dimensionless correlation functions for two characteristic regions of variation of r

$$\Phi(r,\ t) = \frac{\phi_e(\eta,\ R_0)}{t^2}, \quad \eta = \frac{r}{Vt}, \tag{16}$$

$$\Phi(r,\ t) = \frac{\phi_i(\zeta,\ R_0)}{t^2}, \quad \zeta = \frac{r}{\sqrt{\nu t}} = \eta\sqrt{\tau}, \tag{17}$$

on large scales and the dissipation scale, respectively.

Suppose that there exists an inertial range of scales in real space, i.e., such a range of r that is much smaller than the outer scale of the flow but much larger than the viscous enstrophy dissipation scale, where Φ depends neither on outer parameters nor on viscosity. Then representations (16) and (17) must satisfy the matching condition [12]

$$\phi_e(0,\ R_0) = \phi_i(\infty,\ R_0) = D(R_0), \tag{18}$$

where D is a certain function.

Because of spatial uniformity of the flow

$$\langle[\omega(\mathbf{x},\ t) - \omega(\mathbf{x}+\mathbf{r},\ t)]^2\rangle = 2\langle\omega^2\rangle - 2\Phi,$$

which implies that $\Phi \le \langle\omega^2\rangle$ and, therefore, in view of Eq. (4), $D < 2f$. Thus, the function Φ qualitatively has the form shown in Fig. 1.

The second- and third-order velocity correlation functions

$$B_{LL}(r,\ t) = \langle u_1(\mathbf{x},\ t)u_1(\mathbf{x}+\mathbf{r},\ t)\rangle,$$

$$B_{LL,\,L}(r,\ t) = \langle u_1(\mathbf{x},\ t)u_1(\mathbf{x},\ t)u_1(\mathbf{x}+\mathbf{r},\ t)\rangle$$

(the x_1 axis is directed along the vector $\mathbf{r}$) satisfy the Kármán-Howarth equation [14]

$$r^3\frac{\partial B_{LL}}{\partial t} = \frac{\partial}{\partial r}r^3\left(B_{LL,\,L} + 2\nu\frac{\partial B_{LL}}{\partial r}\right). \tag{19}$$

The relation between the velocity and vorticity correlation functions [14]

$$\Phi(r,\ t) = -\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}r^2 B_{LL}(r,\ t) \tag{20}$$

implies that in the inertial range

$$B_{LL}(0,\ t) - B_{LL}(r,\ t) = \frac{D(R_0)r^2}{16t^2}. \tag{21}$$

Hence, on the basis of Eq. (19),

$$B_{LL,\,L}(r,\ t) = \frac{D(R_0)r^3}{48t^3}. \tag{22}$$

Relations (21) and (22) are valid in the inertial range of distances between two selected points in freely decaying two-dimensional self-similar turbulence and differ from the corresponding relation obtained for stationary forced two-dimensional turbulence [15] (see also Ref. [14]).

V. CALCULATION OF THE SPECTRAL ENSTROPHY DENSITY

Substituting (16) and (17) into the integral (15) gives

$$\Omega(k,\ t) = \frac{V}{2t}\left[K\int_{\tau^{-1/4}}^{\infty}\eta\phi_e(\eta,\ R_0)J_0(K\eta)\,d\eta + \frac{\varkappa}{\sqrt{\tau}}\int_0^{\tau^{1/4}}\zeta\phi_i(\zeta,\ R_0)J_0(\varkappa\zeta)\,d\zeta\right]. \tag{23}$$

In view of the relation between the integration variables $\zeta = \eta\sqrt{\tau}$, the first integral (23) encompasses the interval of variation of η, which lies entirely in the region of large-scale motion, where representation (16) is valid, while the second one encompasses the interval of variation of ζ, which lies entirely in the enstrophy dissipation region, where representation (17) is valid. Adding and subtracting the integral

$$\frac{VK}{2t}\int_0^{\tau^{-1/4}}\eta\phi_e(\eta,\ R_0)J_0(K\eta)\,d\eta = \frac{V\varkappa}{2t\sqrt{\tau}}\int_0^{\tau^{1/4}}\zeta\phi_e(\zeta\tau^{-1/2},\ R_0)J_0(\varkappa\zeta)\,d\zeta,$$

we transform (23) into

$$\Omega(k,\ t) = \frac{V}{2t}\left\{K\int_0^{\infty}\eta\phi_e(\eta,\ R_0)J_0(K\eta)\,d\eta + \frac{\varkappa}{\sqrt{\tau}}\int_0^{\tau^{1/4}}\zeta \times\left[\phi_i(\zeta,\ R_0) - \phi_e(\zeta\tau^{-1/2},\ R_0)\right]J_0(\varkappa\zeta)\,d\zeta\right\}. \tag{24}$$

Let us pass in (24) to the limit $\tau \to \infty$ with $K = O(1)$. In view of the equality $\varkappa = K\tau^{-1/2}$, a rough estimate for the second term is $O(\tau^{-1/2})$, and the spectral density representation in the range of wave numbers corresponding to large-scale motion is

$$g_e(K,\ R_0) = \frac{K}{2}\int_0^{\infty}\eta\phi_e(\eta,\ R_0)J_0(K\eta)\,d\eta. \tag{25}$$

A. Asymptotic behavior of the spectral density as $K \to \infty$

We are interested in the asymptotic behavior of the integral (25) as $K \to \infty$, which can be calculated if the asymptotic behavior of the function ϕ_e as $\eta \to 0$ is known [16,17]. In accordance with Eq. (18), we assume the following form of this asymptotics:

$$\phi_e(\eta,\ R_0) = D(R_0) - E(R_0)\eta^{\beta(R_0)} + \cdots, \quad \eta \to 0, \quad (26)$$

where E and β are positive functions. According to Ref. [16], it is necessary to expand $\eta\phi_e$ into an asymptotic series in powers of η and integrate it term by term using the equality

$$\int_0^\infty r^s J_\mu(kr)\,dr = \frac{2^s\Gamma\left[\frac{1}{2}(\mu+s+1)\right]}{k^{s+1}\Gamma\left[\frac{1}{2}(\mu-s+1)\right]}, \qquad (27)$$

where Γ is the gamma function. In our case, $\mu = 0$. Formula (27) is applied for all $s > -\mu - 1$ without taking into account the domain of convergence of the integral. [The integral converges in the strip $-\mu - 1 < \mathrm{Re}\, s < \frac{1}{2}$, but the right-hand side of Eq. (27) gives its analytic continuation to the entire complex s plane.] As a result, on the basis of Eq. (26), we get

$$g_e(K,\ R_0) = A(R_0)K^{-1-\beta(R_0)} + \cdots, \quad K \to \infty,$$
$$A = \frac{\beta^2}{\pi}2^{\beta-2}\sin\left(\frac{1}{2}\pi\beta\right)\Gamma^2\left(\frac{1}{2}\beta\right)E. \qquad (28)$$

The first term in asymptotic expansion (26), as formula (27) shows, gives a zero contribution to the asymptotic behavior of the integral. In Eq. (28) $\beta \neq 2$, since for $\beta = 2$, the function $A = 0$ and the asymptotics of the integral is determined by the next term in expansion (26). If we assume a more general form of the asymptotic expansion for the function ϕ_e at zero:

$$\phi_e(\eta,\ R_0) = D(R_0) - E(R_0)\eta^{\beta(R_0)}(-\ln\eta)^{\beta_1(R_0)} + \cdots,$$
$$\eta \to 0, \quad \beta_1 > 0,$$

the leading term in the expansion of the integral is of order $K^{-1-\beta}(\ln K)^{\beta_1}$ [17].

Similarly, it is possible to calculate the asymptotic behavior of the spectral enstrophy flux, which is expressed in terms of the triple correlation by the formula [13]

$$\Pi(k,\ t) = k^6\int_0^\infty \frac{\partial}{\partial r}[r^3 B_{LL,\,L}(r,\ t)]\left[\frac{J_3(kr)}{(kr)^3} - \frac{J_2(kr)}{2(kr)^2}\right]dr. \qquad (29)$$

We are interested in the range of large wave numbers that correspond to small values of r lying in the inertial range, where $B_{LL,\,L}$ has asymptotics (22). The calculation of the asymptotics of the integral (29) as $k \to \infty$ on the basis of asymptotic representation (22) and formula (27) gives

$$\Pi_* = \Pi(k \to \infty,\ t) = \frac{D(R_0)}{t^3},$$

where Π_* is the spectral enstrophy flux at the boundary of the enstrophy dissipation range. Since $D < 2f$, this flux is not equal to the enstrophy dissipation rate χ. Relation (22) can thus be written as

$$B_{LL,\,L} = \frac{\Pi_* r^3}{48}.$$

This formula is valid for both freely decaying and stationary forced turbulence [15]. The difference is that $\Pi_* = \chi$ only in the case of stationary forced turbulence [15].

B. Small-scale motion: Asymptotic behavior of the spectral density as $\varkappa \to 0$

Consider the second integral (24). By virtue of the matching condition (18) for $\tau \to \infty$, the expression in square brackets on the interval of integration uniformly in ζ tends to the quantity

$$\phi_i(\zeta,\ R_0) - D(R_0),$$

which according to (18) has a zero limit as $\zeta \to \infty$. In the limit $\tau \to \infty$ with $1/\varkappa = O(1)$, in view of asymptotic representation (28) and the relation between the dimensionless wave numbers $K = \varkappa\sqrt{\tau}$, the first integral (24) tends to zero, and the second one on the basis of Eq. (18) is

$$g_i(\varkappa,\ R_0) = \frac{\varkappa}{2}\int_0^\infty \zeta[\phi_i(\zeta,\ R_0) - D(R_0)]J_0(\varkappa\zeta)\,d\zeta. \quad (30)$$

This is the representation of the spectral density in the range of wave numbers corresponding to small-scale motion. Let us investigate the properties of the function ϕ_i entering the integral (30). In the enstrophy dissipation region, we introduce the dimensionless structure functions

$$B_{LL}(0,\ t) - B_{LL}(r,\ t) = \frac{\nu}{t}b(\zeta,\ R_0),$$
$$B_{LL,\,L}(r,\ t) = \left(\frac{\nu}{t}\right)^{3/2}h(\zeta,\ R_0).$$

Substituting these expressions into Eq. (19) gives the ordinary differential equation

$$\zeta^3\left(b + \tfrac{1}{2}\zeta b' - 2f\right) = [\zeta^3(h - 2b')]'. \qquad (31)$$

Here we took into account the equality

$$\frac{d}{dt}B_{LL}(0,\ t) = -\nu\langle\omega^2\rangle = -\frac{2\nu f}{t^2}$$

and Eq. (4).

In the inertial range on the basis of Eqs. (21) and (22), the leading-order terms in the asymptotic representations of the functions b and h are

$$b = \frac{D\zeta^2}{16} + \cdots, \qquad (32)$$
$$h = \frac{D\zeta^3}{48} + \cdots, \quad \zeta \to \infty. \qquad (33)$$

We have to make some assumptions about the next terms in these asymptotic expansions. The asymptotics of the right-hand side of Eq. (31) in the leading term is determined by representation (33). On the basis of (32), $b' = O(\zeta)$. It is natural to assume that the next term in the asymptotic expansion for h has the same (or smaller) order. Then Eq. (31) implies that the next term in asymptotic expansion (32) is some function $D_1(R_0)$. Now Eq. (20) can be written as

$$\zeta(\phi_i - D) = \frac{\partial}{\partial\zeta}\zeta\frac{\partial}{\partial\zeta}\frac{1}{\zeta}\frac{\partial}{\partial\zeta}\zeta^2\left(b - \frac{D\zeta^2}{16} - D_1\right), \qquad (34)$$

which after integrating gives the equality

$$\int_0^\infty \zeta(\phi_i - D)\,d\zeta = 0, \qquad (35)$$

which implies that the function $\phi_i - D$ is nonmonotonic and changes sign. Hence, the function Φ, as shown in Fig. 1, has at least two local maxima and minima.

We now consider the integral (30). Using the expansion for the Bessel function at zero

$$J_0(\varkappa\zeta) = 1 - \tfrac{1}{4}(\varkappa\zeta)^2 + O(\varkappa^4) \tag{36}$$

and equality (35), we obtain

$$g_i(\varkappa,\ R_0) = o(\varkappa),\quad \varkappa \to 0. \tag{37}$$

This estimate can be refined by assuming the convergence of the integral

$$B(R_0) = \frac{1}{8}\int_0^\infty \zeta^3(D - \phi_i)\,d\zeta. \tag{38}$$

Then, substituting expansion (36) into the integral (30) and integrating term by term, we have

$$g_i(\varkappa,\ R_0) = B(R_0)\varkappa^3 + \cdots,\quad \varkappa \to 0. \tag{39}$$

It follows from Eq. (34) that the integral (38) converges, provided that the term in asymptotic expansion (32) following D_1 has the form $D_2(R_0)\zeta^{-2}$. In this case, $B = \frac{1}{2}D_2$.

Let us compare the integrals (35) and (38). When ζ changes from zero to infinity, the integrand (35) is positive at first and then changes sign at least once. The integrals of the positive and negative parts of the function add up to zero. Multiplying the integrand by a positive monotonically increasing function ζ^2, in general, e.g., in the case when the integrand changes sign only once, makes the integral negative. Therefore, we can expect that $B > 0$.

C. Spectrum shape for large wave numbers

For the compensated spectrum based on (24)

$$t^2 k\Omega = K g_e(K,\ R_0) + \varkappa g_i(\varkappa,\ R_0). \tag{40}$$

So, proceeding from the assumption of the existence of the inertial range of scales in real space, we came to the conclusion that there is no such range in the spectrum of self-similar two-dimensional turbulence. In other words, turbulent eddies having a size belonging to the inertial range do not contribute to the spectral density, which is the sum of the two functions (40), depending on the outer parameter V and viscosity by the definition of the dimensionless wave numbers K and $\varkappa$.

According to Eq. (40), the compensated spectrum depends nonmonotonically on the wave number and for sufficiently large values of τ in the range of wave numbers for which the conditions $K \gg 1$ and $\varkappa \ll 1$ are simultaneously satisfied must have a minimum. The value of the compensated spectrum at the minimum point for $\tau \to \infty$ tends to zero. The spectral energy density calculated in Refs. [7,10] behaves in accordance with relation (40). At large wave number values, it decreases in a power law with the exponent that diminishes with time, which corresponds to the growth of β from zero to a value approximately equal to two.

In view of asymptotic representations (28) and (37), Eq. (9) in the limit as $\tau \to \infty$ gives the equality

$$\frac{1}{2}\langle\omega^2\rangle = \frac{1}{t^2}\left[\int_0^\infty g_e\,dK + \int_0^\infty g_i\,d\varkappa\right],$$

which now agrees with (4) and (2).

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